UNITS AND DIMENSIONS

PHYSICAL QUANTITIES

All quantities that can be measured are called physical quantities. eg. time, length, mass, force, work done, etc. In physics we study about physical quantities and their inter relationships.

MEASUREMENT

Measurement is the comparison of a quantity with a standard of the same physical quantity.

UNITS

All physical quantities are measured w.r.t. standard magnitude of the same physical quantity and these standards are called UNITS. eg. second, meter, kilogram, etc.

So the four basic properties of units are:—

1. They must be well defined.
2. They should be easily available and reproducible.
3. They should be invariable e.g. step as a unit of length is not invariable.
4. They should be accepted to all.

SET OF FUNDAMENTAL QUANTITIES

A set of physical quanties which are completely independent of each other and all other physical quantities can be expressed in terms of these physical quantities is called Set of Fundamental Quantities.

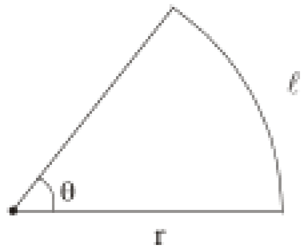
Physical Quantity	**Units(SI)**	**Units(CGS)**	**Notations**
Mass	kg (kilogram)	g	M
Length	m (meter)	cm	L
Time	s (second)	s	T
Temperature	K (kelvin)	°C	θ
Current	A (ampere)	A	I or A
Luminous intensity	cd (candela)	—	cd
Amount of substance	mol	—	mol

Physical Quantity (SI Unit)	**Definition**
Length (m)	The distance travelled by light in vacuum in $\frac{1}{299,792,458}$ second is called 1 metre.
Mass (kg)	The mass of a cylinder made of platinum-iridium alloy kept at International Bureau of Weights and Measures is defined as 1 kilogram.
Time (s)	The second is the duration of 9,192,631,770 periods of

	the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.
Electric Current (A)	If equal currents are maintained in the two parallel infinitely long wires of negligible cross-section, so that the force between them is 2×10^{-7} newton per metre of the wires, the current in any of the wires is called 1 Ampere.
Thermodynamic Temperature (K)	The fraction $\frac{1}{273.16}$ of the thermodynamic temperature of triple point of water is called 1 Kelvin
Luminous Intensity (cd)	1 candela is the luminous intensity of a blackbody of surface area $\frac{1}{600,000}\,m^2$ placed at the temperature of freezing platinum and at a pressure of 101,325 N/m^2, in the direction perpendicular to its surface.
Amount of substance (mole)	The mole is the amount of a substance that contains as many elementary entities as there are number of atoms in 0.012 kg of carbon-12.

There are two supplementary units too:

1. Plane angle (radian) angle = arc / radius
   $\theta = \ell / r$



2. Solid Angle (steradian)

DERIVED PHYSICAL QUANTITIES

The physical quantities those can be expressed in terms of fundamental physical quantities are called derived physical quantities.eg. speed = distance/time.

DIMENSIONS AND DIMENSIONAL FORMULA

All the physical quantities of interest can be derived from the base quantities.

DIMENSION

The power (exponent) of base quantity that enters into the expression of a physical quantity, is called the dimension of the quantity in that base.

To make it clear, consider the physical quantity "force".

Force = mass × acceleration

$$= \text{mass} \times \frac{\text{length / time}}{\text{time}}$$

$$= \text{mass} \times \text{length} \times (\text{time})^{-2}$$

So the dimensions of force are 1 in mass, 1 in length and –2 in time. Thus

$$[\text{Force}] = MLT^{-2}$$

Similarly energy has dimensional formula given by

$$[\text{Energy}] = ML^2T^{-2}$$

i.e. energy has dimensions, 1 in mass, 2 in length and -2 in time.

Such an expression for a physical quantity in terms of base quantities is called dimensional formula.

DIMENSIONAL EQUATION

Whenever the dimension of a physical quantity is equated with its dimensional formula, we get a dimensional equation.

PRINCIPLE OF HOMOGENEITY

According to this principle, we can multiply physical quantities with same or different dimensional formulae at our convenience, however no such rule applies to addition and substraction, where only like physical quantites can only be added or substracted. e.g. If P + Q $\Rightarrow$ P & Q both represent same physical quantity.

Illustration :

Calculate the dimensional formula of energy from the equation $E = \frac{1}{2}mv^2$.

Sol. *Dimensionally,* $E = \text{mass} \times (\text{velocity})^2$.

Since $\frac{1}{2}$ *is a number and has no dimension.*

or, $[E] = M \times \left(\frac{L}{T}\right)^2 = ML^2T^{-2}$.

Illustration :

Kinetic energy of a particle moving along elliptical trajectory is given by $K = \alpha s^2$ *where s is the distance travelled by the particle. Determine dimensions of* α.

Sol. $K = \alpha s^2$

$$[\alpha] = \frac{(ML^2T^{-2})}{(L^2)}$$

$$[\alpha] = M^1 L^0 T^{-2}$$

$$[\alpha] = (M T^{-2})$$

Illustration :

The position of a particle at time t, is given by the equation, $x(t) = \frac{v_0}{\alpha}(1 - e^{-\alpha t})$, *where* v_0 *is a constant and* $\alpha > 0$. *The dimensions of* v_0 & α *are respectively.*

(A) $M^0 L^1 T^0$ & T^{-1}
(B) $M^0 L^1 T^{-1}$ & T
(C*) $M^0 L^1 T^{-1}$ & T^{-1}
(D) $M^1 L^1 T^{-1}$ & LT^{-2}

Sol. $[V_0] = [x]\ [\alpha]$ & $[\alpha]\ [t] = M^0L^0T^0$

$= M^0L^1T^{-1}$ $\quad [\alpha] = M^0L^0T^{-1}$

Illustration :

The distance covered by a particle in time t is going by $x = a + bt + ct^2 + dt^3$ *; find the dimensions of a, b, c and d.*

Sol. *The equation contains five terms. All of them should have the same dimensions. Since [x] = length, each of the remaining four must have the dimension of length.*

Thus, $[a]$ = length = L

$[bt] = L,$ or $[b] = LT^{-1}$

$[ct^2] = L,$ or $[c] = LT^{-2}$

and $[dt^3] = L$ or $[d] = LT^{-3}$

USES OF DIMENSIONAL ANALYSIS

(I) TO CONVERT UNITS OF A PHYSICAL QUANTITY FROM ONE SYSTEM OF UNITS TO ANOTHER :

It is based on the fact that,

Numerical value × unit = constant

So on changing unit, numerical value will also gets changed. If n_1 and n_2 are the numerical values of a given physical quantity and u_1 and u_2 be the units respectively in two different systems of units, then

$$n_1u_1 = n_2u_2$$

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

Illustration

Young's modulus of steel is 19×10^{10} *N/m². Express it in dyne/cm². Here dyne isthe CGS unit of force.*

Sol. *The unit of Young's modulus is N/m².*

This suggest that it has dimensions of $\frac{\text{Force}}{(\text{distance})^2}$.

Thus, $[Y] = \frac{[F]}{L^2} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$.

N/m² is in SI units,

So, $1\ N/m^2 = (1\ kg)(1m)^{-1}\ (1s)^{-2}$

and $1\ dyne/cm^2 = (1g)(1cm)^{-1}\ (1s)^{-2}$

so, $$\frac{1\,N/m^2}{1\,dyne/cm^2} = \left(\frac{1\,kg}{1\,g}\right)\left(\frac{1\,m}{1\,cm}\right)^{-1}\left(\frac{1\,s}{1\,s}\right)^{-2} = 1000 \times \frac{1}{100} \times 1 = 10$$

or, $1\ N/m^2 = 10\ dyne/cm^2$

or, $19 \times 10^{10}\ N/m^2 = 19 \times 10^{11}\ dyne/m^2$.

Illustration :

The dimensional formula for viscosity of fluids is,

$$\eta = M^1L^{-1}T^{-1}$$

Find how many poise (CGS unit of viscosity) is equal to 1 poiseuille (SI unit of viscosity).

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Sol. $\eta = M^1 L^{-1} T^{-1}$

1 *CGS units* $= g\ cm^{-1}\ s^{-1}$

1 *SI units* $= kg\ m^{-1}\ s^{-1}$

$= 1000\ g\ (100\ cm)^{-1}\ s^{-1}$

$= 10\ g\ cm^{-1}\ s^{-1}$

Thus, 1 Poiseuilli = 10 poise

Illustration :

A calorie is a unit of heat or energy and it equals about 4.2 J, where $1\ J = 1\ kg\ m^2/s^2$. *Suppose we employ a system of units in which the unit of mass equals α kg, the unit of length equals β metre, the unit of time is γ second. Show that a calorie has a magnitude* $4.2\ \alpha^{-1}\beta^{-2}\gamma^2$ *in terms of the new units.*

Sol. $1\ cal = 4.2\ kg\ m^2 s^{-2}$

SI	New system
$n_1 = 4.2$	$n_2 = ?$
$M_1 = 1\ kg$	$M_2 = \alpha\ kg$
$L_1 = 1\ m$	$L_2 = \beta$ metre
$T_1 = 1\ s$	$T_2 = \gamma$ second

Dimensional formula of energy is $[ML^2T^{-2}]$

Comparing with $[M^aL^bT^c]$, *we find that* $a = 1,\ b = 2,\ c = -2$

Now, $$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

$$= 4.2\left[\frac{1\ kg}{\alpha\ kg}\right]^1\left[\frac{1\ m}{\beta\ m}\right]^2\left[\frac{1\ s}{\gamma\ s}\right]^{-2} = 4.2\ \alpha^{-1}\beta^{-2}\gamma^2$$

(II) TO CHECK THE DIMENSIONAL CORRECTNESS OF A GIVEN PHYSICAL RELATION:

It is based on principle of homogeneity, which states that a given physical relation is dimensionally correct if the dimensions of the various terms on either side of the relation are the same.

(i) Powers are dimensionless

(ii) $\sin\theta$, e^{θ},, $\cos\theta$, $\log\theta$ gives dimensionless value and in above expression θ is dimensionless

(iii) We can add or subtract quantity having same dimensions.

Illustration :

Let us check the dimensional correctness of the relation $v = u + at$.

Here 'u' represents the initial velocity, 'v' represents the final velocity, 'a' the uniform acceleration and 't' the time.

Dimensional formula of 'u' is $[M^0LT^{-1}]$

Dimensional formula of 'v' is $[M^0LT^{-1}]$

Dimensional formula of 'at' is $[M^0LT^{-2}][T] = [M^0LT^{-1}]$

Here dimensions of every term in the given physical relation are the same, hence the given physical relation is dimensionally correct.

Illustration :

Let us check the dimensional correctness of the relation

$$x = ut + \frac{1}{2}at^2$$

Here 'u' represents the initial velocity, 'a' the uniform acceleration, 'x' the displacement and 't' the time.

Sol. $[x] = L$

$$[ut] = \text{velocity} \times \text{time} = \frac{\text{length}}{\text{time}} \times \text{time} = L$$

$$\left[\frac{1}{2}at^2\right] = [at^2] = \text{accelecration} \times (\text{time})^2$$

$(\because \frac{1}{2}$ *is a number hence dimentionless)*

$$= \frac{\text{velocity}}{\text{time}} \times (\text{time})^2 = \frac{\text{length/tim e}}{\text{time}} \times (\text{time})^2 = L$$

Thus, the equation is correct as far as the dimensions are concerned.

(III) TO ESTABLISH A RELATION BETWEEN DIFFERENT PHYSICAL QUANTITIES :

If we know the various factors on which a physical quantity depends, then we can find a relation among different factors by using principle of homogeneity.

Illustration :

Let us find an expression for the time period t of a simple pendulum. The time period t may depend upon (i) mass m of the bob of the pendulum, (ii) length ℓ of pendulum, (iii) acceleration due to gravity g at the place where the pendulum is suspended.

Sol. *Let (i)* $t \propto m^a$ *(ii)* $t \propto \ell^b$ *(iii)* $t \propto g^c$

Combining all the three factors, we get

$$t \propto m^a \ell^b g^c \qquad or \qquad t = Km^a \ell^b g^c$$

where K is a dimensionless constant of proportionality.

Writing down the dimensions on either side of equation (i), we get

$$[T] = [M^a][L^b][LT^{-2}]^c = [M^a L^{b+c} T^{-2c}]$$

Comparing dimensions, $a = 0,\ b + c = 0,\ -2c = 1$

$\therefore \quad a = 0,\ c = -1/2,\ b = 1/2$

From equation (i) $t = Km^0 \ell^{1/2} g^{-1/2}$ *or* $t = K\left(\frac{\ell}{g}\right)^{1/2} = K\sqrt{\frac{\ell}{g}}$

Illustration :

When a solid sphere moves through a liquid, the liquid opposes the motion with a force F. The magnitude of F depends on the coefficient of viscosity η of the liquid, the radius r of the sphere and the speed v of the sphere. Assuming that F is proportional to different powers of these quantities, guess a formula for F using the method of dimensions.

Sol. *Suppose the formula is* $F = k\,\eta^a\, r^b\, v^c$

Then, $MLT^{-2} = [ML^{-1}\,T^{-1}]^a\, L^b \left(\frac{L}{T}\right)^c$

$= M^a\, L^{-a+b+c}\, T^{-a-c}$

Equating the exponents of M, L and T from both sides,

$a = 1$

$-a + b + c = 1$

$-a - c = -2$

Solving these, $a = 1$, $b = 1$ *and* $c = 1$

Thus, the formula for F is $F = k\eta r v$.

Illustration :

If P is the pressure of a gas and ρ is its density, then find the dimension of velocity in terms of P and ρ.

(A) $P^{1/2}\rho^{-1/2}$ (B) $P^{1/2}\rho^{1/2}$ (C) $P^{-1/2}\rho^{1/2}$ (D) $P^{-1/2}\rho^{-1/2}$

[Sol. $v \propto P^a\,\rho^b$

$v = kP^a\,\rho^b$

$[LT^{-1}] = [ML^{-1}T^{-2}]^a\,[ML^{-3}]^b$ *(Comparing dimensions)*

$a = \frac{1}{2},\ \ b = -\frac{1}{2} \qquad \Rightarrow \qquad [V] = [P^{1/2}\rho^{-1/2}]$

UNITS AND DIMENSIONS OF SOME PHYSICAL QUANTITIES

Quantity	SI Unit	Dimensional Formula
Density	kg/m^3	M/L^3
Force	Newton (N)	ML/T^2
Work	Joule (J)(=N-m)	ML^2/T^2
Energy	Joule(J)	ML^2/T^2
Power	Watt (W) (=J/s)	ML^2/T^3
Momentum	kg-m/s	ML/T
Gravitational constant	$N\text{-}m^2/kg^2$	L^3/MT^2
Angular velocity	radian/s	T^{-1}
Angular acceleration	$radian/s^2$	T^{-2}
Angular momentum	$kg\text{-}m^2/s$	ML^2/T
Moment of inertia	$kg\text{-}m^2$	ML^2
Torque	N-m	ML^2/T^2
Angular frequency	radian/s	T^{-1}
Frequency	Hertz (Hz)	T^{-1}
Period	s	T
Surface Tension	N/m	M/T^2
Coefficient of viscosity	$N\text{-}s/m^2$	M/LT
Wavelength	m	L
Intensity of wave	W/m^2	M/T^3

Temperature	kelvin (K)	K
Specific heat capacity	J/(kg-K)	L^2/T^2K
Stefan's constant	$W/(m^2–K^4)$	M/T^3K^4
Heat	J	ML^2/T^2
Thermal conductivity	W/(m-K)	ML/T^3K
Current density	A/m^2	I/L^2
Electrical conductivity	$1/\Omega$-m(=mho/m)	I^2T^3/ML^3
Electric dipole moment	C-m	LIT
Electric field	V/m (=N/C)	ML/IT^3
Potential (voltage)	volt (V) (=J/C)	ML^2/IT^3
Electric flux	V-m	ML^3/IT^3
Capacitance	farad (F)	I^2T^4/ML^2
Electromotive force	volt (V)	ML^2/IT^3
Resistance	ohm (Ω)	ML^2/I^2T^3
Permittivity of space	$C^2/N\text{-}m^2$ (=F/m)	I^2T^4/ML^3
Permeability of space	N/A^2	ML/I^2T^2
Magnetic field	Tesla (T) (= Wb/m^2)	M/IT^2
Magnetic flux	Weber (Wb)	ML^2/IT^2
Magnetic dipole moment	N-m/T	IL^2
Inductance	Henry (H)	ML^2/I^2T^2

LIMITATIONS OF DIMENSIONAL ANALYSIS

(i) Dimension does not depend on the magnitude. Due to this reason the equation $x = ut + at^2$ is also dimensionally correct. Thus, a dimensionally correct equation need not be actually correct.
(ii) The numerical constants having no dimensions connot be deduced by the method of dimensions.
(iii) This method is applicable only if relation is of product type. It fails in the case of exponential and trigonometric relations.

SI Prefixes : The magnitudes of physical quantites vary over a wide range. The mass of an electron is 9.1×10^{-31} kg and that of our earth is about 6×10^{24} kg. Standard prefixes for certain power of 10. Table shows these prefixes :

Power of 10	Prefix	Symbol
12	tera	T
9	giga	G
6	mega	M
3	kilo	k
2	hecto	h
1	deka	da
–1	deci	d

–2	centi	c
–3	milli	m
–6	micro	μ
–9	nano	n
–12	pico	p
–15	femto	f

ORDER-OF MAGNITUDE CALCULATIONS

If value of phycal quantity P satisfy

$0.5\times10^{x} < P \le 5 \times 10^{x}$

x is an integer

x is called order of magnitude

Illustration :

The diameter of the sun is expressed as 13.9×10^{9} m. Find the order of magnitude of the diameter ?

Sol. *Diameter = 13.9×10^{9} m*

Diameter = 1.39×10^{10} m

order of magnitude is 10.

SYMBOLS AND THERE USUAL MEANINGS

The scientific group in Greece used following symbols.

θ	Theta
α	Alpha
β	Beta
γ	Gamma
δ	Delta
Δ	Delta
μ	Mu
λ	Lambda
ω, Ω	Omega
π	Pi
ϕ, φ	Phi
ε	epsilon

ψ	Psi
ρ	Roh
ν	Nu
η	Eta
σ	Sigma
τ	Tau
κ	Kappa
χ	chi
≅	Approximately equal to

Solved Examples

Q.1 Find the dimensional formulae of follwoing quantities :

(a) The surface tension S,

(b) The thermal conductivity k and

(c) The coefficient of vescosity η.

Some equation involving these quntities are

$$S = \frac{\rho g r h}{2} \qquad Q = k\frac{A(\theta_2 - \theta_1)t}{d} \qquad \text{and} \qquad F = -\eta A \frac{v_2 - v_1}{x_2 - x_1};$$

where the symbols have their usual meanings. (ρ - density, g - acceleration due to gravity, r - radius, h - height, A - area, θ_1 & θ_2 - temperatures, t - time, d - thickness, v_1 & v_2 - velocities, x_1 & x_2 - positions.)

Sol. (a) $S = \frac{\rho g r h}{2}$

or $[S] = [\rho]\,[g]L^2 = \frac{M}{L^3}\cdot\frac{L}{T^2}\cdot L^2 = MT^{-2}$.

(b) $Q = k\,\frac{A(\theta_2 - \theta_1)t}{d}$

or $k = \frac{Qd}{A(\theta_2 - \theta_1)t}$.

Here, Q is the heat energy having dimension ML^2T^{-2}, $\theta_2 - \theta_1$ is temperature, A is area, d is thickness and t is time. Thus,

$$[k] = \frac{ML^2T^{-2}}{L^2KT} = MLT^{-3}K^{-1}.$$

(d) $F = -h A\frac{v_2 - v_1}{x_2 - x_1}$

or $MLT^{-2} = [\eta]L^2\,\frac{L/T}{L} = [\eta]\,\frac{L^2}{T}$

or, $[\eta] = ML^{-1}T^{-1}$.

Q.2 Suppose $A = B^nC^m$, where A has dimensions LT, B has dimensions L^2T^{-1}, and C has dimensions LT^2. Then the exponents n and m have the values:

(A) 2/3; 1/3 (B) 2; 3 (C) 4/5; -1/5 (D*) 1/5; 3/5

(E) 1/2; 1/2

Sol. $LT = [L^2T^{-1}]^n\,[LT^2]^m$

$LT = L^{2n+m}T^{2m-n}$

$2n + m = 1$ ....(i)

$-n + 2m = 1$ ....(ii)

On sovling $n = \frac{1}{5}, m = \frac{3}{5}$

Q.3 If energy (E), velocity (V) and time (T) are chosen as the fundamental quantities, then the dimensions of surface tension will be. (Surface tension = force / length)

(A) $E V^{-2}T^{-1}$ (B) $E V^{-1}T^{-2}$ (C) $E^{-2} V^{-1}T^{-3}$ (D*) $E V^{-2}T^{-2}$

Sol. [surface tension] = [force/length] = $M^1L^0T^{-2}$

suppose [surface tension] = $E^a V^b T^c$

$\therefore M^1L^0T^{-2} = [M^1L^2T^{-2}]^a [L^1T^{-1}]^b [T]^c$

Matching dimensions of M $\Rightarrow a = 1$

Matching dimensions of L $\Rightarrow 2a + b = 0 \Rightarrow b = -2$

Matching dimensions of T $\Rightarrow -2a - b + c = -2 \Rightarrow c = -2$

$\therefore$ [surface tension] = $EV^{-2} T^{-2}$

Q.4 Given that $\ln(\alpha/p\beta) = \alpha z/K_B\theta$ where p is pressure, z is distance, K_B is Boltzmann constant and θ is temperature, the dimensions of β are (useful formula Energy = $K_B \times$ temperature)

(A) $L^0M^0 T^0$ (B) $L^1 M^{-1} T^2$ (C*) $L^2 M^0 T^0$ (D) $L^{-1} M^1 T^{-2}$

Sol. $\ell n\left(\frac{\alpha}{p\beta}\right) = \frac{\alpha z}{k_B\theta}$

$[\alpha z] = [k_\beta\theta]$ Also $[\alpha] = [p\beta]$

$[p\,\beta\,z] = [k_\beta\theta]$

$$[\beta] = \frac{(k_\beta\theta)}{(p\,z)} = \frac{ML^2T^{-2}K^{-1}K}{ML^{-1}T^{-2}L} = L^2$$

Q.5 The SI and CGS units of energy are joule and erg respectively. How many ergs are equal to one joule ?

Sol. Dimensionally, Energy = mass × (velocity)^2

$$= \text{mass} \times \left(\frac{length}{time}\right)^2 = ML^2T^{-2}$$

Thus, 1 joule = (1 kg) (1 m)^2 (1 s)$^{-2}$

and 1 erg = (1 g) (1 cm)^2 (1 s)$^{-2}$

$$\frac{1\,\text{joule}}{1\,\text{erg}} = \left(\frac{1\,kg}{1\,g}\right)\left(\frac{1\,m}{1\,cm}\right)^2\left(\frac{1\,s}{1\,s}\right)^{-2}$$

$$= \left(\frac{1000g}{1g}\right)\left(\frac{1000\,cm}{1\,cm}\right)^2 = 1000 \times 10000 = 10^7.$$

So 1 joule = 10^7 erg.

Q.6 Young's modulus of steel is 19×10^{10} N/m^2. Express it in dyne/cm^2. Here dyne is the CGS unit of force.

Sol. The unit of Young's modulus is N/m^2.

This suggests that it has dimensions of $\frac{Force}{(area)}$.

Thus, $[Y] = \left[\frac{F}{L^2}\right] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$.

N/s^2 is in SI units.

So, 1 N/m^2 = (1kg) (1m)$^{-1}$ (1s)$^{-2}$

and 1 dyne/cm^2 = (1g) (1cm)$^{-1}$ (1s)$^{-2}$

So, $\frac{1N/m^2}{1dyne/cm^2} = \left(\frac{1\ kg}{1\ g}\right) = \left(\frac{1\ m}{1\ cm}\right)^{-1}\left(\frac{1s}{1s}\right)^{-2}$

$$= 1000 \times \frac{1}{100} \times 1 = 10$$

or, 1 N/s^2 = 10 dyne/cm^2

or, 19×10^{10} N/m^2 = 19×10^{11} dyne/cm^2

Q.7 If velcoity, time and force were chosen as basic quantities, find the dimensions of mass.

Sol. Dimensionally, Force = mass × acceleration

$$\text{Force} = \text{mass} \times \frac{velocity}{time}$$

or, $\text{mass} = \frac{Force \times time}{velocity}$

or, [mass] = FTV^{-1}

Q.8 The dimension of $\frac{a}{b}$ in the equation $P = \frac{a-t^2}{bx}$ where P is pressure, x is distance and t is time are _________?

Sol. $P = \frac{a-t^2}{bx}$

$\Rightarrow$ $Pbx = a - t^2$

$\Rightarrow$ $[Pbx] = [a] = [T^2]$

or $[b] = \frac{[T^2]}{[P]\,[x]} = \frac{[T^2]}{[ML^{-1}T^{-2}]\,[L]} = [M^{-1}T^4]$

$\therefore$ $\left[\frac{a}{b}\right] = \frac{[T^2]}{[M^{-1}T^4]} = [MT^{-2}]$

VECTOR

Introduction

History :

There were many scientists involved in developing vectors over several years. Some of them were Caspar Wessel (1745-1818), Jean Robert Argand (1768- 1822), Carl Friedrich Gauss (1777-1855),William Rowan Hamilton. Finally it was Hermann Grassmann (1809-1877) who developed and expanded on the concept of vectors.

Because of this development, physical quantities were divided into two categories

(a) vectors (b) scalars.

Scalars:

Physical quantities are completely specified by a number, which can be positive or negative, and a unit of measure.

An example is temperature, a quantity that is specified completely by a number and a unit (as in 21 °C or 70 F).

Examples of scalar quantities are : mass, distance, average speed, instantaneous speed, energy, pressure, temperature, density, charge etc.

Scalars follow the ordinary arithmetical laws of addition, subtraction, multiplication, and division.

Vectors :

Any physical quantity which in order to be completely specified, requires not only a number and a unit but also a direction in space and also follows laws of vector algebra are known as vectors.

The laws of additioin and substraction of vectors require geometrical techniques.

Examples of vectors are: displacement, velocity, acceleration, force, momentum, impulse, angular momentum, torque, angular impulse, gravitational field, electric field, magnetic field. If we are talking about directions, we should know how to write direction in English. In English sentences, we write directions as 30^o east of north, 60^o south of east, north-east, south-west etc.

Representation of Vector

The representation of vector will be complete if it gives us direction and magnitude.

Symbolic form:

$\vec{v}, \vec{a}, \vec{F}, \vec{s}$ used to separate a vector quantity from scalar quantities (u, i, m). Some books also represent vector by bold letter. Such as **s, v, a**.

Graphical form:

A vector is represented by a directed straight line, having the magnitude and direction of the quantity represented by it.

e.g. if we want to represent a force of 5 N acting 45° N of E

Step:

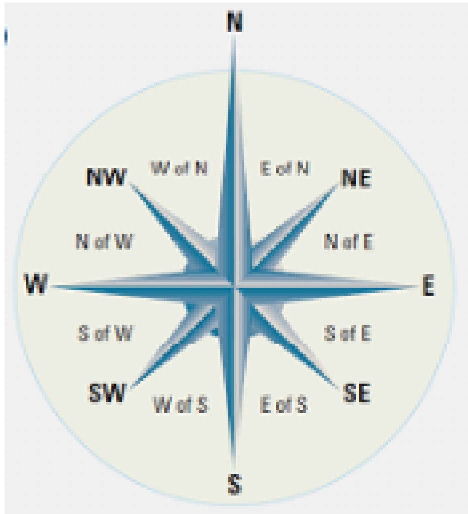
(i) We choose suitable co-ordinates system.

(ii) We choose a convenient scale like $1\text{cm} \equiv 1$ N.

(iii) for the direction 45° N of E, the reference is East then turning 45° towards North.

(iv) We draw a line of length equal in magnitude and in the direction of vector to the chosen quantity.

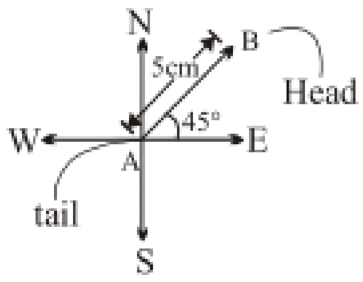
(v) We put arrow in the direction of vector.



$\overrightarrow{AB}$

Magnitude of vector:

$\left|\overrightarrow{AB}\right| = 5\text{N}$

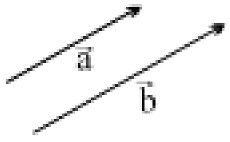


$1 \text{ cm} \equiv 1$ N

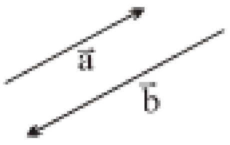
By definition magnitude of a vector quantity is scalar and is always positive.

Terminology of Vectors

Parallel vector: If two vectors have same direction, they are parallel to each other. They may be located anywhere in the space.

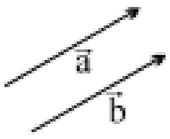


Antiparallel vectors: When two vectors are in opposite direction they are said to be antiparallel vectors.



Equality of vectors: When two vectors have equal magnitude and are in same direction and represent the same quantity, they are equal.

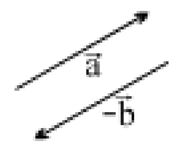
i.e. $\vec{a} = \vec{b}$



Thus when two parallel vectors have same magnitude they are equal. (Their initial point & terminal point may not be same)

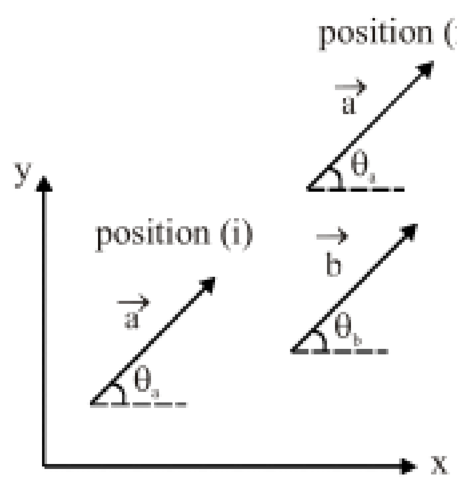
Negative of a vector: When a vector have equal magnitude and is in opposite direction, it is said to be negative vector of the former.

i.e. $\vec{a} = -\vec{b}$ or $\vec{b} = -\vec{a}$



Thus when two antiparallel vectors have same magnitude they are negative of each other.
Vector shifting is allowed without change in their direction.

Let two vectors $\vec{a}$ and $\vec{b}$ are represented as –



Vector shifting is allowed without changing their direction as vector $\vec{a}$ shifted in figure, from position (i) to position (ii).

If $|\vec{a}| = |\vec{b}|$ and $\theta_a = \theta_b$ then both vector are equal vectors.

If $\theta_a = \theta_b$ then both vector are parallel vectors.

Illustration :

The vector $-\vec{A}$ is:

(A) greater than $\vec{A}$ in magnitude *(B) less than $\vec{A}$ in magnitude*

(C) in the same direction as $\vec{A}$ *(D*) in the direction opposite to $\vec{A}$*

(E) perpendicular to $\vec{A}$

Sol. *(D)*

ANGLE BETWEEN VECTORS

Angle between two vectors $\vec{a}$ and $\vec{b}$ is defined as the smaller angle between the tail's or the head's of the two vectors.

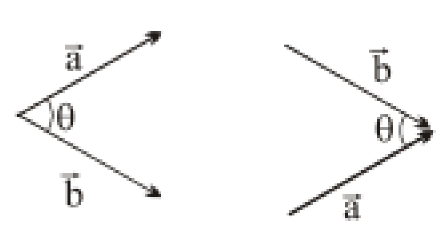
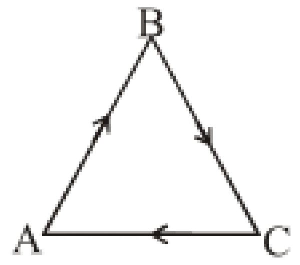


Illustration :

What is the angle between?

(i) $\overline{AB}$ *and* $\overline{AC}$ ***Ans.*** *60°*

(ii) $\overline{AB}$ *and* $\overline{BC}$ ***Ans.*** *120°*



Equilateral triangle

Laws of addition and subtraction of vectors:

Triangle rule of addition:

Steps for adding two vectors representing same physical quantity by triangle law :

(i) Keep vectors such that tail of one vector coincides with head of other.
(ii) Join tail of first to head of the other by a line with arrow at head of the second.
(iii) This new vector is the sum of two vectors. (also called resultant)

For Example :

To add $\overrightarrow{CD}$ to $\overrightarrow{AB}$, place tail of $\overrightarrow{CD}$ at the head of $\overrightarrow{AB}$. The sum is the vector from the tail of $\overrightarrow{AB}$ to head of $\overrightarrow{CD}$ i.e. $\overrightarrow{AD}$.

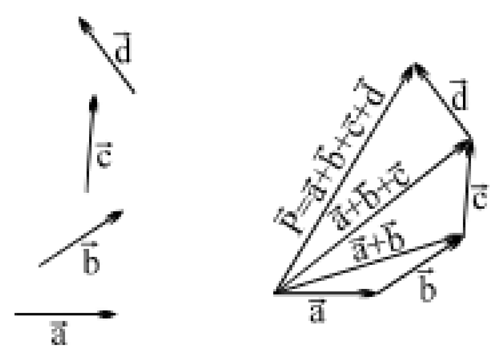
$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$$

Graphical Representation :

Polygon Law of addition:

This law is used for adding more than two vectors. This is extension of triangle law of addition. We keep on arranging vectors s.t. tail of next vector lies on head of former.

When we connect the tail of first vector to head of last we get resultant of all the vectors.



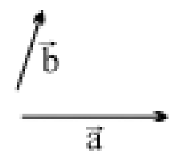
$\vec{P} = \left(\left((\vec{a}+\vec{b})+\vec{c}\right)+\vec{d}\right) = \left((\vec{c}+\vec{a})+\vec{b}\right)+\vec{d}$ [**Associative Law**]

Parallelogram law of addition:

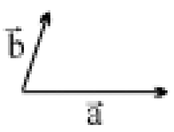
Steps:

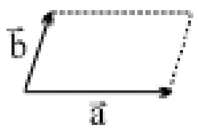
(i) Keep two vectors such that their tails coincide.
(ii) Draw parallel vectors to both of them considering both of them as sides of a parallelogram.
(iii) Then the diagonal drawn from the point where tails coincide represents the sum of two

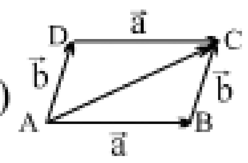
vectors, with its tail at point of coincidence of the two vectors.



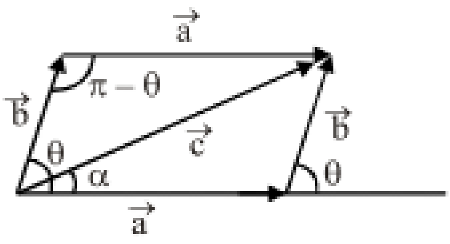
Steps :

(i) 

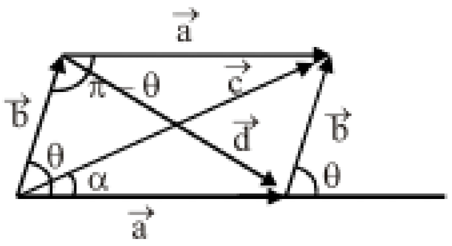
(ii) 

(iii) 

$\overrightarrow{AC} = \vec{a} + \vec{b}$



$\overrightarrow{AC} = \vec{a} + \vec{b}$ and $\overrightarrow{AC} = \vec{b} + \vec{a}$ thus $\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$ [**Commutative Law**]

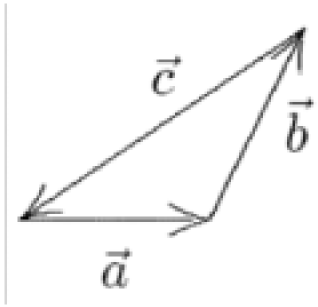


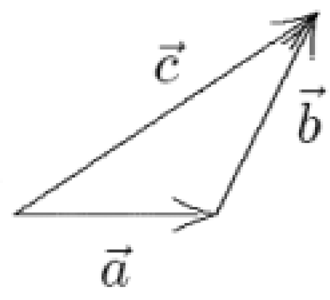
So, the full parallelogram would look like this

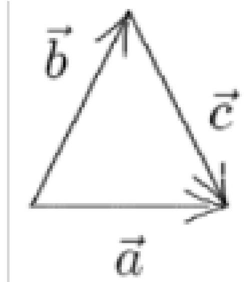
$\vec{c} = \vec{a} + \vec{b}$; $\vec{d} = \vec{a} - \vec{b}$

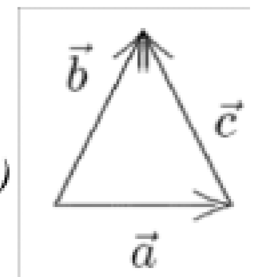
Illustration :

The vectors $\vec{a}$, $\vec{b}$ *and* $\vec{c}$ *are related by* $\vec{c} = \vec{a} + \vec{b}$. *Which diagram below illustrates this relationship?*

(A) 

(B)* 

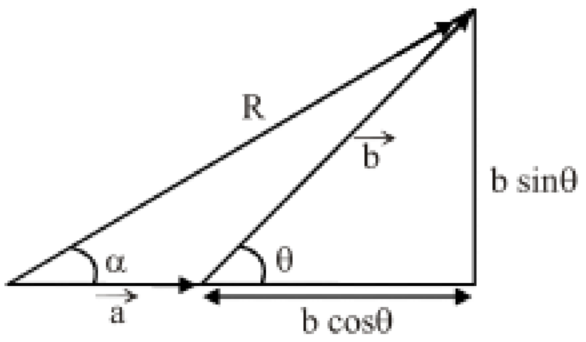
(C) 

(D) 

(E) None of these

Sol. *(B)* $\vec{a} + \vec{b} = \vec{c}$

Derivation of general formula :



$\vec{R} = \vec{a} + \vec{b}$, $|\vec{R}| = R$, $|\vec{a}| = a$, $|\vec{b}| = b$

$R = \sqrt{(a + b\cos\theta)^2 + (b\sin\theta)^2}$

$= \sqrt{a^2 + b^2 + 2ab\cos\theta}$

as in figure angle between $\vec{a}$ and $\vec{R}$ is α.

$\tan\alpha = \dfrac{b\sin\theta}{a + b\cos\theta}$

Illustration :

Two forces P and Q are in ratio P : Q = 1 : 2. If their resultant is at an angle $\tan^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ *to vector P, then angle between P and Q is :*

(A) $\tan^{-1}\left(\dfrac{1}{2}\right)$ (B) 45° (C) 30° (D*) 60°

Sol. $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

$$\frac{\sqrt{3}}{2} = \frac{\sin\theta}{\frac{P}{Q} + \cos\theta} \Rightarrow \frac{\sqrt{3}}{2} = \frac{\sin\theta}{(1/2) + \cos\theta} \Rightarrow \frac{3}{4} = \left(\frac{2\sin\theta}{1 + 2\cos\theta}\right)^2$$

$\Rightarrow \quad 3(1 + 2\cos\theta)^2 = 16\sin^2\theta$

$\Rightarrow \quad 3(1 + 4\cos^2\theta + 4\cos\theta) = 16(1 - \cos^2\theta)$

$\Rightarrow \quad 3 + 12\cos^2\theta + 12\cos\theta = 16 - 16\cos^2\theta$

$\Rightarrow \quad 28\cos^2\theta + 12\cos\theta - 13 = 0 \Rightarrow \quad \cos\theta = 1/2, -0.92$

Important Results:

The angle between $\vec{a}$ and $\vec{b}$ is θ and resultant $\vec{R} = \vec{a} + \vec{b}$.

(i) If $\theta = 0°$ $\Rightarrow \vec{a} \parallel \vec{b}$

then, $|\vec{R}| = |\vec{a}| + |\vec{b}|$

& $|\vec{R}|$ is maximum

(ii) If $\theta = \pi$ $\Rightarrow \vec{a}$ anti $\parallel \vec{b}$

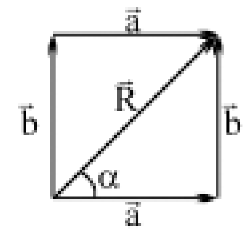
then, $|\vec{R}| = \left||\vec{a}| - |\vec{b}|\right|$

& $|\vec{R}|$ is minimum

(iii) If $\theta = \pi/2 \quad \Rightarrow \vec{a} \perp \vec{b}$

$R = \sqrt{a^2 + b^2}$

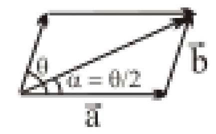
& $\tan\alpha = b/a$ (α is angle with $\vec{a}$)



(iv) If $|\vec{a}| = |\vec{b}| = a$

$|\vec{R}| = 2a\cos\theta/2$

& $\alpha = \theta/2$



(v) If $|\vec{a}| = |\vec{b}| = a$ & $\theta = 2\pi/3$

then $|\vec{R}| = a$

Illustration :

Forces of magnitudes 6N and 4N are acting on the body. Which of the following can be the resultant of the two?

(a) 11 N (b) 2 N (c) 10 N (d) 1 N (e) 8 N (f) 0 N (g) 7 N

Sol. F_{res} *lies between* F_{res} *(max.) and* F_{res} *(min.).*

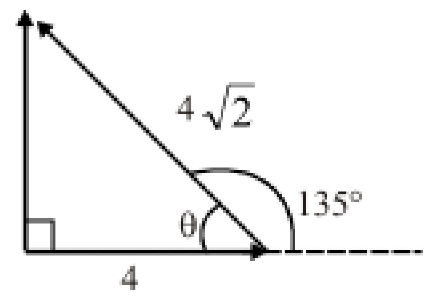
F_{res} *(max.)* $= 10$ *N and* F_{res} *(min.)* $= 2$ *N*

Thus, options b, c, e, and g are correct.

Illustration :

The resultant of two forces of magnitudes 4 N and $4\sqrt{2}$ *N makes 90° with the smaller force. Then angle between those two forces is ?*

Sol.



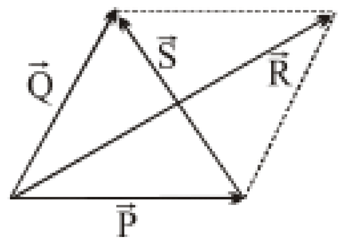
$$\cos\theta = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45°$

angle between forces 4 N and $4\sqrt{2}$ *N is 135°.*

Practice Exercise

Q.1 Two vectors $\vec{P}$ & $\vec{Q}$ are arranged in such a way that they form adjacent sides of a parallelogram as shown in figure



Which of the following options have correct relationship

(A) $\vec{Q} = \vec{R} + \vec{S}$ (B) $\vec{R} = \vec{P} + \vec{Q}$ (C) $\vec{R} = \vec{P} + \vec{S}$ (D) $\vec{S} = \vec{Q} - \vec{P}$

Q.2 Two vectors of 10 units & 5 units make an angle of 120° with each other. Find the magnitude & angle of resultant with vector of 10 unit magnitude.

Answers

Q.1 (B), (D) Q.2 $5\sqrt{3}$, 30°

Multiplication of a vector by a scalar :

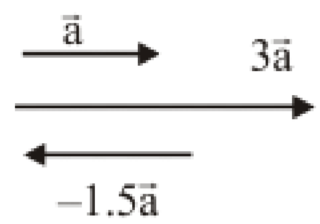
Lets say we have a vector $\vec{a}$ and k is a Scalar. Vector $\vec{b} = k\vec{a}$ is defined as a vector of magnitude |ka|.

If k is a positive then direction of $\vec{b}$ is along $\vec{a}$.

If k is negative then direction of $\vec{b}$ is opposite to $\vec{a}$.

If $|k| > 1 \Rightarrow |\vec{b}| > |\vec{a}|$

$|k| < 1 \Rightarrow |\vec{b}| < |\vec{a}|$



A vector may be multiplied by a pure number or by a scalar.

When a vector is multiplied by a scalar, the new vector may become a different physical quantity for example, when velocity (a vector) is multiplied by time (a scalar) we obtain a displacement (a vector).

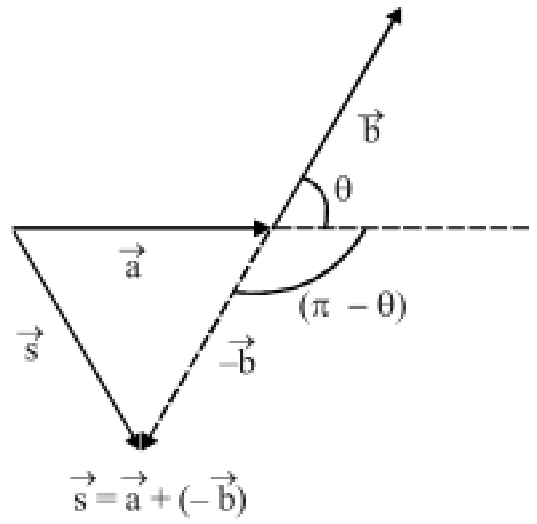
Subtraction of vectors :

To subtract two vectors, reverse the direction of the vector being subtracted and add the inverted vector to the vector from which you are subtracting.

Let say we want to obtain $\vec{a} - \vec{b}$

$$\vec{s} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

i.e. $\vec{a} - \vec{b}$ can be understood as summation of $\vec{a} + (-\vec{b})$



For $\vec{s} = \vec{a} - \vec{b}$

Steps :

(i) Put tail of $\vec{b}$ at head of $\vec{a}$

(ii) Take $-\vec{b}$

(iii) Resultaul vector $\vec{s}$ from tail of $\vec{a}$ to head of $-\vec{b}$.

(iv) Angle between $\vec{a}$ and $\vec{b}$ is θ, then angle between $\vec{a}$ *and* $-\vec{b}$ or between $-\vec{a}$ *and* $\vec{b}$ is (180° – θ).

the angle between $\vec{a}$ and $-\vec{b}$ becomes $\pi - \theta$

so we get $s = \sqrt{(a + b\cos(\pi-\theta))^2 + (b\sin(\pi-\theta))^2}$

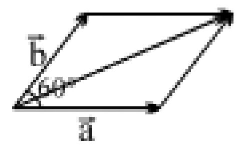
$$= \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

and $\tan\alpha = \dfrac{b\sin(\pi-\theta)}{a + b\cos(\pi-\theta)} = \dfrac{b\sin\theta}{a - b\cos\theta}$

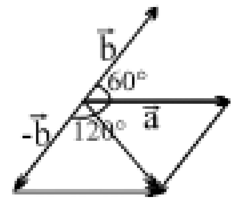
Illustration :

Two vectors of equal magnitude 2 are at an angle of 60° to each other find magnitude of their sum & difference.

Sol. $|\vec{a}+\vec{b}| = \sqrt{2^2 + 2^2 + 2\times2\times2\cos60^\circ} = \sqrt{4+4+4} = 2\sqrt{3}$



$|\vec{a}-\vec{b}| = \sqrt{2^2 + 2^2 + 2\times2\times2\cos120^\circ} = \sqrt{4+4-4} = 2$



Zero vector → *When* $\vec{a} = \vec{b}$ *& if want to find* $\vec{a} - \vec{b}$ *= zero vector. It is a vector with zero magnitude & undefined direction.*

Illustration :

It is given that $\vec{A} + \vec{B} = \vec{C}$ *with* $\vec{A} \perp^{ar} \vec{B}$ *and*

$|\vec{A}| = 10$, $|\vec{C}| = 20$.

Find $|\vec{B}|$ *and angle of* $\vec{C}$ *with* $\vec{A}$

Sol. $\cos\alpha = \dfrac{|\vec{A}|}{|\vec{C}|}$

$\alpha = 60°$ *and* $\theta = 30°$

$\vec{A} + \vec{B} = \vec{C}$

$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2$

$|\vec{B}| = 10\sqrt{3}$

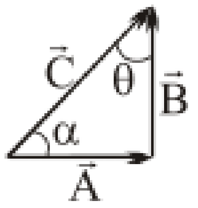


Illustration :

Two vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point and $\vec{C} = \vec{A} + \vec{B}$

(A) If $C^2 = A^2 + B^2$, the angle between vectors $\vec{A}$ and $\vec{B}$ is $90°$*

(B) If $C^2 < A^2 + B^2$, the angle between $\vec{A}$ and $\vec{B}$ is greater than $90°$*

(C) If $C^2 > A^2 + B^2$ then angle between the vectors $\vec{A}$ and $\vec{B}$ is between $0°$ and $90°$.*

(D) If $C = A - B$, angle between $\vec{A}$ and $\vec{B}$ is $180°$.*

Sol. $C^2 = A^2 + B^2 + 2AB \cos\theta$

If $\theta = 90°$

then $C^2 = A^2 + B^2$

if $\theta > 90°$

then $C^2 = A^2 + B^2 + 2AB\cos\theta < A^2 + B^2$

$\therefore$ $\cos\theta$ will be negative

If $\theta < 90°$

then $C^2 = A^2 + B^2 + 2AB\cos\theta > A^2 + B^2$

$\therefore$ If $C = A - B \Rightarrow \theta = 180°$

Unit Vector :

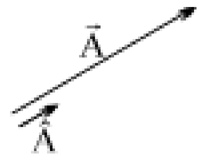
(i) A unit vector is vector of unit length used to specify direction.

(ii) A unit vector is a dimensionless vector with a magnitude of exactly 1.

(iii) Unit vectors are used to specify a direction and have no other physical significance

A unit vector in direction of vector $\vec{A}$ is represented as $\hat{A}$

& $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$



or $\vec{A}$ can be expressed in terms of a unit vector in its direction i.e. $\vec{A} = |\vec{A}|\,\hat{A}$

Unit Vectors along three coordinates axes:

unit vector along x-axis is $\hat{i}$

unit vector along y-axis is $\hat{j}$

unit vector along z-axis is $\hat{k}$

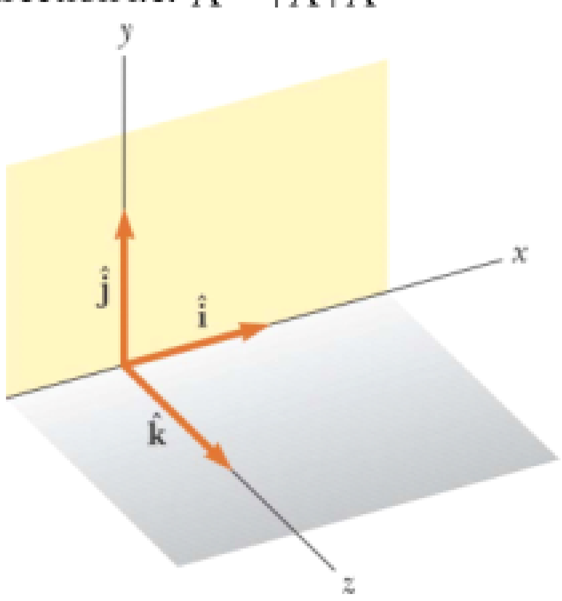


Illustration :

The value of a unit vector in the direction of vector $\vec{A} = 5\hat{i} - 12\hat{j}$ *is* ______*?*

Sol. $|\vec{A}| = \sqrt{5^2 + 12^2} = 13$

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{5\hat{i} - 12\hat{j}}{13}$$

Co-ordinate Systems:

Co-ordinate systems are used to describe the position of a point in space.
Coordinate system consists of :
(i) A fixed reference point called the origin
(ii) Specific axes with scales and labels
(iii) Instructions on how to label a point relative to the origin and the axes

Cartesian Coordinate System :

Also called rectangular coordinate system
x- and y- axes intersect at the origin

Points are labeled (x,y)

In figure, points P and Q are shown as (5, 3) and (–3, 4) respectively.

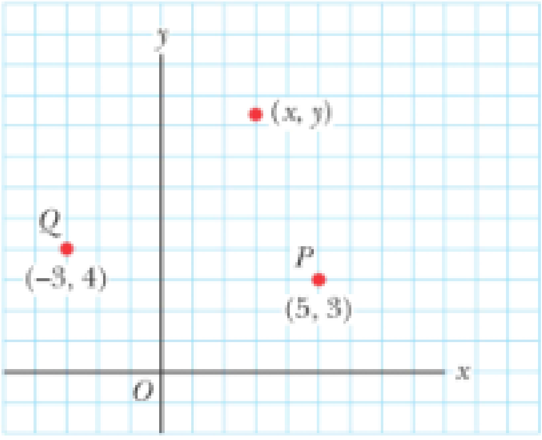


Illustration :

Express the vector in unit - vector notation.

Sol. $2\hat{i}$

Illustration :

Acceleration due to gravity is always downwards. How will you write it vectorially if +Y is (a) downwards (b) upwards.

Sol. *(a) downwards taken as postive Y direcation.*

$$\vec{a} = g\,\hat{j}$$

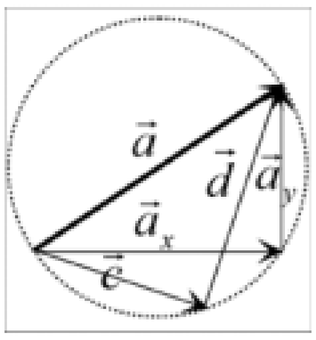
(b) upwards taken as postive Y direcation.

$$\vec{a} = -g\,\hat{j}$$

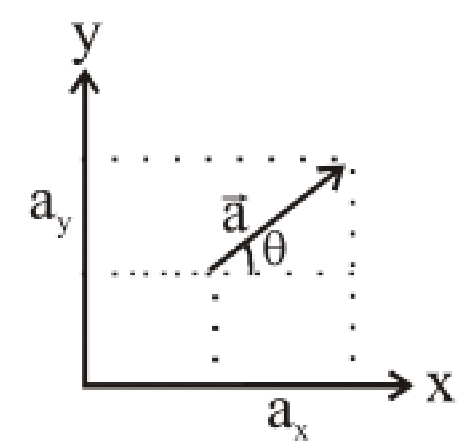
Rectangular Component (resolution) of vectors:

(components means parts)

We can move from tail of $\vec{a}$ to its head via various paths. But, if we move with $\vec{a}$ as the diameter of circle as shown, then the two vectors ($\vec{c}$ & $\vec{d}$ and $\vec{a}_x$ & $\vec{a}_y$) would be perpendicular to each other. Such perpendicular vectors are called rectangular components of $\vec{a}$. But, if



we choose $\vec{a}_x$ & $\vec{a}_y$. Then we can write them in terms of standard unit vectors $\hat{i}$ & $\hat{j}$ respectively. Then we would say that a_x is the component of vector along the x-axis & a_y is the component of vector along y-axis. Now according to triangle law of addition:



$\vec{a} = \vec{a}_x + \vec{a}_y$

So, we write $\vec{a} = a_x(\hat{i}) + a_y(\hat{j}) = a\cos\theta(\hat{i}) + a\sin\theta(\hat{j})$

This is a very convenient form of representing vectors.

$a\cos\theta$ is known as component of $\vec{a}$ along x-axis (a_x)

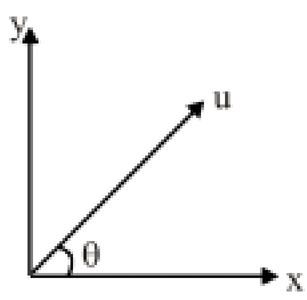
$a\sin\theta$ is known as component of $\vec{a}$ along y-axis (a_y)

This is best form of representing vectors because we can do exact addition and subtraction using simple laws of algebra without needing to draw vectors

Illustration :

A body is thrown from ground making an angle θ with speed u from horizontal. Write its initial velocity in unit vector notation.

Sol.



$$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$$

Results :

1. Unit vector along $\vec{A}$ is $\hat{A}$.

Since, $\vec{A} = |\vec{A}|\hat{A}$

& $\vec{A} = |\vec{A}|(\cos\theta\hat{i} + \sin\theta\hat{j})$

$\hat{A} = \hat{i}\cos\theta + \hat{j}\sin\theta$

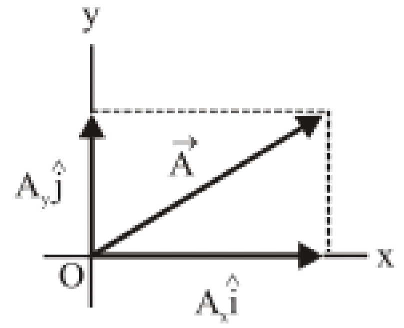
2. If components of a vector along x & y-axis are known, then that vector can be completely represented as

$\vec{A} = A_x\hat{i} + A_y\hat{j}$

3. $|\vec{A}| = \sqrt{A_x^2 + A_y^2}$

4. $\tan\theta = \left(\dfrac{A_y}{A_x}\right)$

θ is angle with x-axis



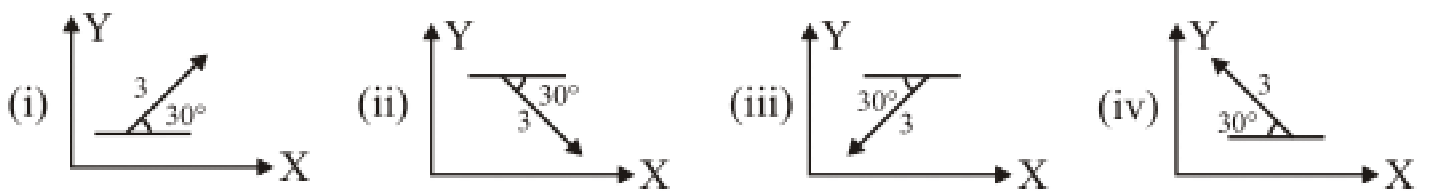
5. The components can be positive or negative and will have the same units as the original vector

6. The signs of the components will depend on the angle

A_x negative, A_y positive	A_x positive, A_y positive
A_x negative, A_y negative	A_x positive, A_y negative

Illustration :

Express the given vector $\vec{A}$ (shown graphically) in unit vector notation.



Sol. *(i)* $\vec{A} = 3(\cos 30\hat{i} + \sin 30\hat{j})$ *(ii)* $\vec{A} = 3(\cos 30\hat{i} - \sin 30\hat{j})$ *(iii)* $\vec{A} = 3(-\cos 30\hat{i} - \sin 30\hat{j})$

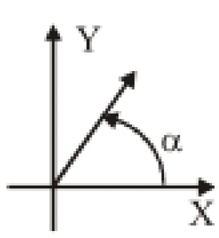
(iv) $\vec{A} = 3(-\cos 30\hat{i} + \sin 30\hat{j})$

Illustration :

Angle made by the vector with positive direction of X-axis

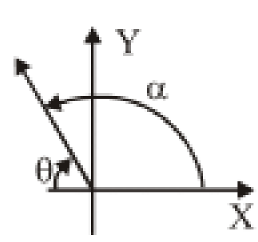
Find angle (α) made by the vectors with the positive direction of X-axis.

$\vec{a} = \hat{i} + \sqrt{3}\hat{j}$



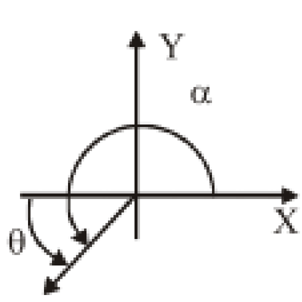
$tan\alpha = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|$, $\therefore \alpha = 60°$ *Ans.* $\alpha = 60°$

$\vec{a} = -\hat{i} + \sqrt{3}\hat{j}$



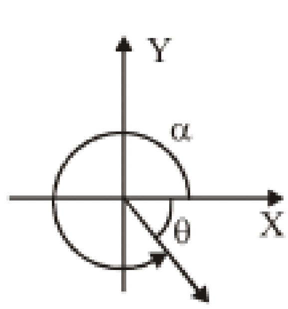
$tan\theta = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|$, $\therefore \theta = 60°,\ \alpha = 180° - \theta$ *Ans.* $\alpha = 120°$

$\vec{a} = -\hat{i} - \sqrt{3}\hat{j}$



$tan\theta = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|$, $\therefore \theta = 60°,\ \alpha = 180° + \theta$ *Ans.* $\alpha = 240°$

$\vec{a} = \hat{i} - \sqrt{3}\hat{j}$

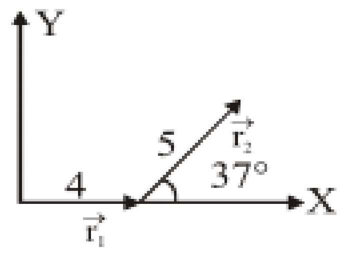


$tan\theta = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|$, $\therefore \theta = 60°,\ \alpha = 360° - \theta$ or $\alpha = -\theta$

Ans. $\alpha = 300°, -60°$

Illustration :

A man moves in the following manner in X-Y plane. Find the magnitude of displacement of man from origin as shown in figure.



Sol.

$$\vec{r}_1 = 4\hat{i}$$

$$\vec{r}_2 = 5\cos 37°\hat{i} + 5\sin 37°\hat{j}$$

$$\vec{r}_2 = 4\hat{i} + 3\hat{j}$$

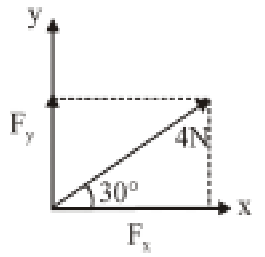
Resultant vector

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

$$= 8\hat{i} + 3\hat{j}$$

Illustration :

A force of 4 N is acting at an angle of 30^o to the horizontal. Find its component along axeses.



Sol $F_y = 4\sin 30° = 2\ N$

$F_x = 4\cos 30° = 2\sqrt{3}\ N$

Illustration :

Find a vector $\vec{F}$ of magnitude 50N parallel to $-4\hat{i}+3\hat{j}$.

Sol. $\vec{F} = 50 \times \dfrac{(-4\hat{i}+3\hat{j})}{5} = -40\hat{i} + 30\hat{j}$

Illustration :

A particle is moving with speed 6 m/s along the direction of $2\hat{i}+2\hat{j}-\hat{k}$ find the velocity vector of a particle?

Sol. $\vec{v} = |\vec{v}|\,\hat{v} = 6\dfrac{(2\hat{i}+2\hat{j}-\hat{k})}{3} = 2(2\hat{i}+2\hat{j}-\hat{k})$

Illustration :

Find a vector of magnitude twice of $12\hat{i}-5\hat{j}$ and anti-parallel to $3\hat{i}-4\hat{j}$

Sol. Suppose $\vec{a} = 12\hat{i}-5\hat{j}$, $\vec{b} = 3\hat{i}-4\hat{j}$ required vector $\vec{r}$.

$$\vec{r} = 2|\vec{a}|(-\hat{b})$$

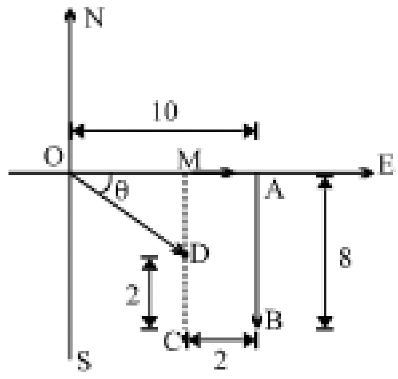
$$= 2\times(13)\left(\frac{-3\hat{i}+4\hat{j}}{5}\right)$$

$$= \frac{26}{5}\left(-3\hat{i} + 4\hat{j}\right)$$

Illustration :

An insect crawls 10 m towards east, turns to its right, crawls 8 m, and again turns to its right, Now crawling a distance of 2 m it turns to its right and stop after moving 2 m more. Find its net displacement.

Sol. Net displacement is $\overrightarrow{OD}$



$$\overrightarrow{OD} = \overrightarrow{OM} + \overrightarrow{MD}$$

$$OD = \sqrt{(OM)^2 + (MD)^2}$$

$$= \sqrt{(OA - BC)^2 + (AB - CD)^2}$$

$$= \sqrt{8^2 + 6^2}$$

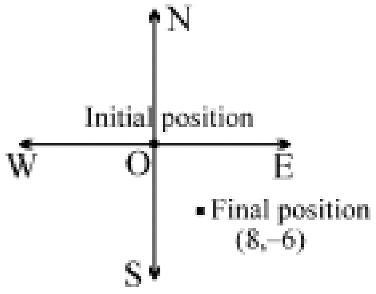
$$OD = 10\ m$$

In $\Delta\, OMD$

$$\tan\theta = \frac{MD}{OM} = \frac{6}{8} = \frac{3}{4}$$

Displacement is 10 m at $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 37°$ S of E

Alternate



$$\vec{r}_i = o\,\hat{i} + o\,\hat{j}\ ,\ \vec{r}_f = 8\,\hat{i} - 6\hat{j}$$

$$\vec{d} = \vec{r}_f - \vec{r}_i = (8\,\hat{i} - 6\hat{j}) - (0\,\hat{i} + 0\,\hat{j})$$

$$\vec{d} = 8\hat{i} - 6\,\hat{j}$$

Concept of Equilibrium :

Equilibrium means net force acting on body is zero.

i.e. $\Sigma\,\vec{F} = \vec{0}$ (for translational equilibrium)

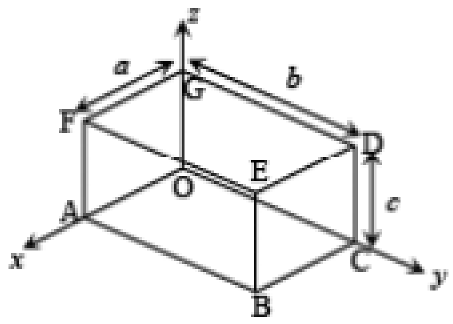
State of rest or moving with constant velocity are the condition of translational equilibrium.

Illustration :

Three forces ($\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$) are acting on a particle moving vertically up with constant speed. Two forces $\vec{F}_1 = -10\,\hat{j}$ N, and $\vec{F}_2 = -6\,\hat{i} + 8\,\hat{j}$ N are acting on particle respectively find $\vec{F}_3$.

Sol. $\Sigma\vec{F} = \vec{0}$ i.e

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$$



To find $\overrightarrow{OE}$, You can move in the path $OA \rightarrow AB \rightarrow BE$

Find the vector $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$,, $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$, $\overrightarrow{OG}$

$\therefore \overrightarrow{OE} = a\hat{i} + b\hat{j} + c\hat{k}$ similarly, $\overrightarrow{OA} = a\hat{i}$ $\quad \overrightarrow{OB} = a\hat{i} + b\hat{j}$

$$\overrightarrow{OC} = b\hat{j} \quad \overrightarrow{OD} = b\hat{j} + c\hat{k}$$

$$\overrightarrow{OG} = c\hat{k} \quad \overrightarrow{OF} = a\hat{i} + c\hat{k}$$

All these vectors are called position vectors as it defines the position of a particle in space with respect to origin.

Illustration :

Can you tell the co-ordinates of A, B, C, D, E, F & G.

Sol. *(A) (a, 0, 0) B(a, b, 0) C(0, b, 0) (D)(0, b, c) E(a, b, c) F(a,, 0, c) G(0, 0, c)*

Illustration :

What is the magnitude of the diagonal $\overrightarrow{OE}$ *?*

Sol. $\overrightarrow{OE} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\left|\overrightarrow{OE}\right| = \sqrt{a^2 + b^2 + c^2}$$

Illustration :

What is the magnitude of the $\overrightarrow{GB}$ *?*

Sol. $\overrightarrow{GB} = \overrightarrow{OB} - \overrightarrow{OG}$

$= a\hat{i} + b\hat{j} - c\hat{k}$

Thus, from now on we shall understand that $|\overrightarrow{GB}| = \sqrt{a^2 + b^2 + c^2}$

Displacement Vector:

Suppose that a particle displaces from position $\vec{r}_1$ to $\vec{r}_2$, then the particle's displacement is given by:

$$\vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$
$$= -\left(-6\hat{i} - 2\hat{j}\right)$$
$$\vec{F}_3 = 6\hat{i} + 2\hat{j}\ N$$

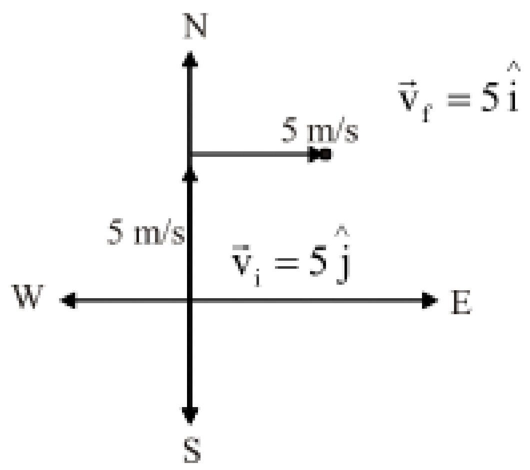
Subtraction of vectors (applications):

To find change in velocity.

Illustration :

A car is moving northwards at a constant speed of 5 m/s. it makes a right turn and continues to move with a constant speed of 5 m/s. Find the magnitude of change in velocity.

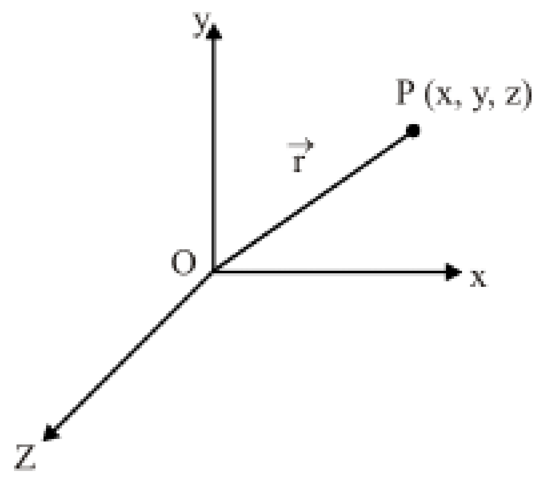
Sol.



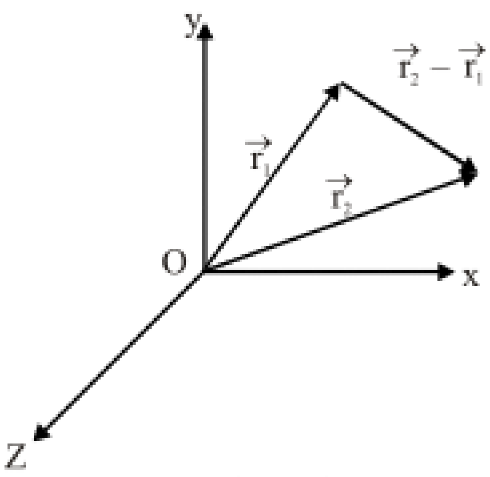
$$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$$
$$\Delta\vec{v} = 5\hat{i} - 5\hat{j}$$
$$|\Delta\vec{v}| = \sqrt{5^2 + 5^2}$$
$$|\Delta\vec{v}| = 5\sqrt{2}\ m/s$$

Position Vector:

A general way of locating a particle is with a position vector $\vec{r}$, which is a vector that extends from a reference point to the particle. This reference point is usually the origin.



In unit vector notation: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$



$\vec{s}$ = p.v. of $\vec{r}_2$ – p.v. of $\vec{r}_1$ $= \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$

$\therefore\ \vec{s} = \left(\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}\right)$

Illustration :

Find the displacement of tip of hour hand of the clock between 1 pm to 5 pm where the length of hour hand is 6 cm.

Sol. $2 \times 6\sin 60°$

$= 2 \times \left(6 \times \frac{\sqrt{3}}{2}\right) = 6\sqrt{3}\ cm$

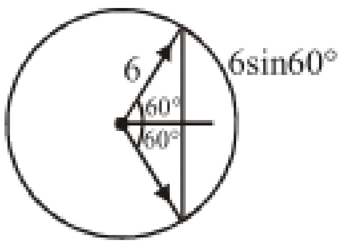


Illustration :

The position vectors of two balls are given by $\vec{r}_1 = 2\hat{i} + 7\hat{j}$, $\vec{r}_2 = -2\hat{i} + 4\hat{j}$ What will be the distance between the two balls?

Sol. $\vec{r} = \vec{r}_2 - \vec{r}_1$

$\vec{r} = -4\hat{i} - 3\hat{j}$

Distance $= \sqrt{4^2 + 3^2} = 5$

***Now**, so far we have learnt about different kinds of vectors. They were velocity vector, displacement vector, position vector etc. Similarly, we can have acceleration vector as well. One would have studied earlier about basic laws of kinematics.*

$v = u + at$

$S = ut + \frac{1}{2}at^2$

In the above equation, other than time, rests physical quantities are vectors. So, a better representation would be:

$\vec{v} = \vec{u} + \vec{a}t$

$\vec{S} = \vec{u}t + \frac{1}{2}\vec{a}t^2$

And we should remember that the above equations are valid when the acceleration is constant. If

the acceleration is zero, then equation will be:

$$\vec{S} = \vec{u}t \text{ or } \vec{r}_2 - \vec{r}_1 = \vec{u}t$$

Illustration :

A particle has initial velocity of $2\hat{i}$ and has constant acceleration of $3\hat{j}$. Find its displacement and velocity after 3s. If initially, the particle is located at $3\hat{i} + 4\hat{j}$, find its final location.

Sol. $\vec{v} = \vec{u} + \vec{a}t = 2\hat{i} + (3\hat{j})3 = (2\hat{i} + 9\hat{j})$

$$\vec{s}_2 - \vec{s}_1 = \vec{u}t + \frac{1}{2}\vec{a}t^2 = (2\hat{i})(3) + \frac{1}{2}(3\hat{j})(9)$$

$$\vec{s}_2 = 6\hat{i} + \frac{27}{2}\hat{j} + 3\hat{i} + 4\hat{j}$$

$$= \left(9\hat{i} + \frac{35}{2}\hat{j}\right)$$

Illustration :

A particle who has a constant speed of 50 m/s. it moves along a straight line from A(2,1) to B(9,25). Find its velocity vector. If at initial instant of time it is located at (2, 0); find its final location after 3 s.

Sol. $\overrightarrow{AB} = \vec{B} - \vec{A} = 7\hat{i} + 24\hat{j}$

$$\vec{v} = (\hat{AB})|\vec{v}| = \frac{(7\hat{i} + 24\hat{j})}{25} \times 50 = 14\hat{i} + 48\hat{j}$$

$$\vec{s}_2 - \vec{s}_1 = \vec{v}(3), \text{ and } \vec{s}_1 = 2\hat{i}$$

$$\vec{s}_2 = (42\hat{i} + 144\hat{j}) + 2\hat{i} = (44\hat{i} + 144\hat{j})$$

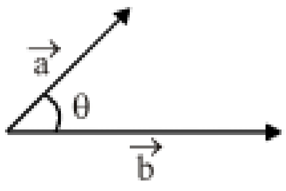
Product of Vectors

There are two ways in which vectors can be multiplied :

(1) Scalar product or dot product.

(2) Vector product or cross product

(1) Scalar product



$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|\cos\theta$, where θ is angle between $\vec{a}$ and $\vec{b}$

It is called a scalar product because its product is a scalar quantity.

(i) $\hat{i}\cdot\hat{i} = |\hat{i}|.|\hat{i}|\cos 0° = 1 = \hat{j}.\hat{j} = \hat{k}.\hat{k}$

$\hat{i}\cdot\hat{j} = |\hat{i}|.|\hat{j}|\cos 90° = 0 = \hat{j}.\hat{k} = \hat{k}.\hat{i}$

(ii) $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$: used to test orthogonality. (means two vectors are mutually perpendicular to each other).

(iii) If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ $\quad \vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$a_x b_x + a_y b_y + a_z b_z = \sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}\cos\theta$$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ (this is used to find the angle between two vectors).

(iv) Work done by vector $\vec{F}$ is defined as $w = \vec{F}\cdot\vec{d}$

Where $\vec{d}$ displacement vector.

Illustration :

Find the dot product of vectors $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k},\ \vec{b} = -\hat{i} + 3\hat{j} + \hat{k}$

Sol. $\vec{a}.\vec{b} = -2 - 9 + 1 = -10$

Illustration :

If $\hat{i} + 2\hat{j} + n\hat{k}$ *is perpendicular to* $4\hat{i} + 2\hat{j} + 2\hat{k}$ *then find the value of n ?*

Sol. $(1\hat{i} + 2\hat{j} + n\hat{k}).(4\hat{i} + 2\hat{j} + 2\hat{k}) = 0$

$4 + 4 + n(2) = 0 \Rightarrow n = -4$

Illustration :

A force of $(-3\hat{i} - \hat{j} + 2\hat{k})$ *N displaced the body from a point (4, - 3, - 5) m to a point (- 1, 4, 3) m in a straight line. Find the work done by the force.*

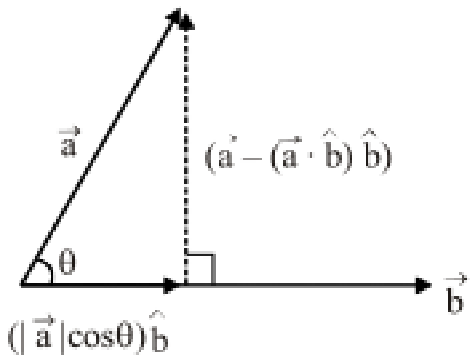
Sol. $\vec{A} = (4\hat{i} - 3\hat{j} - 5\hat{k}),\ \vec{B} = -\hat{i} + 4\hat{j} + 3\hat{k}$

$\overrightarrow{AB} = \vec{B} - \vec{A} = (-5\hat{i} + 7\hat{j} + 8\hat{k})$

$W = \vec{F}\cdot\overrightarrow{AB} = (-3\hat{i} - \hat{j} + 2\hat{k})\cdot(-5\hat{i} + 7\hat{j} + 8\hat{k}) = 24 joule.$

Projection (Component) of Vector : $\vec{a}$ on $\vec{b}$

Find the vector component of (i) $\vec{a}$ along $\vec{b}$ (ii) $\vec{a} \perp$ to $\vec{b}$



(i) *Magnitude of Projection of* $\vec{a}$ on $\vec{b} = |\vec{a}|\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \vec{a}.\hat{b}$

Thus, vector Component of vector $\vec{a}$ *in the direction of* $\vec{b}$ *is* $(\vec{a}.\hat{b})\hat{b}$

(ii) *Vector component of vector* $\vec{a}$ *in the perpendicular direction of* $\vec{b}$ *is* $(\vec{a} - (\vec{a}.\hat{b})\hat{b})$.

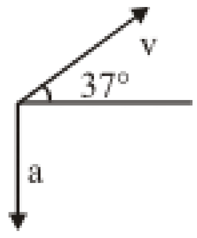
Illustration :

For vector $\vec{a} = 3\hat{i} + 3\hat{j} - 2\hat{k}$, *what are its component along X- axis, Y-axis & Z-axis.*

Sol. *3 along X-axis, 3 along Y-axis, - 2 along Z-axis.*

Illustration :

Just after firing, a bullet is found to move at an angle of 37º to the horizontal. Its acceleration is 10 m/s² downwards. Find the component of its acceleration in the direction of its velocity. ***Ans: - 6 m/s²***



Ans. $\hat{v} = \frac{1}{5}(4\hat{i} + 3\hat{j})$

$\vec{a} = -g\hat{j}$

vector Component of vector $\vec{a}$ *in the direction of* $\vec{v}$ *is* $(\vec{a}.\hat{v})\hat{v}$. *Thus* $-6\ m/s^2$.

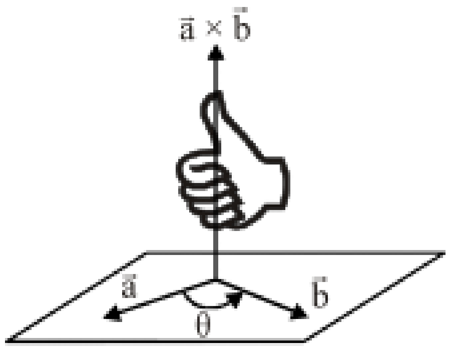
(2) Cross product or vector product of two vectors :

The vector or cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \times \vec{b} = ab\ \sin\theta\ \hat{n}$$

The product is defined to be a vector of magnitude ab sin θ that points in the direction of the unit vector $\hat{n}$ normal (perpendicular) to the plane of $\vec{a}$ and $\vec{b}$. The angle θ is the smaller angle between the vector.

The direction of $\hat{n}$ is still ambiguous. This ambiguity is removed by using a convention called the right-hand rule. Curl the fingers of your right hand and stick out your thumb as if you were hitch-hiking as in figure. The sense of rotation of the fingers should be from the first vector $\vec{a}$ to the second vector $\vec{b}$ through the smaller angle between them. The thumb indicates the direction $\hat{n}$.



Because of the right-hand rule, the order of the vectors in the cross product is important. The vector product is **noncommutative** :

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$

Properties of vector (or Cross Product)

(i) Cross product non-commutative :

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \text{ i.e. } \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$

(ii) Follows distributive law :

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

(iii) If a and b are any vectors, and m is any real number (positive or negative) then

$$(m\,\vec{a}) \times \vec{b} = m\,(\vec{a} \times \vec{b}) = \vec{a} \times (m\,\vec{b})$$

(iv) Does not follow associative law :

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

(v) $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

and $\hat{i} \times \hat{i} = 0, \hat{j} \times \hat{j} = 0, \hat{k} \times \hat{k} = 0$

(vi) $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$

$= (a_y b_z - a_z b_y)\,\hat{i} + (a_z b_x - a_x b_z)\,\hat{j} + (a_x b_y - a_y b_x)\,\hat{k}$

(vii) Angle between two vectors

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

$$\sin\theta = |\vec{a} \times \vec{b}| / |\vec{a}|\,|\vec{b}|$$

Let

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}$

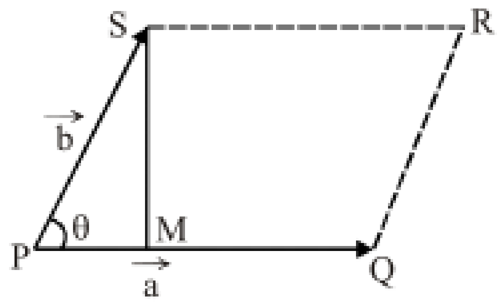
$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\,,\ |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$$\sin\theta = \frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

(ix) The cross product of a vector with itself is a NULL vector *i.e.,*

$$\vec{a} \times \vec{a} = |\vec{a}|\,|\vec{a}| \sin 0^\circ\, \hat{n} = \vec{0}$$

(x) The cross product of two vectors represents the area of the parallelogram formed by them, (Figure., shows a parallelogram *PQRS* whose adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ are represented by vectors $\vec{a}$ and $\vec{b}$ respectively.



Now, area of parallelogram = $|\overrightarrow{PQ}| \cdot |\overrightarrow{SM}| = |\overrightarrow{PQ}| \cdot |\overrightarrow{PS}| \sin\theta = |\vec{a}| \cdot |\vec{b}| \sin\theta = |\vec{a} \times \vec{b}|$ hence cross product of two vectors represents the area of parallelogram formed by it. It is worth noting that area vector $\vec{a} \times \vec{b}$ acts along the perpendicular to the plane of two vectors $\vec{a}$ and $\vec{b}$.

Unit Vector Perpendicular to two given vectors

Let be a unit vector perpendicular to two (non-zero) vectors a, b and positive for right handed rotation from a to b and θ be the angle between the vectors a, b then

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\,\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

Thus we get $= \vec{a} \times \vec{b} \,/\, |\vec{a} \times \vec{b}| = \hat{n}$.

Illustration :

Prove that

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0$$

Sol. $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$$

$$= \vec{a} \times \vec{b} - \vec{c} \times \vec{a} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} + \vec{c} \times \vec{a} - \vec{b} \times \vec{c} = 0$$

Illustration :

Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ if

(i) $\vec{a} = 3\hat{k} + 4\hat{j},\ \vec{b} = \hat{i} + \hat{j} - \hat{k}$

(ii) $\vec{a} = (2, -1, 1)\ ;\ \vec{b} = (3, 4, -1)$

Sol. *(i)* $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -7\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 4 & 3 \end{vmatrix} = 7\hat{i} - 3\hat{j} + 4\hat{k}$$

Thus $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(ii) $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 11\hat{k}$$

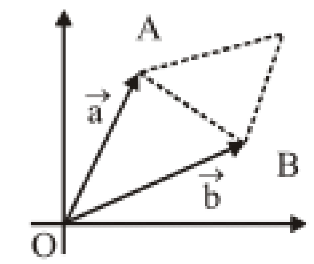
Illustration :

If $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ *and* $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

(i) find the magnitude of $\vec{a} \times \vec{b}$

(ii) find a unit vector perpendicular to both a and b.

(iii) find the cosine and sine of the angle between the vectors a and b



Sol. *(i)* $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8\hat{i} - 8\hat{j} - 8\hat{k}$

$\therefore$ *Magnitude of* $\vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|$

$$\sqrt{(8)^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$$

(ii) $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \dfrac{8\hat{i} - 8\hat{j} - 8\hat{k}}{8\sqrt{3}} = \dfrac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

There are two unit vectors perpendicular to both $\vec{a}$ *and* $\vec{b}$ *they are*

$$\pm\hat{n} = \pm\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

(iii) To find $\cos\theta$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = (3\hat{i} + \hat{j} + 2\hat{k}).(2\hat{i} - 2\hat{j} + 4\hat{k})$$

$$= (3)(2) + (1)(-2) + 2(4) = 12$$

$$|\vec{a}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{24}$$

$$\cos\theta = \vec{a} \cdot \vec{b} / |\vec{a}||\vec{b}| = \frac{12}{\sqrt{14}\sqrt{24}} = \frac{12}{\sqrt{2}\sqrt{7}.2\sqrt{2}\sqrt{3}} = \sqrt{\frac{3}{7}}$$

$$\text{Also, } \sin\theta = |\vec{a} \times \vec{b}| / |\vec{a}||\vec{b}| = \frac{8\sqrt{3}}{\sqrt{14}\sqrt{24}} = \frac{2}{\sqrt{7}}$$

$$\text{Also, } \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{3}{7}} = \sqrt{\frac{4}{7}} = \frac{2}{\sqrt{7}}$$

Illustration :

The vectors from origin to the points A and B are $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ *and* $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$ *respectively. Find the area of :*

(i) the triangle OAB

(ii) the parallelogram formed by $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$ *as adjacent sides.*

Sol. Given $\overrightarrow{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\overrightarrow{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \quad (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$= (12 - 2)\hat{i} - (-6 - 4)\hat{j} + (3 + 12)\hat{k}$

$= 10\hat{i} + 10\hat{j} + 15\hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$

(i) Area of $OAB = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2} \cdot 5\sqrt{17}$ sq. units

$= \frac{5}{2}\sqrt{17}$ sq. units

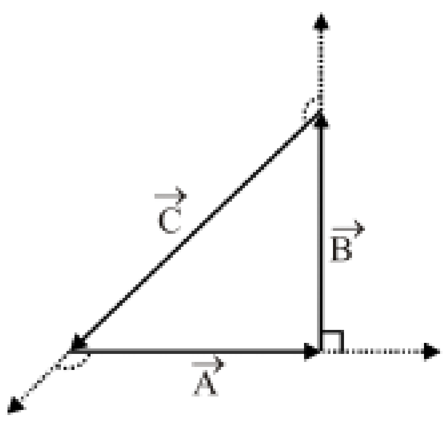
(ii) Area of parallelogram formed by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as adjacent sides $= |\vec{a} \times \vec{b}| = 5\sqrt{17}$ sq. units.

Solved Example

Q.1 Given that $\vec{A}+\vec{B}+\vec{C}=\vec{0}$, but of three two are equal in magnitude and the magnitude of third vector is $\sqrt{2}$ times that of either of the vectors two having equal magnitude. Then the angles between vectors are given by -

(A) 30°, 60°, 90° (B) 45°, 45°, 90° (C) 45°, 60°, 90° (D) 90°, 135°, 135°

Sol. (D) From polygon law, there vectors having summation zero, should from a closed polygon (triangle). Since the two vectors are having same magnitude and the third vector is $\sqrt{2}$ times that of either of two having equal magnitude. i.e. the triangle should be right angled triangle.



Angle between A and B is 90°

Angle between B and C is 135°

Angle between A and C is 135°

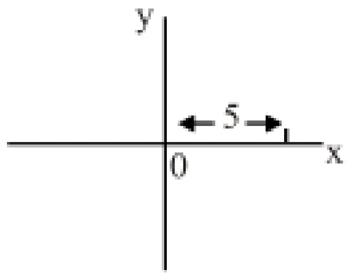
Q.2 If a particle moves 5m in + x-direction. Show the displacement of the particle-

(A) $5\hat{j}$ (B) $5\hat{i}$ (C) $-5\hat{j}$ (D) $5\hat{k}$

Sol. Magnitude of vector = 5

Unit vector in + x direction is $\hat{i}$

displacement = $5\hat{i}$



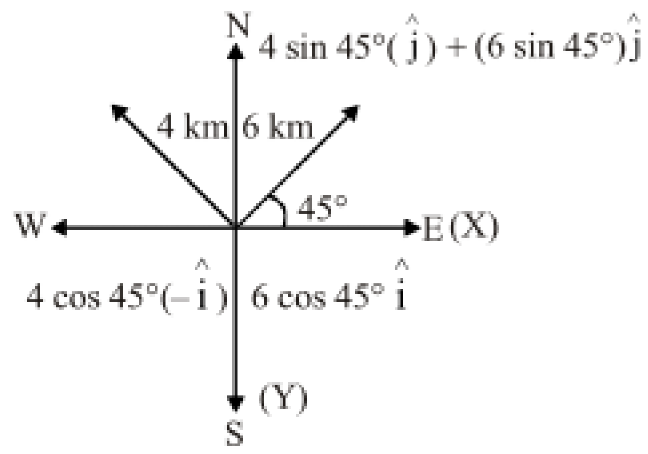
Hence correct answer is (B).

Q.3 A car travels 6 km towards north at an angle of 45° to the east then travels distnace of 4 km towards north at an angle of 135° to the east. How far is its final position due east and due north? How far is the point from the strating point ? What angle does the straight line joining its initial and final position makes with the east ? What is the total distnace travelled by the car ?

Sol. Net movement along X - direction

$$= (6-4)\cos 45^\circ\, \hat{i}$$

$$= 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}\ \text{km}$$



Net movement along Y – direction

$$= (6+4)\sin 45^\circ\, \hat{j}$$

$$= 10 \times \frac{1}{\sqrt{2}} = 5\sqrt{2}\ \text{km}$$

Net movement form starting point (Total distance travelled)

$$= 6 + 4 = 10\ \text{km}$$

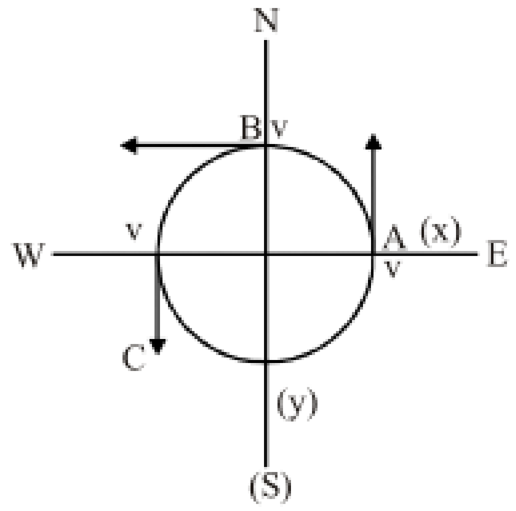
Angle which makes with the east direction

$$\tan\theta = \frac{\text{Y – component}}{\text{X – component}}$$

$$= \frac{5\sqrt{2}}{\sqrt{2}}$$

$$\theta = \tan^{-1}(5)$$

Q.4 A body is moving with uniform speed v on a horizontal circle in anticlockwise direction from A as shown in figure. What is the change in velocity in (a) half revolution (b) first quarter revolution.



Sol. Change in velocity in half revolution

$$\Delta\vec{v} = \vec{v}_C - \vec{v}_A$$

$$= v(-\hat{j}) - v(\hat{j})$$

$$\Delta \vec{v} = -2v\hat{j}$$

$|\Delta \vec{v}| = 2v$ directed towards negative y-axis

change in velocity in first quarter revolution

$$\Delta \vec{v} = \vec{v}_B - \vec{v}_A$$

$$= v(-\hat{i}) - v(\hat{j})$$

$$= -v(\hat{i} + \hat{j})$$

$|\Delta \vec{v}| = \sqrt{2}\,v$ and directed towards south-west.

Q.5 The sum of the magnitudes of two forces acting at a point is 18 and the magnitude of their Resultant is 12. if the resultant is at 90° with the force of smaller magnitude, what are the magnitudes of forces?

(A) 12, 5 (B) 14, 4 (C) 5, 13 (D) 10, 8

Sol. Let P be the smaller force then it is given that

$P + Q = 18$ ....(1)

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = 12$ ....(2)

$Q\sin\theta / P + Q\cos\theta = \tan\phi = \tan 90° = \infty$

$P + Q\cos\theta = 0$ ....(3)

Substituting the value of P

$Q(1 - \cos\theta) = 18$ ....(4)

and subtracting square of equation (2) from (1)

$2PQ[1 - \cos\theta] = 18^2 - 12^2 = 180$ ....(5)

Dividing equation (5) by (4)

$2P = 10$ i.e. $P = 5$, so $Q = 13$

So the magnitude of forces are (5 and 13)

Hence correct answer is (C)

Q.6 Let $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$; $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$. Evaluate

(i) $|\vec{a}|$; $|\vec{b}|$

(ii) $\vec{a}.\vec{b}$

(iii) the angle between the vectors $\vec{a}$ and $\vec{b}$

(iv) the projection of $\vec{a}$ on $\vec{b}$

(v) the projection of $\vec{b}$ on $\vec{a}$

(vi) area of the ΔAOB where O is origin

Sol. Give $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$

(i) $|\vec{a}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14}$

$|\vec{b}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{1+9+16} = \sqrt{26}$

(ii) $\vec{a} \cdot \vec{b} = 2(-1) + 3 \times 3 + (-1)(4) = 3$

(iii) The angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{2\sqrt{91}}$$

(iv) the projection of $\vec{a}$ on $\vec{b} = |\vec{a}|\cos\theta$

$$\sqrt{14} \times \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{26}}$$

(v) The projection of $\vec{b}$ on $\vec{a} = |\vec{b}|\cos\theta$

$$\sqrt{26} \times \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{14}}$$

(vi) Area of $\Delta AOB = \frac{1}{2}|\vec{a}||\vec{b}|\sin\theta$

Now $\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{3}{2\sqrt{91}}\right)^2$

$$= 1 - \frac{9}{364} = \frac{355}{364}$$

Area of $\Delta AOB = \frac{1}{2}\sqrt{14}\sqrt{26} \cdot \sqrt{\frac{355}{364}}$

$$= \frac{\sqrt{355}}{2} = 9.42 \text{ sq. unit approx.}$$

Q.7 The torque of force $\vec{F} = -3\hat{i} + \hat{j} + 5\hat{k}$ acting at the point $\vec{r} = 7\hat{i} + 3\hat{j} + \hat{k}$ is ______?

(A) $14\hat{i} - 38\hat{j} + 16\hat{k}$ (B) $4\hat{i} + 4\hat{j} + 6\hat{k}$ (C) $-21\hat{i} + 4\hat{j} + 4\hat{k}$ (D) $-14\hat{i} + 34\hat{j} - 16\hat{k}$

Sol. (A) The torque is defined as $\vec{\tau} = \vec{r} \times \vec{F}$

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} + \hat{j}\begin{vmatrix} 1 & 7 \\ 5 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 7 & 3 \\ -3 & 1 \end{vmatrix}$$

$$= \hat{i}(15-1) + \hat{j}(-3-35) + \hat{k}(7-(-9))$$

$$= 14\hat{i} - 38\hat{j} + 16\hat{k}$$

Thus the answer is (A)

Q.8 The vectors from origin to the points A and B are $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$ respectively. Find the area of :

(i) the triangle OAB

(ii) the parallelogram formed by OA and OB as adjacent sides.

Sol. Given $\overrightarrow{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

and $\overrightarrow{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \quad (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (12-2)\hat{i} - (-6-4)\hat{j} - (3+12)\hat{k}$$

$$= 10\hat{i} + 10\hat{j} + 15\hat{k}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$$

(i) Area of $\Delta OAB = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}.5\sqrt{17}$ sq. units

$= \frac{5}{2}\sqrt{17}$ sq. units

(ii) Area of parallelogram fromed by OA and OB as adjacent sides $= |\vec{a} \times \vec{b}| = 5\sqrt{17}$ sq. units.

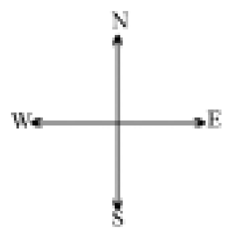
Q.9 A buoy is attached to three tugboats by three ropes. The tugboats are engaged in a tug–of–war. One tugboat pulls west on the buoy with a force $\vec{F}_1$ of magnitude 1000 N. The second tugboat pulls south on the buoy with a force $\vec{F}_2$ of magnitude 2000 N. The third tugboat pulls northeast (that is, half way between north and east), with a force $\vec{F}_3$ of magnitude 2000 N.

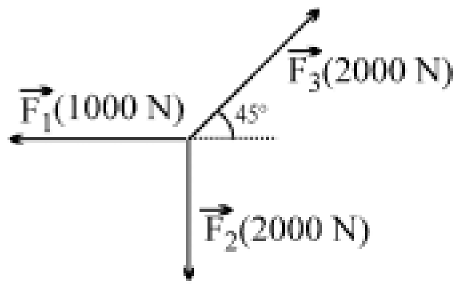
(a) Draw a diagram of forces acting on the buoy to represent this situation.

(b) Express each force in unit vector form ($\hat{i}$, $\hat{j}$).

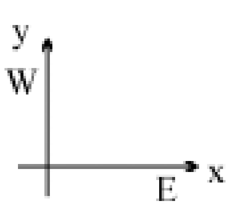
(c) Calculate the magnitude of the resultant force.

Sol. (a)





(b)



$$\vec{F}_1 = -1000\,\hat{i}\ \text{(N)},$$

$$\vec{F}_2 = -2000\ \hat{j}\,\text{(N)},$$

$$\vec{F}_3 = 2000(\cos 45°\ \hat{i} + \sin 45°\,\hat{j})\ \text{(N)} = 1000\sqrt{2}\ \hat{i} + 1000\sqrt{2}\ \hat{j}$$

(c) $\vec{F}_{resultant} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 1000(\sqrt{2}-1)\hat{i} - 1000\ (2-\sqrt{2})\hat{j}$ N

$F_x = 1000\ (\sqrt{2} - 1)$ N

$F_y = -1000(2 - \sqrt{2}\,)$ N

$$|\vec{F}| = \sqrt{F_x^{\ 2} + F_y^{\ 2}} = 1000(\sqrt{2} - 1) \times \sqrt{3} = 1000\ (\sqrt{2} - 1)\ \sqrt{3}\ \text{N}$$

Trigonometry

Angle

The angle is defined as the amount of revolution that the revolving line makes with its initial position. From θ is positive if it is traced by revolving line in anticlockwise direction and is negative if it is covered in clockwise direction.

$1° = 60'$ (minute), $1'$ (min) $= 60''$ (sec)

1 right angle = 90° (degrees) also 1 right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle subtended at the centre of a circle, whose length is equal to the radius of the circle.

$$1 \text{ rad} = \frac{180°}{\pi} \approx 57°\ 17'\ 45'' \approx 57.3°$$

$$\pi = \left(\frac{22}{7}\right)$$

Trigonometric ratios (or T ratios)

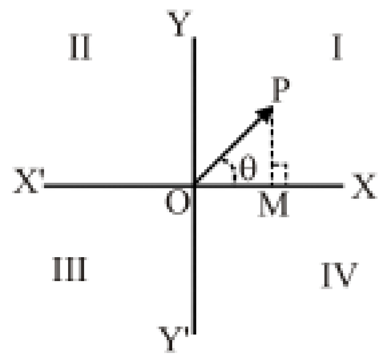
Let two fixed lines XOX' and YOY' intersecting at right angles to each other at point O. Then

(i) Point O is called origin.

(ii) XOX' known as X-axis and YOY' are Y-axis.

(iii) Portions XOY, YOX', Y'OX' and XOY' are called I, II, III and IV quadrant respectively.

Consider that the revolving line OP has traced out angle θ.



(in I quadrant) in anticlockwise direction. From P, perpendicular PM on OX. Then, side OP (in front of right angle) is called hypotenuse, side MP (in front of angle θ) is called opposite side or perpendicular and side OM (making angle θ with hypotenuse) is called adjacent side or base.

The three sides of a right angled triangle are connected to each other through six different ratios, called trigonometric ratios or simply T-ratios.

$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{MP}{OP} \qquad \cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP}$$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{MP}{OM} \qquad \cot\theta = \frac{\text{base}}{\text{perpendicular}} = \frac{OM}{MP}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM} \qquad \operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{MP}$$

It can be easily proved that

$$\text{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta + \sec^2\theta \qquad 1 + \cot^2\theta = \text{cosec}^2\theta$$

Illustration :

Given $\sin\theta = \frac{3}{5}$. Find all the other T-ratios, if θ lies in the first quadrant.

Sol. In ΔOMP, $\sin\theta = \frac{3}{5}$ So $MP = 3$ and $OP = 5$

$$\because OM = \sqrt{(5)^2 - (3)^2} = \sqrt{25-9} = \sqrt{16} = 4$$

Now, $\cos\theta = \frac{OM}{OP} = \frac{4}{5}$ $\quad \tan\theta = \frac{MP}{OM} = \frac{3}{4}$ $\quad \cot\theta = \frac{OM}{MP} = \frac{4}{3}$

$\sec\theta = \frac{OP}{OM} = \frac{5}{4}$ $\quad \text{cosec}\,\theta = \frac{OP}{MP} = \frac{5}{3}$

The T-ratios of a few standard angles ranging from 0° to 180°

Angle (θ)	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

$$\boxed{\begin{aligned}\sin(90^\circ-\theta)&=\cos\theta\\ \cos(90^\circ-\theta)&=\sin\theta\\ \tan(90^\circ-\theta)&=\cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(90^\circ+\theta)&=\cos\theta\\ \cos(90^\circ+\theta)&=-\sin\theta\\ \tan(90^\circ+\theta)&=-\cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(-\theta)&=-\sin\theta\\ \cos(-\theta)&=\cos\theta\\ \tan(-\theta)&=-\tan\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(180^\circ-\theta)&=\sin\theta\\ \cos(180^\circ-\theta)&=\cos\theta\\ \tan(180^\circ-\theta)&=\tan\theta\end{aligned}}$$

$$\boxed{\begin{aligned}\sin(180^\circ+\theta)&=-\sin\theta\\ \cos(180^\circ+\theta)&=-\cos\theta\\ \tan(180^\circ+\theta)&=\tan\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(270^\circ-\theta)&=-\cos\theta\\ \cos(270^\circ-\theta)&=-\sin\theta\\ \tan(270^\circ-\theta)&=\cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(270^\circ+\theta)&=-\cos\theta\\ \cos(270^\circ+\theta)&=\sin\theta\\ \tan(270^\circ+\theta)&=-\cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(360^\circ-\theta)&=-\sin\theta\\ \cos(360^\circ-\theta)&=\cos\theta\\ \tan(360^\circ-\theta)&=-\tan\theta\end{aligned}}$$

Illustration :

Find the value of

(i) cos (–60°) *(ii) tan 210°* *(iii) sin 300°* *(iv) cos 120°*

Sol. (i) $\cos(-60°) = \cos 60° = \frac{1}{2}$ (ii) $\tan 210° = (\tan 180° + 30°) = \tan 30° = \frac{1}{\sqrt{3}}$

(iii) $\sin 300° = \sin(270° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

(iv) $\cos 120° = \sin(180° - 60°) = -\cos 60° = -\frac{1}{2}$

A few important trigonometric formulae

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ $\sin(A-B) = \sin A\cos B - \cos A\sin B$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$	$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$
$\sin 2A = 2\sin A\cos A$	$\cos 2A = \cos^2 A - \sin^2 A$
$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$	$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Small angle approximation :

θ is very small and it must be in radian when you are taking approximation.

$\sin\theta \simeq \theta$, $\tan\theta \simeq \theta$

$\sin\theta \simeq \tan\theta$.

$\cos\theta \simeq 1$

Illustration :

Evaluate sin 2°

Sol. $2° = 2 \times \frac{\pi}{180}\text{ rad} = \frac{\pi}{90}\text{ rad}$

Now

$$\sin 2° = \sin\left(\frac{\pi}{90}\text{rad}\right) \simeq \left(\frac{\pi}{90}\right)$$

Illustration :

Evaluate sin 2° (1 – cos2°)

Sol. Sin 2° (1 – 1 + 2 sin^2 *1°)*

$$2\sin 2^\circ \sin^2 1^\circ \simeq 2\left(2\times\frac{\pi}{180^\circ}\right)\left(\frac{\pi}{180^\circ}\right)^2$$

$$= 4\left(\frac{\pi}{180^\circ}\right)^3$$

Practice Exercise

Q.1 Find the value of

(i) cos (–30°) (ii) sin 120° (iii) sin 135° (iv) cos 120° (v) sin 270° (vi) cos 270°

Ans. (i) $\frac{1}{\sqrt{3}}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $-\frac{\sqrt{3}}{2}$ (v) – 1 (vi) 0

CALCULUS

Introduction:

Suppose on your 8^{th} birthday at midnight you measured your height and found it to be 120 cm. One year later at midnight on your Eight birthday you again measure your height & find it to be 132 cm. Now if we say rate of increase of your height is 12 cm/year it does not mean that at the exact 11:59:59 PM you were 120 cm and as soon as clock moved to 12:00 midnight you streched and became 132 cm.
Now if we express your rate of growth as 1 cm/month we are not implying that every month at midnight of 1^{st} you suddenly strech by 1 cm/month is just a unit and the growth is a continuous process. If we assume you grow uniformly at all times the rate simply implies that if you grow Δh in time Δt at some point of time, the ratio $\left(\frac{\Delta h}{\Delta t}\right)$ will be 12 cm/year or 1 cm/month or whatever unit you may choose.

So we can calculate the rate of growth of height in a finite time internal. If we calculate rate of growth $\left(\frac{\Delta h}{\Delta t}\right)$ in different time interval like

(i) Rate of growth per day $\left(\frac{\Delta h}{\Delta t}\right)=\frac{1}{30}\text{cm / day}$

(ii) Rate of growth per hour $\left(\frac{\Delta h}{\Delta t}\right)=\frac{1}{30\times 24}\text{cm / hour}$

(iii) Rate of growth per min $\left(\frac{\Delta h}{\Delta t}\right)=\frac{1}{30\times 24\times 60}\text{cm / min}$

Similarly if we want to calculate the rate of growth exactly at 12 O'clock mid night then we take time interval Δt is zero. So in that time Δh is also zero becasue nothing can change in zero time. So, rate of growth becomes $\frac{0}{0}$, which is meaningless.

Now the question is how two find the rate of growth exactly at 12 O'clock or at any instant of time.

Differential calculus play an important role to find the rate of growth at any instant. The rate of growth at any instant is known as instantaneous rate. In many situations in physics, it is sometimes necessary to use the concept of instantaneous rate. The basic tool for this is calculus, invented by **Newton and Liebnitz** independently. The use of calculus is fundamental in the treatment of various problems in physics.

Numerical interpretation:

Illustration :

A car moving on a horizontal road whose position changes with time t as $x = 3t^2 + 1$ compute its average speed (average rate of change in position) $\left(v_{avg}=\frac{\text{change in position}}{\text{time interval}}=\frac{\Delta x}{\Delta t}\right)$ *between*

(i) From 2 sec to 3 sec *(ii) From 2 sec to 2.1 sec* *(iii) From 2 sec to 2.001 sec*
(iv) From 2 sec to 2.0001 sec *(v) at 2 sec (instantaneous rate of change)*

Sol. For $t = 2$ sec to 3 sec, we have $\Delta t = 1sec$

$x(\text{at } t = 3 \text{ sec.}) = 3(3)^2 + 1 = 28\ m$

$x(\text{at } t = 2 \text{ sec.}) = 3\ (2)^2 + 1 = 13\ m$

$\Delta x = 28\ m - 13\ m = 15m$

Thus $v_{average} = \dfrac{\Delta x}{\Delta t} = 15\ m/1sec = 15\ ms^{-1}$

(ii) 2 sec and 2.1 sec

Sol. For $t = 2.1$ sec, we have $\Delta t = 0.1$ sec

$x = 3\ (2.\ 1)^2 + 1 = 14.23\ m$ and $\Delta x = 1.23\ m$

Thus $v_{average} = \dfrac{\Delta x}{\Delta t} = 1.23\ m/0.1\ sec = 12.3\ ms^{-1}$

(iii) 2 sec and 2.001 sec,

Sol. for $t = 2.001$ sec, we have $\Delta t = 0.001$ sec

$x = 3(2.001)^2 + 1 = 13.012003\ m$ and $\Delta x = 0.012003\ m$

Thus $v_{average} = \dfrac{\Delta x}{\Delta t} = 0.012003\ m/0.001\ sec = 12.003\ ms^{-1}$

(iv) 2 sec and 2.00001 sec.

Sol. The student may verify that for $t = 2.00001$ sec

$v_{avg} = 12.00003\ ms^{-1}$

(v) Also find instantaneous rate of change in position (instantaneous speed) at 2 sec.

$$v_{ins} = \frac{x_2 - x_1}{t_2 - t_1}$$

here $t_1 = 2$ sec. and also $t_2 = 2$ sec.

$$v_{ins} = \frac{3(2)^2 + 1 - 3(2)^2 - 1}{2 - 2} = \frac{0}{0},\ \text{It is undefined.}$$

So Liebnitz suggest try it like that

$t_1 = 2$ sec., $t_2 = 2 + \Delta t$

$$v_{avg} = \frac{3(2+\Delta t)^2 + 1 - 3(2)^2 - 1}{2 + \Delta t - 2} = \frac{\Delta t^2 + 12\Delta 2}{\Delta t} = \Delta t + 12$$

Then he said let's make $\Delta t \to 0$, means Δt approaches zero not equal to zero. So, t_2 almost becomes 2 sec. and the v_{avg} becomes v_{ins}.

Liebnitz conclude that althrough the method is approximate but result is exact.

In calculus notation

$$v_{ins} = \lim_{\Delta t \to 0} v_{avg}$$

$$v_{ins} = \lim_{\Delta t \to 0}(\Delta t + 12) = 12m/s$$

$$\because \quad v_{avg} = \frac{\Delta x}{\Delta t}$$

$$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$ (it is knows as derivative of x with respect to t)

Illustration :

Suppose displacement y (t) (that is, y as a function of t) is given by

$$y(t) = t^3$$

Sol. *Find velocity (dy/dt) at t = 3 sec.*

Displacement at $t + \Delta t$ *is*

$$y(t + \Delta t) = (t + \Delta t)^3$$

$$= (t^3 + 3t^2\,\Delta t + 3t\,\Delta t^2 + \Delta t^3)$$

hence displacement from t *to* $t + \Delta t$ *is* Δy

$$\Delta y = y(t + \Delta t) - y(t) = (3t^2\,\Delta t + 3t\,\Delta t^2 + \Delta t^3)$$

Substituting this into Equation (i) gives

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} [3t^2 + 3t\,\Delta t + \Delta t^2]$$

Here we can see that if we take very small value of Δt *then value of* dy/dt *will approach* $3t^2$ *as all other terms will become negligible and impossible to measure by any instrument available in this world.*

hence, $\dfrac{dy}{dt} = 3t^2 \quad \Rightarrow \quad \dfrac{dy}{dt} = v\ (\text{at } t = 3\text{ sec}) = 3(3)^2 = 27\ \text{m/s}$

Leibneiz derived some general formula for differentiation which is written below.

Differentiation Formulae

1. **Power rule :** $\dfrac{d}{dt}(t^n) = nt^{n-1}$
2. $\dfrac{d}{dt}(t) = 1$
3. $\dfrac{d}{dt}(\sin t) = \cos t$
4. $\dfrac{d}{dt}(\cos t) = -\sin t$
5. $\dfrac{d}{dt}(e^t) = e^t$
6. $\dfrac{d}{dt}(\log_e t) = \dfrac{1}{t}$; $[\log_e t \equiv \ln t]$
7. $\dfrac{d}{dt}(\tan t) = \sec^2 t$
8. $\dfrac{d}{dt}(\sec t) = \sec t \tan t$

Differentiation Rules : u and v are functions of t i.e. $u = f(t)$ and $v = g(t)$

1. $\dfrac{d}{dt}(c) = 0$
   where c : constant
2. ***Constant multiple rule:*** $\dfrac{d}{dt}(cu) = c\dfrac{du}{dt}$
3. ***Sum rule:*** $\dfrac{d}{dt}(u+v) = \dfrac{du}{dt} + \dfrac{dv}{dt}$
4. ***Difference rule:*** $\dfrac{d}{dt}(u-v) = \dfrac{du}{dt} - \dfrac{dv}{dt}$
5. ***Product rule :*** $\dfrac{d}{dt}(uv) = u\dfrac{dv}{dt} + v\dfrac{du}{dt}$
6. ***Quotient rule:*** $\dfrac{d}{dt}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dt} - u\dfrac{dv}{dt}}{v^2}$

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Example : Differentiate w.r.t. time.

(i) $y = t^2$ (ii) $x = t^{3/2}$ (iii) $y = \frac{1}{\sqrt{t}}$

(iv) $x = 4t^3$ (v) $y = 2\sqrt{t}$ (vi) $y = 2t^2 + t - 1$

(vii) $y = 3\sqrt{t} + \frac{2}{\sqrt{t}}$ (viii) $y = t^3 \sin t$ (ix) $x = te^t$

(x) $x = \sqrt{t}\,(1-t)$

Sol. (i) $\frac{dy}{dt} = 2t$ (ii) $\frac{dx}{dt} = \frac{3}{2}t^{1/2}$ (iii) $\frac{dy}{dt} = -\frac{1}{2}t^{-3/2}$

(iv) $\frac{dx}{dt} = 12t^2$ (v) $\frac{dy}{dt} = 2\left(\frac{1}{2}t^{-1/2}\right) = t^{-1/2}$

(vi) $\frac{dy}{dt} = 4t + 1 + 0 = 4t + 1$ (vii) $\frac{dy}{dt} = 3t^{-1/2} + 2\left(-\frac{1}{2}t^{-3/2}\right)$

$$= \frac{3}{\sqrt{t}} - t^{-3/2}$$

(viii) $\frac{dy}{dt} = 3t^2 \sin t + t^3 \cos t$ (ix) $\frac{dx}{dt} = e^t + te^t = e^t(1+t)$

(x) $\frac{dx}{dt} = \frac{1}{2}t^{-1/2} - \frac{3}{2}t^{1/2} = \frac{1}{2\sqrt{t}} - \frac{3}{2}\sqrt{t}$

Example : Differentiate with respect to x

(i) $y = x\,\ell n\,x$ (ii) $y = x^2 e^x$ (iii) $y = \frac{\sin x}{x}$ (iv) $y = \frac{3x^2 + 2\sqrt{x}}{x}$

Sol. (i) $\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ell n\,x = 1 + \ell n\,x$ (ii) $\frac{dy}{dx} = 2xe^x + x^2e^x = xe^x(2+x)$

(iii) $y = x^{-1}\sin x$

$$\frac{dy}{dx} = \left(-x^{-2}\right)\sin x + x^{-1}\cos x$$

(iv) $y = 3x + 2x^{-½}$

$$\frac{dy}{dx} = 3 + 2\left(-½\,x^{-3/2}\right) = 3 - x^{-3/2}$$

The Chain Rule :

Suppose you are asked to differentiate the function

$$y = \sqrt{x^2 + 1}$$

The differentiation formula you learned in the previous section of this chapter do not enable you to calculate $\frac{dy}{dx}$

In fact, if we let $y = \sqrt{u}$ and let $u = x^2 + 1$,

Then we can evaluate $\frac{dy}{du}, \frac{du}{dx}$ easily using the formula that we have learned in previous section.

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \qquad \frac{du}{dx} = 2x$$

but our aim is to calculate $\frac{dy}{dx}$ so we can write.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

and its value will become

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x$$

Substituting value of 'u' we will get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

Ex.1 $x = (at + b)^n$ where a, b and n are a real number then $\frac{dx}{dt} = an(at+b)^{n-1}$.

Sol. $u = at + b$ and $x = u^n$

$$\frac{dx}{du} = n\,u^{n-1}$$

$$\frac{du}{dt} = a$$

$$\frac{dx}{dt} = \frac{dx}{du} \times \frac{du}{dt} = n\,(u)^{n-1} \times a = an\,(at+b)^{n-1}$$

Ex.3 It $x = \sin^2\theta$, then find $\frac{dx}{dt}$ where $\frac{d\theta}{dt} = \omega$

Sol. $\frac{dx}{d\theta} = 2\,(\sin\theta)\,(\cos\theta) = \sin 2\theta$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \sin 2\theta \cdot \omega$$

Application in physics :

velocity $(v) = \frac{dx}{dt}$, acceleration $(a) = \frac{dv}{dt}$; $\left(a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = v\frac{dv}{dx}\right)$

Force (F) = rate of change of momentum $= \frac{dp}{dt}$;

Current $(i)=\frac{dq}{dt}$; angular speed $(\omega)=\frac{d\theta}{dt}$;

angular acceleration $(\alpha)=\frac{d\omega}{dt}$; $\left(\alpha=\frac{d\omega}{dt}=\frac{d\omega}{d\theta}\frac{d\theta}{dt}=\omega\frac{d\omega}{d\theta}\right)$

Ex. The radius of a circle is increasing at a rate $\frac{dr}{dt}=\alpha$. Find the rate at which its area is increasing when radius is equal to 3 m.

Sol area of cricle (A) = πr^2

$$\frac{dA}{dt}=\pi(2r)\frac{dr}{dt}=2\pi r\left(\frac{dr}{dt}\right)=2\pi r(\alpha)$$

$$\left(\frac{dA}{dt}\right)_{at\, r=3m}=6\pi\alpha\ m^2/sec.$$

DERIVATIVE OF A VECTOR :

$$\vec{v}=\frac{d\vec{r}}{dt}, \vec{a}=\frac{d\vec{v}}{dt} ; \vec{F}=\frac{d\vec{p}}{dt} ; \vec{\alpha}=\frac{d\vec{\omega}}{dt}$$

Ex. $\vec{r}=2t\hat{i}+3t^2\hat{j}$. Find $\vec{v}$ and $\vec{a}$ where $\vec{v}=\frac{d\vec{r}}{dt}$, $\vec{a}=\frac{d\vec{v}}{dt}$

Sol. $\vec{v}=\frac{d\vec{r}}{dt}=2\hat{i}+6t\hat{j}$ m/s

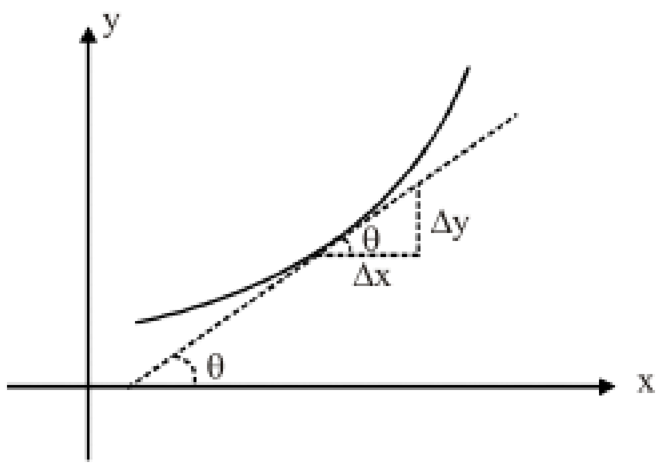
$\vec{a}=\frac{d\vec{v}}{dt}=0+6\hat{j}$ m/s^2

Physical meaning of $\frac{dy}{dx}$

1. The ratio of small change in the function y and the varibalbe x is called the average rate of change of y w.r.t.x. For example, if a body cover a small distance Δs in small time Δt, then average velocity of the body, $v_{av}=\frac{\Delta s}{\Delta t}$ Also, if the velocity of a body changes by a small amount Δv in small time Δt, then average acceleration of the body, $a_{av}=\frac{\Delta v}{\Delta t}$

2. When $\Delta x \to 0$ The limiting value of $\frac{dy}{dx}$ is $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=\tan\theta$

= slope of the tangent



It is called the instantaneous rate change of y w.r.t.x.
The differentiation of a function w.r.t. a veriable imples the instantaneous rate change of the funcaiton w.r.t. that variable.

Like wise, instantaneous velocity of the body, $(v) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$

and instantaneous acceleration of the body $(a) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$

Maxima & Minima :

Maxima & minima of a function $y = f(x)$

for maximum value $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} =$ negative

for minimum value $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} =$ positive

Ex. Find minimum value of $y = 25x^2 - 10x + 5$.

Sol. For maximum / minimum value $\frac{dy}{dx} = 0 \Rightarrow 50x - 10 \Rightarrow x = \frac{1}{5}$

Now at $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = 50$, which is positive

So $y_{min} = 25\left(\frac{1}{5}\right)^2 - 10\left(\frac{1}{5}\right) + 5 = 1 - 2 + 5 = 4$

Ex. A body is moving vertically upwards under gravity such that its position from ground is given as $y = ut - \frac{1}{2}gt^2$. Find the max height reached by body.

Sol. $\frac{dy}{dt} = u - gt = 0$, $t = \frac{u}{g}$

$\frac{d^2y}{dx^2} = -g < 0$ (which is negative). so we get maximum height y_{max} at $t = \frac{u}{g}$ sec.

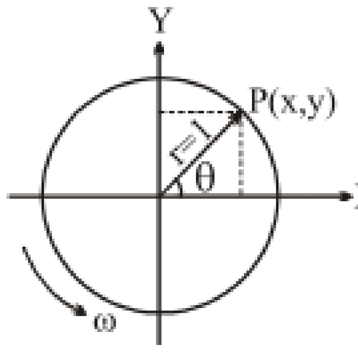
$$y_{max} = u\left(\frac{u}{g}\right) - \frac{1}{2}g\left(\frac{u}{g}\right)^2$$

$$= \frac{u^2}{2g}$$

Equation of Trajectory :

It is path traversed by a particle, independent of time parameter.

Ex. Find the equation of trajectory for the particle moving i n circular path as shown.



$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2 + y^2 = r^2 \quad \Rightarrow \quad x^2 + y^2 = 1$

Integration or Antiderivative :

So far we have studied as how to find velocity of a particle when its position is given. Now if we know the velocity of a particle and we might wish to know its position at any given time or an engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. This reverse process is called antiderivative of integration.

There are two types of integration :

(a) Indefinite integration **(b)** Definite integration.

(a) Indefinite integration

Since we know that , $\Delta t = t_2 - t_1$

Now, when time t_2 approaches t_1 ; $\Delta t \to dt$

To remind you, $v = \lim\limits_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

So, If we integrate dt, we should get $t_2 - t_1$; i.e. $\int dt = t_2 - t_1$

Similarly, $\int dx = x_2 - x_1$

but since the differentiation of a constant becomes zero, so i.e. why we can not retrive the value of that constant.

$\therefore$ We should write $\int dx = x + c$.

Where 'c' is integration constant.

Integration Formulae :

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \;\; [n \neq -1]$
2. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
3. $\int \cos x \, dx = \sin x + c$
4. $\int \sin x \, dx = -\cos x + c$

5. $\int e^x dx = e^x + c$

6. $\int \frac{1}{x} dx = \ln|x| + c$, $[x \neq 0]$

7. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

8. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

9. $\int e^{(ax+b)}dx = \frac{1}{a}e^{(ax+b)} + c$

10. $\int \frac{1}{(ax+b)} dx = \frac{1}{a} ln|(ax+b)| + c$

Rules of Integration :

1. $\int k\,u\,dx = k\int u\,dx$
   where is k constant.
2. $\int (u+v)dx = \int udx + \int vdx$

Ex.1 Evaluate indefinite integration.

(i) $x = \int dt$ (ii) $x = \int t\,dt$ (iii) $x = \int (2t)\,dt$ (iv) $x = \int (t^2)\,dt$

(v) $x = \int\left(-\frac{2}{t^3}\right)dt$

Sol. (i) $x = t + c$ (ii) $x = \frac{t^2}{2} + c$ (iii) $x = \frac{2t^2}{2} + c = t^2 + c$

(iv) $x = \frac{t^3}{3} + c$ (v) $x = -2\int t^{-3}dt \Rightarrow x = -2\left(\frac{t^{-2}}{-2}\right) + c = t^{-2} + c$

Where 'c' is integration constant.

(b) Definite integration.

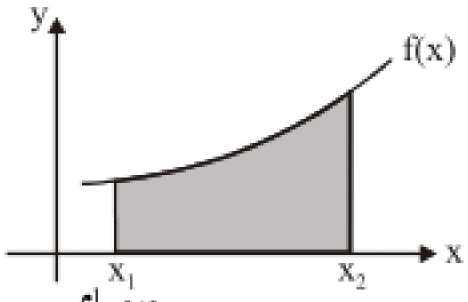
When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

If $\frac{d}{dx}(f(x)) = g(x)$

then $\int g(x)dx$ is called indefinite integral and $\int_a^b g(x)dx = [f(b) - f(a)]$ is called definite integral.

Here, a and b are called lower and upper limits of the variable x.
After carrying out integration, the result is evaluated between upper and lower limits as explained below :

$\int_{x_1}^{x_2} g(x)dx = f(x_2) - f(x_1) =$ Area under the curve.



Ex. Evaluate the integral :

(i) $\int_1^5 x^2 dx$ (ii) $\int_0^1 t^2 dt$ (iii) $\int_3^5 t\,dt$ (iv) $\int_0^1 t^{3/2}\,dx$

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Sol. (i) $\int_1^5 x^2 dx = \left[\frac{x^3}{3}\right]_1^5 = \frac{1}{3}[x^3]_1^5 = \frac{1}{3}\left((5)^3 - (1)^3\right) = \frac{1}{3}(125-1) = \frac{124}{3}$

(ii) $\int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$ **(iii)** $\int_3^5 t\, dt = \left[\frac{t^2}{2}\right]_3^5 = \frac{5^2-3^2}{2} = 8$

(iv) $\int_0^1 t^{3/2} dx = \left[\frac{t^{5/2}}{5/2}\right]_0^1 = \frac{2}{5}$

Practise Exercise :

(i) $\int_R^\infty \frac{GMm}{x^2} dx$ (ii) $\int_u^v Mv dv$ (iii) $\int_0^{\pi/2} \cos x\, dx$

Ans. (i) $\frac{GMm}{R}$ (ii) $\frac{1}{2}M(v^2 - u^2)$ (iii) 1

Ex.3 Find displacement of a particle in 1-D if its velocity is v = (2t – 5) m/s, from t = 0 to t = 4 sec.

Sol. $\frac{dx}{dt} = 2t - 5$

$$\int_{x_1}^{x_2} dx = \int_0^4 (2t-5)\, dt$$

$$x_2 - x_1 = \left(t^2 - 5t\right)_0^4$$

displacement = 16 – 20 = – 4 m

Graphical Interpretation :

Since $\frac{dx}{dt} = v \Rightarrow cdx = v\, dt \quad \int dx = \int v\, dt$

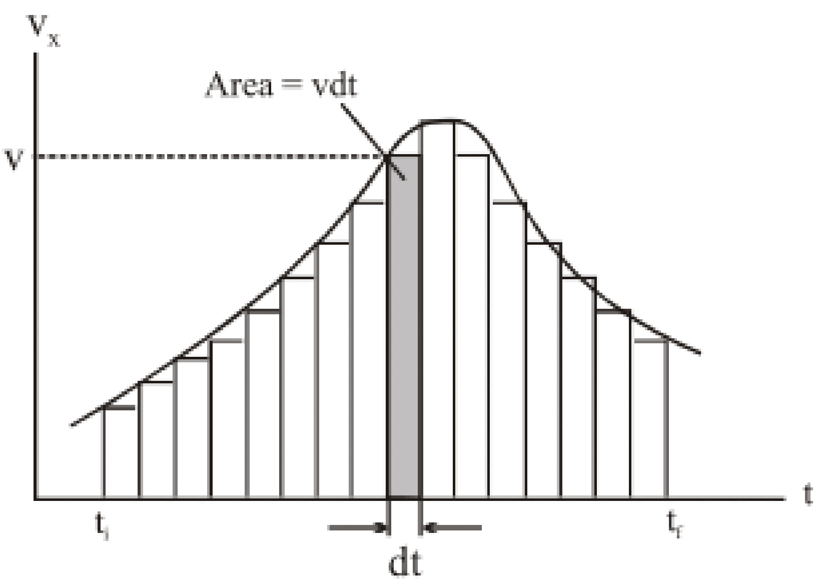
$\therefore \; x_2 - x_1 = \int v\, dt$ ..... (1)

Now, we have to understand the meaning of $\int v\, dt$

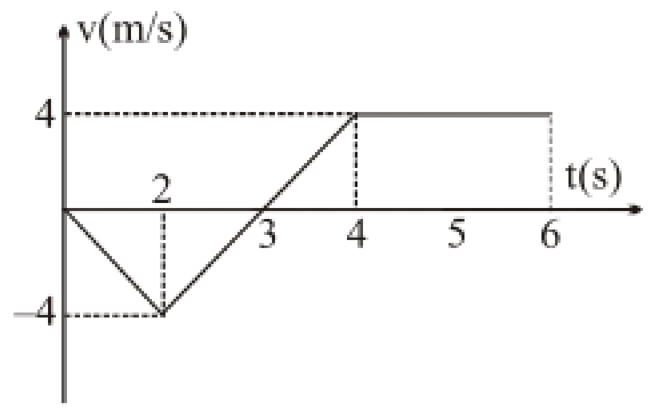
Graphically, it is equivalent to finding the area under a curve. Suppose the v – t graph for a particle is as shown in Fig. given below. We want to find the displacement for time interval $t_i - t_f$. Let us divide the time interval $t_i - t_f$ into many small intervals, each of duration Δt and if $\Delta t \rightarrow dt$. We can find the displacement 'dx' of the particle during any small interval, such as the one shaded in figure given below, by dx = v dt, where 'v' is the velocity in that interval. Therefore, the displacement during this small interval is simply the area of the shaded rectangle. The total displacement for the interval t_i - t_f is the sum of the areas of all the rectangles from t_i to t_f. But by adding up it means integrating.

Thus, Area under curve = $\int v\, dt$ ... (2)

Hence from (1) & (2), Area under curve = $x_2 - x_1$ = displacement.



Ex.1 From the v versus t graph of figure (a) the time(s) at which the particle is at rest (b) at what time, if any does the particle reverse the direction of its motion? (c) The distance and displacement of the particle from t = 0 s to t = 6s.



Sol. (a) t = 0, 3 sec (b) t = 3 sec

(c) displacement $= -\frac{1}{2}(3\times 4) + \frac{1}{2}(3+2)(4)$

$= -6 + 10 = 4$ m

distance $= \left|-\frac{1}{2}(3\times 4)\right| + \left|\frac{1}{2}(3+2)(4)\right|$

$= 6 + 10 = 16$ m

Finding position and trajectory if velocity is known:

Lets take velocity vector $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

$$\Rightarrow\ v_x = \frac{dx}{dt};\ v_y = \frac{dy}{dt};\ y_z = \frac{dz}{dt}$$

$$\therefore\quad x = \int v_x dt\ ;\ y = \int v_y dt \qquad z = \int v_z dt$$

Ex.1 Find the equation of trajectory of a particle whose velocity components are
$v_x = 2x + 1$, $v_y = 2y + 3$
Given that particle starts from rest from origin.

Sol. $v_x = 2x + 1,$

$$\frac{dx}{dt} = 2x + 1$$

$$\int_0^x \frac{dx}{2x+1} = \int_0^t dt$$

$$\frac{1}{2}\ln(2x+1) = t \qquad \text{.... (1)}$$

$$v_y = 2y + 3$$

$$\int_0^y \frac{dy}{2y+3} = \int_0^t dt$$

$$\frac{1}{2}\ln\frac{(2y+3)}{3} = t \qquad \text{.... (2)}$$

from (1) and (2)

$$\frac{1}{2}\ln(2x+1) = \frac{1}{2}\ln\frac{(2y+3)}{3}$$

$$2x + 1 = \frac{(2y+3)}{3}$$

$$y = 3x$$

Binomial Approximation :

For $x << 1$, the following approximation can be used :

$$(1+x)^n \approx 1 + nx$$

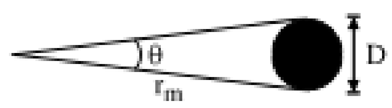
Ex Find $(1.03)^{1/3}$

Sol. $(1 + 0.03)^{1/3} \quad \left(x = 0.03 \;\&\; n = \frac{1}{3}\right)$

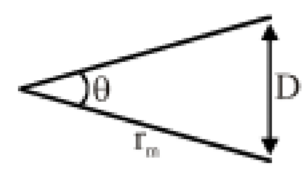
$$\simeq 1 + \frac{0.03}{3} = 1.01$$

Solved Example

Q.1 The angle subtended by the moon's diameter at a point on the earth is about 0.50°. Use this and the fact that the moon is about 384000 km away to find the approximate diameter of the moon.



Sol. $\tan\theta \simeq \theta = \dfrac{D}{r_m}$ θ is radians



$180° = \pi$-radians

$$\Rightarrow \quad 0.5° = \frac{\pi}{180} \times \frac{1}{2} \text{rad} = \frac{\pi}{360} \text{rad}$$

$$\Rightarrow \quad D = \theta \,.\, r_m$$

$$= \frac{\pi}{360} \times 384000 \text{ km} = 3350 \text{ km}$$

Q.2 Find $\dfrac{dx}{dt}$ (derivation of x with respect to t)

(i) $x = (t^2 + 1)^3$ (ii) $x = \sin 2t$

Sol. (i) $\dfrac{dx}{dt} = 3\,(t^2 + 1)^2\,(2t)$

$= 6t\,(t^2 + 1)^2$

(ii) $\dfrac{dy}{dt} = 2\cos 2t$

Q.3 Find the derivative of $y(x) = x^3/(x + 1)^2$ with respect to x.

Sol. We can rewrite this function as $y(x) = x^3\,(x + 1)^{-2}$ and apply Equation ( )

$$\frac{dy}{dx} = (x + 1)^{-2}\,\frac{d}{dx}\,(x^3) + x^3\,\frac{d}{dx}\,(x + 1)^{-2}$$

$$= (x + 1)^{-2}\,3x^2 + x^3\,(-2)\,(x + 1)^{-3}$$

$$\frac{dy}{dx} = \frac{3x^2}{(x+1)^2} - \frac{2x^3}{(x+1)^3}$$

Q.4 The velocity of particle is given by $v = \sqrt{gx}$. Find its acceleration.

Sol. $\dfrac{dv}{dt} = \dfrac{1}{2}(gx)^{-1/2}\left(g\dfrac{dx}{dt}\right)$

$= \dfrac{1}{2}\,(gx)^{-1/2}\,gv$

$= \dfrac{1}{2}\,g$

Q.5 If $\vec{r} = \left[u\cos\theta(\hat{i}) + u\sin\theta(\hat{j})\right]t + \frac{1}{2}g(-\hat{j})t^2$ then calculate equation of trajectory.

Sol. $x = u\cos\theta\, t$ & $y = u\sin\theta\, t - \frac{1}{2}gt^2$

$$y = u\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Q.6 Find the value of definite integral : $\int_0^{\pi}\left(\frac{\pi t}{2} - \frac{t^2}{2}\right)dt$

Sol. $\int_0^{\pi}\left(\frac{\pi t}{2} - \frac{t^2}{2}\right)dt$

$$= \left(\frac{\pi}{2}\left(\frac{t^2}{2}\right) - \frac{t^3}{6}\right)_0^{\pi}$$

$$= \frac{\pi^3}{4} - \frac{\pi^3}{6} = \frac{\pi^3}{12}$$

Q.7 The velocity of a body moving in a straight line is given by $v = (3x^2 + x)$ m/s. Find acceleration at $x = 2$m.

Sol. $v = (3x^2 + x)$

$$\frac{dv}{dx} = 6x + 1$$

$$a = v\frac{dv}{dx} = (3x^2 + x)(6x + 1)$$

at $x = 2$ m

$a = (3 \times 2^2 + 2)(6 \times 2 + 1) = 182$ m/s^2

Q.8 Find $(104)^{1/2}$

Sol. $(100 + 4)^{1/2}$

$= 10\,[1 + 0.04]^{1/2}$

$\simeq 10[1 + 0.02] = 10.2$

KINEMATICS

Introduction

The branch of physics in which motion and the forces causing motion are studied is called mechanics.

As a first step in studying mechanics, we describe the motion of particles and bodies in terms of space and time without studying the cause of motion. This part of mechanics is called kinematics. We first define displacement, velocity and acceleration. Then, using these concepts, we study the motion of the objects moving under different conditions. The forces causing motion will be discussed later in Dynamics.

From everyday experience, we recognize that motion represents continuous change in position, so we begin our study with change in position i.e. with displacement.

Various quantities used in Kinematics

Displacement ($\vec{S}$ or $\overrightarrow{\Delta r}$) :

Change in position vector is called **displacement**.

Its magnitude is minimum distance between final and initial point, and is directed from initial position to final position.

For a particle moving along x axis, motion from one position x_1 to another position x_2 is displacement, Δx where, $\Delta x = x_2 - x_1$

If the particle moves from $x_1 = 4m$ to $x_2 = 12$ m, then $\Delta x = (12m) - (4m) = +8m$. The positive result indicates that the motion is in the positive direction. If the particle then returns to x = 4m, the displacement for the full trip is zero. The actual number of meters covered for the full trip is irrelevant ***displacement involves only the original and final position.***

In general if initial position vector and final position vector are $\vec{r}_{in}$ and $\vec{r}_f$ respectively, then

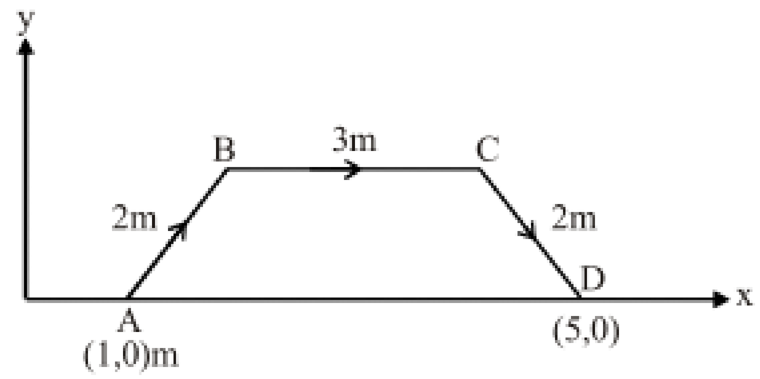
$$\vec{S} = \vec{r}_f - \vec{r}_{in} = \overrightarrow{\Delta r}$$

Distance:

Length of path traversed by a body is called **distance**.

It is dependent on the path chosen, thus for motion between two fixed points A and B we can have many different values of distance traversed. It is a **scalar** quantity, as length of path has no indication of direction in it. Its SI unit is meter (m) and dimensions is (L).

eg. Suppose a particle moves from position A to B as shown after travelling from A to B to C to D.



Here Displacement $\vec{S} = \overrightarrow{AD} = 5\hat{i} - \hat{i} = 4\hat{i}$ m

$\therefore \quad |\text{displacement}| = 4m$

Also distance covered ,

$$l = |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}| = 2 + 3 + 2 = 7 \text{ m}$$

Note : Here |Displacement| < Distance

Magnitude of displacement would be equal to distance travelled if there is no change in direction during the whole motion.

In general, |Displacement| ≤ Distance

Average Velocity :

The average velocity $\vec{V}_{avg}$ is the ratio of the total displacement $\overrightarrow{\Delta r}$, and total time ($\Delta t$) taken to complete that displacement. It should be noted that $\vec{V}_{avg}$ is independent of path as displacement is independent of path.

$$\vec{V}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_{in}}{\Delta t}$$

Unit for $\mathbf{V}_{avg}$ is the meter per second (m/s). The average velocity $\mathbf{V}_{avg}$ always has the same sign as the displacement $\overrightarrow{\Delta r}$.

Average Speed :

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time interval}} = \frac{l}{\Delta t}$$

It is a scalar and always has positive sign.

|Average velocity| ≤ |Average speed|

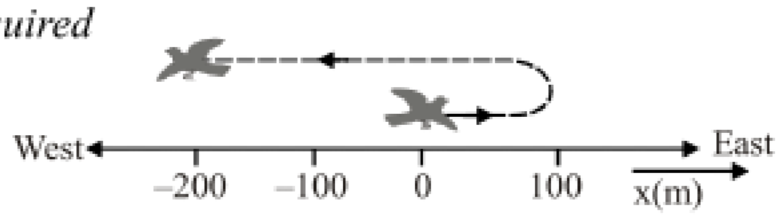
Illustration :

A bird flies east at 10 m/s for 100 m. It then turns around flies at 20 m/s for 15 s. Neglect time taken for turning, find

(a) its average speed

(b) its average velocity

Sol. *Let us take the x axis to point east. A sketch of the path is shown in the figure. To find the required quantities, we need the total time interval.*



The first part of the journey took

$\Delta t_1 = (100 \text{ m}) / (10 \text{ m/s}) = 10 s$,

and we are given $\Delta t_2 = 15$ s for the second part. Hence the total time interval is

$$\Delta t = \Delta t_1 + \Delta t_2 = 25 \text{ s}$$

The bird flies 100 m east and then (20 m/s) (15s) = 300 m west

(a) Average speed $= \dfrac{\text{Distance}}{\Delta t} = \dfrac{100\text{m} + 300\text{m}}{25\text{s}} = 16 \text{ m/s}$

(b) The net displacement is

$$\Delta x = -200 \text{ m}$$

So that

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{-200}{25s} = -8\ m/s$$

The negative sign means that v_{av} is directed toward the west.

Illustration :

A particle moves with speed v_1 along a particular direction. After some time it turns back and reaches the starting point again travelling with speed v_2. Find (for the whole journey)

(a) Average velocity *(b) Average speed*

Sol. *(a) Since the particle reaches the starting point again, its displacement is zero*

$$\therefore \quad \text{Average velocity} = \frac{\text{displacement}}{\text{total time}} = 0$$

(b) Let it travelled distance x while moving away as well as while moving towards the starter point.

Time taken to go away is $t_1 = \dfrac{x}{v_1}$

Time taken while return journey $t_2 = \dfrac{x}{v_2}$

$$\therefore \quad \text{Average speed} = \frac{x+x}{t_1+t_2} = \frac{2x}{\dfrac{x}{v_1}+\dfrac{x}{v_2}} = \frac{2v_1v_2}{v_1+v_2}$$

Instantaneous Velocity :

Instantaneous Velocity is defined as the value approached by the average velocity when the time interval for measurement becomes closer and closer to zero, i.e. $\Delta t \to 0$. Mathematically

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Thus Instantaneous velocity function is the derivative of the displacement with respect to time.

$$v(t) = \frac{dx(t)}{dt}$$

Instantaneous Speed

It is measure of how fast a particle or a body is moving at a particular instant. It is the magnitude of instantaneous velocity. Thus particle moving with instantaneous velocity of + 5m/s and another moving with –5m/s will have same instantaneous speed of 5 m/s.

The speedometer in a car measure the instantaneous speed not the instantaneous velocity, because it cannot determine the direction.

Average Acceleration

. For any change in velocity either in its magnitude or direction or both, acceleration must be present. Without acceleration neither direction nor magnitude of velocity can be changed.

When a particle's velocity changes, the particle is said to undergo acceleration (or to accelerate).

The **Average Acceleration ($\vec{a}_{avg}$)** over a time interval Δt is

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

where the particle has velocity v_1 at time t_1 and then velocity v_2 at time t_2.

Instantaneous Acceleration

The **Instantaneous Acceleration** (or simply acceleration) is the derivative of the velocity with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

In words, the acceleration of a particle at any instant is the rate at which its velocity is changing at that instant.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

In words, the acceleration of a particle at any instant is the second derivative of its position vector with respect to time.

Acceleration has both magnitude and direction (it is yet another vector quantity). For motion on a straight line its algebraic sign represents its direction on an axis just as for displacement and velocity; that is, acceleration with a positive value is in the positive direction of an axis, and acceleration with a negative value is in the negative direction.

Illustration :

The position of a particle moving along x-axis is given by $x = (5t^2 - 4t + 20)$ meter, where t is in second.

(a) Find average velocity between 1s & 3s

(b) Find velocity as a function of time v(t) and its value at t = 3s

(c) Find acceleration at t = 2 sec.

(d) When is the particle at rest ?

Sol. (a) At $t = 1s$; $x_{in} = 5(1)^2 - 4(1) + 20 = 21\ m$

At $t = 3s$; $x_f = 5(3)^2 - 4(3) + 20 = 53\ m$

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{53-21}{3-1} = 16\ m/s$$

(b) $v(t) = \frac{dx}{dt} = (10t - 4)\ m/s$

at $t = 3s$, $v = 10(3) - 4 = 26\ m/s$

(c) $a = \frac{dv}{dt} = 10\ m/s^2$ (constant at any instant)

(d) particle at rest i.e. $v = 0 = 10t - 4$

$\Rightarrow t = 0.4s$

Note :

(1) Here we can observe at t = 0.4s, particle has zero velocity but acceleration of 10 m/s^2. Thus particle having zero velocity need not have zero acceleration.

(2) For t < 0.4s, velocity is negative and for t > 0.4 s, velocity is in positive direction i.e. its velocity

changes its direction at t = 0.4 sec, when becoming zero.

Illustration :

Position of a particle moving along a straight live is given by

$x = (t^2 - 4t)$ *meters. (t is in sec.)*

Find Displacement and distance travelled between

$t = 0$ *and* $t = 3sec$

Sol. *Displacement* $= \Delta x = x_3 - x_0$

$= [(3)^2 - 4(3)] - [(0)^2 - 4\ (0)]$

$= -\ 3m$

Now, velocity $v = \frac{dx}{dt} = 2t - 4$

i.e. $v = 0$ *at* $t = 2$ *sec*

for $t < 2$ *sec,* $V = -ve$ *& for* $t >$ *sec,* $v = +ve$

Distance travelled = |*Displacement in –ve direction*| + |*Displacement in +ve direction*|

$= |\Delta x \text{ for } t = 0 \text{ to } t=2s| + |\Delta x \text{ for } t = 2s \text{ to } t = 3s|$

$= |x_2 - x_0| + |x_3 - x_2|$

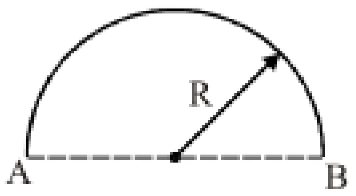
$= |[(2)^2 - 4(2) - (0)]| + |3^2 - 4(3) - 2^2 + 4\ (2)|$

$= |-4| + |1|$

$= 5m$

Practice Exercise

Q.1 If a particle traverses on a semicircular path of radius R from A to B as shown in time T, find average speed and average velocity.



Q.2 A man runs for first 120 m at 6m/s and then next 120m at 3m/s in the same direction. Find

(a) Total time of run (b) Average velocity

Q.3 Position of particle moving along x-axis is given by

$x = (3t^2 - 2t^3)$m (t is in sec)

Find

(a) its average velocity form t = 0 s to t = 2s

(b) v (t) and a (t)

(c) The time at which its acceleration is zero and find velocity at the instant.

Q.4 Position of a particle moving along straight line is given by $x = (-t^2 + 6t + 5)$ m (t is in sec) Find

(a) The time at which velocity of particle is zero.

(b) Average velocity from t = 0 to t = 4 sec

(c) Average speed from t = 0 to t = 4sec

Answers

1. Average speed = $\frac{\pi R}{T}$; Average velocity = $\frac{2R}{T}$ (from A to B)

2. (a) 60s (b) 4 m/s 3. (a) – 2m/s (b) $v(t) = 6t - 6t^2$ m/s (c) $\frac{1}{2}$ s ; $\frac{3}{2}$ m/s

4. (a) 3 sec (b) – 2 m/s (c) 2.5 m/s.

One Dimensional or Rectilinear Motion

We may divide this topic in the following different situations.

(i) Motion with constant velocity

(ii) Motion with variable velocity but constant acceleration

(iii) Motion with variable acceleration.

Motion with constant velocity

$$v = \frac{dx}{dt} \quad \Rightarrow \quad \int_{x_0}^{x} dx = \int_0^t v\,dt$$

Since velocity is constant, it comes out of the integration

$$\Rightarrow \quad \int_{x_0}^{x} dx = v\int_0^t dt$$

$$[x]_{x_0}^{x} = v\,[t]_0^t$$

$x - x_0 = vt$ i.e., displacement $\mathbf{\Delta x = vt}$

Motion with variable velocity but constant acceleration

Basic formulae

(i) $a = \frac{dv}{dt}$

(ii) $a = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$ (By chain rule)

From formula (i)

$$a = \frac{dv}{dt} \Rightarrow dv = a\,dt; \quad \int_u^v dv = \int_0^t a\,dt$$

Since acceleration is constant so it comes out of the integration

$$[v]_u^v = a\int dt$$

$$\therefore v - u = at$$

$$\Rightarrow \quad \mathbf{v = u + at} \qquad \text{............(i)}$$

$$\frac{dx}{dt} = u + at$$

$$dx = udt + at\,dt$$

on further integrating

$$\int_{x_0}^{x} dx = u\int_0^t dt + a\int_0^t t\,dt$$

$$[x]_{x_0}^{x} = ut + \frac{at^2}{2}$$

$$x - x_0 = ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad \mathbf{\Delta x = ut + \frac{1}{2}at^2} \qquad \text{............(ii)}$$

From formula (ii)

$$a = v\frac{dv}{dx}$$

$$\int_u^v vdv = a\int_{x_0}^{x} dx$$

$$\frac{v^2}{2} - \frac{u^2}{2} = a(x - x_0)$$

$$\mathbf{v^2 = u^2 + 2a(\Delta x)} \qquad \text{............(iii)}$$

Taking $a = \frac{v-u}{t}$ from equation (i) and putting it in equation (ii), we get

$$\Delta x = ut + \frac{1}{2}\left(\frac{v-u}{t}\right)t^2$$

$$\Rightarrow \quad \mathbf{\Delta x = \left(\frac{v+u}{2}\right)t} \qquad \text{............(iv)}$$

Displacement in nth second

Displacement in nth second = Displacement in n sec. – Displacement in (n–1) sec.

$$S_{n^{th}} = S_n - S_{n-1^{th}} = [u(n) + \frac{1}{2}\, an^2] - [u\,(n-1) + \frac{1}{2}\, a\,(n-1)^2]$$

$\therefore$ $$\mathbf{S_{n^{th}} = u + \frac{a}{2}\,(2n-1)} \qquad \text{.............(v)}$$

Illustration :

On seeing a board of speed limit, you brake a car from speed of 108 km/h to a speed of 72 km/h. covering a distance of 100m at a constant acceleration.

(a) What is that acceleration ?

(b) How much time is required for the given decrease in speed ?

Sol. *Initial speed,* $u = 108\ km/h = 108 \times \frac{5}{18} m/s = 30\ m/s$

final speed, $v = 72\ km/h = 72 \times \frac{5}{18} m/s = 20\ m/s$

(a) By 3rd equation of motion

$$v^2 = u^2 + 2as$$

$$\therefore a = \frac{v^2 - u^2}{2s} = \frac{(20)^2 - (30)^2}{2 \times 100}$$

$$\therefore a = -\,2.5\ m/s^2$$

(b) By 1st equation, $v = u + at$

$$\Rightarrow t = \frac{v-u}{a} = \frac{20-30}{(-2.5)} = 4sec$$

Illustration :

The time taken between observation of an event and taking action according to that is called reaction time. Suppose a person having reaction time of 0.3 sec is driving the car as stated in above example. Find the distance travelled by him after seeing the board till the car reaches 72 km/h.

Sol. *Till the reaction time i.e. till the brakes are applied speed of car remains uniform. So distance travelled during that time is*

$$S_1 = 30 \times 0.3 = 9\ m$$

Distance travelled after that time is $S_2 = 100$

$\therefore$ *Total distance travelled* $= S_1 + S_2 = 109\ m.$

Illustration :

A train is moving with 108 km/h. On a straight track. receiving red signal its brakes are applied and it retards at the rate of $3 m/s^2$. Find its displacement and average velocity for next 15 sec.

Sol. *Initial velocity, $u = 100$ km/h = 30 m/s*

Let time reqd. for the velocity to become zero is t.

$$V_{final} = u + at$$

$$\therefore \quad 30 - 3t = 0 \Rightarrow t = 10 \text{ sec.} < 15 \text{ sec.}$$

i.e., it covers no distance after t = 10 sec.

$\therefore$ Displacement till 15 sec = displacement till 10 sec

$$= 30\,(10) + \frac{1}{2}\,(-3)\,(10)^2$$

$$= 150 \text{ m}$$

$$V_{av} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{150}{15} = 10 \text{ m/s}$$

Note : In above example, for finding V_{av}, we have taken total time of 15 sec, which actually was required.

If we have to find V_{av} for 10 sec, it would be

$$V_{av} = \frac{150}{10} = 15 \text{ m/s}$$

Although displacement in 15 sec = Displacement in 10 sec., but times are different.

Thus V_{av} for 15 sec. is not same as V_{av} for 10 sec.

Motion with variable acceleration

Relations :

(i) $\dfrac{dv}{dt} = a \quad \Rightarrow \quad \int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt$

(iii) $\dfrac{dx}{dt} = v \quad \Rightarrow \quad \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt$

(iii) $a = \dfrac{dv}{dt} = \dfrac{dv}{dx}\cdot\dfrac{dx}{dt}$ (By chain rule)

$$\therefore \quad a = v\frac{dv}{dx}$$

$$\therefore \quad \int_{v_1}^{v_2} V\,dv = \int_{x_1}^{x_2} a\,dx$$

Illustration :

The acceleration of a particle is given by $a = 2t^2\ m/s^2$. If it is at rest at the origin at time $t = 0$, find its position, velocity, and acceleration at time $t = 1s$.

Sol. $a = 2t^2$

$\therefore$ $a = 2 \times 1^2 = 2\ m/s^2$ (at $t = 1$ sec.)

Formula for v,

$$\frac{dv}{dt} = 2t^2$$

or, $$\int_0^v dv = \int_0^t 2t^2\, dt$$

or, $$v = \frac{2t^3}{3}$$

At $t = 1$ sec $\quad t = 1,\ v = \dfrac{2\times 1^3}{3} = \dfrac{2}{3}\ m/s$

Formula for x,

$$\frac{dx}{dt} = \frac{2}{3}t^3 \qquad \text{or,} \qquad \int_0^x dx = \int_0^t \frac{2}{3}t^3 dt$$

or, $$x = \frac{t^4}{6}$$

At $t = 1$ sec, $\quad x = \dfrac{1}{6}\ m$

Illustration :

A particle located at $x = 0$ at time $t = 0$ starts moving along the positive x direction with a velocity v that varies as $v = \alpha\sqrt{x}$. How do the velocity and acceleration of the particle vary with time? What is the average velocity of the particle after the first s meter of its path ?

Sol. $$v = \alpha\sqrt{x} \qquad \text{or,} \qquad \frac{dx}{dt} = \alpha\sqrt{x}$$

or $$\int_0^x \frac{dx}{\sqrt{x}} = \alpha\int_0^t dt \qquad \text{or,} \qquad 2\sqrt{x} = \alpha t \quad ....(i)$$

$$\therefore \quad x = \left(\frac{1}{4}\right)\alpha^2 t^2$$

$$\therefore \quad v = \frac{dx}{dt} = \frac{1}{4}\alpha^2(2t) = \frac{1}{2}\alpha^2 t$$

$$\therefore \quad a = \frac{dv}{dt} = \frac{1}{2}\alpha^2$$

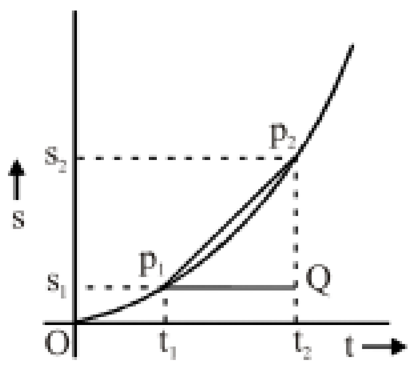
Time to cover a path of length s is obtained from (i) as $t = \dfrac{2\sqrt{s}}{\alpha}$ $(\because\ x = z)$

$\therefore$ *The average velocity over the path s is given by* $v_{av} = \dfrac{s}{t}\ \dfrac{s}{\left(\dfrac{2\sqrt{s}}{\alpha}\right)} = \dfrac{1}{2}\alpha\sqrt{s}$

Graphical Representation of Motion in one Dimension

S-t curve :

If we put **s** on y-axis and **t** on x-axis then for every value of **t** we have a specific value of **s**.



The **Average velocity** from time t_1 to t_2 will be

$$V_{avg} = \frac{s_2 - s_1}{t_2 - t_1} = \text{slope of line joining the points } p_1 \text{ and } p_2$$

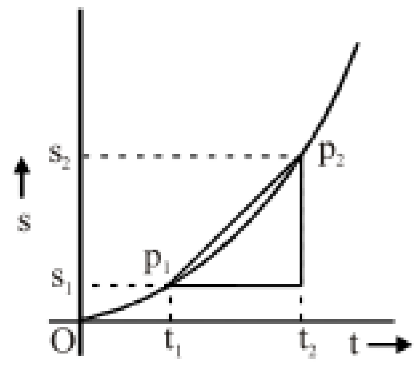
For a particle moving along a straight line when we plot a graph of **s** versus **t**, $\mathbf{V}_{avg}$ is the slope of the straight line that connects two particular points on the **s(t)** curve : one is the point that corresponds to s_2 and t_2, and the other is the point that corresponds to s_1 and t_1. Like displacement, $\mathbf{v}_{avg}$ has both magnitude and direction (it is another vector quantity). Its magnitude is the magnitude of the line's slope. A positive $\mathbf{v}_{avg}$ (and slope) tells us that the line slants upward to the right ; a negative v_{avg} (and slope), that the line slants downward to the right.

Instantaneous velocity

According to defination

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

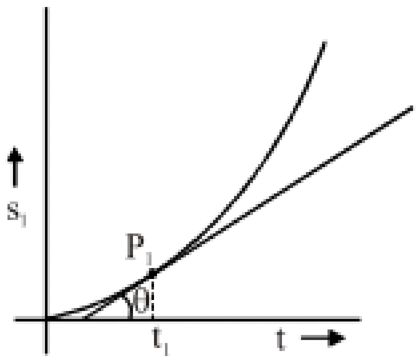
In curve if $\Delta t \to 0$ the point p_2 comes very close to point p_1.



Note : The instantaneous velocity can be found by determining the slope of the tangent to the displacement time graph at that instant.

Velocity at point p_1 or time t_1 is V

$V = \tan\theta$



Cases :

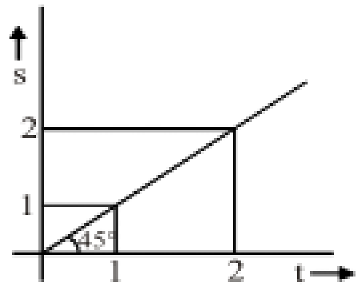
(A) **Uniform velocity :**

If velocity is uniform slope of curve must remain unchanged.

Curve with uniform slope is straight line

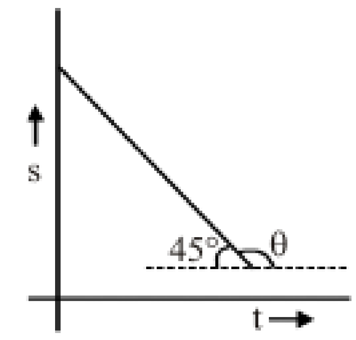
e.g. (i) $S = Vt$, If Velocity is $1 ms^{-1} \Rightarrow S = t$

$\tan\theta = 1$



e.g. (ii) If velocity is -1 m/s $\Rightarrow S = -t$

$\tan\theta = -1$

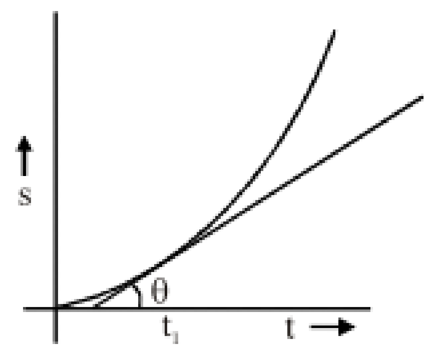


For negative velocity

(B) **Uniform acceleration**

We have a particle moving with uniform acceleration a and initial velocity u. Its displacement s at any time t can be represented as

$$s = ut + \frac{1}{2}at^2$$



Curve is parabola

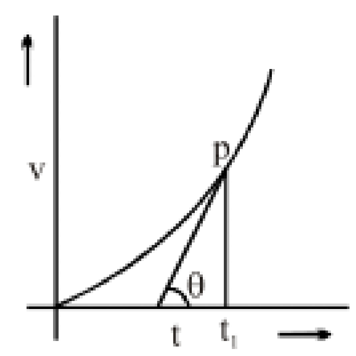
Velocity at t_1 is $\tan\theta$

Velocity Vs time curve

By using dependence of v on t we can plot a V s t graph.

Slope of v Vs t curve at any point represents acceleration at that instant.

$\tan\theta$ = acceleration at time t_1



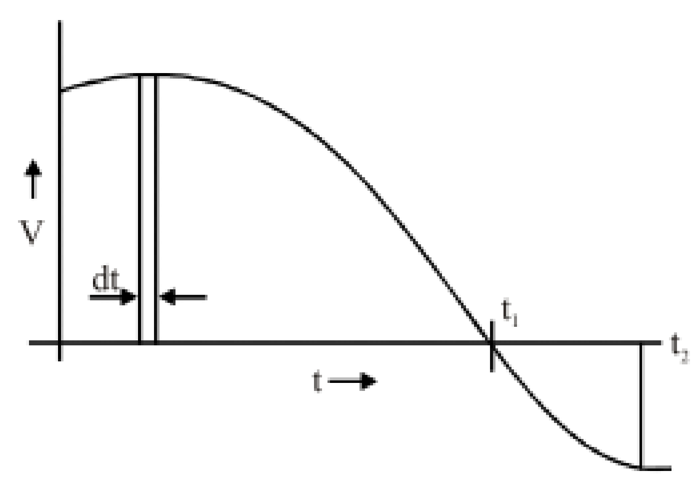
<u>Area under v Vs t graph</u> :

As we know $dx = Vdt$

$$\Rightarrow \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} vdt$$

$$\Rightarrow \Delta x = \int_{t_1}^{t_2} vdt$$

= Area under v Vs t graph.



Thus area under curve will represent displacement in that time period.

Note : (1) Area above t-axis +ve displacement.

(2) Area below t-axis is –ve displacement.

Thus,

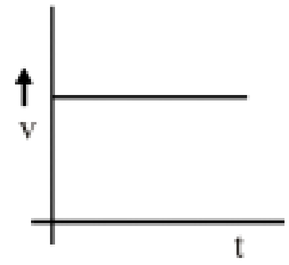
1. Total displacement will be sum of areas with appropriate signs.
2. Total distance travelled will be sum of areas without sign.

Cases :

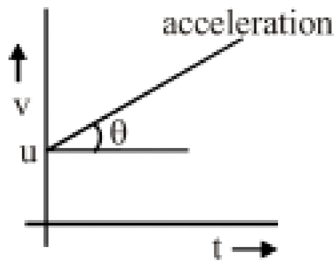
(1) For uniform velocity :

acceleration = 0

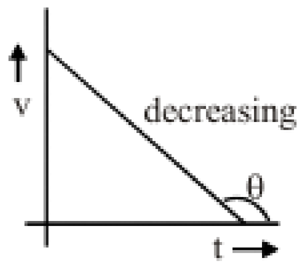
slope = 0



(2) For uniform st. line curve
$\tan\theta$ = acceleration
For increasing velocity



$\tan\theta$ = acceleration
for decreasing velocity
(slope is –ve) i.e. $\theta > 90°$



Note θ is always with + ive x - axis

Table for :

Variation of Displacement (s), velocity (v) and acceleration (a) with respect to time for different type of motion,

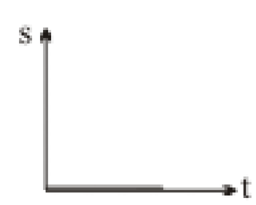
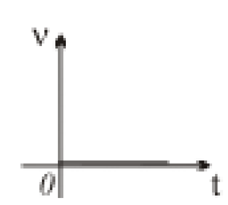
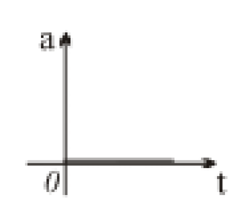
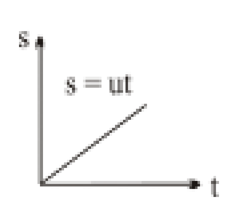
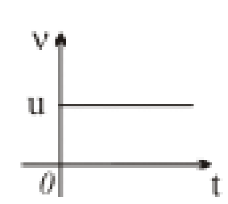
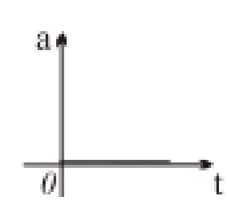
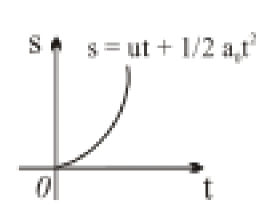
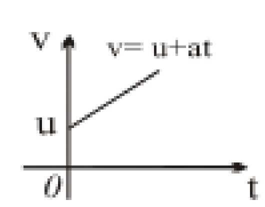
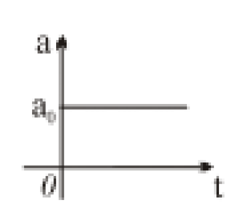
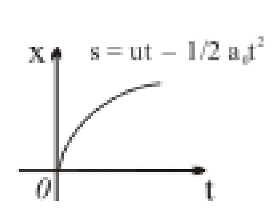
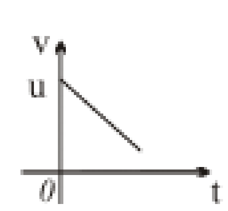
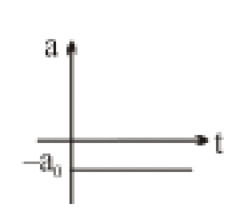
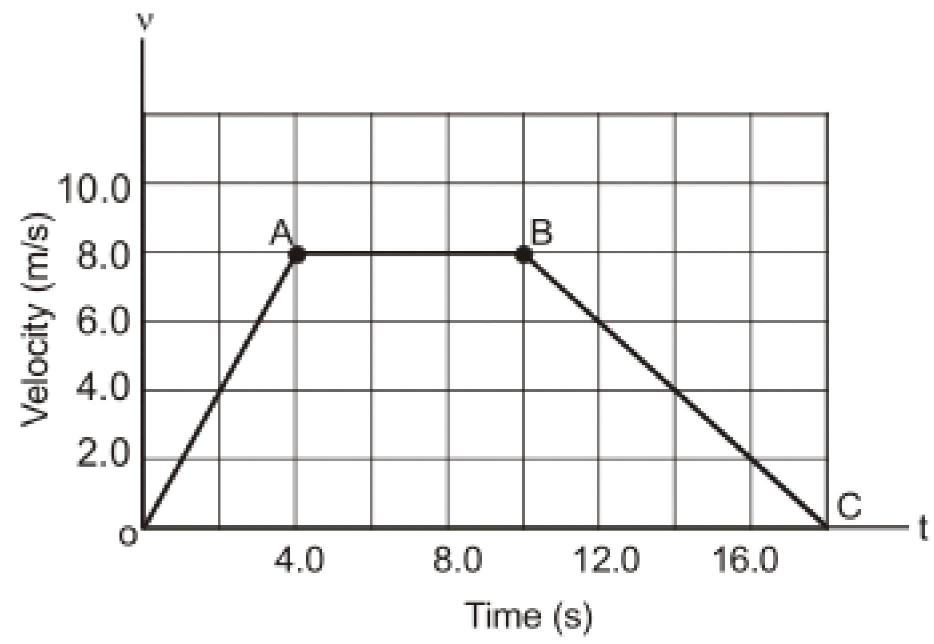
	Displacement	Velocity	Acceleration
1. At rest			
2. Motion with constant velocity			
3. Motion with constant acceleration			
4. Motion with constant deceleration			

Illustration :

What is the average acceleration for each graph segment in figure? Describe the motion of the object over the total time interval. Also calculate displacement.



Sol. Segment OA; $a = \dfrac{8-0}{4-0} = 2\ m/s^2$

Segment AB; graph horizontal i.e., slope zero i.e., $a = 0$

Segment BC; $a = \dfrac{0-8}{18-10} = -1 m/s^2$

The graph is trapezium. Its area between $t = 0$ to $t = 18s$ is displacement

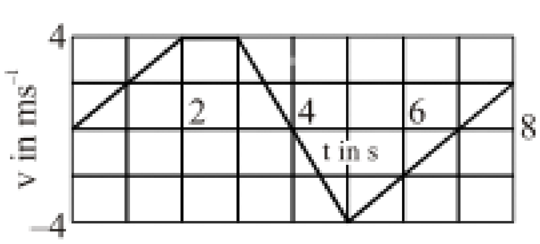
Area = displacement = $\dfrac{1}{2}(18 + 6) \times 8 = 96m$

Particle accelerates uniformly for first 4 sec., then moves uniformly for 6 sec. and then retards uniformly to come to rest in next 8 sec.

Illustration :

Figure here gives the velocity time graph for a body. Find the displacement and distance travelled between $t = 0s$ and $t = 7.0$:

Sol Area between $t = 0$ sec. to $t = 4$ sec.



$$= \frac{1}{2} \times (4 + 1) \times 4 = 10\ m$$

Area between $t = 4$ sec. to $t = 7$ sec.

$$= \frac{1}{2} \times 3 \times (-4) = -6$$

Net displacement = total area = $10 - 6 = 4\ m$

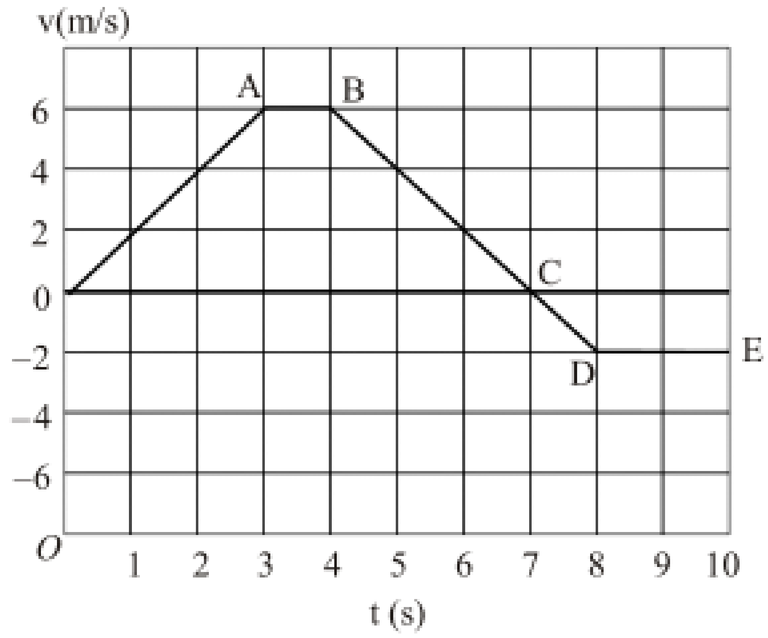
Distance = $|10| + |-6| = 16\ m$

Illustration :

Figure is a graph of V versus t for a particle moving along a straight line. The position of the particle at time $t = 0$ is $x_0 = 0$.

(a) *Find x for various times t and sketch x versus t.*

(b) *Sketch the acceleration a versus t.*



Sol. ***Segment OA***;

Displacement $= x_A - x_0 = x_A - 0 = x_A$

Also, displacement = area between O and A $= \frac{1}{2} \times 6 \times 3 = 9\ m$

$\therefore \qquad X_A = 9\ m$

Again, acceleration = slope of segment OA $= \frac{6}{3} = 2\ m/s^2$

In this segment v and a both are positive, so speed increases.

Segment AB;

Displacement $= x_B - x_A = x_B - 9$

Also, displacement = area between A and B $= 6 \times 1 = 6m$

$\therefore \qquad x_B - 9 = 6 \qquad or, \qquad x_B = 15$

Again, acceleration = slope of segment AB = 0

In this segment acceleration is zero, so speed is constant.

Segment BC ;

Displacment $= x_C - x_B = x_C - 15$

Also, displacement = area between B and C $= \frac{1}{2} \times 3 \times 6 = 9m$

$\therefore \qquad x_C - 15 = 9 \qquad or, \qquad x_c = 24\ m$

Again, acceleration = slope of segment BC $= \frac{0-6}{7-4} = -2\ m/s^2$

In this segment velocity is positive but acceleration is negative, so particle decreases its speed.

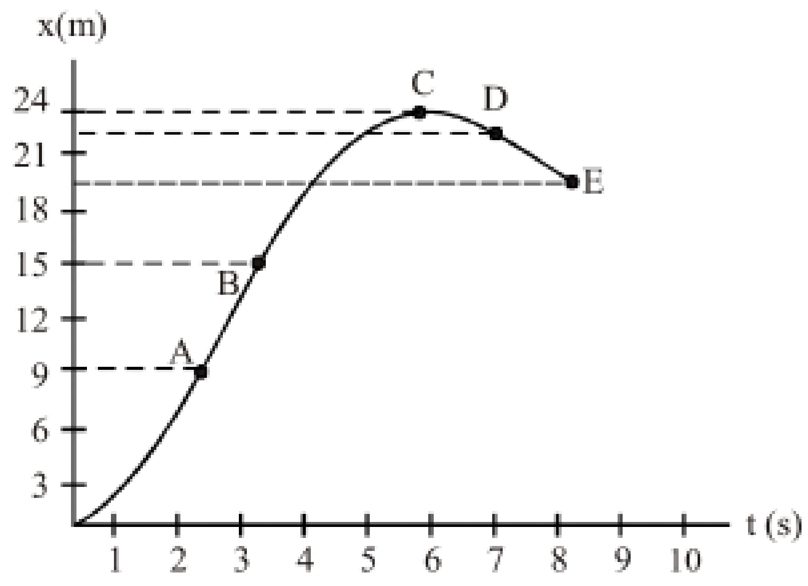
Similarly, for ***segment CD****; we have*

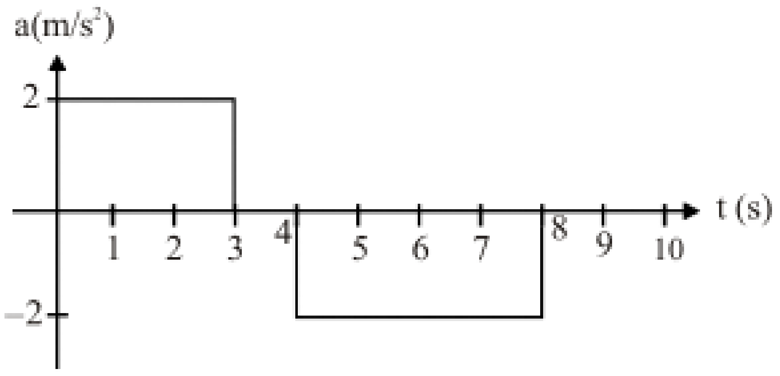
$$x_D = 23\ m \quad \text{and} \quad a = -2\ m/s^2$$

and for segment DE;

$x_E = 19\ m$ *and* $a = 0\ m/s^2$

The graphs are shown below :



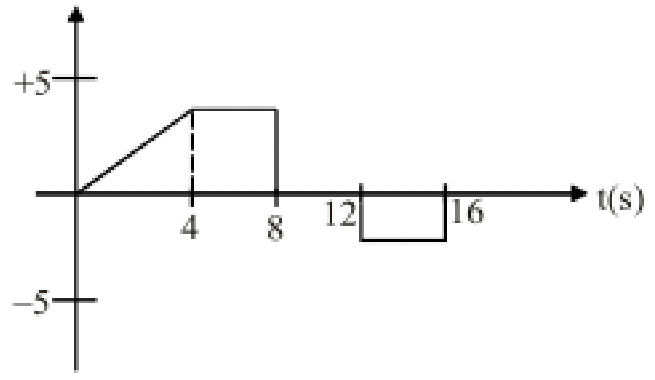


Practice Exercise

Q.1 A particle starts moving with speed 3 m/s and accelerates for 5sec. with acceleration $2m/s^2$. Find the displacement of the particle.

Q.2 A particle has an initial velocity of 9 m/s due east and has a constant acceleration of $2m/s^2$ due west. Find the distance covered by the particle in the 5^{th} second of its motion.

Q.3 The acceleration of a particle traveling along a straight line is shown in the figure. What is the maximum speed of the particle ?



Q.4 A runner is at the position x = 0 m when time t = 0s. One hundred meters away is the finish line. Every ten seconds, this runner runs half the remaining distance to the finish line. During each ten-second segment, the runner has a constant velocity. For the first thirty seconds of the motion, construct

(a) the position-time graph.

(b) the velocity-time graph.

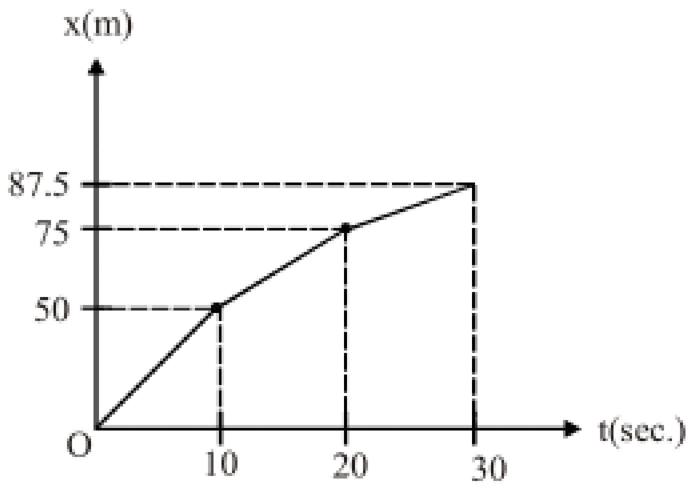
Q.5 Velocity of particle starting from rest varies with position according to equation $v = \sqrt{\alpha x}$. What is distance travelled by particle in t second from start?

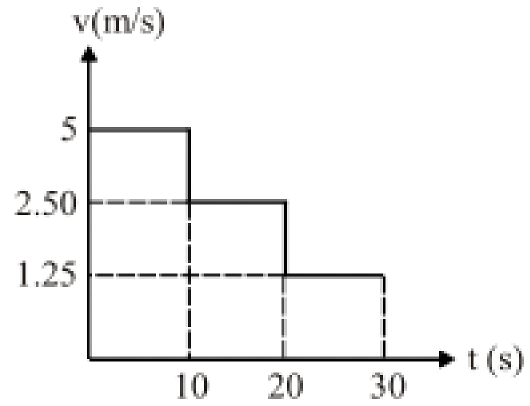
Q.6 A body starts from origin and moves along x-axis such that at any instant velocity is $v = 4t^3 - 2t$. Find the acceleration of the particle when it is 2 m from the origin.

Answers

Q.1 2sec Q.2 0.5m Q.3 30 m/s

Q.4





Q.5 $\frac{1}{4}\alpha t^2$ Q.6 22 m/s^2

Vertical motion under gravity (Free fall)

Motion that occurs solely under the influence of gravity is called free fall. Thus a body projected upward or downward or released from rest are all under free fall.

In the absence of air resistance all falling bodies have the same acceleration due to gravity, regardless of their sizes or shapes.

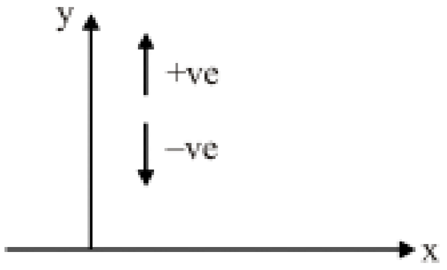
The value of the acceleration due to gravity depends on both latitude and altitude. It is approximately 9.8 m/s^2 near the surface of the earth. For simplicity a value of 10 m/s^2 is used. To do calculations regarding motion under gravity, we follow a proper sign convention. We are taking upward direction as positive and downward as negative, Thus acceleration is taken $a = -g = 10\ m/s^2$ no matter whether body is moving upwards or downwards, since **g** always acts downward.

Thus the equation of kinematics may be modified as

$v = u - gt$ .... (i)

$\Delta y = y - y_0 = ut - \frac{1}{2}gt^2$ .... (ii)

$v^2 = u^2 - 2g\,(y - y_0)$ .... (iii)



These y_0 = position of particle at time $t = 0$

y = position of particle at time t.

u = velocity of particle at time $t = 0$

v = velocity of particle at time t.

Illustration :

A man is standing on the top of a building, throws a ball with speed 5m/s from 30 height above the ground level. How much time it takes to reach the ground.

Sol. $u = 5m/s$

when it reaches the ground, $\Delta y = -30m$

$\therefore$ *from above equation (ii)*

$$-30 = 5t - \frac{1}{2}(10)\, t^2$$

$\Rightarrow \quad t^2 - t - 6 = 0$

On solving, we get $t = 3$ & -2

Rejecting $t = -2$ *sec, we get* $t = 3$ *sec*

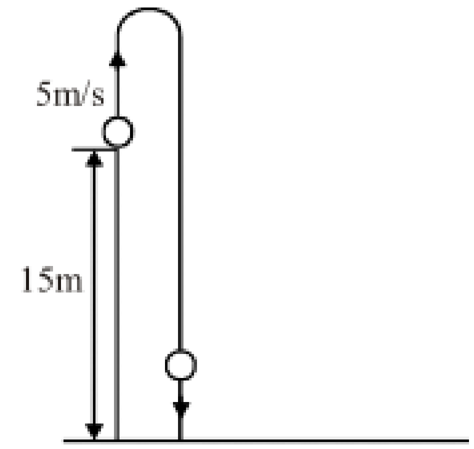
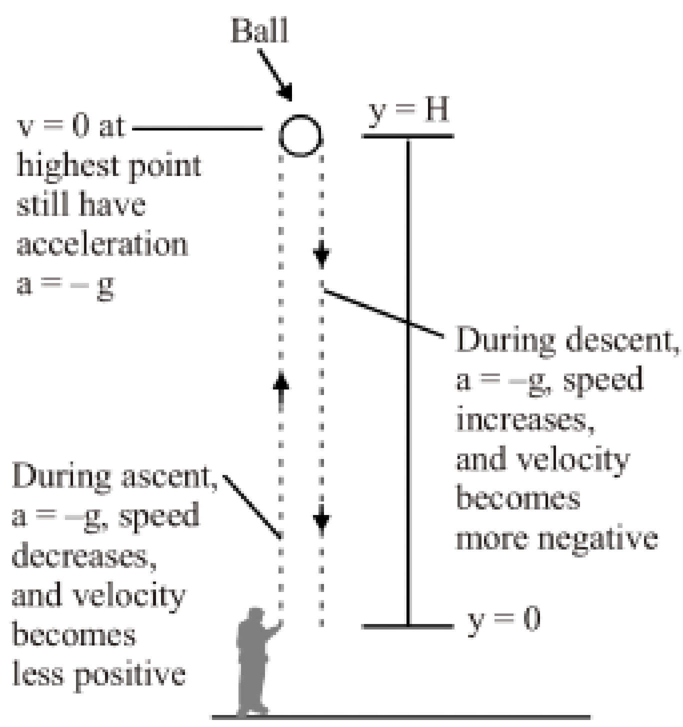


Illustration :

A kid throws a ball up, with some initial speed. Comment on magnitudes and signs of acceleration and velocity of the ball.

Sol.



Here : *(i) During ascent,* $a = -g$*, velocity becomes less positive i.e., speed decreases*

(ii) During descend, $a = -g$*, but now it is in the direction of velocity so it is not retardation. It makes velocity becomes more negative i.e. increases v in negative direction.*

Some results

1. *Maximum Height : -* $\mathbf{H = \frac{u^2}{2g}}$

Derivation : At maximum height $v = 0$

$\therefore$ from equation (iii), $v^2 = u^2 - 2gH = 0 \Rightarrow H = \frac{u^2}{2g}$

2. *Time to reach maximum height : -* $\boldsymbol{t = \frac{u}{g}}$

Derivation : At maximum height $v = o = u - gt$ [equation (i)]

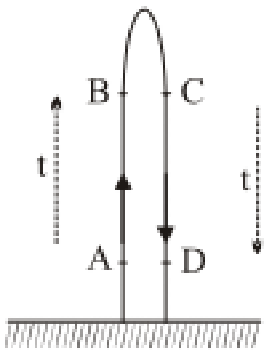
$\therefore \quad t = \frac{u}{g}$

3. Total time of flight = time to go up + time to move down (to reach the same horizontal level again)

$$T = 2t$$

$$\mathbf{T = \frac{2u}{g}}$$

4. Time of ascent = Time of descent for motion between two points at same horizontal level for example between A & B and between C & D shown in the figure.



5. If an object is dropped (means initial velocity is zero) from Height h. Its speed on reaching ground is $v = \sqrt{2gh}$ and time taken to reach ground is $t = \sqrt{\frac{2h}{g}}$

Derivation : From equation (iii) $0 - 2g(-h) = v^2$ $\quad [\therefore \Delta y = -h]$

Also from equation (ii) $\Delta y = -h = 0 - \frac{1}{2} gt^2$

$\therefore \quad t = \sqrt{\frac{2h}{g}}$

6. A particle has the same speed at a point on the path. While going vertically up and down.

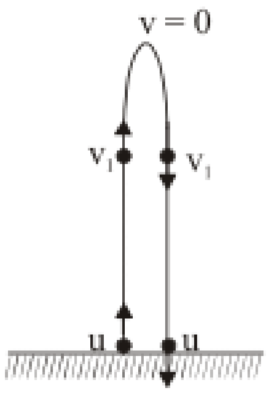


Illustration :

A ball is released from the top of a building. It travels 25 m in last second of its motion before striking the ground. Find height of the building. Take g = 10 m/s^2.

Sol. *Let it takes 't' time to strike the ground.*

$|\Delta y \text{ in } t \text{ sec}| - |\Delta y \text{ in } (t-1) \text{ sec}| = 25$

$$\frac{1}{2} gt^2 - \frac{1}{2} g (t-1)^2 = 25$$

on solving , $t = 3$ *sec*

$\therefore$ *height of the building,* $h = \frac{1}{2} g (3)^2$

$$h = 45 \text{ m}$$

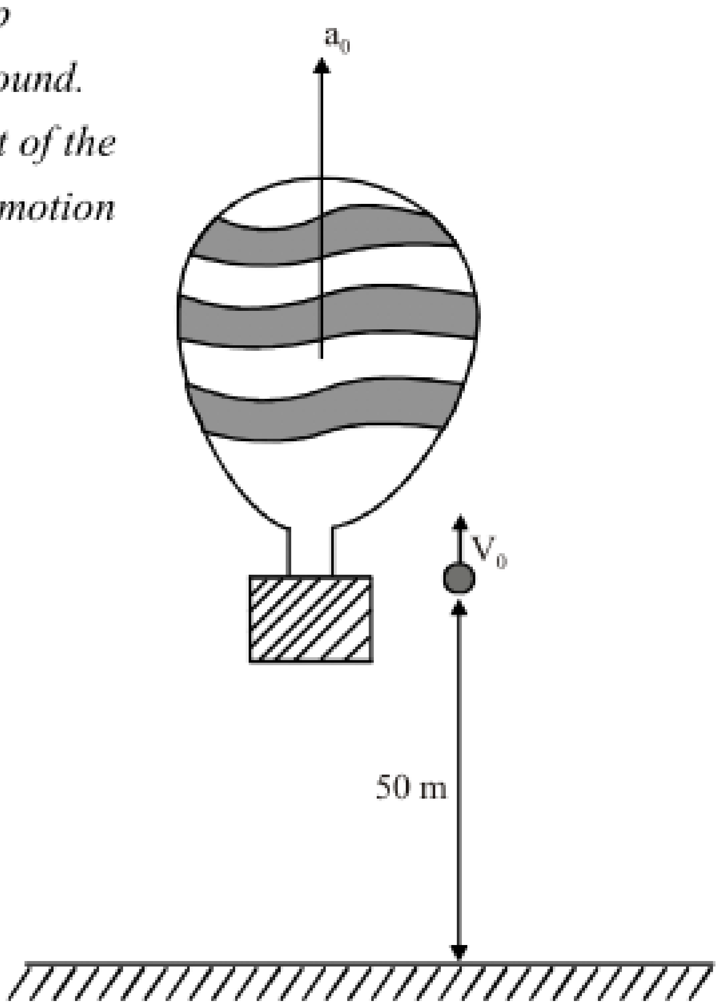
Illustration :

A Baloon is moving up with an acceleration $a_0 = 4 \text{ m/s}^2$ *starting from rest. A coin is dropped from the baloon 5 sec after the start balloon. Find:*

(a) The initial velocity of the dropped coin.

(b) The height attained by the lift till the time of drop

(c) The time after the drop when the coin reaches ground.



Sol. *Till t = 5sec, the coin shares the same motion as that of the baloon and for t > 5sec (after release) the coin has motion under gravity only.*

(a) Velocity of the coin just after it is dropped

V_0 = *velocity of the lift at 5 sec*

$= 0 + a_0 (5) = 20 \text{ m/s}$

(b) The height attained by the lift till 5 sec.

$$h = \frac{1}{2} a_0 (5)^2 = 50 \text{ m}$$

(c) Let it takes t_o *time to reach the ground after the drop i.e. for the time* t_0 *its displacement is 50 m in downword direction.*

$$\therefore \quad \Delta y = -50 = 20t_0 - \frac{1}{2} g t_0^2$$

on solving, $t_0 = 5.74$ *sec.*

Motion In Two Dimensions

Whatever we have studied in kinematics of one dimensional motion, we apply the same for motion in two and three dimensional motion, for x, y & Z components separately.

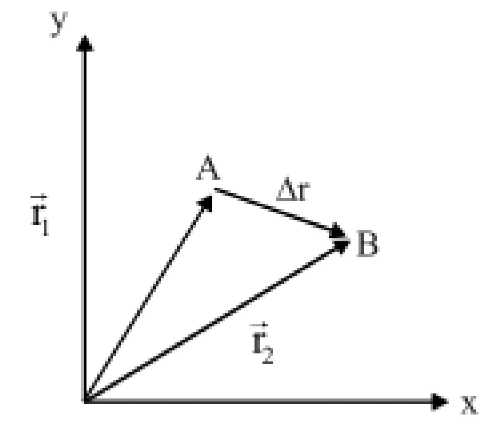
Suppose a particle has position coordinates (x,y) at any instant, then its position vector is

given by, $\vec{r} = x\hat{i} + y\hat{j}$

If particle moves from point A to B, through any path, then its displacement is

$$\Delta \vec{r} = \vec{r}_2 - \vec{r}_1$$

$= \Delta x\,\hat{i} + \Delta y\,\hat{j}$



Now at any instant, its velocity is given by

$$\vec{V} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j}$$

i.e $\quad V_x = \dfrac{dx}{dt}$ $\quad$ i.e. x - component of velocity.

and $\quad V_y = \dfrac{dy}{dt}$ $\quad$ i.e. y - component of velocity

Similarly $\vec{a} = \dfrac{d\vec{v}}{dt} = a_x\hat{i} + a_y\hat{j}$

Where $a_x = \dfrac{dv_x}{dt}$ & $a_y = \dfrac{dv_y}{dt}$

Projectile Motion

It consists of two independent motions, a horizontal motion at constant velocity and a vertical motion under acceleration due to gravity.

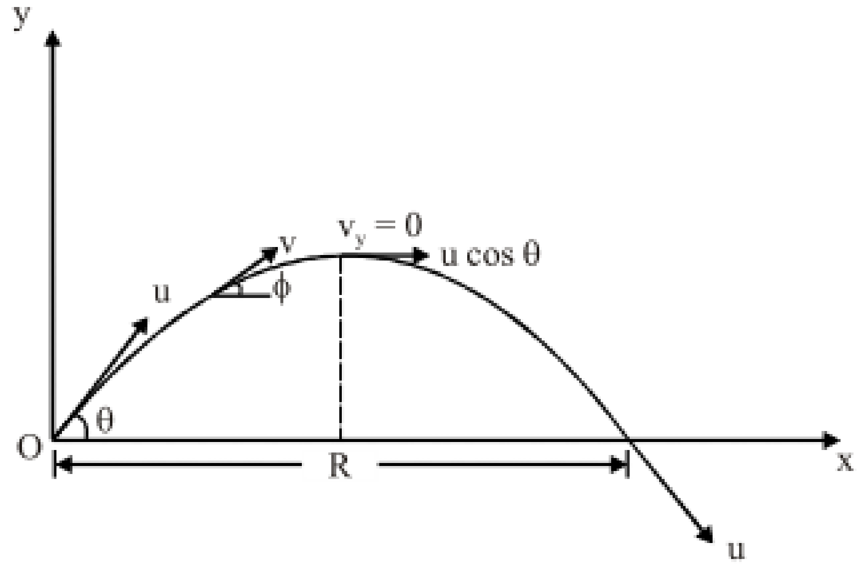
In order to deal with problems in projectile motion, one has to choose a coordinate system. Let's take horizontal as x-axis and vertical upward direction as y-axis, then

$\mathbf{a_x = 0}$ and $\mathbf{a_y = -g}$; since there is only one force "mg" downward (negative air resistance)

Equation along x-axis	**Equations along y-axis**
$v_x = u_x$ (constant)	$v_y = u_y - gt$
$\Delta x = u_x t$	$\Delta y = u_y t - \frac{1}{2}gt^2$
	$v_y^2 = u_y^2 - 2g(\Delta y)$

If an object is dropped from rest or projected up or down, it follows straight line path. If its initial velocity is not along the line of force it follows parabolic path which is proved mathematically in this topic later on.

Projectile Thrown from the Ground Level



A particle is projected from ground level at an angle θ from horizontal with speed u.

$\therefore\ u_x = u\cos\theta$ and $u_y = u\sin\theta$

At any instant, $v_x = u_x = u\cos\theta$ & $v_y = u_y - gt = u\sin\theta - gt$

$$\Delta x = (u\cos\theta)\,t \qquad \& \qquad \Delta y = (u\sin\theta)\,t - \frac{1}{2}gt^2$$

Time of flight (T) : Let it strikes the ground again at time T.

i.e. for $t = T$, $\Delta y = 0$ $u_y = 0 = u_yT - \frac{1}{2}gT^2$

$$\therefore \quad \boxed{T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}}$$

Horizontal Range (R) It is horizontal displacement till time $t = T$

i.e. $R = u_x\,T$

$$\boxed{R = \frac{2u_xu_y}{g}}$$

i.e. $$R = \frac{2(u\cos\theta)(u\sin\theta)}{g} = \frac{u^2\sin(2\theta)}{g}$$

Maximum height (H)

$H = \Delta y$ when $v_y = 0$

$\therefore\ v_y^{\,2} = u_y^{\,2} - 2g\,(H) = 0$

$$\therefore \quad \boxed{H = \frac{u_y^2}{2g} = \frac{u^2\sin^2\theta}{2g}}$$

From above formulae, we can observe

(i) $T \propto u_y$ i.e. depends only on vertical component of initial velocity

(ii) $H \propto u_y^{\,2}$ i.e. depends only on vertical component of initial velocity

(iii) $R \propto u_xu_y$ i.e. depends both on horizontal and vertical components of initial velocity.

Velocity at any general point

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2} = \sqrt{u^2 + g^2t^2 - 2(u\sin\theta)gt}$$

If angle which direction of motion makes at an instant is ϕ, then

$$\tan\phi = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$$

$\tan\phi$ is positive during its upward notion i.e. before reaching highest point and after that $\tan\phi$ is negative.

Illustration :

A particle is projected with 20 m/s at an angle 60° with the horizontal. At what time it is moving at an angle 45° with the horizontal while moving downwards.

Sol. $u_x = 20\cos 60° = 10\ m/s$

& $u_y = 20\sin 60° = 10\sqrt{3}\ m/s$

At required instant, $\tan\phi = -1$

i.e. $$\frac{u_y - gt}{u_x} = -1$$

i.e. $\frac{10\sqrt{3}-10t}{10} = -1$

on solving, me get $t = (\sqrt{3}+1)$sec

Illustration :

A particle is projected in the X-Y plane with y-axis along vertical. At 2 sec after projection the velocity of the particle makes an angle 45° with the X - axis and 4 sec after projection, it moves horizontally. Find the velocity of projection.

Sol. At $t = 2$ sec, $\tan\phi = \frac{u_y - 10(2)}{u_x} = 1$ $(\because \phi = 45°)$

$\Rightarrow \quad u_y - 20 = u_x \quad$ .... (i)

Also $\frac{1}{2}$ (time of flight) = 4 sec

$\Rightarrow \quad \frac{1}{2}\left(\frac{2u_y}{g}\right) = 4 \quad \Rightarrow \quad u_y = 40$ m/s

$\therefore$ from equation (i), $u_x = 20$ m/s

$\therefore \quad u = \sqrt{u_x^2 + u_y^2} = 20\sqrt{5}$ m/s

Equation of Trajectory :

$$y = u \sin\theta\, t - (1/2)\, gt^2$$

and $\quad x = (u\cos\theta)\, t \quad \Rightarrow \quad t = \frac{x}{u\cos\theta}$

From these equations, (eliminating t)

$$\mathbf{y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2}$$

The above relation between x and y is equation of parabola, which proves that the trajectory i.e. path of projectile is parabolic.

Illustration :

The path followed by a body projected along y axis is given as by $y = \sqrt{3}x - (1/2)x^2$. If $g = 10$ m/s^2 then the initial velocity of projectile will be- (x and y are in m)

(A) $3\sqrt{10}$ m/s (B) $2\sqrt{10}$ m/s (C) $10\sqrt{3}$ m/s (D) $10\sqrt{2}$ m/s

Sol. Given, that $y = \sqrt{3}\,x - (1/2)\,x^2$ ....(1)

The above equation is similar to equation of trajectory of the projectiles

$y = \tan\theta\ x - 1/2\ \frac{g}{u^2\cos^2\theta}\ x^2$ ....(2)

Comparing (1) & (2) we get

$\tan\theta = \sqrt{3}$

$\Rightarrow \quad \theta = 60°$

and $\quad 1/2 = (1/2)\dfrac{g}{u^2\cos^2\theta}$

$\Rightarrow \quad u^2\cos^2\theta = g$

$\Rightarrow \quad u^2\cos^2 60 = 10$

$\Rightarrow \quad u^2 (1/4) = 10$

$\Rightarrow \quad u = 2\sqrt{10}$ m/s

Hence correct answer is (B)

Other points of remember

* $$R = \frac{u^2\sin(2\theta)}{g}$$

Range is maximum, when $\theta = 45°$

and $\quad R_{max} = \dfrac{u^2}{g}$

* For two objects projected with **same speed** Range is same for two angles of projection θ & $(90–\theta)$

Proof : Let $R_1 = R_2$ for θ and α

i.e. $\dfrac{u^2\sin(2\theta)}{g} = \dfrac{u^2\sin(2\alpha)}{g}$

i.e. $\sin(2\theta) = \sin(2\alpha)$

i.e. $2\theta = 180° - 2\alpha$

$\Rightarrow \alpha = 90° - \theta$

* $H = \dfrac{u^2\sin^2\theta}{g}$ is maximum i.e. $\left(\dfrac{u^2}{2g}\right)$ if projected at $\theta = 90°$

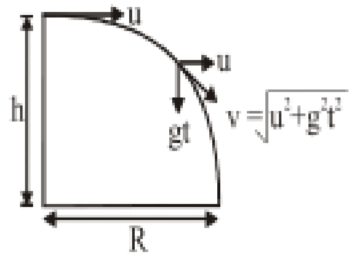
Horizontal Projection :

	In Horizontal Direction		In Vertical Direction
(i)	Initial velocity $u_x = u$	(i)	Initial velocity $u_y = 0$
(ii)	Acceleration $= 0$	(ii)	Acceleration = 'g' downward
(iii)	Horizontal velocity of particle remains same after time t horizontal velocity $= v_x = u$	(iii)	Velocity of particle after time t $v_y = 0 + (-g)t$ $= -gt = gt$ (downward)
(iv)	Range $x = ut$	(iv)	Displacement $y = (1/2)\,gt^2$ (downward)

Velocity at a general point P(x, y):

$v = \sqrt{v_x^2 + v_y^2} \qquad \tan\phi = \dfrac{v_y}{v_x}$

ϕ is angle made by v with horizontal in clockwise direction



Time of flight:

$$-h = u_y t - (1/2)\ gt^2 = 0 - \frac{1}{2}\ gt^2$$

$$t = \pm\sqrt{\frac{2h}{g}}$$

$$t = +\sqrt{\frac{2h}{g}}$$

(negative time is not possible)

Range:

$$R = u_x\ t = u\sqrt{\frac{2h}{g}}$$

Note :

If a projectile is projected with initial velocity u and another particle is dropped from same height at the same time, both the projectile would strike the ground at the same instant velocity. Both will have same vertical components of velocity but their net velocities would be different.

Illustration :

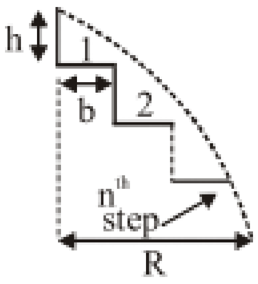
A ball rolls off top of a stair way with a horizontal velocity u m/s. If the steps are h m high and b meters wide, the ball will just hit the edge of n^{th} step if n equals to-

(A) $\frac{hu^2}{gb^2}$ (B) $\frac{u^2g}{gb^2}$ (C) $\frac{2hu^2}{gb^2}$ (D) $\frac{2u^2g}{hb^2}$

Sol. *If the ball hits the n^{th} step, the horizontal and vertical distances traversed are nb and nh respectively. Let t be the time taken by the ball for these horizontal and vertical displacement. Then velocity along horizontal direction remains constant = u and initial vertical velocity is zero*

$\therefore\quad nb = ut$ ....(1)

$\&\ nh = 0 + (1/2)\ gt^2$ ....(2)



From (1) & (2) we get

$n\ h = (1/2)\ g\ (nb/u)^2$

$\Rightarrow\quad n = \frac{2hu^2}{gb^2}$ *(eliminating t)*

Hence correct answer is (C)

Illustration :

An aeroplane is flying horizontally with a velocity of 720 km/h at an altitude of 490 m. When it is just vertically above the target a bomb is dropped from it. How far horizontally it missed the target? (Take g = 9.8 m/s²)

(A) 1000 m (B) 2000 m (C) 100 m (D) 200 m

Sol. *Horizontal component of velocity*

$$= 720 \times 5/8 = 200 \ m/s$$

Let t be the time taken for a freely falling body from 490. Then

$$y = (1/2)\ gt^2$$

$$\Rightarrow \quad 490 = (1/2) \times 9.8 \times t^2$$

$$\Rightarrow \quad t = 10 \text{ second}$$

Now horizontal distance

= Velocity × time = 200 × 10 = 2000 m

Hence the bomb missed the target by 2000 m

Hence correct answer is (B)

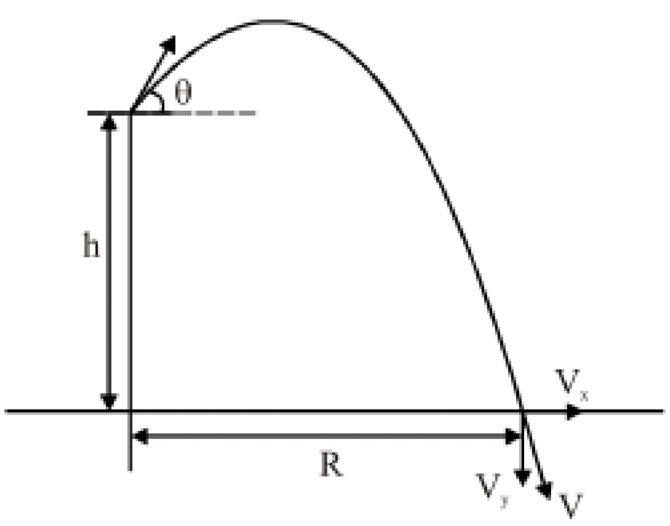
Projected from some height at some angle

Case I :

When projected at some angle θ with the horizontal towards upward direction.

Let it takes time $\mathbf{t_1}$ (time of flight) to strike the ground

$\Delta y = -h$ When $t = \mathbf{t_1}$



$$\therefore \quad -h = (u \sin\theta)\, t_1 - \frac{1}{2} g t_1^2$$

$$\Rightarrow \quad t_1^2 - \left(\frac{2u\sin\theta}{g}\right) t_1 - \frac{2h}{g} = 0$$

$$\therefore \quad \mathbf{t_1} = \frac{T + \sqrt{T^2 + 8h/g}}{2}$$

$$\left(\text{where } T = \frac{2u\ \sin\theta}{g}\right)$$

Also $\mathbf{R} = \Delta x = u_x t_1$ (putting values of t_1 & u_x we can find **R** whenever required)

When it reaches ground $v_x = u \cos\theta$

$$\& \ v_y^2 = u_y^2 - 2g\,(\Delta y) \Rightarrow v_y = \sqrt{(u \sin\theta)^2 - 2g(-h)}$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$$

Case II :

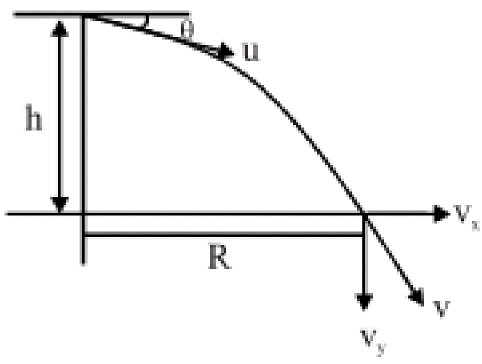
When projected at angle θ with horizontal towards downward direction

Here $u_y = -u \sin\theta$

Thus, if it takes time

$\mathbf{t_2}$ to strike the ground then

$$-h = -(u\sin\theta)\,t_2 - \frac{1}{2}g\,t_2^2$$



$$\Rightarrow \quad t_2 + \left(\frac{2u\sin\theta}{g}\right)t_2 - h = 0$$

$$\therefore \quad \mathbf{t_2} = \frac{\sqrt{T^2 + 8h/g} - T}{2}$$

Also $\mathbf{R} = u_x\, t_2$

Here on reaching ground, $v_x = u\cos\theta$

and $v_y = \sqrt{(u\sin\theta)^2 - 2g(-h)}$

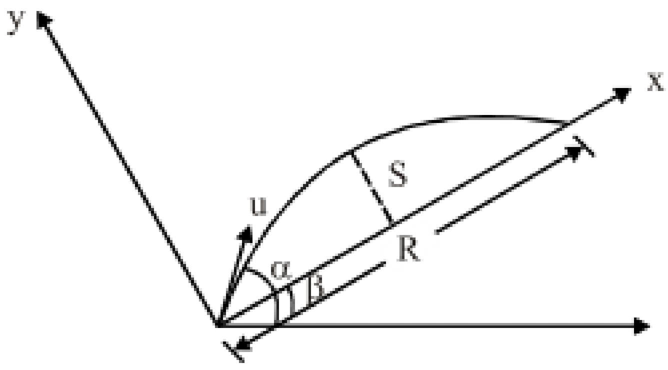
$$\therefore \quad v = \sqrt{v_y^2 + v_x^2} = \sqrt{u^2 + 2gh}$$

Thus we can observe if some particles are projected from same height with same speed, they reach the ground with same speed whatever may be the angles of their projection.

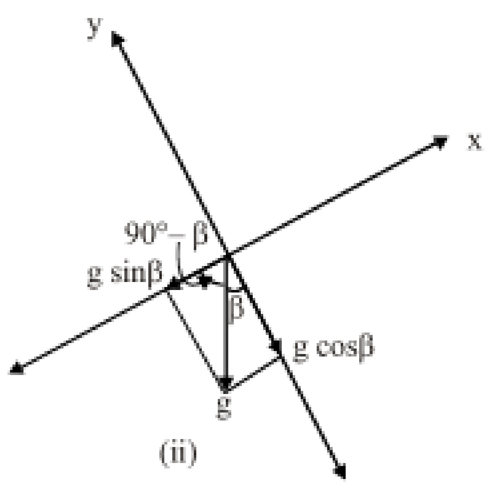
Projectile thrown On An Inclined plane

To deal with problems of projectile thrown along an incline we choose the x-axis along the plane and y-axis perpendicular to the plane.

Let a particle is projected at an angle α with the horizontal on an incline plane which has angle of inclination β with the horizontal



(i)



(ii)

From fig (i) ; $u_x = u\cos(\alpha - \beta)$; $u_y = u\sin(\alpha - \beta)$

[∴ angle of projection with the incline is $(\alpha - \beta)$]

From fig(ii) ; $a_x = -g\sin\beta$; $a_y = -g\cos\beta$

(a) Time of flight (T) on the incline

At t = T, it strikes the incline

i.e. $\Delta y = 0$

$$u_y T + \frac{1}{2} a_y T^2 = 0$$

$$\Rightarrow \quad u \sin(\alpha - \beta)\, T - \frac{1}{2} g \cos\beta\, T^2 = 0$$

$$\Rightarrow \quad \mathbf{T = \frac{2u\sin(\alpha - \beta)}{g\cos\beta}}$$

(b) Range (R) Along the incline

$R = \Delta x$ till $t = T$

i.e. $R = u_x T + \frac{1}{2} a_x T^2$ (Here $a_x = -g \sin\beta \neq 0$)

Putting values of u_x, a_x & T, we get

$$\mathbf{R = \frac{2u^2\sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}}$$

(c) Maximum range

From above formula, $R = \dfrac{u^2[\sin(2\alpha - \beta) - \sin\beta]}{g\cos^2\beta}$

R is maximum when $\sin(2\alpha - \beta) = 1$

i.e. $2\alpha - \beta = 90°$

$$\Rightarrow \quad \alpha = 45° + \frac{\beta}{2}$$

Also, $R_{max} = \dfrac{u^2(1 - \sin\beta)}{g\cos^2\beta} = \dfrac{u^2(1 - \sin\beta)}{g(1 - \sin^2\beta)}$

$$\therefore \quad R_{max} = \frac{u^2}{g(1 + \sin\beta)}$$

(d) Greatest distance from incline

$S = \Delta y$ when $v_y = 0$ i.e. when $u_y + at = 0$

i.e. when $u \sin(\alpha - \beta) - g \cos\beta\, t = 0$

$$\Rightarrow \quad t = \frac{u\sin(\alpha - \beta)}{g\cos\beta}$$

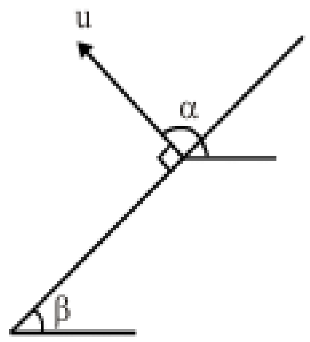
Now, $S = u_y t + \frac{1}{2} a_y t^2$

Putting values of u_y, a_y and t, we get

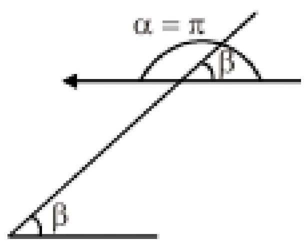
$$S = \frac{u^2 \sin(\alpha - \beta)}{2g \cos \beta}$$

Special cases :

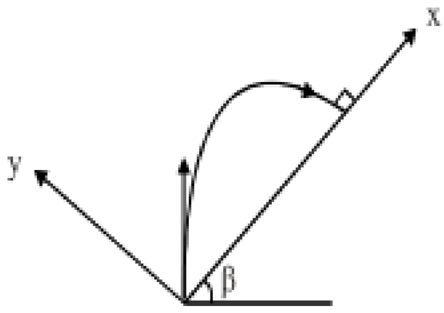
(1) If projected normally (i.e. perpendicular) to the plane, i.e. angle with plane $(\alpha - \beta) = 90°$
i.e. $\alpha = (90° + \beta)$



(2) If projected horizontally
i.e. $\alpha = 180°$ and angle with the incline $= \alpha - \beta = 180° - \beta$



(3) If the particle strikes normally to the plane i.e. at the moment of strike, $V_x = 0$
i.e. $u \cos(\alpha - \beta) - g \sin \beta\, t = 0$



Practice Exercise

Q.1 A juggler throws a boll vertically upward and catches it after 6 sec. Determine

(i) the initial velocity of the ball.

(ii) the maximum height attained by the ball.

(iii) the position of ball at t = 2 sec.

(iv) the time at which ball is 20 m below the topmost point

Q.2 A healthy yeoman standing at a distance of 7 m from a 11.8 m high building sees a kid slipping from the top floor. With what uniform speed should he run to catch the kid at his arms height (1.8m) ?

Q.3 A balloon starts rising from the ground with an acceleration of $1.25\ m/s^2$. After 8s a stone is released from the balloon. Find the time taken by the stone (after its fall) to reach ground.

Q.4 A particle is projected in such a way that its position coordinates vary with time as $y = 8t - 5t^2$ and $x = 6t$ (taking point of projection as origin). What is the range of projectile?

(A) 48 m (B) 4.8 m (C) 9.6 m (D) 24 m

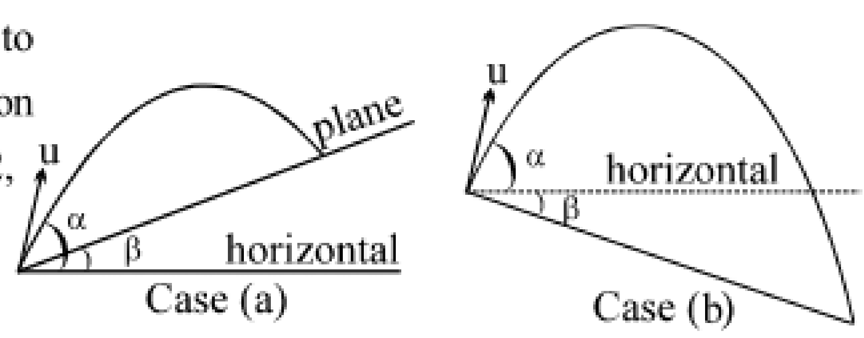
Q.5 A ball is projected upwards from the top of tower with a velocity 50 m/s making an angle 30° with the horizontal. The height of the tower is 70 m. After how many seconds from the instant of throwing will the ball reach the ground ?

(A) 2 s (B) 5 s (C) 7 s (D) 9 s

Q.6 A particle is projected upwards with a velocity of 100 m/s at an angle of 60°, with the vertical. Then time taken by the particle when it will move perpendicular to its initial direction-

(A) 10 sec (B) $\frac{20}{\sqrt{3}}$ sec (C) 5 sec (D) $10\sqrt{3}$ sec

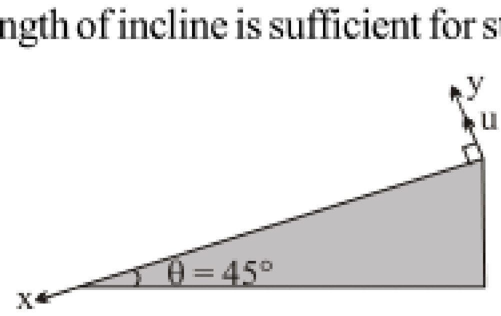
Q.7 A particle is projected at an angle 'α' to the horizontal. Up and down there is a plane in case (a) & case (b), inclined at an angle β to the horizontal. If the ratio of time of flights on these plane in case (a) & case (b) be 1 : 2, then the ratio $\frac{\tan\alpha}{\tan\beta}$ is equal to:



(A) $\frac{2}{1}$ (B) $\frac{3}{1}$ (C) $\frac{4}{1}$ (D) $\frac{5}{3}$

Paragraph for question no. 8 to 10

An inclined plane makes an angle $\theta = 45°$ with horizontal. A stone is projected normally from the inclined plane, with speed u m/s at $t = 0$ x and y axis are drawn from point of projection along and normal to inclined plane as shown. The length of incline is sufficient for stone to land on it and neglect air friction.



Q.8 The instant of time at which velocity of stone is parallel to x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.9 The instant of time at which velocity of stone makes an angle $\theta = 45°$ with positive x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.10 The instant of time till which (starting from $t = 0$) component of displacement along x-axis is half the range on inclined plane is

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Answers

Q.1 (i) 30m/s (ii) 45m (iii) 40 m (iv) 1sec. and 5 sec

Q.2 $\frac{7}{\sqrt{2}}$ m/s Q.3 4sec Q.4 (C) 9.6 m

Q.5 (C) 7 s Q.6 (B) $\frac{20}{\sqrt{3}}$ sec

Q.7 (B) $\frac{3}{1}$ Q.8 (C) $\frac{\sqrt{2}u}{g}$

Q.9 (D) $\frac{u}{\sqrt{2}\,g}$ Q.10 (B) $\frac{2u}{g}$

Q.8 The instant of time at which velocity of stone is parallel to x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.9 The instant of time at which velocity of stone makes an angle $\theta = 45°$ with positive x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.10 The instant of time till which (starting from $t = 0$) component of displacement along x-axis is half the range on inclined plane is

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Answers

Q.1 (i) 30m/s (ii) 45m (iii) 40 m (iv) 1sec. and 5 sec

Q.2 $\frac{7}{\sqrt{2}}$ m/s Q.3 4sec Q.4 (C) 9.6 m

Q.5 (C) 7 s Q.6 (B) $\frac{20}{\sqrt{3}}$ sec

Q.7 (B) $\frac{3}{1}$ Q.8 (C) $\frac{\sqrt{2}u}{g}$

Q.9 (D) $\frac{u}{\sqrt{2}\,g}$ Q.10 (B) $\frac{2u}{g}$

Solved Example

Q.1 The motion of an object falling from rest in a viscous medium can be described by the equation $a = \alpha - \beta$ v. Where a and v are the acceleration and velocity of the object and α and β are constants. Find.

(i) the initial acceleration.

(ii) the velocity at which acceleration becomes zero.

(iii) the velocity as a function of time.

Sol. (i) The initial velocity of object $v = 0$

So, initial acceleration $a = \alpha - (\beta \times 0) = \alpha$

(ii) For acceleration to be zero

$$0 = \alpha - \beta v \qquad \text{or,} \qquad v = \frac{\alpha}{\beta}$$

Note : The velocity at which acceleration reduces to zero is the maximum velocity with which an object will fall in a viscous medium. This velocity is called Terminal velocity.

(iii) $a = \frac{dv}{dt}$ or, $\alpha - \beta v = \frac{dv}{dt}$ or, $\frac{dv}{\alpha - \beta v} = dt$

Integrating the expression with boundary conditions : $t = 0$, $v = 0$ and $t = t$, $v = v$

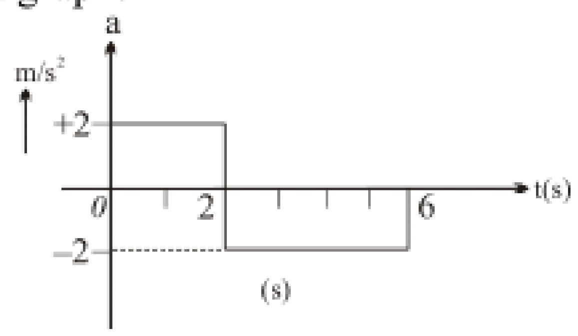
$$\int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt \qquad \text{or} \qquad \left(\frac{-1}{\beta}\right) ln\,(\alpha - \beta v)\Big|_0^v = t\Big|_0^t$$

$$\Rightarrow \quad -\frac{1}{\beta}\left[\ell n\,(\alpha - \beta v) - \ell n\,(\alpha)\right] = t - 0 \quad \Rightarrow \ln\left(\frac{\alpha - \beta v}{\alpha}\right) = -\beta t$$

Rearranging the terms, we get

$$v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

Q.2 At $t = 0$, a particle is at rest at origin. Its acceleration is $2 m/s^2$ for first 2 sec. and $-2\ m/s^2$ for next 4 sec as shown in a versus t graph.



Plot graphs for

(i) Velocity versus time

(ii) speed versus time

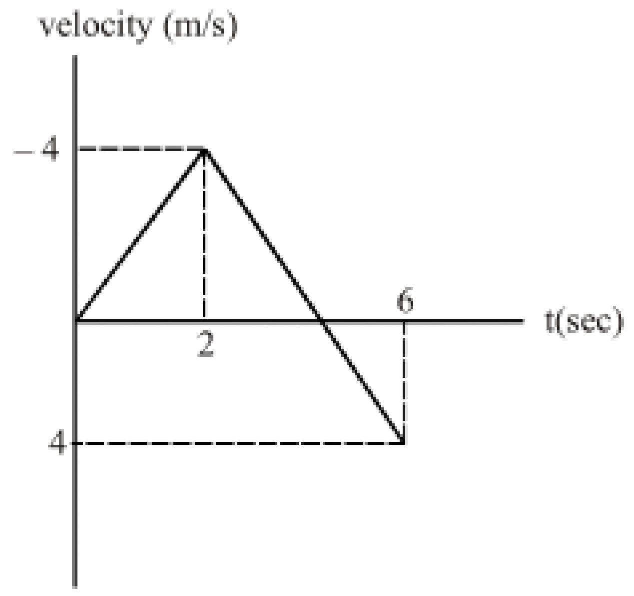
(iii) Displacement versus time

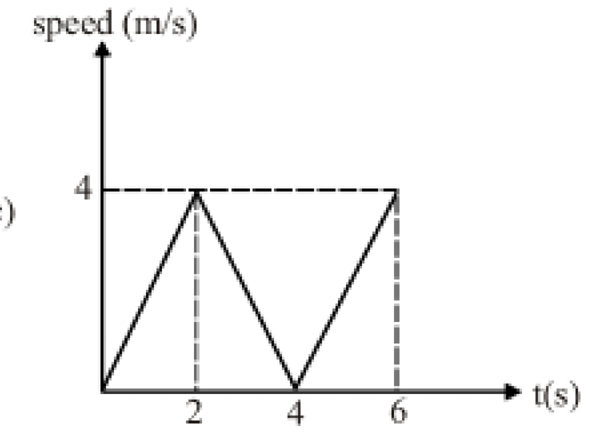
(iv) Distance versus time

Sol. (i) $V_2 - V_0$ = Area of a Vs t graph for $t = 0$ to $t = 2$ sec

$V_2 - 0 = 2 \times 2 \Rightarrow \quad V_2 = +4$ m/s

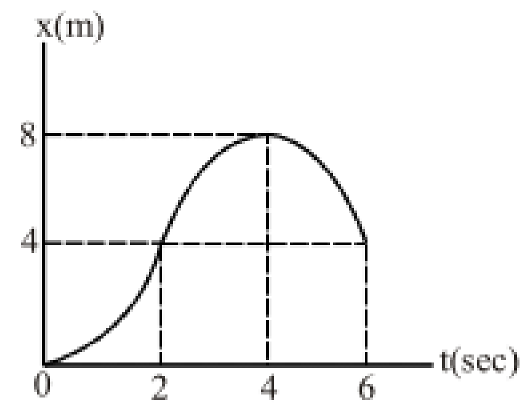
Now $V_6 - V_2 = -2 \times 4 \quad \Rightarrow \quad V_6 = -4$ m/s





(ii) Since slope of a Vs t graph from t = 2 to 6 sec. is constant, we can observe its speed i.e. magnitude of its velocity is zero at = 4 sec. and after that magnitude of velocity increases in negative direction up to 4 m/s at the same rate.

(iii) Displacement (x) Vs t



$x_2 - x_0$ = area of v vst graph for t = 0, t = 2sec

$x_2 - 0 = \frac{1}{2}(2)(4) \Rightarrow x_2 = +4\text{m}$

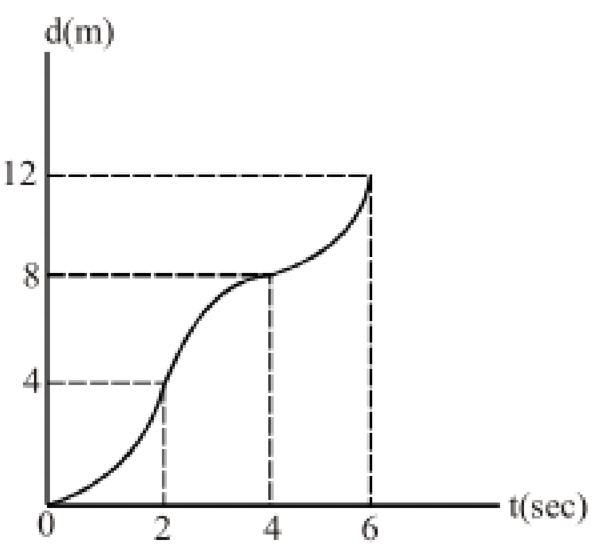
$x_4 - x_2 = \frac{1}{2}(4)(2)$

$x_4 = 8$ m

also $x_6 - x_4 = \frac{1}{2}(-4)(2) = -4$ m

$\therefore \quad x_6 = +4\text{m}$

(iv) Distance (d) vs t



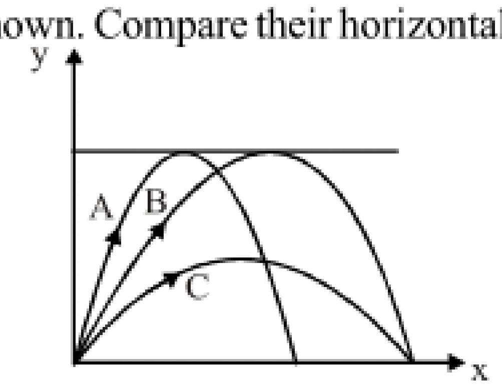
$d_2 - d_0 = \frac{1}{2}(2)(4) \quad \Rightarrow \quad d_2 = 4$ m

$d_4 - d_2 = \frac{1}{2}(2)(4) \quad \Rightarrow \quad d_4 = 8\text{m}$

Also $d_6 - d_4 = \left|\frac{1}{2}(2)(-4)\right| = 4$

$\Rightarrow d_6 = 12$ m

Q.3 Three particles are projected from same point and their paths are as shown. Compare their horizontal and vertical component of velocities of projection



Sol. $H_A = H_B > H_C \Rightarrow (u_y)_A = (u_y)_B > (u_y)_C$

$\therefore$ $R_B > R_A$

i.e. $(u_x u_y)_B > (u_x u_y)_A$ [$\because$ their u_x are equal]

$\therefore$ $(u_x)_B > (u_x)_A$

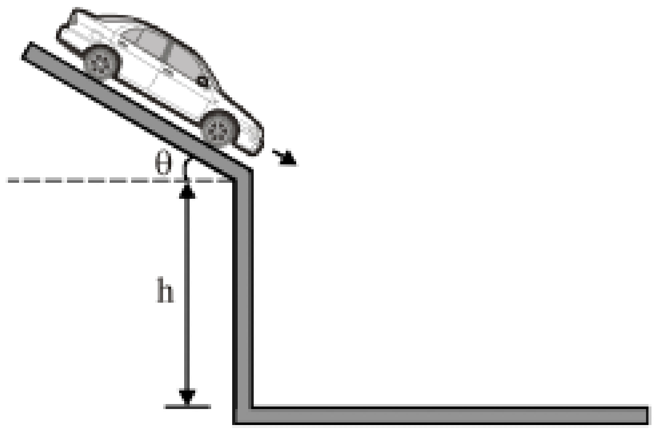
Also $R_B = R_C$

i.e. $(u_x u_y)_B = (u_x u_y)_C$

but $(u_y)_B > (u_y)_C$

$\therefore$ $(u_x)_C > (u_x)_B > (u_x)_C$

Q.4 A car goes out of control and slides off a steep embankment of height h at θ to the horizontal. It lands in a ditch at a distance R from the base. Find the speed at which the car leaves the slope. (Take h = 12.5 m ; R = 10 m ; θ = 45°)



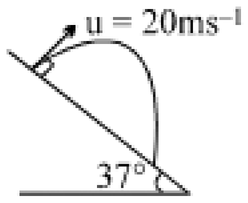
Sol. $\Delta x = 10 = (u \cos 45°)t \quad \Rightarrow t = \dfrac{10\sqrt{2}}{u}$

$\Delta y = -(u \sin 45°)\, t - \dfrac{1}{2}\, gt^2 = -12.5$

$$\Rightarrow \left(\frac{u}{\sqrt{2}} \times \frac{10\sqrt{2}}{u}\right) + \frac{1}{2}\,(10)\left(\frac{10\sqrt{2}}{u}\right)^2 = 12.5$$

On solving we get u = 20 m/s

Q.5 Find range of projectile on the inclined plane which is projected perpendicular to the incline plane with velocity 20m/s as shown in figure.



Sol. $\beta = 37°$

$\alpha - \beta = 90°$ & $\alpha = 90° + \beta = 90° + 37°$

$$\therefore \quad \text{Range}, R = \frac{2(20)^2 \sin(90°)\cos(90° + 37°)}{10 \times \cos^2(37°)}$$

$$= \frac{2(400)}{10(4/5)^2} \times \left(-\frac{3}{5}\right) \qquad [\because \cos(90° + \theta) = -\sin\theta]$$

$\therefore$ R = – 75 m

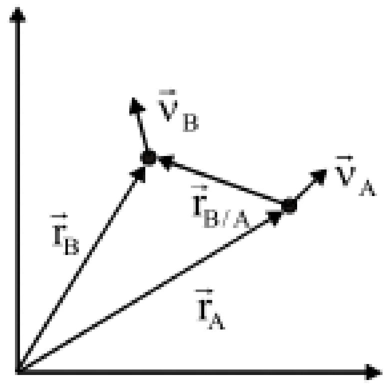
$\therefore$ |R| = 75 m

Here negative sign shown that particle strikes the plane along down the incline

Relative Velocity

It is given by the time rate of change of position of one object w.r.t. another. Relative velocity of a body B with respect to some other body A means velocity of B is recorded by an observer sitting on A. Mathematically.

Relative velocity of B w.r.t. A : $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$



Proof : $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$. Differentiation this equation w.r.t. time, we get

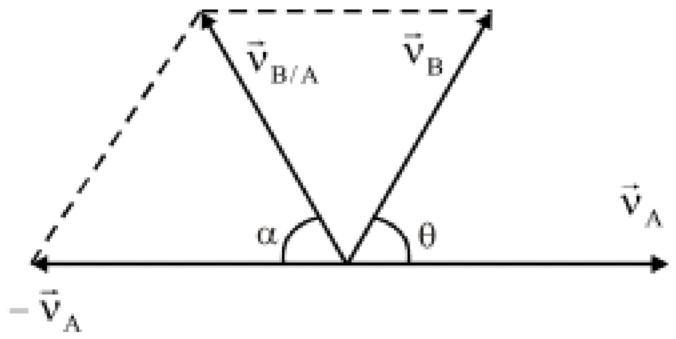
$$\frac{d(\vec{r}_{B/A})}{dt} = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} \qquad \text{but} \qquad \frac{d\vec{r}_A}{dt} = \vec{v}_A\ ,\ \frac{d\vec{r}_B}{dt} = \vec{v}_B \text{ and } \frac{d(\vec{r}_{B/A})}{dt} = \vec{v}_{B/A}$$

putting these values we get $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$. Hence proved.

Similarly, we can prove that relative velocity of A w.r.t. B :

$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B$

Graphical Method to find Relative Velocity



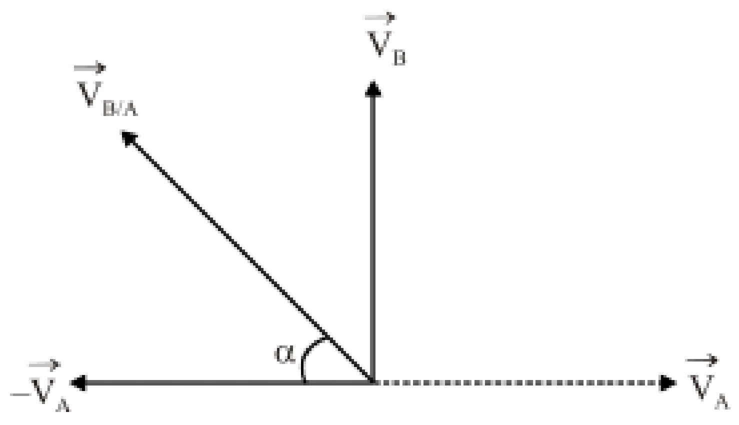
When two bodies move at angle θ with each other then their relative velocity is given by :

Magnitude : $|\vec{v}_{B/A}| = |\vec{v}_B - \vec{v}_A| = \sqrt{v_A^2 + v_B^2 + 2v_A v_B \cos(180-\theta)} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$

Direction : $\tan\alpha = \dfrac{v_B \sin(180-\theta)}{v_A + v_B \cos(180-\theta)}$

$\Rightarrow \qquad \tan\alpha = \dfrac{v_B \sin\theta}{v_A - v_B \cos\theta}$

If $\theta = 90°$ then $|\vec{v}_{B/A}| = \sqrt{v_A^2 + v_B^2}$ and $\tan\alpha = \dfrac{v_B}{v_A}$



We can find the velocity of a particle in a frame if we know the particle's velocity in some other frame and the relative velocity of frames w.r.t each other.

In two observers are watching a moving particle P from the origins of reference A and B, while B moves at a constant velocity $\vec{v}_{B/A}$ relative to A.

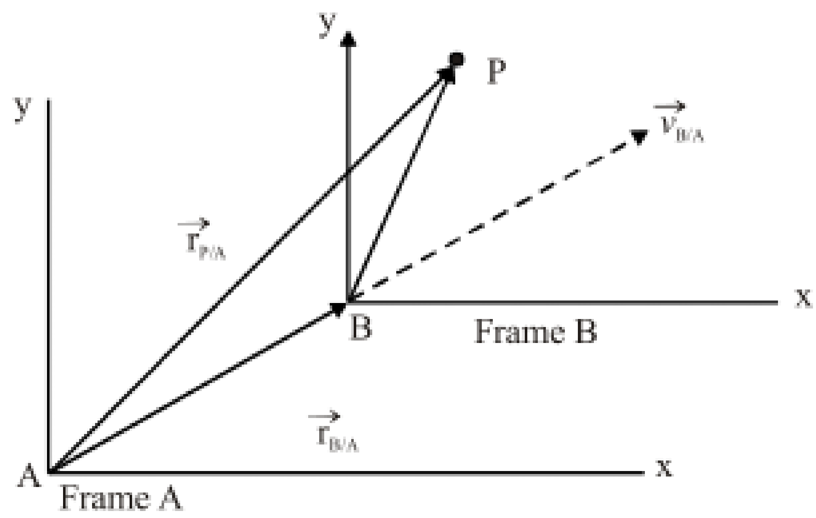


Fig. shows a certain instant during the motion. At this instant, the position vector of B relative to A is $\vec{r}_{BA}$. Also, the position vectors of particle P are $\vec{r}_{P/A}$ relative to A and $\vec{r}_{P/B}$ relative to B. From the arrangement of heads and tails of those three position vectors, we can relate the vectors with $\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}$.

By taking the time derivative of this equation, we can relate the velocities $\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$.

We can understand the concept of relative velocity by a simple situation as follows :

Illustration:

Assume two cars A and B each 5 m long. Car A is travelling at 84 km/h and overtakes another car B which travelling at low speed of 12 km/h. Find out the time taken for overtaking.

Sol. *To analyses the motion in case of overtaking we need relative velocity of object which overtakes w.r.t. the other object. Therefore, we need to find relative velocity of car A w.r.t car B which is*

$84 - 12 = 72 \text{ km/h} = 20 \text{ ms}^{-1}$

Total relative distance covered with this velocity = sum of lengths of car A and car B = 5 + 5 10 m.

$$\therefore \quad \text{the time taken} = \frac{\text{Distance covered}}{\text{Relative velocity}} = \frac{10}{20} = 0.5 \text{ s}$$

Illustration:

(a) *Find velocity of tree, bird and old man as seen by boy.*

(b) *Find velocity of tree, bird, boy as seen by old man*

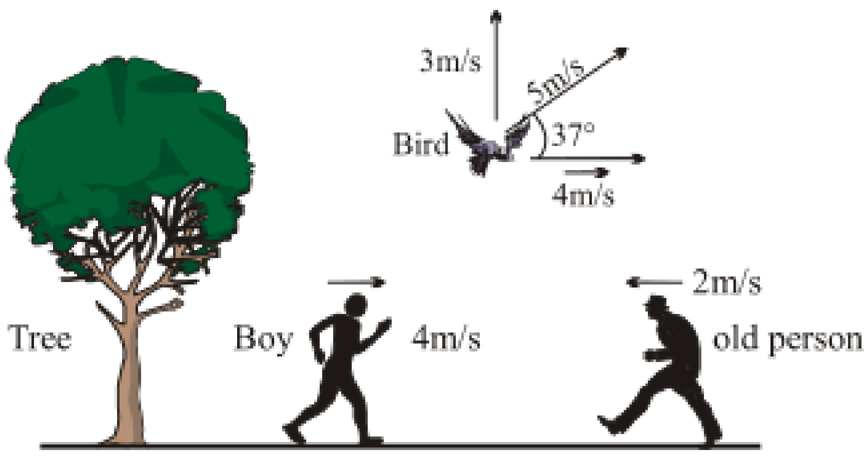
(c) *Find velocity of tree, boy and old man as seen by bird.*

[Sol. (a) *With respect to boy :*

$V_{tree} = 4\ m/s\ (\leftarrow)$

$V_{bird} = 3\ m/s\ (\uparrow)\ \&\ 0\ m/s\ (\rightarrow)$

$V_{old\ man} = 6\ m/s\ (\leftarrow)$



(b) *With respect to old man :*

$V_{boy} = 6\ m/s\ (\rightarrow)$

$V_{tree} = 2\ m/s\ (\rightarrow)$

$V_{bird} = 6\ m/s\ (\rightarrow)$ and $3\ m/s\ (\uparrow)$

(c) *With respect to Bird :*

$V_{tree} = 3\ m/s\ (\downarrow)$ and $4\ m/s\ (\leftarrow)$

$V_{old\ man} = 6\ m/s\ (\leftarrow)$ and $3\ m/s\ (\downarrow)$

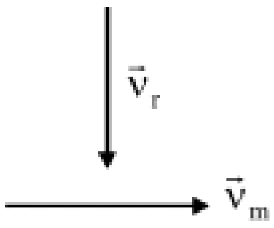
$V_{boy} = 3\ m/s\ (\downarrow)$

Rain - Man Problems

Formula to be applied : $\vec{v}_{r/m} = \vec{v}_r - \vec{v}_m$, where $\vec{v}_{r/m}$ is velocity of rain w.r.t. man, $\vec{v}_r$ is the velocity of rain (w.r.t. ground), and $\vec{v}_m$ is the velocity of man (w.r.t. ground).

If rain is falling vertically downwards with a speed v_r and a man is running horizontally towards east with a speed v_m.

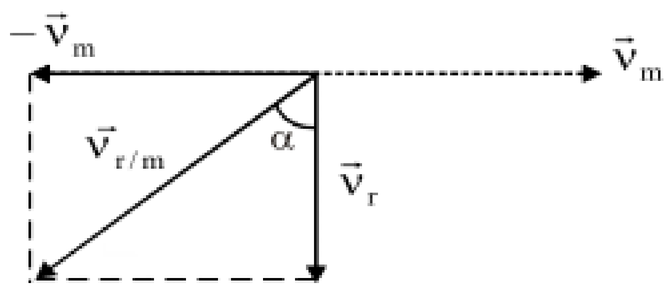
What is the relative velocity of rain w.r.t. man ?



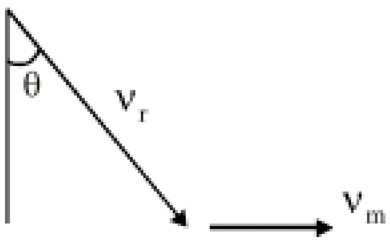
Given : $\vec{v}_r = -v_r\hat{j}$, $\vec{v}_m = v_m\hat{i}$,

Now $\vec{v}_{r/m} = \vec{v}_r - \vec{v}_m = -v_r\hat{j} - v_m\hat{i} \Rightarrow \vec{v}_{r/m} = -v_m\hat{i} - v_r\hat{j}$.

Magnitude : $\sqrt{v_m^2 + v_r^2}$ and direction : $\tan\alpha = \dfrac{v_m}{v_r}$



Example : If rain is already falling at some angle θ with horizontal, then with what velocity the man should travel so that the rain appears vertically downwards to him ?



Here, $\vec{v}_m = v_m \hat{i}$, $\vec{v}_r = v_r \sin\theta \, \hat{i} - v_r \cos\theta \, \hat{j}$

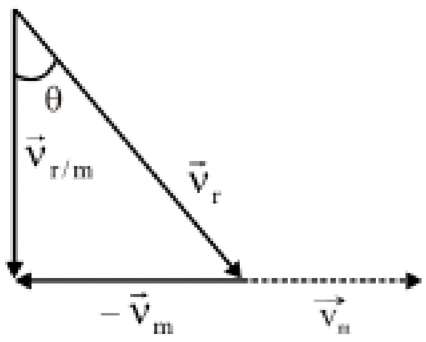
Now, $\vec{v}_{r/m} = \vec{v}_r - \vec{v}_m = (v_r \sin\theta - v_m)\,\hat{i} - v_r \cos\theta \, \hat{j}$

Now for rain to appear falling vertically, the horizontal component of $\vec{v}_{r/m}$ should be zero, i.e.,

$$v_r \sin\theta - v_m = 0 \Rightarrow \sin\theta = \frac{v_m}{v_r} \text{ and } |v_{r/m}| = v_r \cos\theta$$

$$= v_r\sqrt{1-\sin^2\theta} = v_r\sqrt{1-\frac{v_m^2}{v_r^2}}$$

$$\text{or } v_{r/m} = \sqrt{v_r^2 - v_m^2}$$



We can illustrate the whole situation by the diagrams.

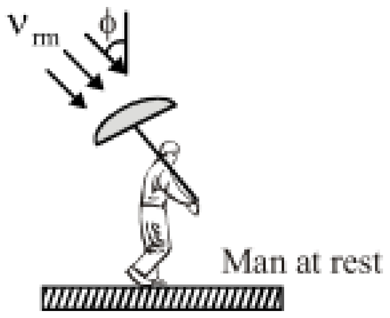
It is quite interesting to notice the steady rainfall sitting in a vehicle such as bus, car, etc. While moving on a straight track the direction of rainfall changes when the vehicle changes its velocity. That means, the velocity of the rain you observe is the velocity of the rain relative to you. Therefore, your observed velocity of rainfall (both magnitude and direction of velocity of rainfall) is the velocity of the rain with respect to the vehicle (you). If you measure the velocity of the rainfall while the vehicle is stationary, that gives actual velocity of rainfall.

Remember following points regarding relative motion :

- If the velocity is mentioned without specifying the frame, assume it is with respect to the ground.
- In many cases, a body travels on water or in air. Depending on the context you will have to figure out whether the velocity is with respect to the water/air or with respect to the ground.

Let us analyse following situation.

The man is stationary and the rain is falling at his back to an angle φ with the vertical

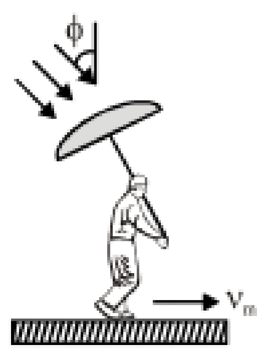


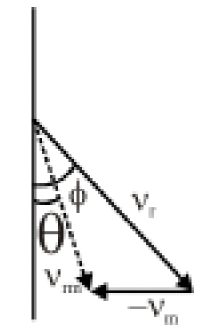
$v_{rm} = v_r \quad v_m = 0$

$\boxed{\theta = \phi}$

here θ = Angle at which rain appears to man

Now man starts moving forward with speed v_m. The relative velocity of rain w.r.t. man shifts towards vertical direction.

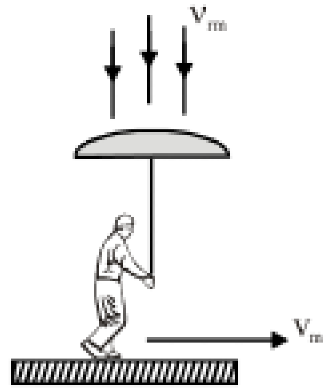


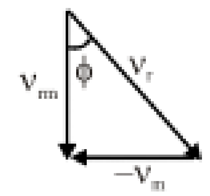


$\boxed{\theta < \phi}$

Velocity vector diagram

As the man further increase his speed, then at a particular value the rain appears to be falling vertically.

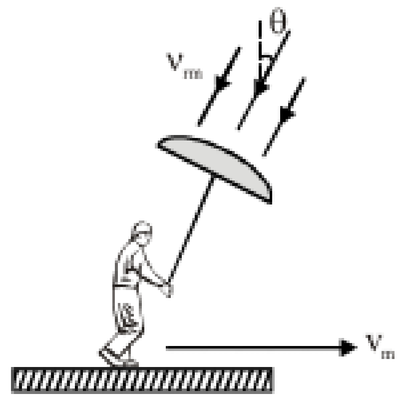


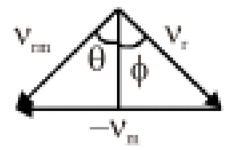


$\boxed{\theta = 0°}$

Velocity vector diagram

If the man increases his speed further more, then rain appear to be falling from the forward direction.





Velocity Vector Diagram

Notice in above figure how man changes orientation of umbrella to prevent himself from rain

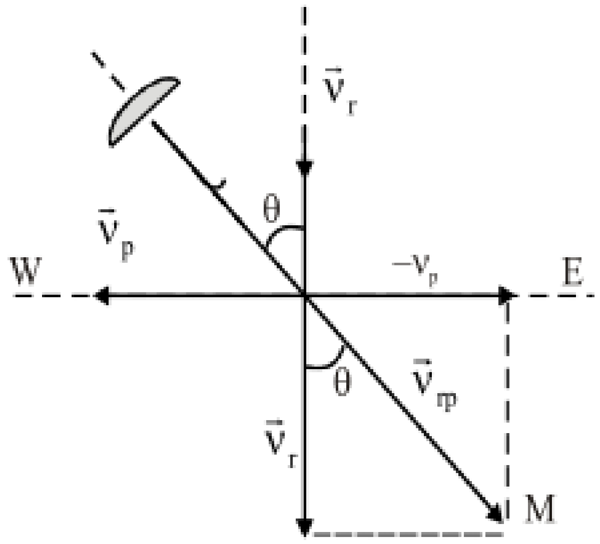
Illustration:

Rain is falling vertically with a speed of 12 ms^{-1}. A cyclist is moving east to west with a speed of $12\sqrt{3}$ *ms^{-1}. In order to protect himself from rain at what angle he should hold his umbrella ?*

Sol. ***Method 1 :*** *In the case of rain falling vertically with a velocity of tan* $\theta = \dfrac{\upsilon_{re}}{\upsilon_{br}}$ *and a person (cyclist, bikers, etc) is moving horizontally with a speed* $\vec{\upsilon}_m$, *the person can protect himself from rain by keeping umbrella in the direction of relative velocity of rain w.r.t. person* $\vec{\upsilon}_{rp}$. *If θ is the angle that* $\vec{\upsilon}_{rp}$ *makes with vertical or rain*

∴ *velocity of rain w.r.t. cyclist*

$$\vec{\upsilon}_{rp} = \vec{\upsilon}_r - \vec{\upsilon}_p$$



Here, $v_r = 12\ ms^{-1}$ *and* $v_p = 12\sqrt{3}$, $\tan\theta = \left(\dfrac{\upsilon_p}{\upsilon_r}\right)$

and $\theta = \tan^{-1}\left(\sqrt{3}\right) = 60°$

So the cyclist has to hold the umbrella at an angle 60° to the vertical.

Method 2 : $\vec{v}_P = -12\sqrt{3}\,\hat{i}$ *(m/s)*

$\vec{\upsilon}_{rain} = -12\,\hat{j}$ *(m/s);* $\vec{\upsilon}_{rain,person} = \vec{\upsilon}_{rain} - \vec{\upsilon}_{person}$

$= [(-12\,\hat{j})] - (-12\sqrt{3}\,\hat{i})]$ (m/s) $= (12\sqrt{3}\,\hat{i} - 12\,\hat{j})$ m/s

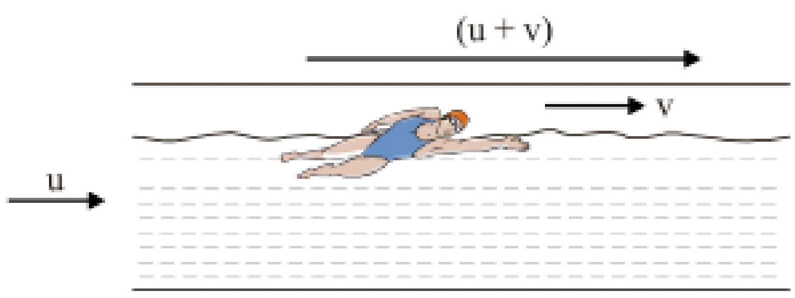
Hence, the direction of orientation of umbrella with vertical is

$$\tan\theta = \frac{12\sqrt{3}}{12} = \sqrt{3} \Rightarrow \quad \theta = 60°$$

River-Swimmer Problems

When a man or a boat is swimming in water, he generates a velocity relative to water (v m/s) by his own efforts. Actual velocity of man in water will be a resultant of man's effort and the river velocity (u m/s).

Down stream : Man makes efforts in direction of flow, the velocity of man w.r.t. ground is (u + v) m/s as shown below.



Up stream : Man makes efforts opposite to the direction of flow, the velocity of man w.r.t. ground is (v – u) m/s as shown below.

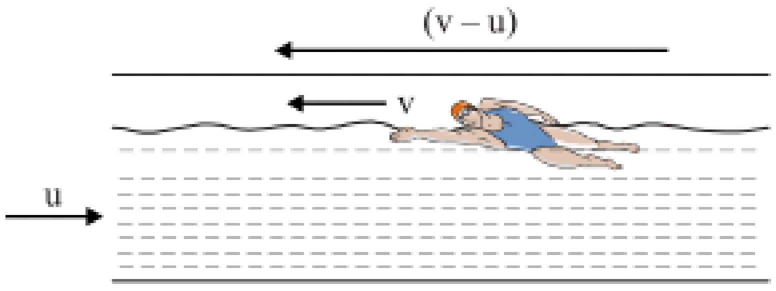


Illustration:

A man whose velocity in still water is 5m/s swims from point A to B (100m downstream of A) and back to A. velocity of river is 3m/s. Find the time taken in going down stream and up stream and the average speed of the man during the motion ?

Sol. *In down stream velocity of man* = $\vec{v}_m = \vec{v}_{m/w} + \vec{v}_w = 3 + 5 = 8 \text{ m/s}$

In down stream time : 100/8 = 12.5 sec

In upstream velocity of man = $\vec{v}_m = \vec{v}_{m/w} + \vec{v}_w$

$= -5 + 3 = -2$ *m/s.*

In up stream time : 100/2 = 50 sec

average speed = 200/62.5 = 3.2 m/s

In a similar manner, when a boat is rowed across a river, the river tries to carry it down stream whereas the boatman makes an effort at an angle to the river bank. The natural consequence is that he reaches somewhere in between. Here also, the velocity of man in still water refers to velocity due to his own efforts. This is fixed in magnitude, but the direction can be changed at will.

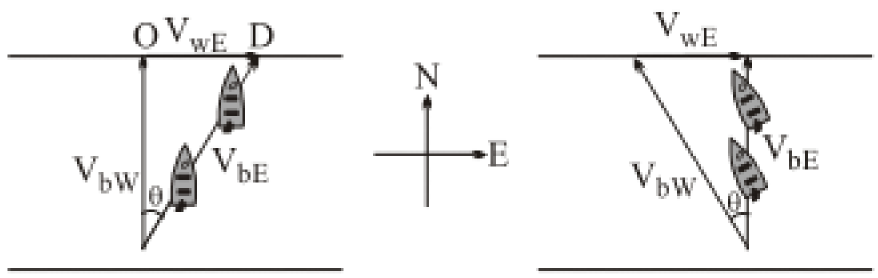


Fig. (a) Fig. (b)

For example, in the figure (a), the boat is rowed directly across in the north direction, but it will reach somewhere in the northeast direction due to the river flow. Similarly in figure (b), the boat is rowed in the north west direction, whereas it will reach in the north direction due to the effect of river flow.

Drift is the distance down stream from the point exactly opposite to the starting point where a person finally reaches. In figure (a) DO = drift. In figure (b) drift = 0

Note following points :

(i) Swimmer keeps himself at an angle of 30° with river flow mean the velocity of swimmer is w.r.t river flow.

(ii) A man swims in water ⇒ velocity of man w.r.t. water.

(iii) A swimmer heads to means (velocity is not w.r.t. ground)

(iv) Person wants to go to destination then direction of velocity is w.r.t. ground.

Let discuss a situation when swimmer & river velocity are known

Suppose velocity of river is u and swimmer can swim with a velocity 'v' w.r.t. river flow.

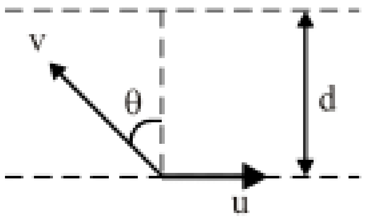
(a) What should be the angle θ with the river flow at which the man should swim so that the time taken to cross the river be minimum ?

Sol. *Let man storts swimming at an angle as shown in figure.*

$$\vec{v}_m = \vec{v}_m + \vec{v}_r$$

$$= (-V\sin\theta\,\hat{i} + v\cos\theta\,\hat{j}) + u\,\hat{i}$$

$$= (u - v\sin\theta)\,\hat{i} + (v\cos\theta)\,\hat{j}$$



= *If width of river is 'd' then time to cross.*

$$t = \frac{d}{v\cos\theta}$$

for t_{min}, $\cos\theta = 1$ *at* $\theta = 0°$

$$t_{min} = \frac{d}{v}$$

So the man should try to swim perpendicular to the river flow to minimize the time in each case.

(b) *What should be the angle* θ *at which the man should swim so that the length of path be minimum ?*
for minimum length of the path, drift x should be minimum.

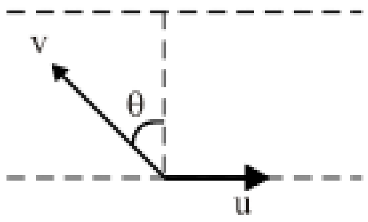
Sol. *Drift for given situation = time × {*$\vec{v}_m$ *along the flow}*

$$x = \frac{d}{v\cos\theta} \times (u - v\sin\theta)$$

$$x = \frac{du}{v}\sec\theta - d\tan\theta \quad ....(A)$$

Case-I

v > u or the river flow is less than the velocity of man's effort.
In such case the minimum possible drift will be zero. So the man should has swim at the angle.



$$x = 0 \Rightarrow u - v\sin\theta = 0$$

$$\sin\theta = \frac{u}{v}$$

Case-II

v < u or the river flow is greater than velocity of man's effort.
In such a case, boat cannot reach the point directly opposite to its starting point. i.e. drift can never be zero.

∴ for x to be minimum

$$\frac{dx}{d\theta} = 0$$

Differentialing equation (A)

$$\therefore \qquad \frac{dx}{d\theta} = \frac{du}{v}\sec\theta\tan\theta - d\sec^2\theta = 0$$

$$\therefore \qquad \sin\theta = \frac{v}{u} \qquad\qquad \theta = \sin^{-1}\frac{v}{u}$$

Thus, to minimize the drift, boat starts at an angle θ from the river flow.

In this case minimum drift can be calculated by putting value of

$$\theta = \sin^{-1}\left(\frac{v}{u}\right) \text{ in equation (A)}$$

$$x_{min} = \frac{d\sqrt{u^2 - v^2}}{v}$$

Illustration:

A boat heading due north crosses a wide river with a speed of 12 km/h relative to the water. The water in the river has uniform speed of 5 km/h due east relative to the earth.

(a) *Determine the velocity of the boat relative to an observer standing on either bank and the direction of boat.*

(b) *If the boat travels with the speed of 13 kh/h relative to the river and is to travel due north, what should its angle of direction be ?*

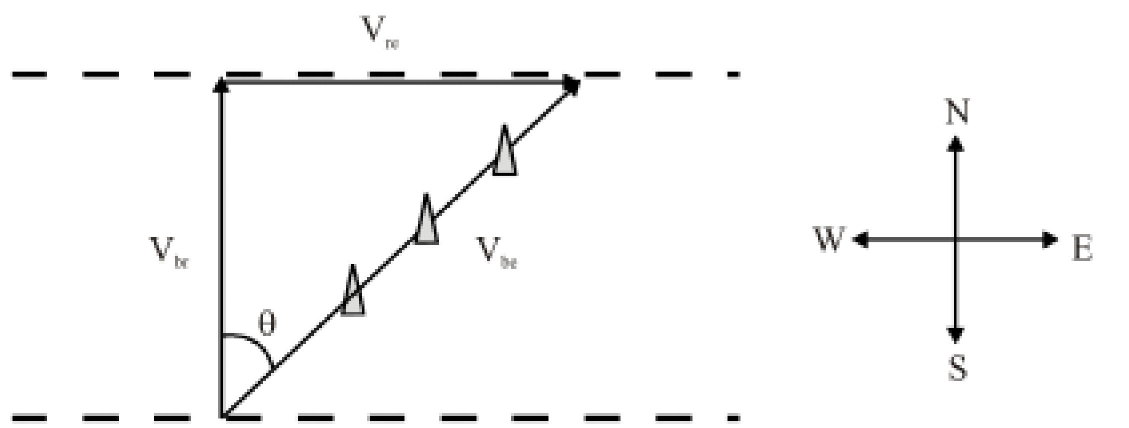
Sol.

(a) *Imagine a situation in your mind of a boat moving across the river. The boat is heading north, which means it wants to go straight, where the current pushes the boat along the direction of current, i.e., east We are given*

Velocity of boat relative to the river $\upsilon_{br} = 10$ *km/h*

Velocity of river = velocity of river relative to earth

$= \upsilon_{re} = 5$ *km/h*



Velocity of the river can be taken as relative to the earth as the velocity measured has only earth as reference. We have to find out the velocity of the boat relative to an observer standing on the bank. Since the observer is stationary with respect to earth, so the velocity of boat relative to observer will be same as the velocity of boat relative to earth.

Let us suppose due to push of current the boat gets drifted by an angle θ form the straight line path.

As seen from velocities in situation from a right angled triangle and we have the values of two sides. Therefore, the third side can be calculated which represents the desired velocity.

From pythagorous theorem, $\upsilon_{be} = \sqrt{\upsilon_{br}^2 + \upsilon_{rE}^2}$

$= \sqrt{(12)^2 + (5)^2} = \sqrt{144 + 25} = 13\ km/h$

To find out the direction, we need to find the angle θ through which boat has deviated.

$$\tan\theta = \frac{\upsilon_{re}}{\upsilon_{br}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{\upsilon_{re}}{\upsilon_{be}}\right) \quad = \tan^{-1}\left(\frac{5}{12}\right)$$

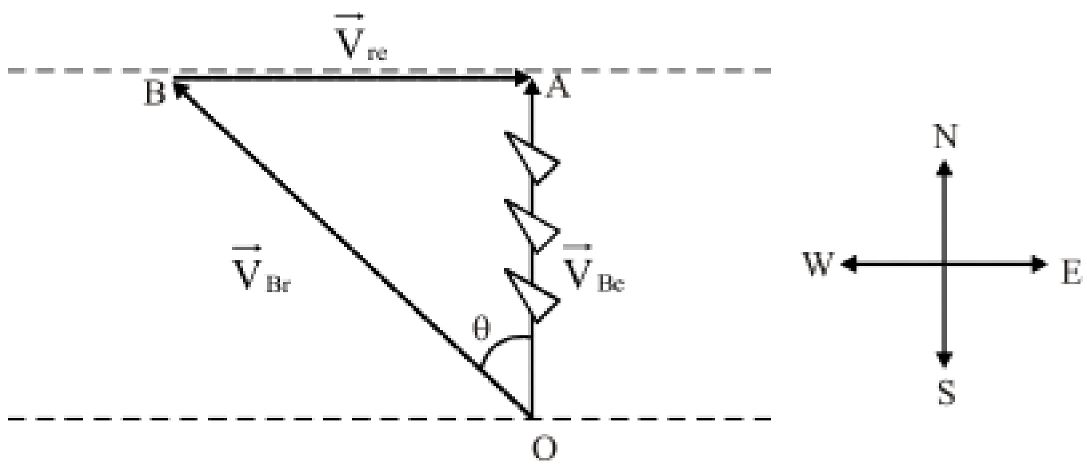
Hence, the boat is moving with a velocity 13 km/h in the direction $\tan^{-1}\left(\frac{5}{12}\right)$ *east of north relative to earth.*

Sol.

(b) *Given* $\upsilon_{br} = 10\ km/h$

As the boat has to move due north, so it needs to start at an angle θ move upward direction of the river.

This is necessary because the boat during the motion will be drifted downwards due to the push of current.



υ_{be} = *velocity of boat w.r.t. earth is along hypotenuse = 13 km/h*

υ_{re} = *velocity of river w.r.t. earth is along perpendicular 5 km/h*

υ_{be} = *velocity of boat w.r.t. earth is along base = ?*

$$\vec{\upsilon}_{Br} = \vec{\upsilon}_{Be} - \vec{\upsilon}_{re} \quad \Rightarrow \quad \vec{\upsilon}_{Br} = 13\ km/hr$$

$$\vec{\upsilon}_{Be} = \vec{\upsilon}_{Br} + \vec{\upsilon}_{re} \quad \Rightarrow \quad \vec{\upsilon}_{re} = 5\ km/hr$$

Using Pythagorous theorem we have, $\upsilon_{br}^2 = \upsilon_{be}^2 + \upsilon_{re}^2$

$$\Rightarrow \quad \upsilon_{be}^2 = \upsilon_{br}^2 + \upsilon_{re}^2 \quad \Rightarrow \quad \upsilon_{be} = \sqrt{\upsilon_{br}^2 - \upsilon_{re}^2}$$

$$\therefore \quad \upsilon_{be} = \sqrt{(13)^2 - (5)^2} = \sqrt{169 - 25} = \sqrt{144} = 12\ m/s$$

Now to find the right direction of movement of boat so that it goes straight in north direction, the

angle θ needs to be obtained

$$\tan\theta = \frac{v_{re}}{v_{be}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{v_{re}}{v_{be}}\right) = \tan^{-1}\left(\frac{5}{12}\right)$$

Hence the boat has to start at an angle $\tan^{-1}\left(\frac{5}{12}\right)$ *in order to move due north.*

Practice Exercise

Q.1 A man is trying to cross the river 100m wide by a boat. The river is flowing with the velocity 5m/s and the boat's velocity in still water is 3m/s. Find the minimum time in which he can cross the river and the drift in this case?

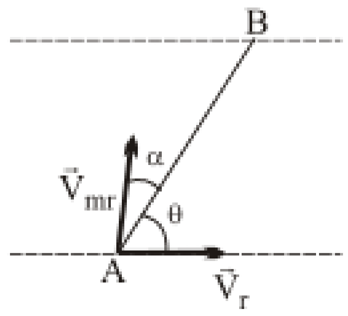
Q.2 Find the direction in which the man (of above illstration) should row so as to have minimum drift. Also find the minimum possible drift and the time taken to cross the river in this case ?

Answers

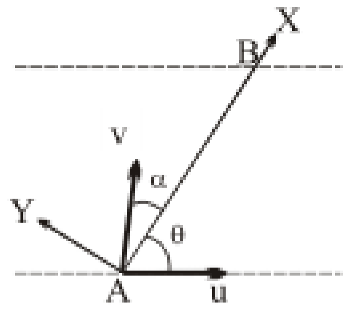
Q.1 $\frac{100}{3}$ sec, $\frac{500}{3}$ m Q.2 $\sin\theta = \frac{3}{5}$, $x_{min} = \frac{400}{3}$ m, $t = \frac{125}{3}$ sec.

Swimming in a desired direction:

Many times the person is not interested in minimizing the time or drift. But he has to reach a particular place. This is common in the cases of an airplane or motor boat.



The man desires to have this final velocity along AB in other words he has to move from A to B. We wish to find the direction in which he should make an effort so that his actual velocity is along line AB, w.r.t. ground. In this method we assume AB to be the reference line the resultant of v and u is along line AB. Thus the components of v and u in a direction perpendicular to line AB should cancel each other.



So $v \sin\alpha = u \sin\theta$

or $\sin\alpha = \dfrac{u\sin\theta}{v}$

here θ, u & v are given in a problem, so we can calculate α by above relation

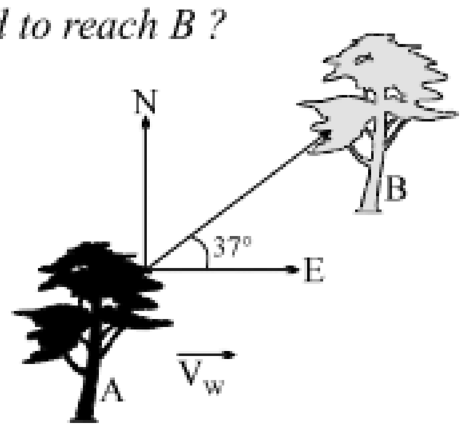
Illustration:

Wind is blowing in the east direction with a speed of 2m/s. A bird wishes to travel from tree A to tree B. Tree B is 100m away from A in a direction 37° north of east the velocity of bird in still air is 4m/s.

(a) Find the direction in which bird should fly so that it can reach from A to B directly.

(b) Find the actual velocity of the bird during the flight ?

(c) Find the time taken by the bird to reach B ?



Sol.

(a) $4 \sin\alpha = 2\sin 37°. \Rightarrow \qquad \alpha = \sin^{-1}\left(\dfrac{3}{10}\right)$

$\Rightarrow 37° + \sin^{-1}\left(\dfrac{3}{10}\right)$ *with east.*

(b) $\vec{v}_b = \vec{v}_{bw} + \vec{v}_w$

$= v_w \cos 37° + 4\cos\alpha.$

$= 2 \times \dfrac{4}{5} + 4 \times \dfrac{\sqrt{91}}{10} \quad = \dfrac{8+2\sqrt{91}}{5}$

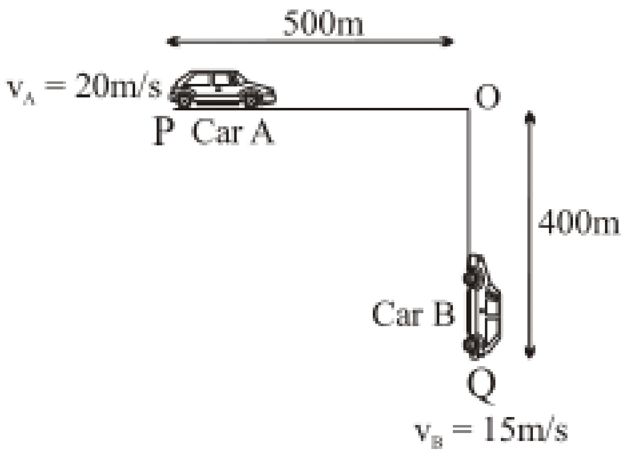
(c) $t = \dfrac{100\times 5}{8+2\sqrt{91}} = \dfrac{250}{4+\sqrt{91}}$ *sec.*

Closest distance of approach between two bodies

Illustration :

Two roads intersect at right angles. Car A is situated at P which is 500m from the intersection O on one of the roads. Car B is situated at Q which is 400m from the intersection on the other road. They start out at the same time and travel towards the intersection at 20m/s and 15m/s respectively. What is the minimum distance between them ? How long do they take to

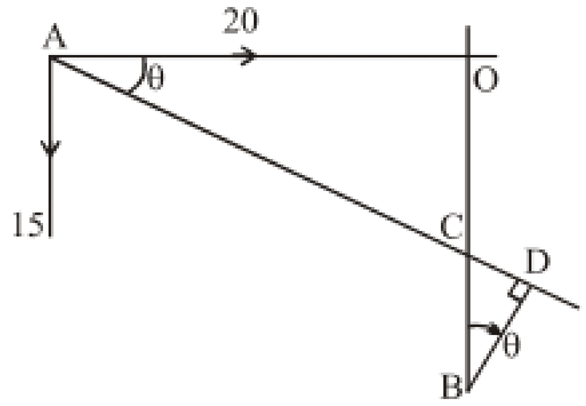
reach it.



Sol. *First we find out the velocity of car A relative to B*

As can be seen from (fig.), the magnitude of velocity of B with respect to A

$v_A = 20\ m/s,\ V_B = 15\ m/s,\ OP = 500\ m;\ OQ = 400m$

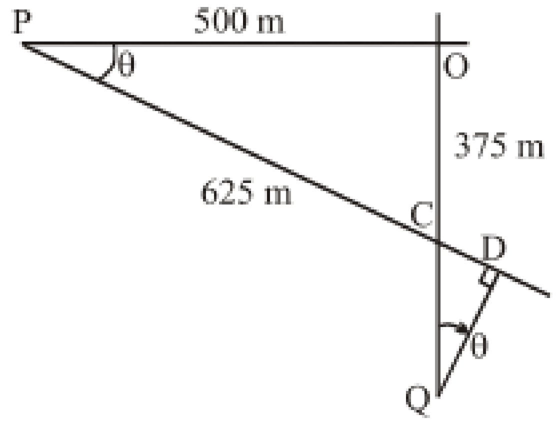


$$\tan\theta = \frac{15}{20} = \frac{3}{4};\qquad \cos\theta = \frac{4}{5};\qquad \sin\theta = \frac{3}{5}$$

$$OC = AO\tan\theta = 500 \times \frac{3}{4} = 375\ m$$

$$BC = OB - OC = 400 - 375 = 25\ m$$

$$BD = BC(\cos\theta) = 25 \times \frac{4}{5} = 20\ m$$



shortest distance = 20 m

$$PD = PC + CD = 625 + 15 = 640$$

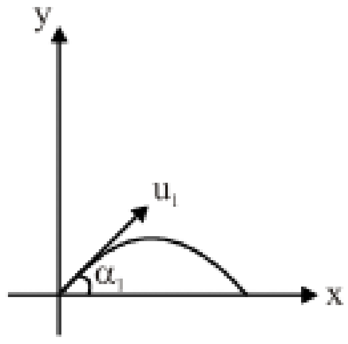
$$|\vec{v}_{AB}| = 25\ m/s$$

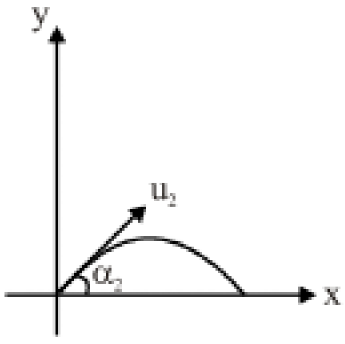
$$t = \frac{640}{25} = 25.6\ sec.$$

Relative Motion Between two Projectiles

Let us now discuss the relative motion between two projectiles or the path observed by one projectile of the other. Suppose that two particles are projected from the ground with speeds u_1 and u_2 at angles α_1 and α_2 as shown in figure. Acceleration of both the particles is g downwards. So, relative acceleration between them is zero because

$a_{12} = a_1 - a_2 = g - g = \text{zero}$



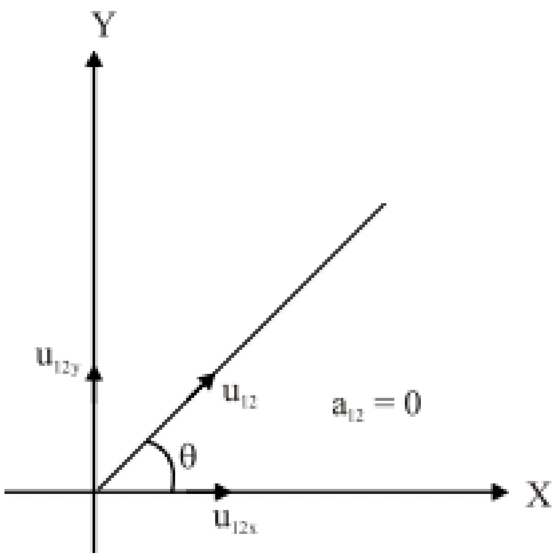


i.e., the relative motion between the two particles is uniform. Now

$$u_{1x} = u_1 \cos\alpha_1, \qquad u_{2x} = u_2 \cos\alpha_2$$
$$u_{1y} = u_1 \sin\alpha_1, \text{ and } \qquad u_{2y} = u_2 \sin\alpha_2$$

Therefore, $u_{12x} = u_{1x} - u_{2x} = u_1 \cos\alpha_1 - u_2 \cos\alpha_2$

and $u_{12y} = u_{1y} - u_{2y} = u_1 \sin\alpha_1 - u_2 \sin a_2$



u_{12x} and u_{12y} are the x and y components of relative velocity of 1 with respect to 2.

Hence, relative motion of 1 with respect to 2 is a straigh line at an angle $\theta = \tan^{-1}\left(\dfrac{u_{12y}}{u_{12x}}\right)$ with positive x-axis.

Now, if $u_{12x} = 0$ or $u_1 \cos\alpha_1 = u_2 \cos\alpha_2$, the relative motion is along y-axis or in vertical direction (as $\theta = 90°$). similarly, if $u_{12y} = 0$ or $u_1 \sin\alpha_1 = u_2 \sin\alpha_2$, the relative motion is along x-axis or in horizontal direction (as $\theta = 0°$).

Condition of Collision of two Projectiles

From the above discussion, it is clear that relative motion between two projectiles is uniform and the path of one projectile as observed by the other is a straight line. Now let the particles are projected simultaneously from two different heights h_1 and h_2 with speeds u_1 and u_2 in the directions shown in figure. Then the particles collide in air if relative velocity of 1 with respect to 2 $(\vec{u}_{12})$ is along line AB or the relative velocity of 2 with respect to 1 $(\vec{u}_{12})$ is along the line BA. Thus,

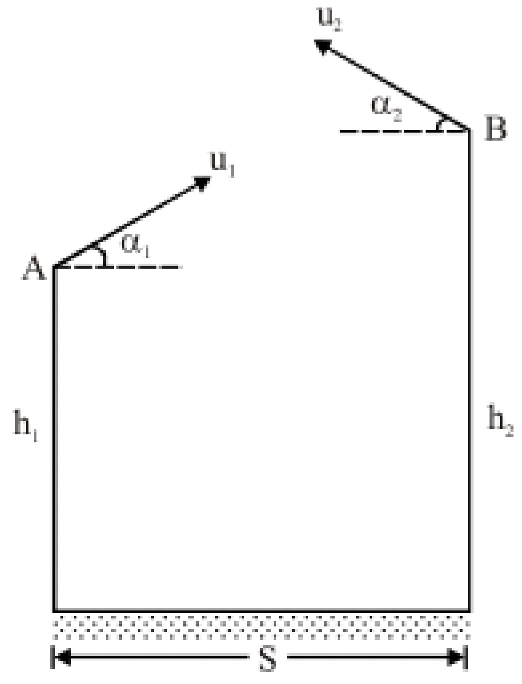
$$\tan\theta = \frac{u_{12y}}{u_{12x}} = \left(\frac{h_2 - h_1}{s}\right)$$

Here $u_{12y} = u_1 \sin\alpha_1 - u_2 \sin a_2$

and $u_{12x} = (u_1 \cos\alpha_1) - (-u_2 \cos\alpha_2) = u_1 \cos\alpha_1 + u_2 \cos\alpha_2$

If both the particles are initially at the same level $(h_1 = h_2)$, then for collision

$$u_{12y} = 0 \qquad \text{or} \qquad u_1 \sin\alpha_1 = u_2 \sin\alpha_2$$



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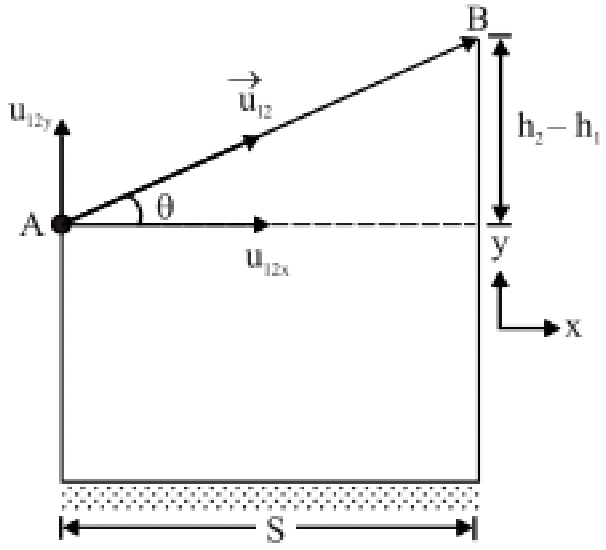


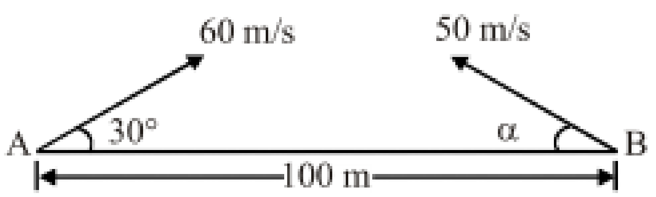
The time of collision of the two particles will be

$$t = \frac{AB}{|\vec{u}_{12}|} = \frac{AB}{\sqrt{(u_{12x})^2 + (u_{12y})^2}}$$

Further, the above conditions are not merely sufficient for collision to takes place. For example, the time of collision discussed above should be less than the time of collision of either of the particles with the ground.

Illustration:

A particle A is projected with an initial velocity of 60 m/s at an angle 30° to the horizontal. At the same time a second particle B is projected in opposite direction with initial speed of 50 m/s from a point at a distance of 100 m from A. If the particles collide in air, find



(a) The angle of projection α of particle B (b) time when the collision takes place and (c) the distance of P from A, where collision occurs. ($g = 10\ m/s^2$)

Sol. *(a) Taking x and y directions as shown in figure.*

Here $\vec{a}_A = -g\hat{j}$

$$\vec{a}_B = -g\hat{j}$$

$$u_{Ax} = 60 \cos 30° = 30\sqrt{3}\ m/s$$

$$u_{Ay} = 60 \sin 30° = 30\ m/s$$

$$u_{Bx} = -50 \cos \alpha$$

and $u_{By} = 50 \sin \alpha$

Relative acceleration between the two is zero as $\vec{a}_A = \vec{a}_B$. Hence the relative motion between the two is uniform. Condition of collision is that $\vec{u}_{AB}$ should along AB. This is possible only when

$$u_{Ay} = u_{By}$$

i.e., component of relative velocity along y-axis should be zero.

or $30 = 50 \sin \alpha$

$\therefore$ $\alpha = \sin^{-1}(3/5)$

(b) Now, $|\vec{u}_{AB}| = u_{Ax} - u_{Bx}$

$$= (30\sqrt{3} + 50 \cos \alpha)\ m/s$$

$$= \left(30\sqrt{3} + 50 \times \frac{4}{5}\right) m/s$$

$$= (30\sqrt{3} + 40)\ m/s$$

Solved Example

Q.1 Ram crossing a 2.5m wide conveyor belt moves with a speed of 1.6 m/s. The conveyor belt moves at uniform speed of 1.2 m/s.

(A) If the Ram walks straight across the belt, determine the velocity of the Ram relative to an observer standing on ground.

Sol. If you walk across a conveyor belt while the conveyor belt takes you along the length, you will not be able to move directly across the conveyor belt, but will end up down the length.

Here the velocity of the Ram will be net effect of his own motion and due to motion of conveyor belt

The velocity of the Ram relative to the conveyor belt v_{rc}, is same as velocity of Ram if conveyor belt was still ,

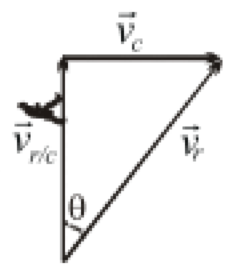
v_c is the velocity of the conveyor belt

we need to find $\mathbf{v}_r$, the velocity of the Ram relative to the Earth.

Writing Equation of net motion $\mathbf{v}_r = \mathbf{v}_{rc} + \mathbf{v}_c$.

three vectors are shown in Figure (a). The quantity $\mathbf{v}_{rc}$ is due y ; $\mathbf{v}_c$ is due x ; and the vector sum of the two, $\mathbf{v}_r$, is at an angle θ as defined in Figure (a).

(a)

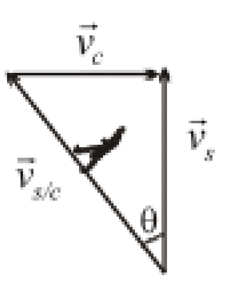


the speed υ_r of the Ram relative to the Earth is

$$v_r = \sqrt{v_{r/c}^2 + v_c^2}$$

(B) If Shyam has same speed on a still conveyor belt, and is to reach directly across the same moving conveyor belt . At what angle should he walk?

(b)



Sol. To go straight across the conveyor belt he has to walk at some angle.

Writing Equation of net motion $\mathbf{v}_s = \mathbf{v}_{s/c} + \mathbf{v}_c$.

three vectors are shown in Figure (b)

As in part (b), we know $\mathbf{v}_c$ and the magnitude of the vector $\mathbf{v}_{s/c}$, and we want $\mathbf{v}_c$ to be directed across the conveyor belt.

$$\upsilon_s = \sqrt{v_{s/c}^2 - v_c^2}$$

Note the difference between the triangle in Figure (a) and the one in Figure (b)

Q.2 An aeroplane pilot wishes to fly due west. A wind of 100 km/h is blowing toward the south

(A) What is the speed of the plane with respect to ground ?

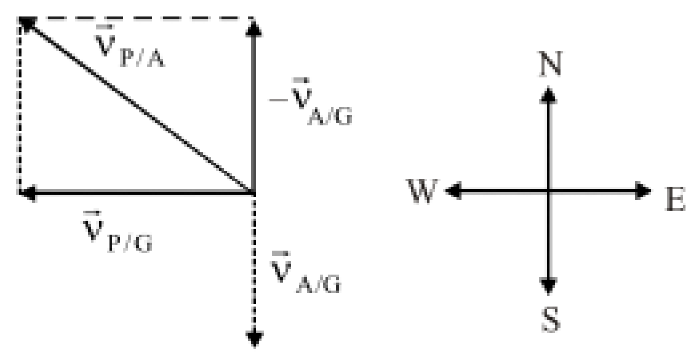
(B) If the airspeed of the plane (its speed in still air) is 300 km/h, in which direction should the pilot head ?

Sol.

(A) Given,

Velocity of air with respect of ground $\vec{v}_{A/G} = 100$ km/hr

Velocity of plane with respect to air $\vec{v}_{P/A} = 300$ km/hr



(B) As the plane is to move towards west, due to air in south direction, air will try drift the plane in south direction., air will try to drift the plane in south direction. Hence, the plane has to make an angle θ towards north-west, south west direction, in order to reach at point on west.

$$\vec{v}_{P/A} = \vec{v}_{P/G} - \vec{v}_{A/G} \text{ and } V_{P/A} \sin\theta = V_{AG}$$

Q.3 A river flows due south with a speed of 2.0 m/s. A man steers a motorboat across the diver: his velocity relative to the water is 4 m/s due east. The river is 800 m wide.

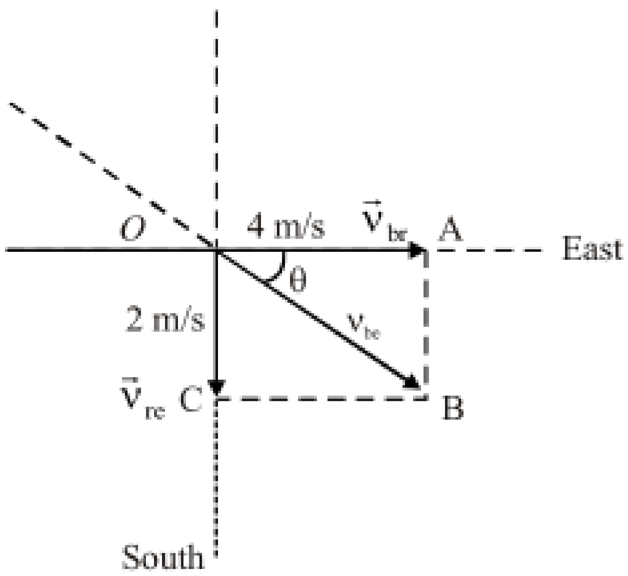
(A) What is his velocity (magnitude direction) relative to the earth ?

(B) How much time is required to cross the river ?

(C) How far south of his starting point will be reach the opposite bank ?

Sol. Velocity of river (i.e., speed of river w.r.t. earth) $\vec{\upsilon}_{re} = 2$ m/s

Width of the river = 800 m



According to the given statement the diagram will be as given

(A) When two vectors are acting at an angle of 90°, their resultant can be obtained by pythagorous theorem,

$$\vec{v}_{be} = \sqrt{\upsilon_{br}^2 + \upsilon_{re}^2} = \sqrt{16+4} = \sqrt{20} = 4.6 \text{ m/s}$$

To find direction , we have

$$\tan\theta = \frac{\upsilon_{re}}{\upsilon_{br}} = \frac{2}{4} = \frac{1}{2} \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

(B) Time taken to cross the river $= \dfrac{\text{Displacement of boat w.r.t. river}}{\text{Velocity of boat w.r.t river}}$

$$\Rightarrow \frac{800}{4} = 200 \text{ s}$$

(C) Desired position on other side is A, but due to current of river boat is drifted to position B. To find out this drift we need time taken in all to cross the river (200s) and speed of current (2 ms^{-1})

So the distance AB = Time taken × speed of current = 200 × 2 = 400 m

Hence, the boat is drifted by 400 m away from position A.

Q.4 A person walks up a stationary escalator in t_1 second. If he remains stationary on the escalator, then it can take him up in t_2 second. If the length of the escalator is L, then

(A) Determine the speed of man with respect to the escalator.

(B) Determine the speed of the escalator.

(C) How much time would it take him to walk up the moving escalator ?

Sol.

(A) As the escalator is stationary, so the distance covered in t_1 second is L which is the length of the escalator.

Speed of the man w.r.t. the escalator $\upsilon_{me} = \dfrac{L}{t_1}$

(B) When the man is stationary, by taking man as reference point the distance covered by the escalator is L in time t_2.

Speed of escalator $\upsilon_e = \dfrac{L}{t_2}$

(C) Speed of man w.r.t. the ground

$$v_m = v_{me} + v_e$$

$$\Rightarrow \qquad v_m = \frac{L}{t_1} + \frac{L}{t_2} = L\left[\frac{1}{t_2} + \frac{1}{t_2}\right] = L\left[\frac{t_1 + t_2}{t_1 t_2}\right]$$

$$\Rightarrow \qquad L = \upsilon_m\left[\frac{t_1 t_2}{t_1 + t_2}\right]$$

$\left[\dfrac{t_1 t_2}{t_1 + t_2}\right]$ is the time taken by the man to walk up the moving escalator.

Q.5 A person standing on a road has to hold his umbrella at 60° with the vertical to keep the rain away. He throws the umbrella and starts running at 20 ms^{-1}. He finds that rain drops are falling on him vertically. Find the speed of the rain drops with respect to

(A) the road, and

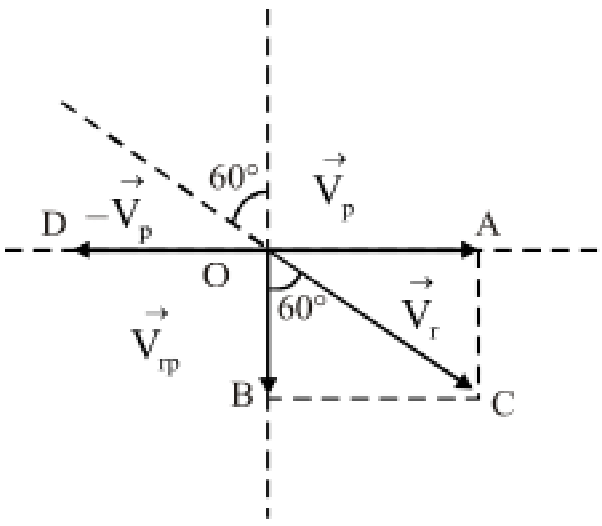
(B) the moving person.

Sol. Given $\theta = 60°$ and velocity person

$\vec{v}_P = \overrightarrow{OA} = 20\ ms^{-1}$.

This velocity is the same as the velocity of person w.r.t ground. First of all let's see how the diagram works out.

$\vec{\upsilon}_{rP} = \overrightarrow{OB}$ = velocity of rain w.r.t. the person.



$\vec{\upsilon}_r = \overrightarrow{OC}$ = velocity of rain w.r.t. earth $\vec{\upsilon}_{rP}$ is along $\overrightarrow{OB}$ as a person has to hold umbrella at an angle with vertical which is the angle between velocity of rain and velocity of rain w.r.t. the person.

Values of $\vec{\upsilon}_r$ and $\vec{\upsilon}_{rP}$ can be obtained by using simple trigonometric relations.

(A) Speed of rain drops w.r.t. earth = $\vec{\upsilon}_r = \overrightarrow{OC}$

From ΔOCM, $\dfrac{CB}{OC} = \sin 60° \Rightarrow \quad OC = \dfrac{CB}{\sin 60°}$

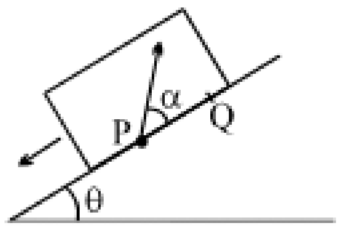
$$= \frac{20}{\sqrt{3}/2} = \frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3}\ ms^{-1}$$

(B) Speed of rain w.r.t. the person $\vec{\upsilon}_{rP} = \overrightarrow{OB}$

From ΔOCM, $\dfrac{OB}{CB} = \cot 60°$

$$\Rightarrow \quad OB = CB \cot 60° = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}\ ms^{-1}$$

Q.6 A large heavy box is sliding without friction down a smooth plane of inclination θ. From a point P on the bottom of a box, a particle is projected inside the box. The initial speed of the particle with respect to box is u and the direction of projection makes an angle α with the bottom as shown in figure.



(a) Find the distance along the bottom of the box between the point of projection P and the point Q where the particle lands. (Assume that the particle does not hit any other surface of the box. Neglect air resistance).

(b) If the horizontal displacement of the particle as seen by an observer on the ground is zero, find the speed of the box with respect to the ground at the instant when the particle was projected.

Sol.

(a) u is the relative velocity of the particle with respect ot the box. Resolve u.

u_x is the relative velocity of particle with respect to the box in x - direction.

u_y is the relative velocity with respect to the box in y - direction.

Since, there is no velocity of the box in the y-direction, therefore this is the vertical velocity of the particle with respect to ground also.

Y - direction motion (Taking relative terms w.r.t. box)

$$u_y = +u \sin \alpha$$

$$a_y = -g \cos \theta$$

$$s_y = 0$$

$$s_y = u_y t + \frac{1}{2} a_y t^2 \Rightarrow 0 = (u \sin \alpha)\, t - \frac{1}{2} g \cos \theta \times t^2$$

$$\Rightarrow t = \frac{2u \sin \alpha}{g \cos \theta}$$

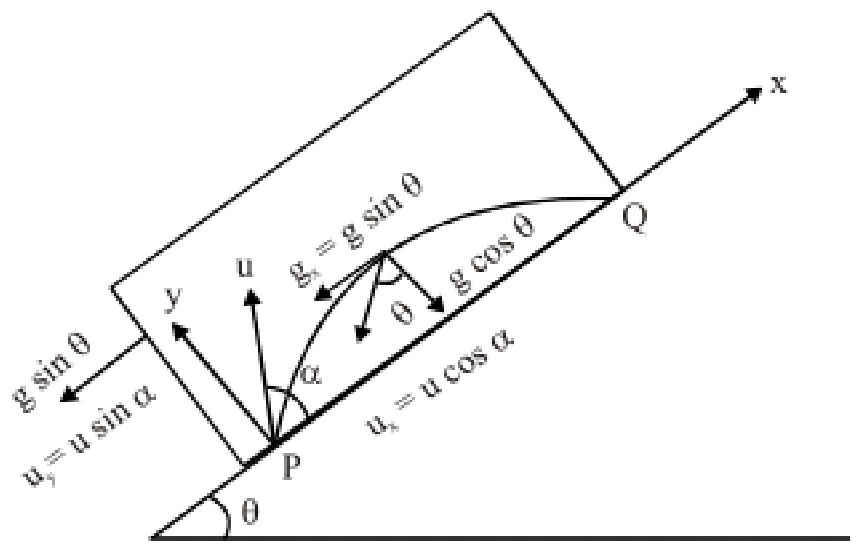
x - direction motion (Taking relative terms w.r.t. box)

$$u_x = +u \cos \alpha \;;\; a_x = 0$$

$$s_x = u_y t + \frac{1}{2} a_y t^2 \Rightarrow s_x = u \cos \alpha \times \frac{2u \sin \alpha}{g \cos \theta} = \frac{u^2 \sin 2\alpha}{g \cos \theta}$$

(b) For the observer (on ground) to see the horizontal displacement to be zero, the distance travelled by box in time $\left(\frac{2u \sin 2\alpha}{g \cos \theta}\right)$ should be equal to the range of the particle w.r.t. box.

Let the speed of the box at the time projection of particle be U. Then for the motion of box with respect to ground.



$$u_x = -U\ ;\ a_x = -g\sin\theta\ ;\ t = \frac{2u\sin\alpha}{g\cos\theta}\ ;\ s_x = \frac{-u^2\sin 2\alpha}{g\cos\theta}$$

$$s_x = u_x t + \frac{1}{2}a_x t^2$$

$$\frac{-u^2\sin 2\alpha}{g\cos\theta} = -U\left(\frac{2u\sin\alpha}{g\cos\theta}\right) - \frac{1}{2}\,g\sin\theta\left(\frac{2u\sin\alpha}{g\cos\theta}\right)^2$$

on solving we get

$$U = \frac{u\cos(\alpha+\theta)}{\cos\theta}$$

Q.7 A man wants to cross a river 500 m wide. Rowing speed of the man relative to water is 3 km/hr and river flows at the speed of 2 km/hr. If man's walking speed on the shore is 5 km/hr, then in which direction he should start rowing in order to reach the directly opposite point on the other bank in shortest time.

Sol. Let he sould start at an angle θ with the normal

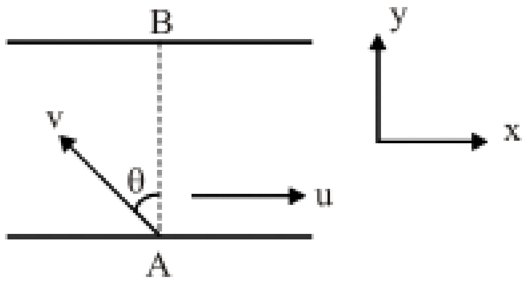
hence

$$\vec{v}_m = (u - v\sin\theta)\,\hat{i} + v\cos\,\hat{j}$$

Here $\vec{v}_m$ = velocity of the man relative to ground.

v = velocity of the man relative to water

u = velocity of water



Hence time taken by the man to cross the river is $t_1 = \dfrac{0.5}{v\cos\theta}$

∴ Drift of the man along the river is

$$x = (u - v\sin\theta)t_1$$

$$x = (u - v\sin\theta)\,\frac{0.5}{v\cos\theta}$$

Time taken by the man to cover this distance is

Therefore, time of collision is

$$t = \frac{AB}{|\vec{u}_{AB}|} = \frac{100}{30\sqrt{3}+40}$$

or $\qquad t = 1.09\ s$

(c) Distance of point P from A where collision takes place is

$$s = \sqrt{(u_{Ax}t)^2 + \left(u_{Ay}t - \frac{1}{2}gt^2\right)^2}$$

$$= \sqrt{(30\sqrt{3}\times 1.09)^2 + \left(30\times 1.09 - \frac{1}{2}\times 10\times 1.09\times 1.09\right)^2}$$

$$s = 62.64\ m$$

$$t_2 = \frac{0.5\left(\frac{u\sec\theta}{v}\tan\theta\right)}{5} = 0.1\left(\frac{u}{v}\sec\theta - \tan\theta\right)$$

Therefore,

total time $T = t_1 + t_2$

$\Rightarrow$ $$T = \frac{0.5}{v}\sec\theta + \frac{0.1u}{v}\sec\theta - 0.1\tan\theta$$

Putting the value of u and v, we get

$$T = \frac{0.5}{3}\sec\theta + \frac{0.1\times 2}{3}\sec\theta - 0.1\tan\theta$$

$$= \frac{0.7}{3}\text{secsec}\,\theta - 0.1\tan\theta$$

$\Rightarrow$ $$\frac{dT}{d\theta} = \frac{0.7}{3}\sec\theta\tan\theta - 0.1\sec^2\theta$$

for T to be minimum

$$\frac{dT}{d\theta} = 0$$

$\Rightarrow$ $\sin\theta = (3/7)$

$\Rightarrow$ $\theta = \sin^{-1}(3/7)$

LAWS OF MOTION

We described motion in terms of position, velocity, and acceleration without considering what might cause that motion. Now we consider what might cause one object to remain at rest and another object to accelerate ? The two main factors we need to consider are the forces acting on an object and the mass of the object. We discuss the three basic laws of motion, which deal with forces and masses and were formulated more than three centuries ago by Isaac Newton. Once we understand these laws, we can answer such questions as "What mechanism changes motion ? And "Why do some objects accelerate more than other ?'

Force

Everyone has a basic understanding of the concept of force. In a vague language, **Force** is push or pull which is an attempt to change the state of rest or motion of an object, or merely deform it. The effect of force on the state of motion of body was first understood by Isaac Newton (1642 – 1727)

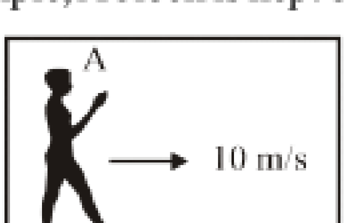
Newtonian Mehanics

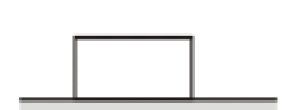
The study of impact of force on the motion of a body as Newton presented it, is called Newtonian mechanics. Newtonian mechanics does not apply to all situations. If the speed of the interacting bodies is very large, an appreciable fraction of the speed of light, we must replace Newtonian mechanics with Einstein's special theory of relativity. If the interacting bodies are on the scale of atomic structure for example, they might be electrons in an atom, We must replace newtonian mechanics with quantum mechanics. Newtonian mechanics is a special case of these two theories, it applies to the motion of objects ranging in size from very small (macroscopic) to astronomical which move with speed neglible w.r.t. speed of light.

Newton's First Law

When net force acting on a body is zero, then we can always find frames of reference in which acceleration of the body is zero. Such frames are called **inertial reference frames.**

A frame moving with constant velocity w.r.t. inertial reference frame is also an inertial reference frame. Thus in an inertial reference frame, if no net force acts on a body, it's velocity connot change. Let us take an example, A block is kept on the floor, such that net force acting on it is zero, and it is at rest



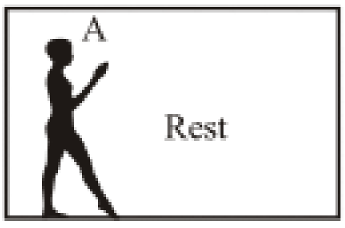


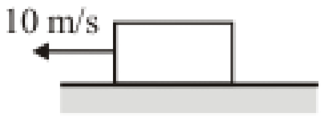
(block is at rest on the floor)

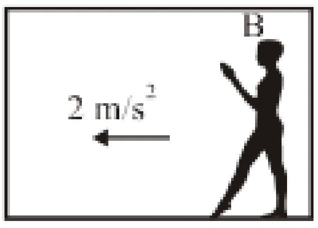
Frame A moving with constant velocity 10 m/s

In the frame A : Observer A observes the block, to be moving with same velocity in opposite direction.

the velocity of block remains 10 m/s towards left. Thus the block has zero acceleration w.r.t. A, Therefore A is an inertial reference frame.



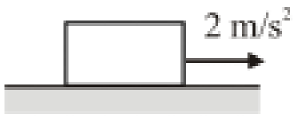


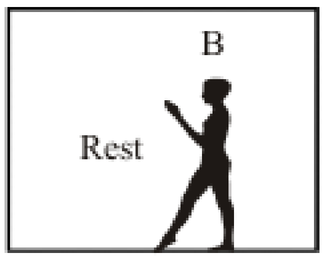


(block is at rest on the floor) (Frame B moving with an acceleration 2 m/s^2.)

In the frame B : observer B observes the block to be accelerated in opposite direction. Thus velocity of block is seen to be changing (w.r.t. B) with time, although no force acts on it here, therefore B is not an inertial reference frame and these type of reference frames are known as non inertial reference frame.

A frame of reference accelerated w.r.t. inertial frame of reference is a non-inertial reference frame.



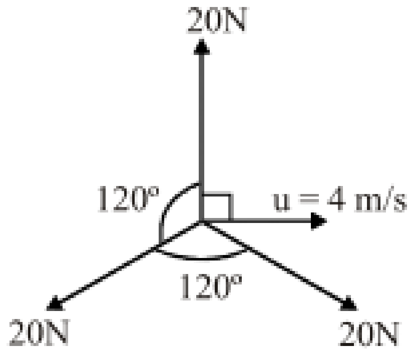


Inertial frame is an ideal concept because only an isolated object can have zero acceleration. However we can assume earth to be an inertial reference frame, if we neglect its rotation & revolution. This can be done because Earth's acceleration due to revolution and rotation is much smaller than the acceleration we notice in daily life i.e acceleration due to gravity ($g = 9.8$ m/s^2).

Illustration :

An object is acted upon by three forces, 20N each as shown in fig. If its initial velocity is 4 m/s in the direction shown in figure. Find its displacement in first 4 sec.

Sol.



As net force (vector sum of all forces) acting on the object is zero, therefore according to Newton's first law,

$$\vec{a} = 0$$

applying, $$S = ut + \frac{1}{2}at^2$$

$$S = 4 \times 4 = 16\ m/s$$

Mass

Mass is that property of a substance which specifies how much resistance an object exhibits to changes in its velocity i.e. it gives an idea of inertia of that body

Suppose a ball is moving in a straight line and a train is moving in a straight line; both with same constant speed. Which one is more difficult to stop? Of course the train.

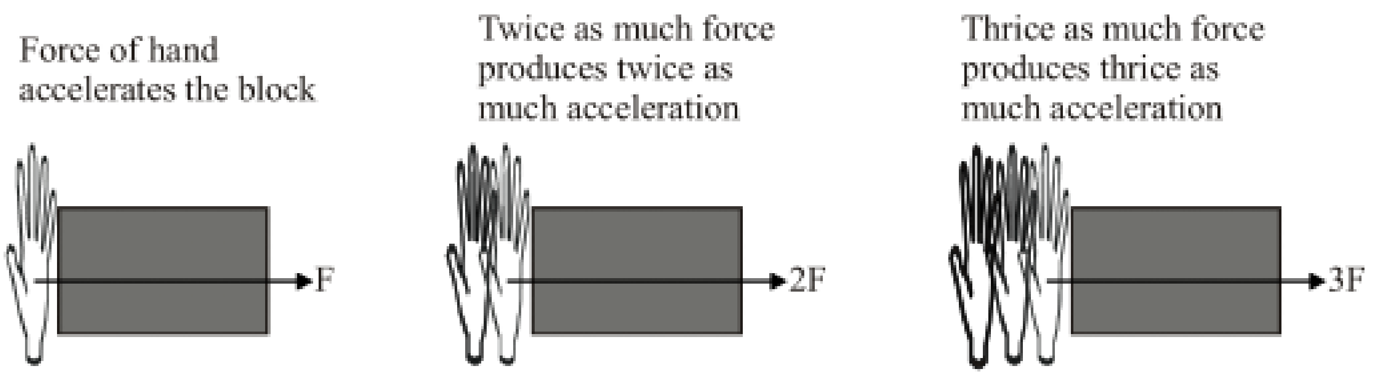
Mass is an inherent property of an object and of the method used to measure it. The mass of a closed system of bodies is independent of the processes going on in the system, no matter what kind these processes are. Also, mass is a scalar quantity and thus obeys the rules of ordinary arithmetic.

Mass should not be confused with weight. Mass and weight are two different quantities. Weight is the force with which Earth attracts the object and is dependent on mass.

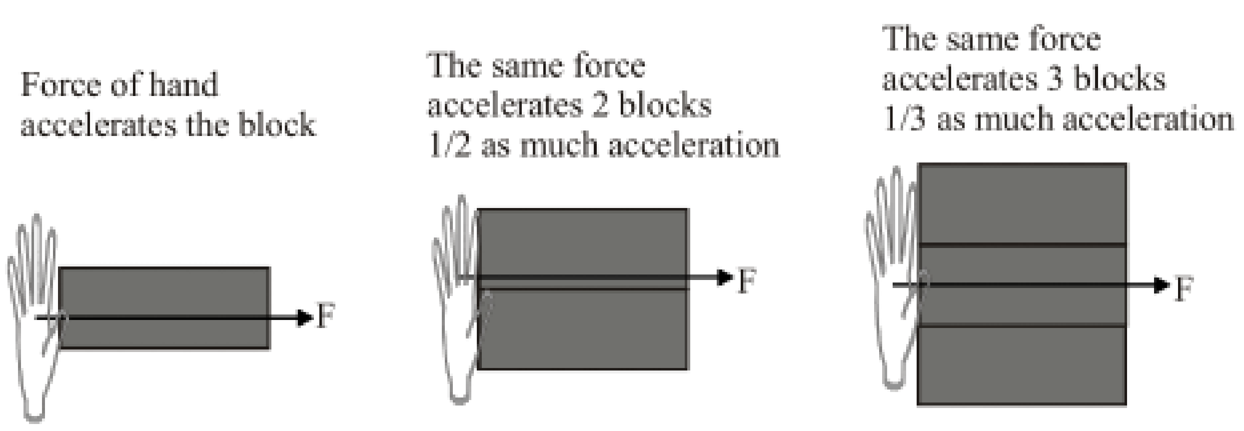
Newton's Second law

Newton's first law explains what happens to an object when no net force acts on it. It either remains at rest or moves in straight line with constant speed. Newton's second law answers the question, what happens to an object that has a nonzero resultant force acting on it.

When viewed from an inertial reference frame, the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass.



$$\textbf{acceleration} \propto \textbf{force}$$



$$\textbf{acceleration} \propto \frac{\textbf{1}}{\textbf{mass}}$$

Thus, we can relate mass, acceleration, and force through the following mathematical statement of Newton's second law, by choosing proper units so that constant of proportionality is 1 :

$\Sigma\ \vec{F} = m\vec{a}$ ....(i)

we have indicated that the acceleration is due to the net force acting on an object. The net force on an

object is the vector sum of all forces acting on the object. In solving a problem using Newton's second law, it is important to determine the correct net force on an object.

What happens when several forces act simultaneously on an object ?

In this case, the object accelerates only if the net force acting on it is not equal to zero.

$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...$: vector sum of all forces. ....(ii)

There may be many forces acting on an object, but there is only one acceleration.

Note that equation 2 is vector expression and hence is equivalent to three component equations

$\Sigma F_x = ma_x \quad \Sigma F_y = ma_y \quad \Sigma F_z = ma_z$

When net force is zero, then the object is said to be in equilibrium. And if net force is zero, acceleration will also be zero, therefore velocity will be constant. **When the velocity of an object is constant (including when object is permanently at rest), the object is said to be in equilibrium.**

Force is the cause of change in motion :

Force does not cause motion. We can have motion in the absence of force, as described in Newton's first law. Force is the cause of change in motion as measured by acceleration.

"ma" is not a force :

Equation (i) does not say that the product ma is a force. All forces on object are added vectorialy as in equation (ii) to get the net force on the left side of the equation. This net force is then equated to the product of the mass of object and the acceleration that results from the net force.

Do not include an "ma force" in your analysis of the forces on an object.

Definition of the newton

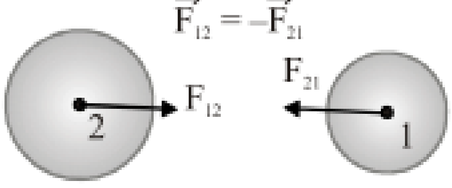
The SI unit of force is the newton, which is defined as the force that, when acting on an object of mass 1 kg, produces an acceleration of 1 m/s^2. From the definition and Newton's second law, we see that the newton can be expressed in terms of the following fundamental units of mass, length, and time

$1 \text{ N} \equiv 1 \text{ kg. m/s}^2$

Newton's Third law

Every action has equal and opposite reaction, and the action reaction act on the different bodies, simultaneously These action reaction pair must be of same type, viz. friction for friction, gravitation for gravitation etc.

$\vec{F}_{12} = -\vec{F}_{21}$

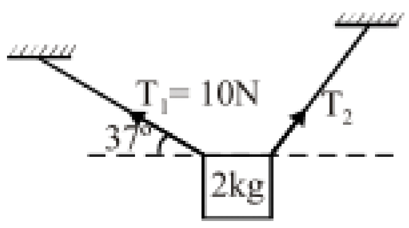


Caution

(i) Action and reaction forces act on different objects. Two forces acting on the same object, even if they are equal in magnitude and opposite in direction, cannot be an action-reaction pair.

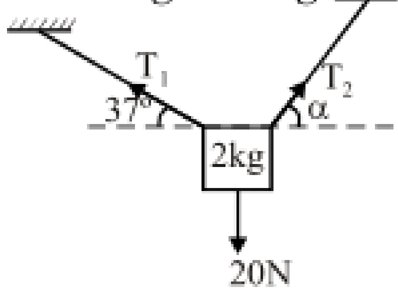
(ii) In an action reaction pair, both forces act simultaneously i.e. we can not say one as action and other reaction.

Illustration :



As in figure, A block of 2 kg hangs by two massless string. The tension in left string is 10N and makes 37° from horizontal. Find tension in right string. Also show action reaction pairs.

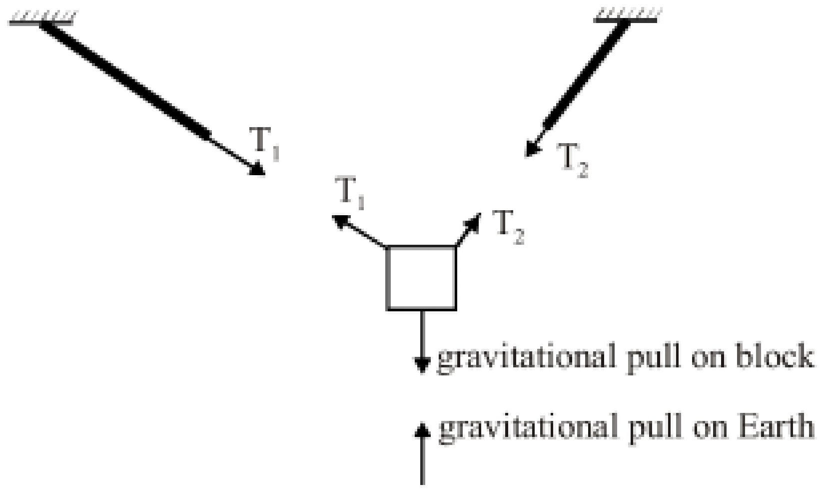
Sol.



block is in equilibrium i.e. rest so the net force on the block must be zero accounding to newton's first law as observed from an inertial reference frame

$\vec{T}_1 + \vec{T}_2 + m\vec{g} = \vec{0} \quad \Rightarrow \quad (-8\hat{i} + 6\hat{j}) + \vec{T}_2 - 20\hat{j} = \vec{0}$

$\vec{T}_2 = 8\hat{i} + 14\hat{j}$, *i.e. tension in right string* $T_2 = \sqrt{8^2 + 14^2} = 16.12\text{N}$ *and string makes angle* $\alpha = \tan^{-1}\left(\frac{7}{4}\right)$ *with the horizontal*



Practice Exercise

Q.1 An object experiences no acceleration. Which of the following cannot be true for the object ?

(A) A single force acts on the object.

(B) No force acts on the object.

(C) Forces act on the object, but the forces cancel.

Q.2 An object experiences net force and exhibits an acceleration in response. Which of the following statements is always true ?

(A) the object moves in the direction of the force.

(B) The acceleration is in the same direction as the velocity.

(C) The acceleration is in the same direction as the net force.

(D) The velocity of the object increases.

Q.3 If a fly collides with the windshield of a fast-moving bus, which object experiences an impact force with a larger magnitude ?

(A) the fly (B) the bus (C) the same force is experienced by both.

Q.4 If a fly collides with the windshield of a fast-moving bus, which object experiences the greater acceleration:

(A) The fly (B) the bus (C) the same acceleration is experienced by both.

Answers

Q.1	A	Q.2	C	Q.3	C	Q.4	A

Free body diagram (FBD)

In this diagram the object of interest is isolated from its surroundings and the interactions between the object and the surroundings are represented in terms of forces.

Remember not to show " M $\vec{a}$ " as force in FBD. $\vec{a}$ is the effect of net force acting on an object.

System

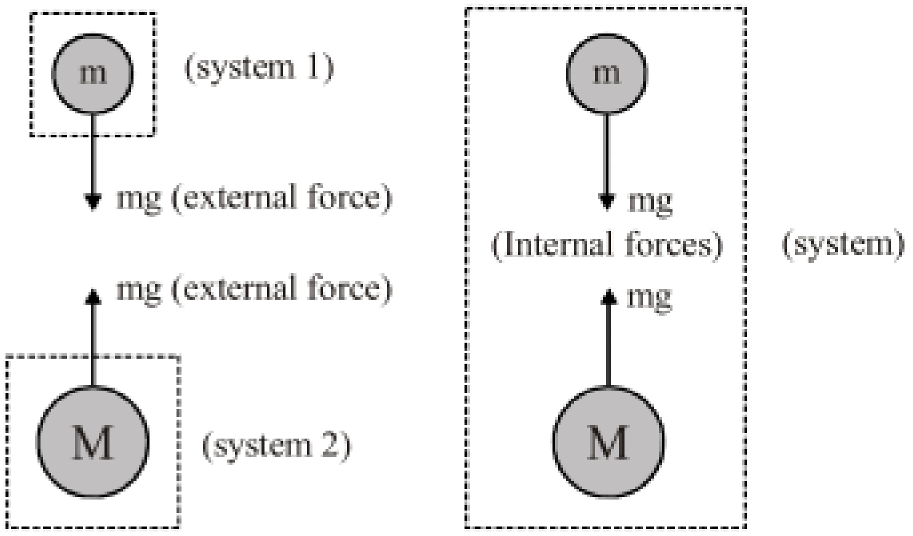
The system is what you choose to analyse motion.

1. According to system chosen, forces can be divided into two categories :

(i) Internal force (ii) External force

Whether a force is external or internal will depend on the system chosen. For example, earth is pulling a mass by applying a force mg on it. The mass is also applying a force mg on earth.

If we chose earth-mass as system, then mg becomes an internal force as it is within the system. But if choose only mass m as system; then mg acting on the mass is an external force.



We do not show internal force in FBD while solving a problem.

2. According to the origin of force, we can classify it into two types :

(i) Field forces

(ii) Contact forces

Field forces

The force experienced by an object without physical contact is known as field force.

e.g. Gravitational force, Electromagnetic force.

Gravitational force :

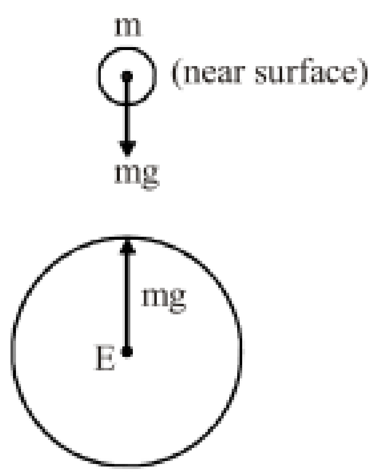
The force by which two bodies attract each other by virtue of their masses.

If two particles haivng masses m_1 & m_2 are seperated by a distance r, then the magnitude of force is given by $F = \frac{Gm_1m_2}{r^2}$.

This is called Newton's law of Gravitation

The force acts along the line joining the particles, this force is always attractive in nature.

For earth mass system, earth applies a force mg on mass m, which always acts towards the center of earth downwards. mg is called weight of the body.



Electromagnetic force :

If two particles having charges q_1 and q_2 are separated by a distance r, then the magnitude of force is given by : $F = \frac{Kq_1q_2}{r^2}$. This is called Coulomb's law and it acts along the line joining the particles. For like charges, the force is repulsive, and for unlike charges, the force is attractive.

Contact Force

The force experienced by an object by physical contact is known as contact force.

e.g. Frictional force (which will be dealt in detail later), Normal force, tension.

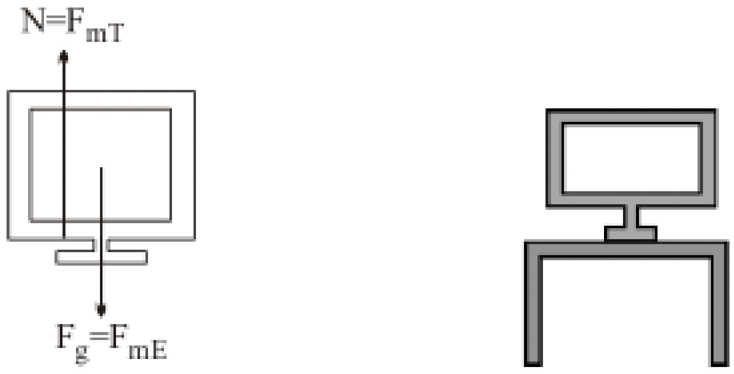
Normal force :

A contact force perpendicular (normal means perpendicular) to the contact surface that prevents two objects from passing through one another is called the normal contact force.

In the situation shown, LCD is kept on the table. Weight mg acts on it downwards still it is at rest. i.e. some force acts in the upward direction (Normal to the table) which balances the weight. This force is the normal force by the table on the LCD.

$F_{mT} \Rightarrow$ F on mass by table

$F_{me} \Rightarrow$ F on mass by Earth



(b)

Here $N = mg$ (not an action reaction pair)

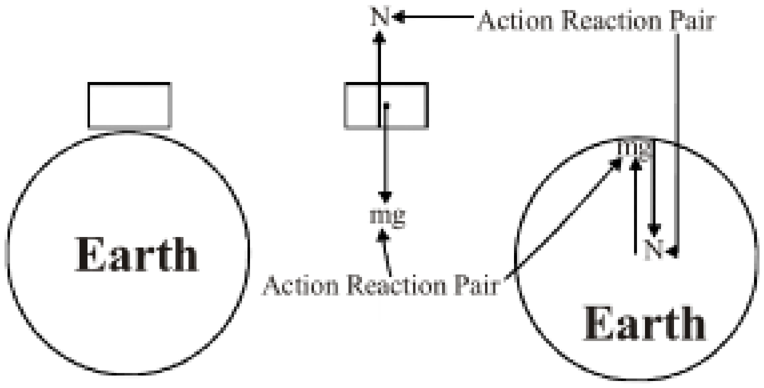
N Does not always equal mg

In the situation shown in figure and in many others, we find that the normal force has the same magnitude as the gravitational force. However, this is not always true. If an object is on an incline, if there are applied forces with vertical components, or if there is a vertical acceleration of the system, then $N \neq mg$. Always apply Newton's second law to find the relationship between N and mg.

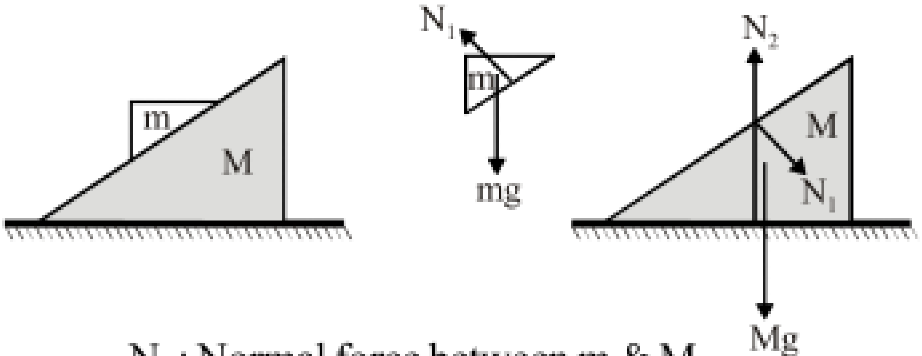
Note : Normal force is always a pushing force.

Normal force acts along the common normal

e.g. (i)



e.g. (ii)



N_1 : Normal force between m & M

N_2 : Normal force between M & floor

Normal force acts perpendicular to the surface in contact.

If one surface is not well defined, then normal force acts perpendicular to the surface of second object.

Here we can not specity the perpenducular to the rod at the point of contact. Thus normal acts perpenducular to the floor and the wall respectively.

e.g.

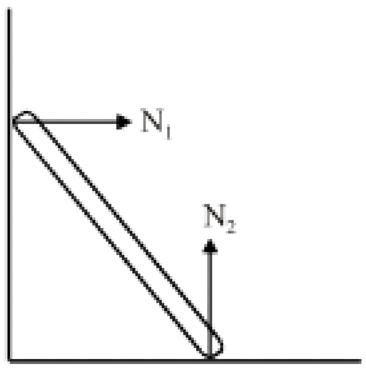
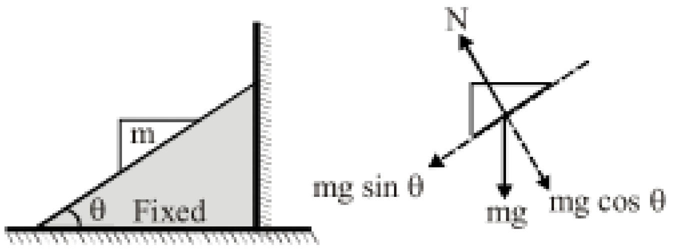


Illustration :

Find normal force at all contact points.

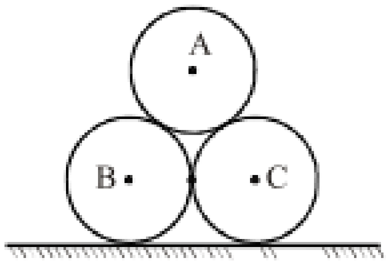
(i)



Force along the perpendicular to the inclined must be zero, as m has no acceleration in this direction

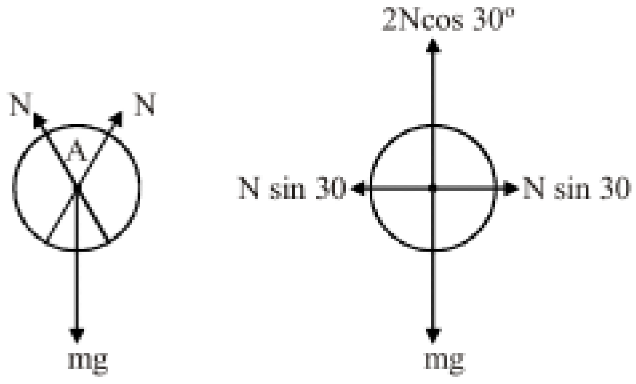
$\therefore \quad N = mg\cos\theta$

(ii)



Cylinders B & C are fixed and each cylinder is of mass m.

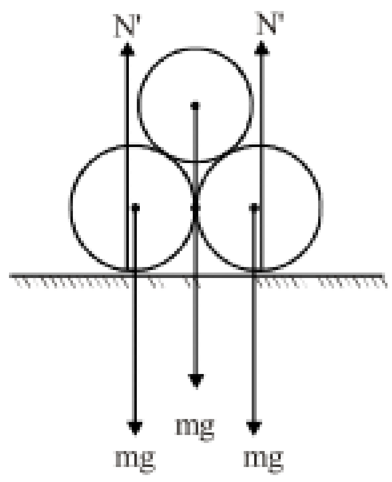
FBD of A



$$2\,N\cos 30^\circ = mg$$

$$N = \frac{mg}{\sqrt{3}}$$

Considering all the three together as system,

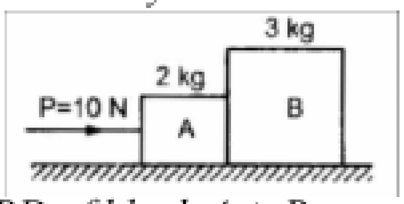


$$2N' = 3mg$$

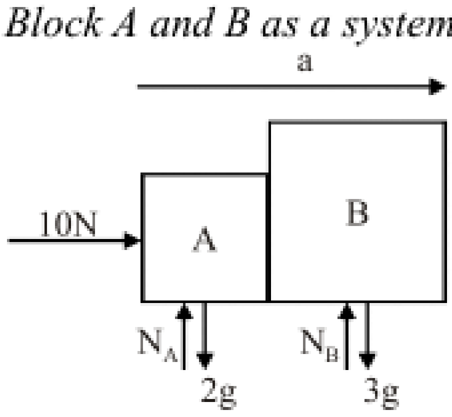
$$N' = \frac{3mg}{2}$$

Illustration :

Blocks A and B have masses of 2kg and 3kg respectively. The ground is smooth. P isan external force of 10 N. Find the force exerted by B on A .



Sol. *Block A and B as a system, FBD of block A + B*

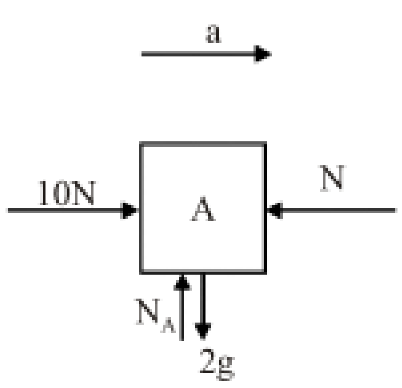


(here normal force between A & B is internal so we can not show it in FBD)

there is no vertical motion so the net force on system in vertical direction $\left(\Sigma \vec{f}_y = \vec{0}\right)$. *In horizotnal direction there is an external force 10N so the system will accelelate with acceleration a.*

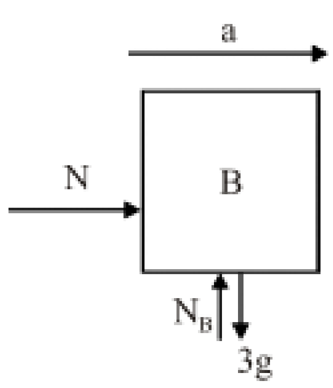
$$a = \frac{10}{5} = 2\,\text{m/s}^2$$

FBD of block A



$10 - N = 2 \times 2 \Rightarrow N = 6\ N$

FBD of block B

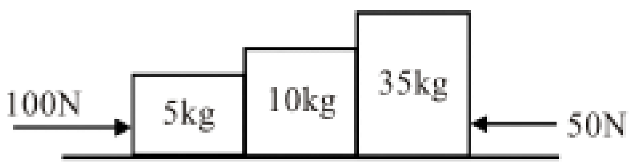


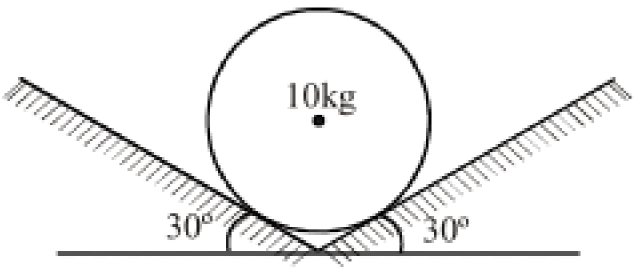
$N = 3 \times 2 = 6\ N$

(Here block A & block B are the seperate systems so the normal force N is external, normal force N must be shown in each FBD)

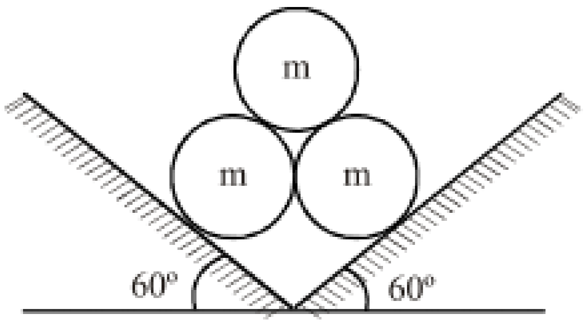
Practice Exercise

Q.1 Find normal forces at all the contact points.

(i) 

(ii) 

Q.2 Find normal contact force between sphere and inclind plane.



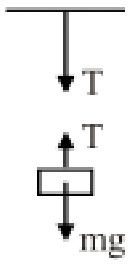
Answers

Q.1 (i) Normal contact force between 5 kg and 10 kg is 95 N. while 10 kg and 35 kg is 85 N.

(ii) $\dfrac{100}{\sqrt{3}}$ N

Q.2 3 mg

Tension force :



- It is self adjusting pulling force, which is electromagnetic in nature.
- We assume string is massless, Unless specified.
- Strings will be tight only when the ends are pulled apart.
- **Tension is a pulling force, acts away from the object along the string**
- Tension throughout the massless string is same usually.
- A pulley can change the direction of the force exerted by a cord.

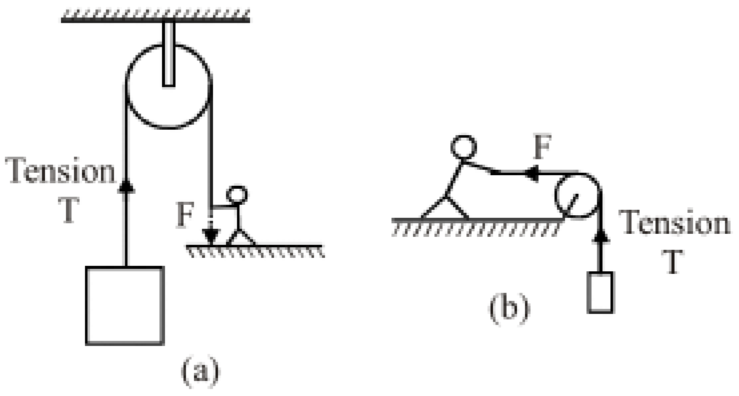
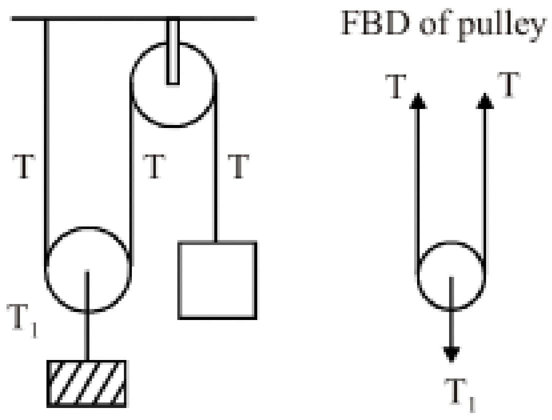


Illustration :

In the situation shown in figure draw the FBD of the moving pulley assume that the pulley is massless. Also relate tension in both strings.

Sol.



According to Newton's second law :

$T_1 - 2T = m_{pulley}\, a$ ($\because$ mass of pulley is negligible i.e $m_{pulley} \to 0$)

$T_1 - 2\,T = 0 \quad \Rightarrow T_1 = 2T$

Illustration :

When two unequal masses are hung vertically over a frictionless pulley of negligible mass the arrangement is called atwood machine. The device is sometimes used in the laboratory to measure the gravitational field strength. Determine the magnitude of the acceleration of the two masses and the tension in the string. Consider $m_2 > m_1$.

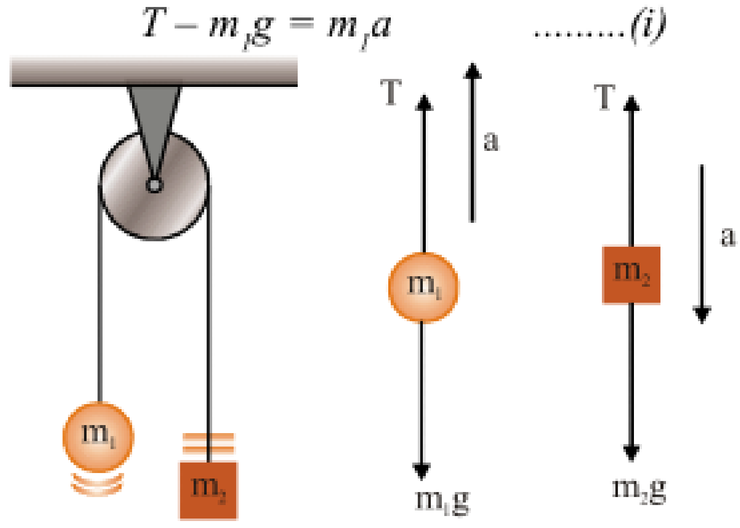
Sol. *The free body diagrams for the two masses are shown in figure. Two forces act on each block: the upward force exerted by the string. T, and the downward force of gravity.*

The magnitude of the net force exrted on m_1 is $T - m_1g$, the magnitude of the net force exerted on m_2 is $m_2g - T$.

Because the blocks are connected by a string, their accelerations must be equal in magnitude. m_1 must accelerate upward, while m_2 must accelerate downward.

When Newton's second law is applied to m_1, with acceleration upward for this mass we find (taking upward to be the positive y direction)

$$T - m_1g = m_1a \qquad ....(i)$$



Similarly, for m_2 we find

$$m_2g - T = m_2a \qquad ....(ii)$$

Adding (i) & (ii)

$$m_2g - T + T - m_1g = m_1a + m_2a$$

or

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g \qquad ....(iii)$$

When (iii) substituted into (i), we get

$$T = \left(\frac{2m_1m_2}{m_1 + m_2}\right)g \qquad ....(iv)$$

The result for the acceleration, in equation (iii), can be interpreted as the ratio of the unbalanced force on the system to the total mass of the system.

Special cases

When $m_1 = m_2$, $a = 0$ *and* $T = mg = m_1g = m_2g$

If $m_2 >> m_1$, $a \approx g$ *(a freely falling body) and* $T \approx 2m_1g$.

Basic steps for applying Newton's Laws

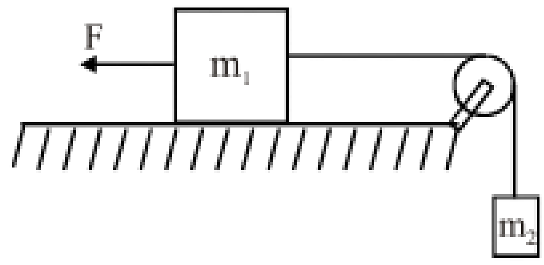
The following procedure is recommended when dealing with problems involving Newton's laws:

- Draw a simple, neat diagram of the system to help conceptualize the problem.
- If any acceleration component is zero, the particle is in equilibrium in this direction and $\Sigma F = 0$ in this direction. If not the particle is undergoing an acceleration, the problem is one of non-equilibrium in this direction and $\Sigma F = ma$.
- Isolate the object whose motion is being analyzed. Draw a free-body diagram for this object. For systems containing more than one object, draw separate free-body diagrams for each object.

  Do not include in the free-body diagram forces exerted by the object on its surroundings.
- Establish convenient coordinate axes for each object and find the components of the forces along these axes. Apply Newton's second law, $\Sigma F = ma$, in component form. Check your dimensions to make sure that all terms have units of force.
- Solve the component equations for the unknowns. Remember that you must have as many independent equations as you have unknowns to obtain a complete solution.
- Make sure, your results are consistent with the free-body diagram. Also check the predictions of your solutions for extreme value of the variables. By doing so you can often detect errors in your results.

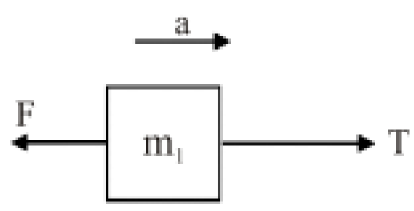
Illustration :

A constant force $F = m_2g/2$ *is applied on the block of mass* m_1 *as shown in figure. The string and the pulley are light and the surface of the table is smooth. Find the acceleration of* m_1.



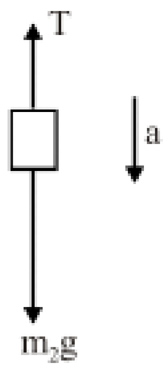
Sol. *When Newton's second law is applied to m_1, with acceleration rightward for this mass (because $m_2g > F$).*

FBD of m_1



$$T - F = m_1 a \qquad \text{...........(i)}$$

similarly for m_2



FBD of m_2

$$m_2 g - T = m_2 a \qquad \text{............(ii)}$$

Adding (i) & (ii)

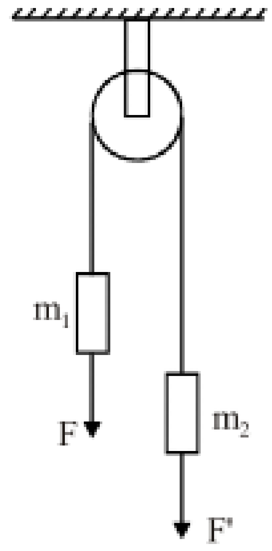
$$m_2 g - T + T - F = (m_1 + m_2)\, a$$

$$a = \frac{m_2 g - F}{m_1 + m_2} \qquad \left(F = \frac{m_2 g}{2}\right)$$

$$a = \frac{m_2 g}{2(m_1 + m_2)}$$

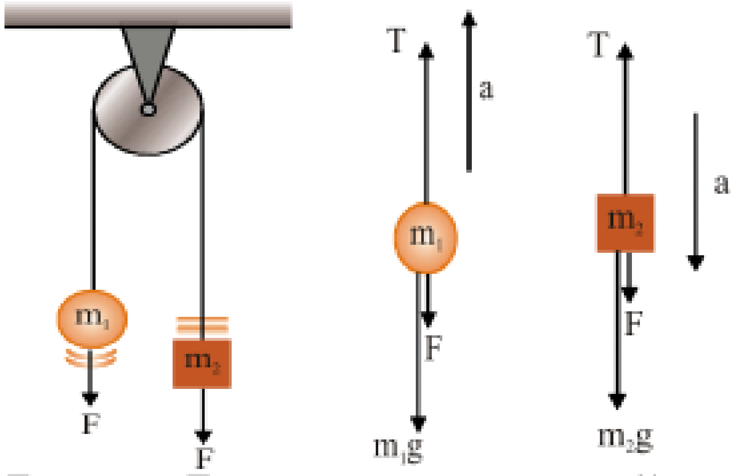
Illustration :

In figure $m_1 = 2$ kg, $m_2 = 5$ kg and $F = 1$ N. Find the acceleration of blocks.



Sol. *The free body diagrams for the two masses are shown in figure. Three forces act on each block: the upward force exerted by the string. T, the downward force of gravity and the downward force F. Thus, the magnitude of the net force ecerted on m_1 is $T - m_1g - F$, while the magnitude of the net force exerted on m_2 is $m_2g + F - T$. Because the blocks are connected by a string, their accelerations must be equal in magnitude. It is given that $m_2 > m_1$, then m_1 must accelerate upward, while m_2 must accelerate downward.*

When Newton's second law is applied to m_1, with a (acceleration) upward for this mass (because $m_2 > m_1$), we find (taking upward to be the positive y direction)



$$T - m_1g - F = m_1a \qquad \text{.........(i)}$$

Similarly, for m_2 we find

$$m_2g + F - T = m_2a \qquad \text{.........(ii)}$$

Adding (i) & (ii)

$$m_2g + F - T + T - m_1g - F = m_1a + m_2a$$

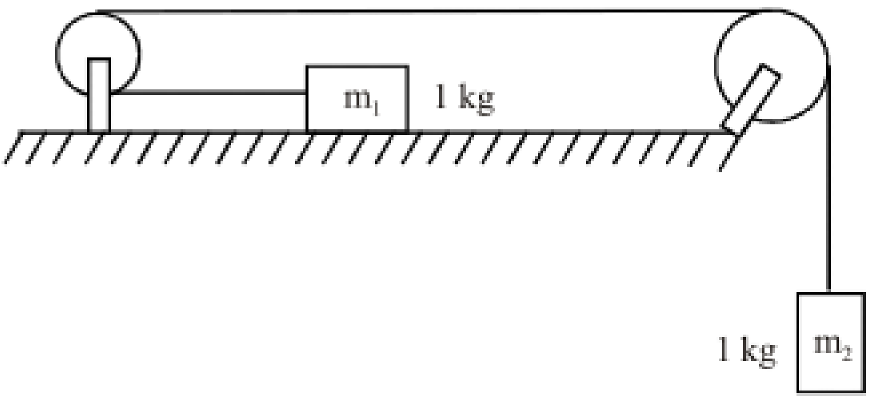
$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g \qquad \text{.........(iii)}$$

Given that $m_1 = 2$ *kg,* $m_2 = 5$ *kg and* $F = 1$ *N*

therefore $$a = \left(\frac{5-2}{5+2}\right)g = \frac{3g}{7}$$

Illustration :

Calculate the acceleration of the two blocks (m_1 and m_2) and tension in the string shown in figure. The pulley and the string are light and all the surfaces are frictionless. Take g = 10 m/s².



Sol. *The free body diagrams for the two masses are shown in figure. Two forces act on m_2: the upward force exerted by the string T, and the downward force of gravity. Thus, the magnitude of the net force exerted on m_2 is $m_2g - T$. Because the blocks are connected by a string, their accelerations must be equal in magnitude.*

When Newton's second law is applied to m_2, with a (acceleration) downward for this mass.

$$m_2 g - T = m_2 a \qquad \text{............(i)}$$

similarly for m_1 $\qquad T = m_1 a \qquad$ ....(ii)

Adding (i) & (ii)

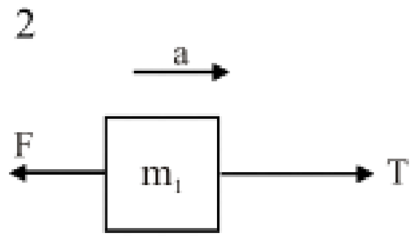
$$m_2 g - T + T = (m_1 + m_2)\, a$$

$$a = \frac{m_2 g}{m_1 + m_2}$$

Given that $m_1 = m_2 = 1$, therefore

$$a = \frac{g}{2}$$

FBD of m_1



FBD of m_2

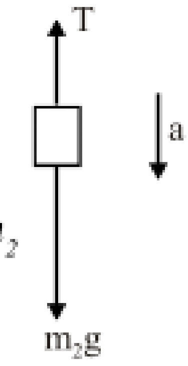
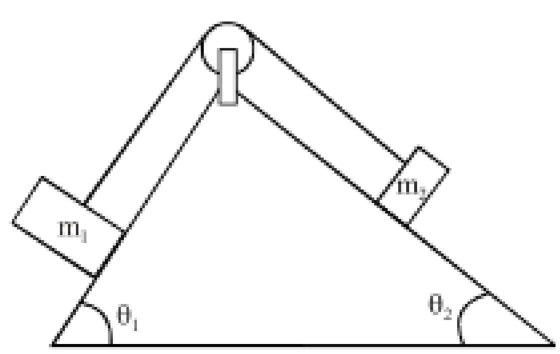


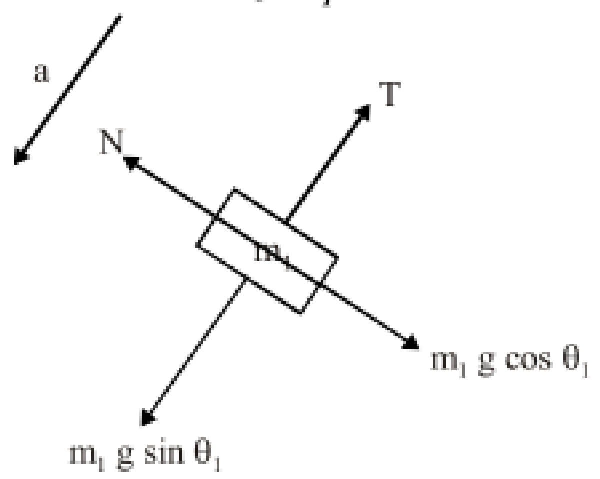
Illustration :

Consider the situation shown in figure. All the surface are frictionless and the string and the pulley are light. Find the magnitude of the acceleration of the two blcoks.

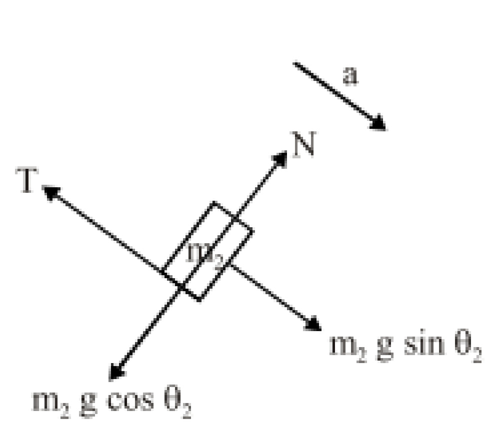


Sol. *The free body diagrams for the two masses are shown in figure. Two forces act on each block: the upward force along the inclined plane exerted by the string T, the downward force of gravity and normal force (N_1 & N_2). Thus, the magnitude of the net force exerted on m_1 along the inclined is $m_1 g \sin\theta_1 - T$, while the magnitude of the net force exerted on m_2 along the inclined is $T - m_2 g \sin\theta_2$. Because the blocks are connected by a string, their accelerations must be equal in magnitude. If we assume that $m_2 > m_1$, then m_1 must accelerate down the inclined, while m_2 must accelerate up the inclined.*

FBD of m_1



FBD of m_2



When Newton's second law is applied to m_1, with a (acceleration) down the inclined for this mass

(because $m_1 g \sin\theta_1 > m_2 g \sin\theta_2$)

$$m_1 g \sin\theta_1 - T = m_1 a \quad \text{.........(i)}$$

Similarly, for m_2 we find

$$T - m_2 g \sin\theta_2 = m_2 a \quad \text{.........(ii)}$$

Adding (i) & (ii)

$$m_1 g \sin\theta_1 - T + T - m_2 g \sin\theta_2 = m_1 a + m_2 a$$

or

$$a = \left(\frac{m_1 g \sin\theta_1 - m_2 g \sin\theta_2}{m_1 + m_2}\right) \text{.........(iii)}$$

Illustration :

A ball of mass 'm' falls from rest under gravity in a resistive medium. The resistive force is given by $f = -kv$. Find the velocity of ball as a function of time. Plot a v-t graph for it.

Sol. $$m\frac{dv}{dt} = mg - kv$$

$$\frac{m}{k}\int\frac{dv}{v} = -\int dt \qquad \frac{m}{k}\frac{dv}{dt} = \frac{mg}{k} - v$$

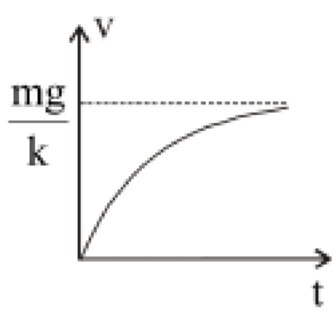
$$\int_0^v \frac{dv}{\frac{mg}{k} - v} = \frac{k}{m}\int_0^t dt$$

$$\left[\frac{\ell n\left(\frac{mg}{k} - v\right)}{-1}\right]_0^v = \frac{k}{m}[t]_0^t$$

$$\ell n\,\frac{mg - kv}{mg} = -\frac{k}{m}t$$

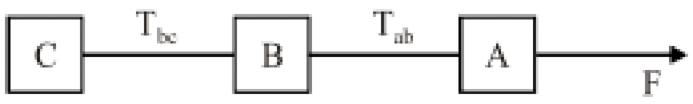
$$1 - \frac{kv}{mg} = e^{-\frac{kt}{m}}$$

$$v = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$$

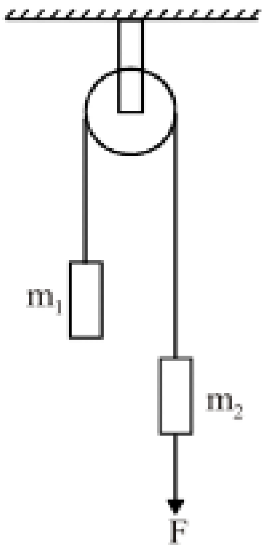


Practice Exercise

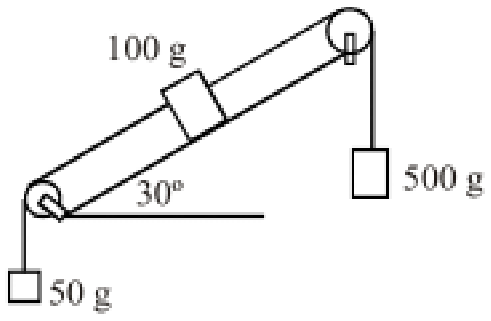
Q.1 Find the accelerations of blocks A, B & C and tentions in the strings. All the blocks are of equal mass 10 kg and F = 60 N



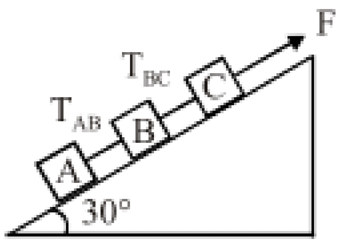
Q.2 In figure $m_1 = 2$ kg, $m_2 = 5$ kg and F = 21N. Find the acceleration of blocks.



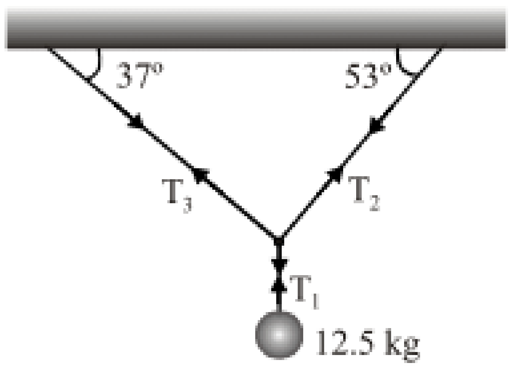
Q.3 Find the acceleration of the 500 g block in figure.



Q.4 Find the accelerations of blocks A, B & C and tensions in the strings. All the blocks are of equal mass 10 kg and F = 300 N



Q.5 A body of mass 12.5 kg is suspended with the help of light inextensible strings as shwon in figure. Find tension in three strings. Strings are light [g = 10 ms^{-2}]



Q.6 Which of the following is the reaction force to the gravitational force acting on the body as you sit in your desk chair ?

(a) The normal force exerted by the chair

(b) The force you exert downward on the seat of the chair

(c) Neither of these forces.

Answers

Q.1 $a_A = a_B = a_C = 2\ m/s^2$, $T_{AB} = 40\ N$, $T_{BC} = 20\ N$ Q.2 $a = 7m/s^2$

Q.3 $a = 80/13\ m/s^2$ Q.4 $a = 5\ m/s^2$ Q.5 $T_1 = 125\ N$, $T_2 = 100\ N$, $T_3 = 75\ N$

Q.6 C

Spring Force

We know that the more force we apply to a spring, the more it stretches. For a spring that obeys Hooke's low, the extension of the spring is proportional to the applied force. If we stretch spring by a distance x from its equilibrium position, it applies a restoring force F, towards its equilbrium position, which is proportional to x, given by

$$F = kx$$

Here k is a proportionality constant, known as spring constant or stiffness. A spring has tendecy of restoring its natural length state, thus whether we stretch it or compress, it always opposes the external force in the direction towards its equilibrium position.

One more point is to be noted is that, a spring applies restoring force equally at both of its ends, doesn't matter whether an end is fixed or not.

If we look at FBD of spring we will note that equal force on spring must act from both ends.

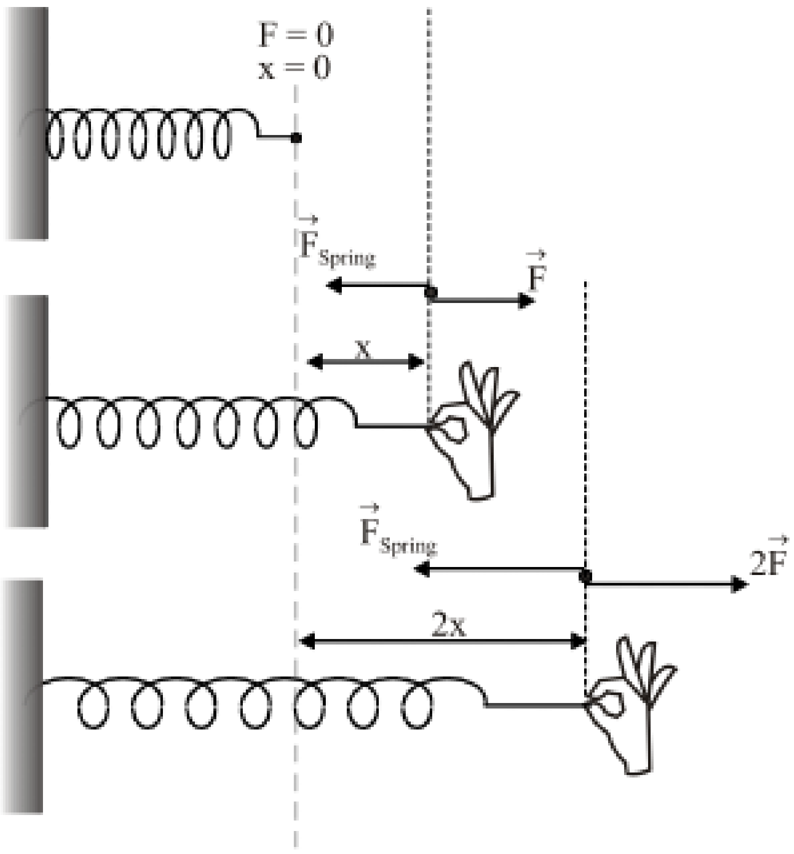
x is the elongation in spring

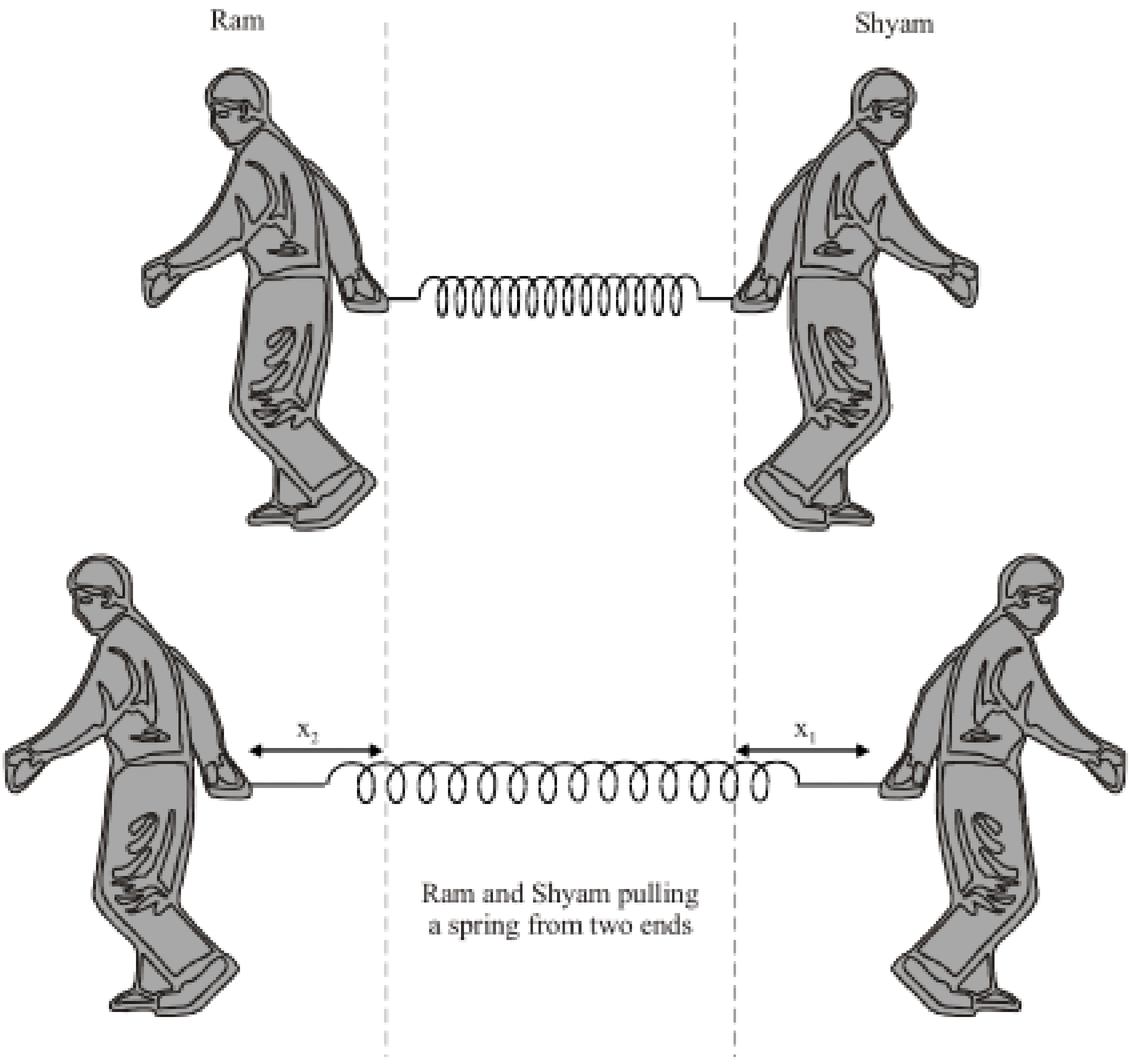
$$F - F' = m_s \times a \qquad \text{(spring is mass less, } m_s \to 0)$$

therefore $F = F'$

In figure an end of the spring is fixed to wall and other is pulled by applying a force. As the restoring force is directly proportional to the deformation in it, for stretching it by x, we apply a force F on it and for stretching it to double the length (2x), we have to apply a force double of the previous value i.e. 2F.

Let's say Ram & Shyam are pulling a spring from two ends as shown in figure Ram moves x_2 and Shyam moves x_1.



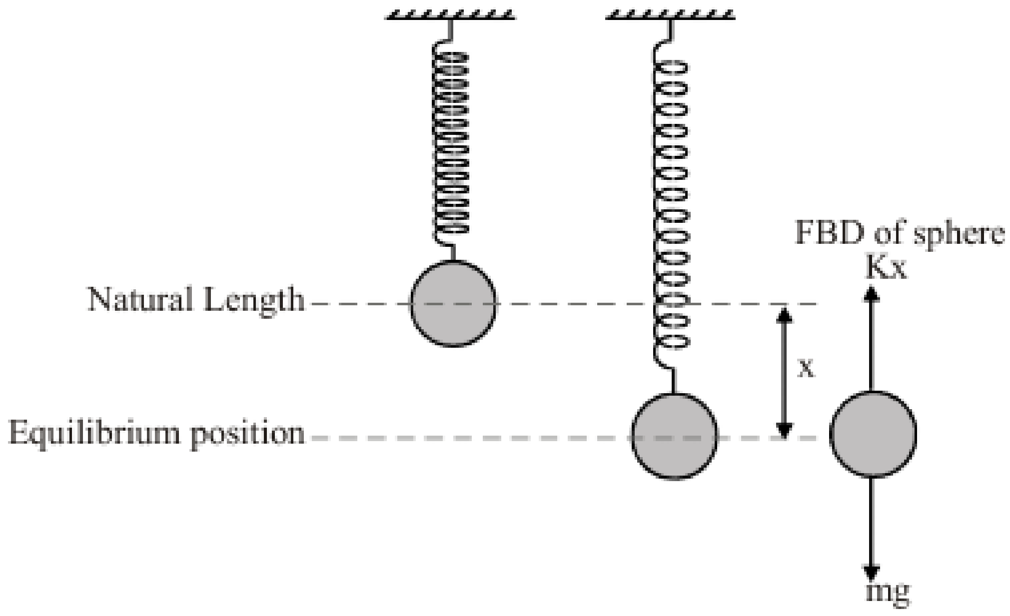


Ram and Shyam pulling a spring from two ends

The force acting on Ram and Shyam is $k(x_1 + x_2)$, not kx_2 on Ram and kx_1 on Shyam. Force due to spring is kx where x is defined as $|l - l_0|$, where l is present length and l_0 is natural length,

Illustration :

The spring is hanging vertically, and a sphere of mass m is attached to its lower end. Under the pull of the weight, the spring stretches a distance x from its natural length. If a spring is stretched 2.0 cm by a suspended object having a mass of 0.5 kg, what is the force constant of the spring ?



Sol. *Free-body diagrams for sphere in Shown in figure. Two force are acting on sphere : the upward force exerted by the spring (T = kx) and downward force of gravity (mg). As the sphere is in equilibrium the two forces will be equal in magnitude.Equilibrium is the condition when the acceleration of the block is zero)*

$$kx = mg \qquad \Rightarrow \qquad k = \frac{mg}{x} = \frac{0.5\times 10}{2.0\times 10^{-2}} = 250 N/m$$

Equivalent spring constant

(In all the discussions we will ignore mass and and damping of spring)

When two or more springs are connected in some manner then the combination can be replaced by a single equivalent spring such that it produces same elongation for same applied force.

Parallel combination

When springs are connected in parallel, then we can replace them by single spring of spring constant k_{eq} where $k_{eq} = k_1 + k_2$. This situation is shown in figure. Here we present the proof.

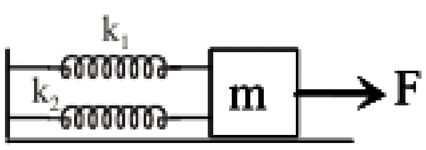
If the force F pulls the mass m by y, the stretch in each spring will be same.

$$y_1 = y_2 = y$$

Now for an equivalent spring $F = k_{eq} \times y$ and as spring constants are not equal so $F_1 \neq F_2$.

For equivalence,

$$F = F_1 + F_2 \Rightarrow K_{eq}\, y = k_1 y + k_2 y$$



this reduces to

$$k_{eq} = k_1 + k_2$$

For more springs $k = k_1 + k_2 + k_3 + ...$

Series combination

When spring are connected in series then we can replace them by a single spring of spring constant k_{eq}

where $\dfrac{1}{k_{eq}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$. This situation is shown in figure.

Here we present the proof. As the spring are massless, so force in the spring will be the same

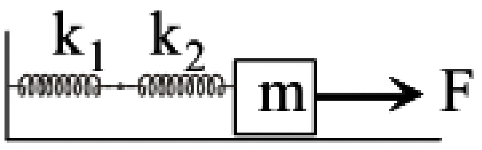
$$F_1 = F_2 = F$$

Now for equivalent spring $F = k_{eq}\,y$, as spring constants are not equal so extensions will not be equal, but total extension y can be written as sum of two extensions $y = y_1 + y_2$

or $$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2} \qquad \left[as\ for\ F = ky, y = \frac{F}{k}\right]$$

For more than two springs Connected in series:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$$



Parts of a spring :

If a spring of force constant k of length l is cut in two parts say of l_1 and l_2, let us assume that new force constants are k_1 and k_2 for the two parts. If we connect these two parts in series, the equivalent force constant must be initial k. Thus we have

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

According to the molecular properties of a spring, the force constant of a part of the spring is inversely proportional to its length, which gives us

$$k_1 = \frac{c}{l_1} \qquad \text{and} \qquad k_2 = \frac{c}{l_2}$$

Where c is a positive constant depending upon the material of spring. Substituting the above values of new force constants k_1 and k_2 in equation, we get

$$\frac{1}{k_{eq}} = \frac{l_1}{c} + \frac{l_2}{c}$$

or $$c = \frac{l}{k_{eq}}$$

Using value of c in equation, we have

$$k_1 = \frac{k_{eq}\,l}{l_1} \qquad \text{and} \qquad k_2 = \frac{k_{eq}\,l}{l_2}$$

Illustration :

A spring of constant K is cut into two parts in ratio 1:3. Find the spring constant of individual springs.

Sol. *Let us suppose natural length of spring is l*

Now, we have to find spring constant of both the parts (of length l/4 and 3l/4)

lets say spring constants of the parts are k_1 and k_2

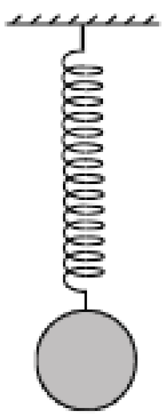
$$\text{we know } k \propto \frac{1}{l} \text{ or } kl = \text{constant i.e. } k_1 l_1 = k_2 l_2 = kl$$

$$k_1 \times \frac{l}{4} = k_2 \times \frac{3l}{4} = kl$$

$$\therefore \quad k_1 = 4k \; \& \; k_2 = \frac{4k}{3}$$

Illustration :

A mass m is connected to a spring of spring constant K as shown. Find the extension of the spring when the block is at its equilibrium position.



Sol. *The free body diagrams for the sphere is shown in figure. Two forces act on sphere: the upward force exerted by the spring (T=kx), and the downward force of gravity (mg).*

Given that sphere is in equilibrimum therefore these two forces should be equal in magnitude.

$$kx = mg \Rightarrow x = \frac{mg}{k}$$

FBD of sphere

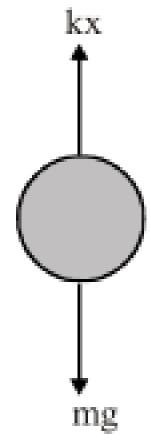
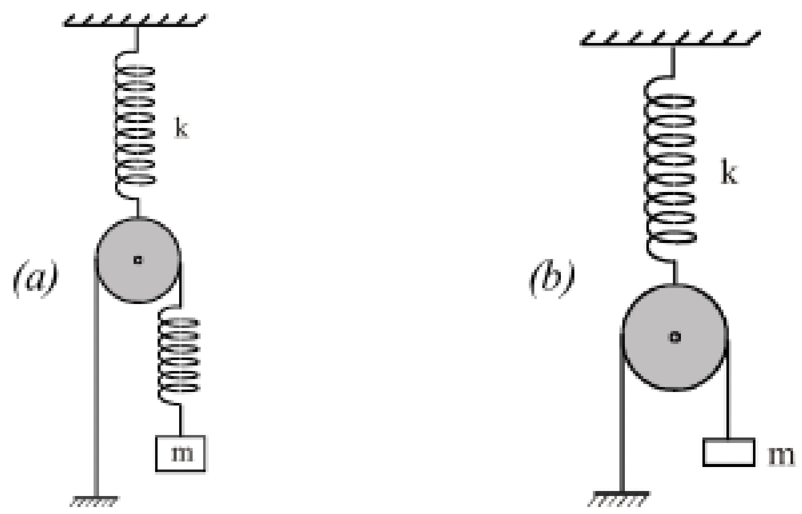


Illustration :

Find extension of spring in equilibrium condition, all the blocks are of mass m and all springs are of spring constant k.

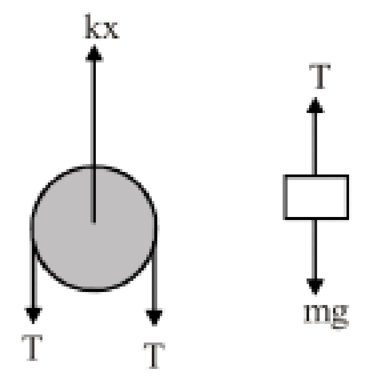


Sol. (a) The free body diagrams for the block and pulley are shown in figure. Three forces act on pulley Two forces act on block: the upward force exerted by the string (T), and the downward force of gravity (mg). Given that block is in equilibrium therefore these two forces should be equal in magnitude.

$kx = 2T$ (force on pulley should be zero because it is massless)

$$\frac{kx}{2} = mg \Rightarrow x = \frac{mg}{2k}$$

FBD of pulley & block



(b) Here again mass M is in equilibrium, thus tension in the string connected to it must be equal to Mg and hence the restoring force in the lower spring will also be Mg. If x_1 be the extension in this spring. we have

$$k_1x_1 = Mg$$
$$x_1 = \frac{Mg}{k_1}$$

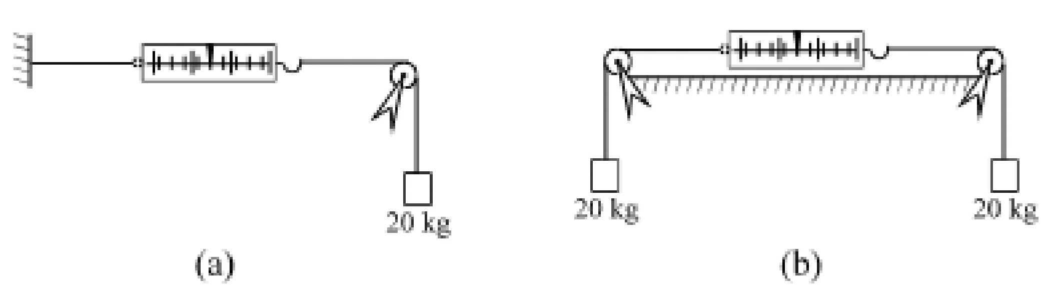
As pulley is light, the tension in upper string must be twice that of the lower string, 2Mg which is equalt to the restioring force in the upper spring. If x_2 is the extention in the upper spring, we have

$$k_2x_2 = 2Mg$$

or $$x_2 = \frac{2Mg}{k_2}$$

Illustration :

Figures show a light spring balance connected in two different arrangements. The graduations in the balance measure the tension in the spring.



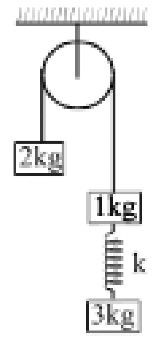
The ratio of reading of balances (T_a/T_b) is __________?

Sol. Spring balnce gives tension in the string
Because tension at both side of the spring balance will be same in case (a) & in case (b), $T_a = T_b = 20\,g$

$$\therefore \quad \frac{T_a}{T_b} = 1$$

Illustration :

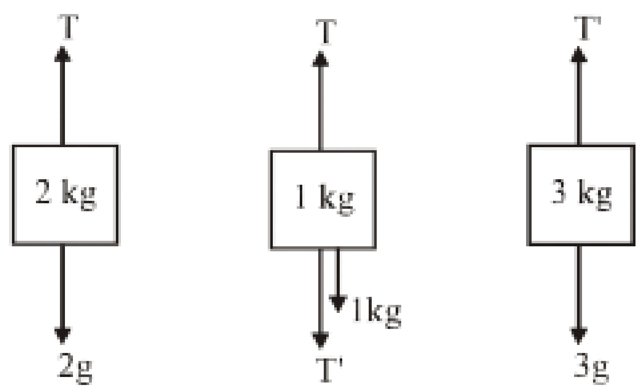
From the fixed pulley, masses 2 kg, 1 kg and 3 kg are suspended as shown in the figure. Find the extension in the spring if k = 100 N/m. (assume 1 kg and 3 kg block move with same acceleration)



Sol. *The FBD for the three block are shown in figure :*

Two force act on 2 kg block upward force exerted by the string T & downward force of gravity 2g thus, the magnitude of net force on 2 kg block is T – 2g.

FBD of blocks



Applying newton's II law on 2 kg block;

$$T - 2g = 2a \qquad \text{.....................(i)}$$

Similarly for 1 kg & 3 kg block;

$$T' + 1g - T = 1a \qquad \text{.....................(ii)}$$

$$3g - T' = 3a \qquad \text{.....................(iii)}$$

Adding (i), (ii) & (iii)

we get $2g = 6a$

$$a = \frac{g}{3}$$

$\therefore$ *substituting value of a in equation (iii); tension in spring* $T' = 3g - 3a \quad (T' = kx)$

$$T' = 2g$$

$$\therefore x = \frac{2g}{k} = \frac{g}{50} = 0.2\ m$$

Cutting of spring and string :

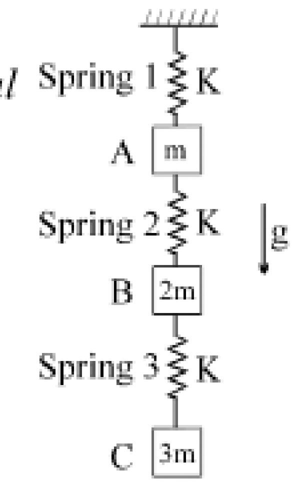
Spring force doesn't change instantaneously, whereas, the tension in the string changes instantaneously.

If a spring is cut, the tension in that spring becomes zero instantaneously.

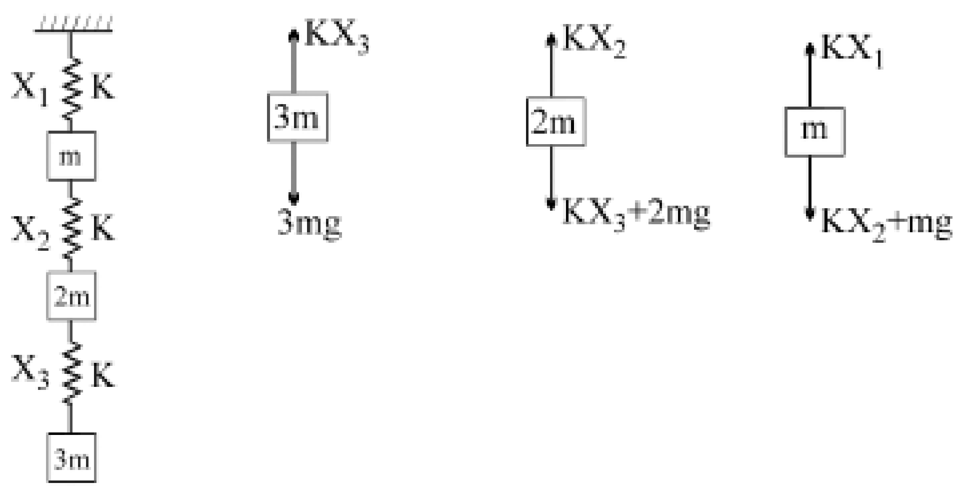
Illustration :

The system shown in the figure is in equilibrium. Find the initial acceleration

of A, B and C just after the spring-2 is cut.



Sol. *FBD of blocks before cutting the spring*



$$3mg = KX_3 \qquad ....(1)$$

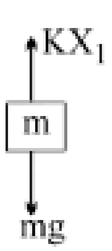
$$2mg + KX_3 = KX_2$$

$$\therefore \quad 2mg + 3mg = KX_2 \Rightarrow 5\,mg = KX_2 \qquad ....(2)$$

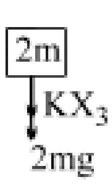
$$KX_1 = 6\,mg \qquad ....(3)$$

when spring 2 is cut

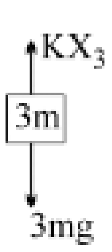
spring force in other two spring remain becomes zero, while unchanged.



$$KX_1 - mg = ma_3 \Rightarrow a_3 = 5g \uparrow$$



$$KX_3 + 2mg = 2ma_2 \Rightarrow a_2 = \frac{5g}{2} \downarrow$$



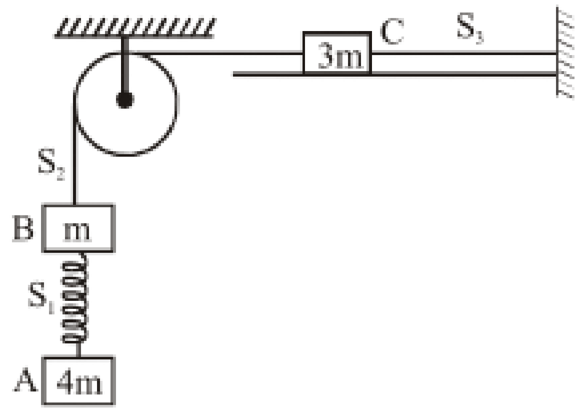
acceleration of 3 m will be zero.

Conclusion :

It is important to remember that ropes can change tension instantaneously while spring need to move to change tension, so in this example tension in spring is not changing instantaneously

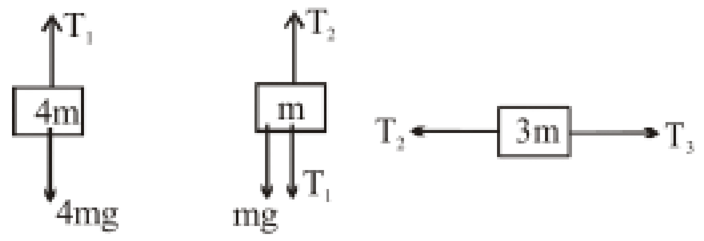
Illustration :

In following setup pulley strings and spring are light. Initially all masses are in equilibrium and at rest.



(a) Find tension in spring and tension in ropes
(b) Find accleration of masses immediately after the string S_3 is cut.

Sol. (a)



applying Newton's 2^{nd} law to block A

$$4mg - T_1 = 0$$

applying Newton's 2^{nd} law to block B

$$mg + T_1 - T_2 = 0$$

applying Newton's 2^{nd} law to block C

$$T_3 - T_2 = 0$$

Solving $T_1 = 4mg$

$T_2 = 5mg$

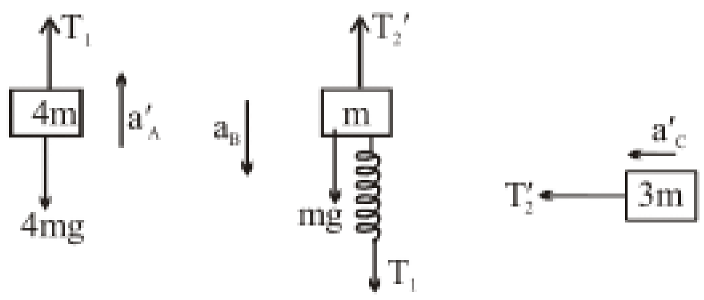
$T_3 = 5mg$

Here spring is behaving same as string except that it is streched while string can not strech.

(b) *The most important point in this problem is that any object of finite mass can not change its position instantaneously, as this require infinite velocity.*

Thus immediately after cutting the string S_3 all masses will remain at same position and force due to spring will not change. As force of spring is k x and x is $l - l_0$. At the same time we would like to emphasize that tension in string S_2 will change instantaneausly (Tension is a self adjusting Force). To maintain constraint relation between blocks B & C have same since they magnitude of accleration. We can identify all the forces acting on all objects. Only tension T in string S_2 is unknown force all other forces are known.

Considering FBD



We know from part (a) that tension in spring is

T_1 *and* $T_1 = 4\,m\,g$

Writing Newton's Second Law for A

$$4\,mg - T_1 = 4m\,a_A'$$

Writing Newton's Second Law for B

$$T_1 + mg - T_2' = ma_B'$$

Writing Newton's Second Law for C

$$T_2' = 3ma_C'$$

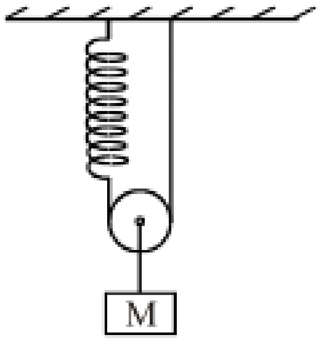
Quantities which may have different value from part (a)are represented using symbol ', for eg a tenstion in string s_2 is T_2', others which have same value as part (a) have been retained with same symbol.

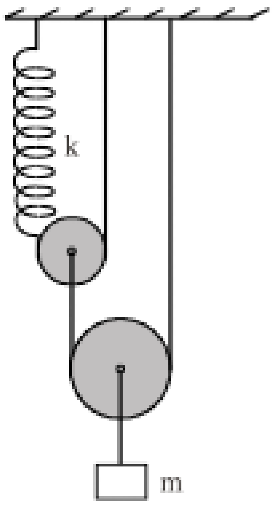
as $a_B' = a_C'$

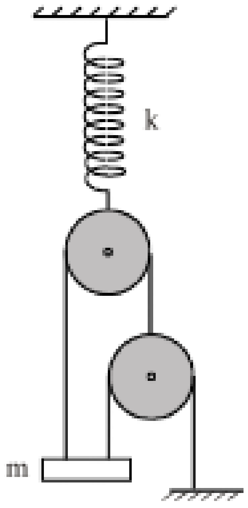
solving we get $a_A' = 0 \quad a_B' = a_C' = \dfrac{5}{4}g$

Practice Exercise

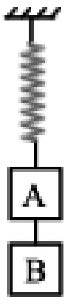
Q.1 Find extension of spring in equilibrimum condition, all the blocks are of mass m and all springs are of spring constant k.

(i) 

(ii) 

(iii) 

Q.2 Two blocks A and B of mass M and 2M respectively, are hanging from a ceiling by means of a light spring and a light string as shown. If the string between the blocks is suddenly cut, tinα, the accelerations of the block A and B.



Answers

Q.1 (i) $\dfrac{Mg}{2k}$; (ii) $\dfrac{mg}{4k}$; (iii) $\dfrac{4mg}{3k}$ Q.2 $a_A = 2g$; $a_B = g$

Pseudo force

Motion in Accelerated Frames :

Till now we have restricted ourselves to apply Newton's laws of motion, only to describe observations that are made in an inertial frame of reference. In this part, we learn how Newton's laws can be applied by an observer in a noninertial reference frame. For example,consider a block kept on smooth surface of a compartment of train.

If the train acclerates, the block accelerates toward the back of the train. When observed from the train we may conclude based on Newton's second law F= ma that a force is acting the block to cause it to accelerate, but the Newton's second law is not applicable from this non- inertial frame. So we can not relate observed accleration with the Force acting on the block.

If we still want to use Newton's second law we need to apply a psuedo force, acting in back-ward direction, ie opposite to the aceleration of noninertial reference frame. This force explains the motion of block towards the back of train. The fictious force is equal to $-\overrightarrow{ma}$, where $\vec{a}$ is the accelera-tion of the non inertial reference frame. Fictitious force appears to act on an object in the same way as a real force, but real forces are always interactions between two objects, on the other hand there is no second object for a fictitious force.

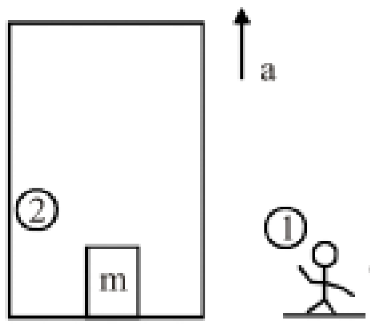
$$\text{pseudo force, } \vec{F}_P = -m\vec{a}_0$$

where a_0 is acceleration of non inertial reference frame

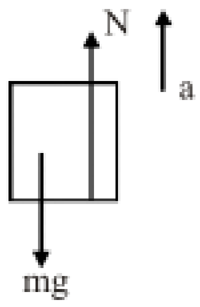
Thus, we may conclude that pseudo force is not a real force. When we draw the free body diagram of a mass, with respect to an inertial frame of reference we apply only the real forces (forces which are actually acting on the mass), but when the free body diagram is drown from a non-inertial frame of reference a pseudo force (in addition to all real forces) has to be applied to make the equation

$\vec{F} = m\vec{a}_0$, valid in this frame also.

Suppose a block A of mass m is placed on a lift ascending with an acceleration a_0. Let N be the normal reaction between the block and the floor of the lift Free body diagram of A is shown in figure.

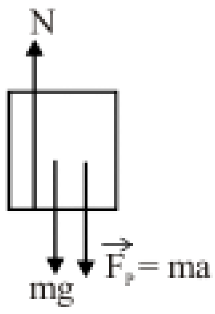


for observer 1 (Inertial reference frame):



$N - mg = ma$

For observer 2 (Non inertial reference frame):



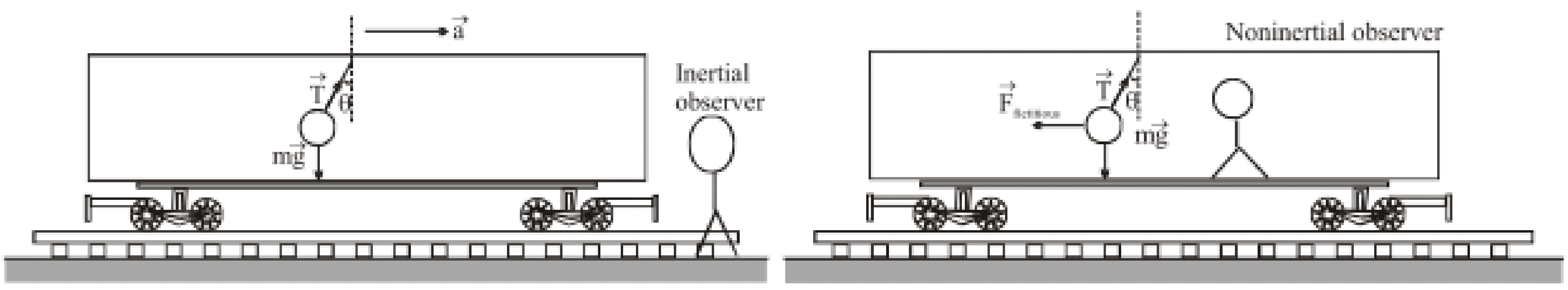
$a = 0$

$\therefore \quad N = mg + ma$

Illustration :

A small ball of mass m hangs by a cord from the ceiling of a compartment of a train that is accelerating to the right as shown in Figure. Analyze the situaions for two observers A & B.

Sol. *The observer A on the ground, is inertial Frame. He sees the compartment is accelerating and knows that the horizontal comp. of tension in the cord provides the ball, required horizontal force. The noninertial observer on the compartment, can not see the car's motion, he is not aware of its acceleration. He will say that Newton's second law is not valid as the object has net horizontal force (the horizontal component of tension) but no horizontal acceleration.*



(a) (b)

For the inertial observer, ball has a net force in the horizontal direction and is in equilibrium in the vertical direction. For the noninertial observer, we apply fictitious force towards left and consider it to be in equilibirium.

According to the inertial observer A, the ball experience two forces, T exerted by the cord and the weight.

Applying, Newton's second law in in horizontal and vertical direction we get

Inertial observer $T\sin\theta - mg = 0$ .....(i)

$T\cos\theta = ma$ .....(ii)

According to the noninertial observer B riding in the car (Fig. b), the ball is always at rest and so its acceleration is zero. The noninertial observer applies a fictitious force in the horizontal direction of magnitude ma towards left. This fictious force balances the horizontal component of T and thus the net force on the ball is zero.

Apply Newton's second law in horizontal and vertical direction we get

Noninertial observer $T\sin\theta - mg = 0$

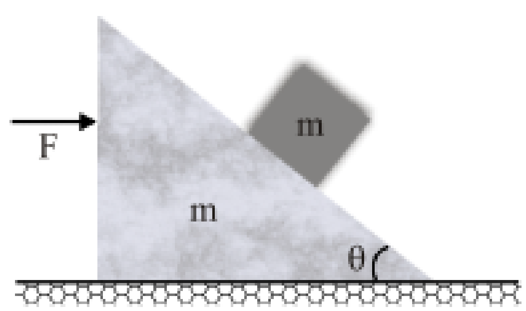
$T\cos\theta - ma = 0$

These expressions are equivalent to Equations (1) and (2).

The noninertial observer B obtains the same equations as the inertial observer. The physical explanation of the cord's deflection, however, differs in the two frames of reference.

Illustration :

All surfaces are smooth in following figure. Find F, such that blcok remains stationary with respect to wedge.



Sol. *Acceleration of (block + wedge)* $a = \dfrac{F}{(M+m)}$

Let us solve the problem by both the methods.

From inertal frame of reference (Guound)

FBD of block w.r.t. ground (Apply real forces):

With respect to ground block is moving with an acceleration 'a'.

$\therefore \qquad \Sigma F_y = 0 \quad \Rightarrow \quad N\cos\theta = mg \qquad$ ....(i)

and $\qquad \Sigma F_x = ma \quad \Rightarrow \quad N\sin\theta = ma \qquad$ ....(ii)

From Equation (i) and (ii)

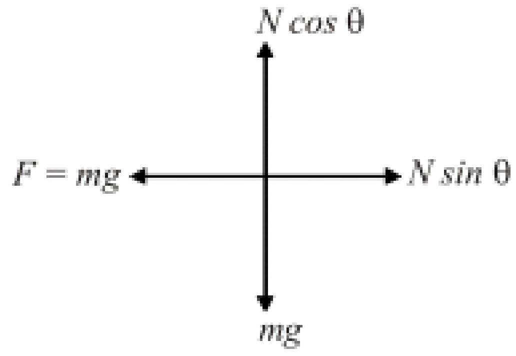
$$a = g\tan\theta$$

$$F = (M+m)a$$

$$= (M+m)g\tan\theta$$

From non-inertial frame of reference (Wedge)

FBD of block w.r.t. wedge (real force + pseudo force)



w.r.t. wedge block is stationary

$\therefore \qquad \Sigma F_y = 0 \quad \Rightarrow \quad N\cos\theta = mg \qquad$ ....(iii)

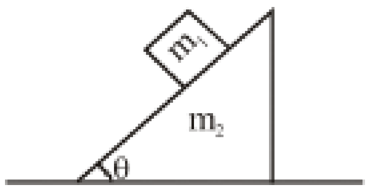
and $\qquad \Sigma F_x = ma \quad \Rightarrow \quad N\sin\theta = ma \qquad$ ....(iv)

From equation (iii) and (iv), we will get the same result

$$F = (M+m)g\tan\theta$$

Illustration:

There is no friction at any contact. Wedge is free to move
Find force acting on wedge due to block. Also find acceleration of wedge.



Sol. *You may want to directly reach to conclusion that answer is* $m_1 g\cos\theta$. *but it is being solved in refrence frame of wedge which may be acclerating. Horizontal component of normal contact force applied by block on wedge will accelerate the wedge. Thus refrence frame attached to wedge is non-inertial refrence frame.*

Acceration vector of block in ground frame is sum of acceleration of wedge and acceleration of block w.r.t. wedge $(\vec{a}_{b/w})$

$$\vec{a}_b = \vec{a}_{b/w} + \vec{a}_w$$

$$\vec{a}_b \equiv$$

Consider F.B.D of wedge. Take horizontal component of normal contact force and apply Newton's 2nd Law

$$N \sin\theta = m_2 a_w$$

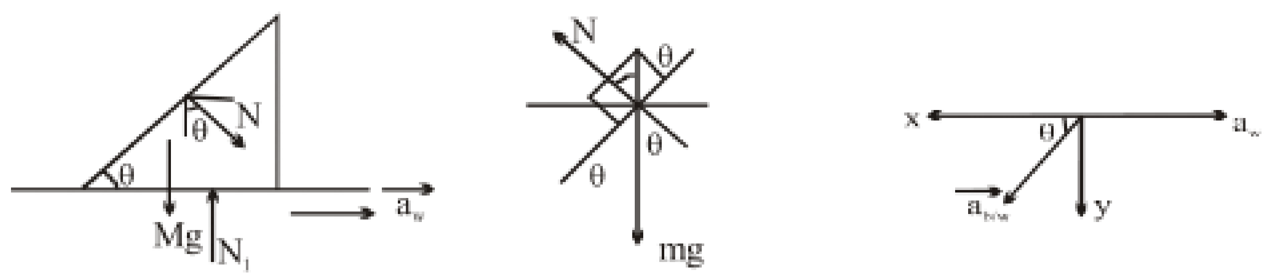
Consider F.B.D. of block and acceleration vector of block, Take horizontal and vertical component of forces and acceleration and apply Newton's second law.

$$a_x = a_{b/w} \cos\theta - a_w$$

$$a_y = a_{b/w} \sin\theta$$

$$N \sin\theta = m_1 (a_{b/w} \cos\theta - a_w)$$

$$m_1 g - N \cos\theta = m_1 (a_{b/w} \sin\theta)$$



Solving we get

$$N = \frac{m_1 m_2 g \cos\theta}{(m_2 + m_1 \sin^2\theta)}$$

$$a_w = \frac{m_1 g \cos\theta \sin\theta}{(m_2 + m_1 \sin^2\theta)}, \quad a_{b/w} = \frac{(m_1 + m_2) g \sin\theta}{(m_1 \sin^2\theta + m_2)}$$

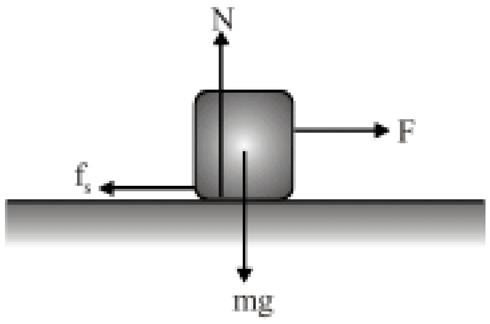
Friction

Frictional forces are unavoidable in our daily lives. If we were not able to counteract them, they would stop every moving object and bring to a halt every rotating shaft. About 20% of the gasoline used in an automobile is needed to counteract friction in the engine and in the drive train on the other hand, if friction were totally absent, we could not get an automobile to go anywhere, and we could not walk or ride a bicycle. We could no hold a pencil, and, if we could, it would not write. Nails and screws would be useless, woven cloth would fall apart, and knots would untie.

When surfaces slide or tend to slide over one another, a force of friction acts. Friction is caused by the irregularities in the surfaces in mutual contact, and it depends on the kinds of material and how much they are pressed together. Even surfaces that appear to be very smooth have microscopic irregularities that obstructs motion. Atoms cling together at many points of contact. When one object slides against another, it must either rise over the irregular bumps or else scrape atoms off. Either way requires force. Although the details of friction are quite complex at atomic level, it ultimately involves the elctromagnetic force between atoms & molecules.

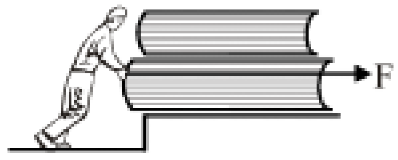
The direction of the friction force is always in a direction opposing relative motion. An object sliding down an incline experiences friction directed up the incline; an object that slides to the right experiences friction toward the left. Thus, if an object is to move at constant velocity, a force equal to the opposing force of friction must be applied so that the two forces exactly cancel each other. The zero net force then results in zero acceleration and constant velocity.

Static Friction



$F = f_s$, As long as the block is not moving

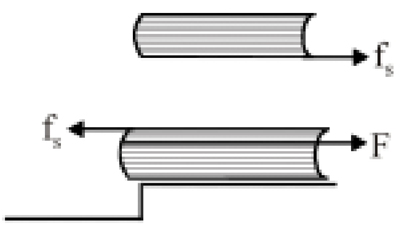
Static friction acts when two contact surfaces are not moving relative to each other. For example, consider a block on a horizontal table, as in figure. If we apply an external horizontal force F to the block, acting to the right, the block remains stationary if F is not too large. The force that counteracts F and keeps the block from moving acts to the left and is the frictional force f. As long as the block is not movng, f = F. Since the block is stationary, we call this frictional force the force of static friction, f_s.



Two books kept on top of each other & the lower book being pushed slowly

If we keep two books one on top the other and now we slowly push the lower book, both the books move together, the force moving the upper book is friction. Since the books are moving with respect to ground but they are not moving with respect to each other, this force of friction between two books is of static nature.

Similarly when we walk on ground the friction force acting on the foot is of static nature. You may be suprised but the foot in contact with ground is not moving while the other foot is moving. During this time friction acting between foot and ground is static friction.



Free body diagram of two books

If we are increasing our speed then static friction is acting in forward direction. This happens because we try to pull our leg backward while walking in forward direction. This means foot is trying to move in backward direction with respect to ground and ground applies friction in forward direction

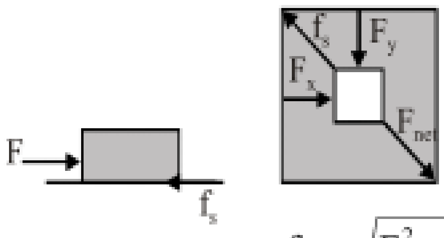
Man is moving but the point in contact of the foot is not moving i.e. static friction is acting on the man

Person is walking due to static friction

Static friction is self-adjusting and it opposes the tendency of relative motion

In fact, in the previous case of books also, we can see that static friction acting in forward direction on upper book has given it some motion so it can move with the lower book. Another point to be noted here is that even by third law, pair of static friction is opposing relative motion by trying to slow down the lower book (which is being accelerated externally).

for example in figure we assume that the block is stationary and we can see that static friction is acting in such a direction so as to oppose the relative motion. F represents external force and f_S represents friction.



$$f_s = \sqrt{F_x^2 + F_y^2}$$

Block is stationary due to static friction

The magnitude of the static friction between any two surfaces in contact can have the values

$$f_s \le \mu F_N$$

where the dimensionless constant μ_s is known as the coefficient of static friction and F_N is the magnitude of the normal contact force exerted by one surface on the other. The equality in equation holds when the surfaces are on the verge of slipping, that is when $f_s = f_{s,max} = \mu_s F_N$, This maximum value of f_s is called limiting friction. This situation is called impending motion. The inequality hold when the surfaces are not on the verge of sliping. Maximum strength of the joints formed is directly proportional to the normal contact force, that is $f_{s,max} \propto F_N$.

Maximum strength also depends on the roughness of contact surface $f_{s,max}$ (also called $f_{limiting}$) = $\mu_s N$. Magnitude of static friction is self-adjusting such that relative motion does not start (but still it has maximum value).

Let us say we are applying force F on a block kept on horizontal rough surface with coefficient of static friction $\mu_s = 0.1$ & mass of block is 5 kg. When applied F is less than 5 N the value of static friction is

equal to the applied force, not 5 N. It is the maximum value of friction. But when applied force F is equal to 5 N, the value of static friction is 5 N

Illustration :

A block lying on a horizontal surface is pulled by a force of 0.1 N but the block does not move i.e., remains at rest. Find friction force

Sol. *To analyse the frictional force on the block we proceed as follows. As the block remains at rest, the force of static friction is balancing the 0.1 N force (the applied force). So the frictional force - 0.1 N*

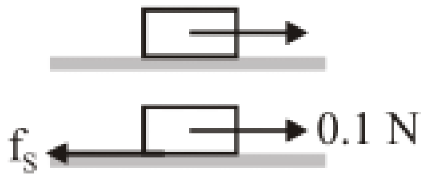
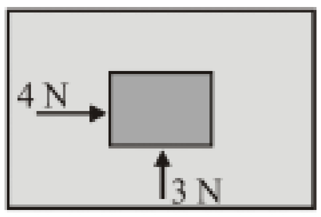


Illustration :

A block lying on a horizontal surface is pushed by forces as shown in figure but the block does not move i.e., it remains at rest.

Find the frictional force on the block.



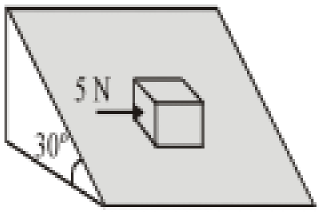
Sol. *Net force acting on the block is* $\sqrt{3^2+4^2}=5N$

As the block remains at rest, the force of static friction is balancing the applied force (5 N)

Illustration :

A block lying on an inclined surface at an angle 30° with the horizontal is pushed by force as shown in figure but the block does not move i.e., it remains at rest.

Find the frictional force on the block.



Sol. *Two forces are acting on the block along the surface of inclined, first one is mg sin 30° down the inclined and second 5 N as shown in figure*

So the net force acting on the block along the surface of inclined is $\sqrt{12^2+5^2}=13N$

As the block remains at rest, the force of static friction is balancing the applied force (13 N)

Illustration :

A block of weight 100 N lying on a horizontal surface just begins to move when a horizontal force of 25 N acts on it. Determine the coefficient of static friction.

Sol. *As the 25 N force brings the block to the point of sliding i.e., limiting friction is 25 N.*

limiting frictional force = $\mu_s N$.

From force diagram : $N = 100\ N$

$\mu_s N = 25 \quad \Rightarrow \quad \mu_s = 0.25.$

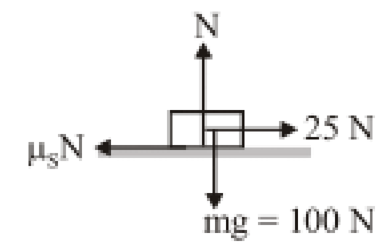
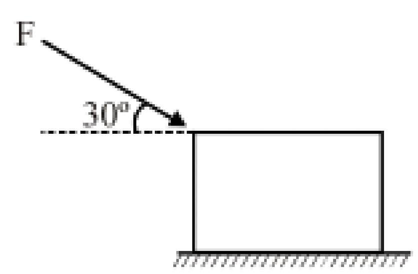


Illustration :

A block of weight 100 N lying on a horizontal surface is pushed by a force F acting at an angle 30° with horizontal. For what value of F will the block begin to move if $\mu_s = 0.25$?

Sol. *Consider the force diagram of the block at the moment when it is just to start moving.*

Balancing force :



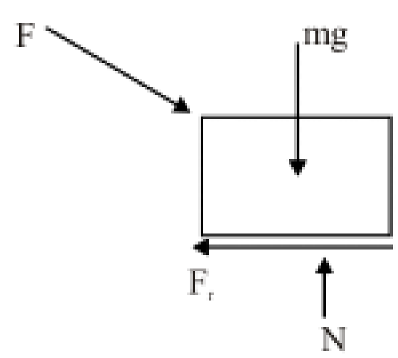
$$N = mg + F \sin 30^o$$

$$F = \cos 30^o = \mu_s N$$

$$F = \cos 30^o = \mu_s (mg + F \sin 30^o)$$

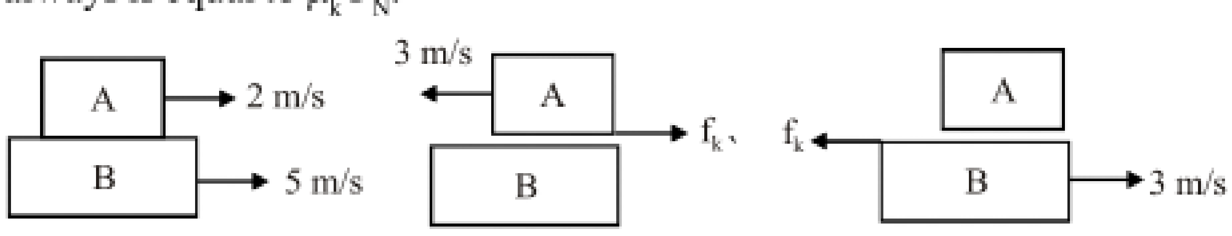
$$F = \frac{\mu_s mg}{\cos 30^o – \mu_s \sin 30^o} = \frac{0.25(100)2}{\sqrt{3} - 0.25}$$

$$F = 33.74\ N$$



Kinetic friction

Kinetic friction acts when there is relative motion between two surfaces in contact. It acts always opposite to the relative velocity as we can see in figure. The magnitude is not self-adjusting as in static friction, it is always is equal to $\mu_k F_N$.



with rspect to B with respect to A

Experimentally, we find that, to a good approximation, both $f_{s,max}$ and f_k are proportional to the magnitude of the normal force. The following empirical laws of friction summarize the experimental observations :

1. The magnitude of the force of kinetic friction acting between two surfaces is

$$f_k = \mu_k N$$

where μ_k is the coefficient of kinetic friction. Although the coefficient of kinetic friction can vary with speed, we shall usually neglect any such variations in problems.

2. The value of μ_k and μ_s depend on the nature of the surfaces.

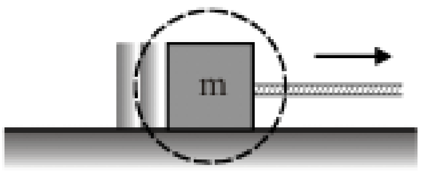
3. μ_k is generally less than μ_s.

The actual value depends on the degree of smoothness and other environmental factors. For example, wood may be prepared at various degrees of smoothness and the friction coefficient will vary. Dust impurities, surface oxidation etc. have a great role in determining the friction coefficient. Suppose we take two blocks of pure copper, clean them carefully to remove any oxide or dust layer at the surfaces, heat them to push out any dissolved gases and keep them in contact with each other in an evacuated chamber at a very low pressure of air. The blocks stick to each other and a large force is needed to slide on over the other. The friction coefficient as defined above, becomes much larger than one this is called cold welding. If a small amount of air is allowed to go into the chamber so that some oxidation takes place at the surface, the friction coefficient reduces to usual values.

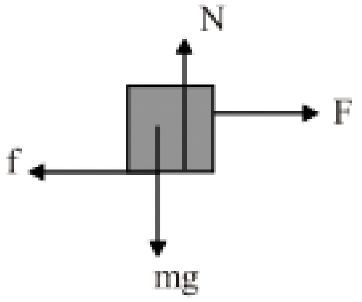
4. The direction of the friction force on an object is parallel to the surface with which the object is in contact and opposite to the motion (kinetic friction) or the tendency of motion (static friction) of the object relative to the surface.

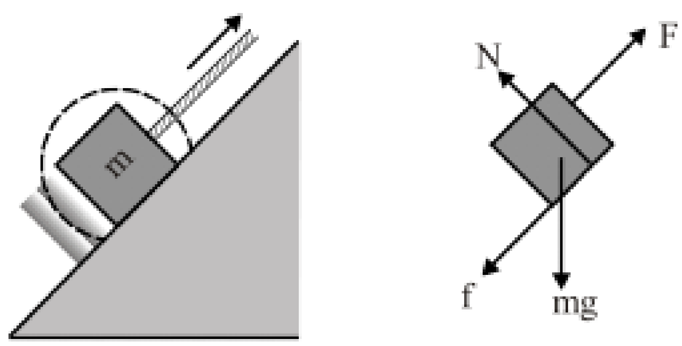
5. The coefficients of friction are nearly independent of the area of contact between the surfaces. We might expect that placing an object on the side having the more area might increase the friction force. While this provides more points in contact, the weight of the object is spread out over a larger area, so that the individual points are not pressed so tightly together. These effects approximately compensate for each other, so that the friction force is independent of the area. So those extra wide tires you see on some cars provide no more friction than narrower tires. The wider tire simply spreads the weight of the car over more surface area to reduce heating and wear. Similarly, the friction between a truck and the ground is the same whether the truck has four tires or eighteen! More tires spread the load over more ground area and reduces the pressure per tire. Interestingly, stopping distance when brakes are applied is not affected by the number of tires.But the wear that tires experience, very much depends on the number of tires.

Various mechanical configurations (left) and the corresponding free-body diagrams (right). The term rough here means only that the surface is not frictionless.

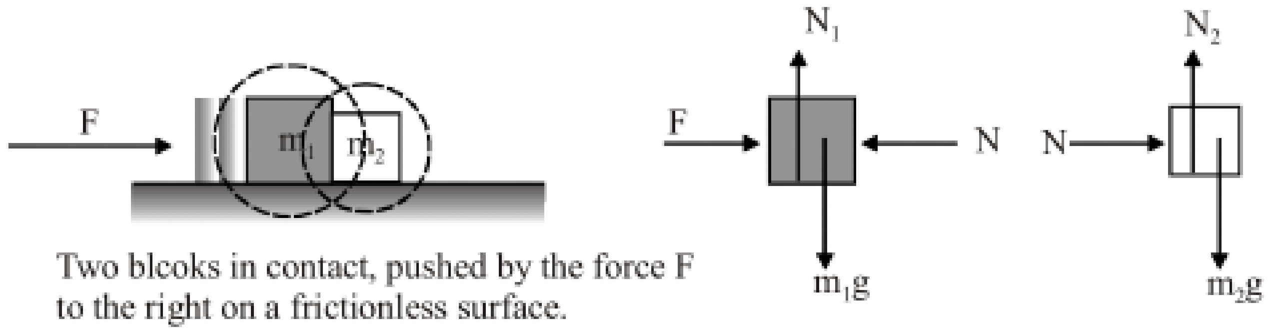


A block is pulled by force F to the right on a rough horizontal surface

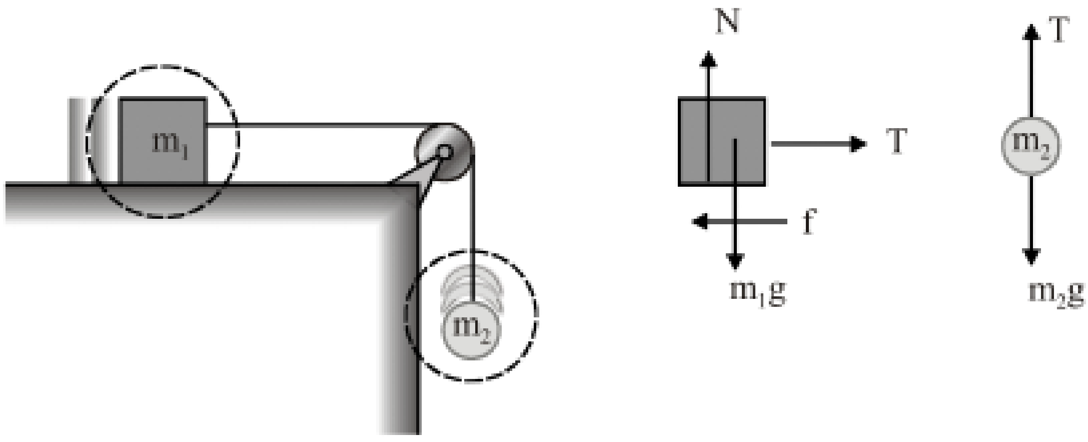




A block is pulled up the rough incline



Two blcoks in contact, pushed by the force F to the right on a frictionless surface.



Two masses connected by a light cord.
The surface is rough and the pulley is frictionless

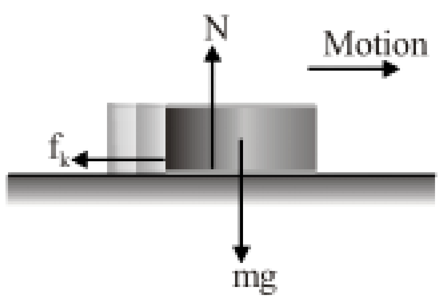
Illustration:

A hockey puck on a frozen pond is given an initial speed of 20.0 m/s. If puck always remains on the ice and slides 115 m before coming to rest, determine the coefficient of kinetic friction between the puck and ice.

Sol. *Imagine that the puck in figure slides to the right and eventually comes to rest. The forces acting on the puck after it is in motion are shown in figure first, we find the acceleration in terms of the coefficient of kinetic friction, using Newton's second law. Knowing the acceleration of the puck and the distance it travels, we can then use the equation of knematics to find the numerical value*

of the coefficient of kinetic friction.

Defining rightward and upward as our positive directions, we apply Newton's second law in component form to the puck and obtain



$$\Sigma F_x = -f_k = ma_x \qquad \text{.................(1)}$$

$$\Sigma F_y = N - mg = 0 \qquad (a_y = 0) \qquad \text{.................(2)}$$

But $f_k = \mu_k N$, *and from (2) we see that* $N = mg$. *Therefore,*

(1) becomes

$$-\mu_k N = -\mu_k mg = ma_x$$

$$a_x = -\mu_k g$$

The negative sign means the acceleration is to the left in figure because the velocity of the puck is to the right, this means that the puck is slowing down. The acceleration is independent of the mass of the puck and is constant because we assume that μ_k *remains constant.*

Because the acceleration is constant, we can use Equation $v^2 = u^2 + 2as$

$$0 = u^2 + 2a\,S = u^2 - 2\mu_k gS \qquad v = 0$$

$$\mu_k = \frac{u^2}{2gS}$$

$$\mu_k = \frac{(20.0 m/s^2)^2}{2(9.80 m/s^2)(115 m)} = 0.136$$

Illustration :

A 5 kg block slides down a plane inclined at 30^o *to the horizontal. Find*

(a) The acceleration of the block if the plane is frictionless.

(b) the acceleration if the coefficient of kinetic friction is 0.2.

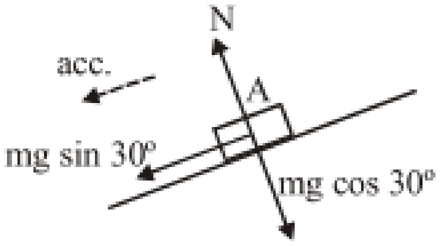
Sol. *(i)* $N = mg\cos 30^o$

$mg\sin 30^o = ma$

$a = g\sin 30^o$,

down the plane if plane is smooth.

$a = g/2 = 4.9\ m/s^2$



(ii) $N = mg \cos 30^o$

$mg \sin 30^o - \mu_k N = ma$

$a = g \sin 30^o - \mu_k g \cos 30^o$

$a = 3.20\ m/s^2$

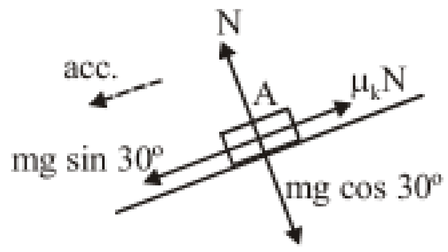


Illustration :

5 kg block projected upwards with an initial speed of 10 m/s from the bottom of a plane inclined at 30° with horizontal. The coefficient of kinetic friction between the block and the plane is 0.2.

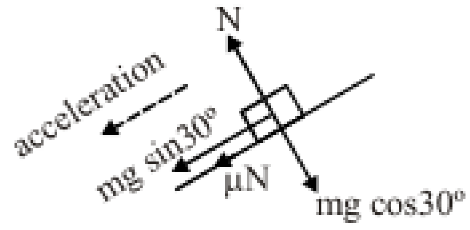
(a) How far does the block move up the plane ?

(b) How long it move up the plane ?

Sol *While the block is moving up the frictional forc acts downward.*

As the block is slowing downs, the velocity and acceleration must be in opposite direction.

Velocity in this case is upwards, so acceleration is in downward direction.



the megnitude of acceleration $= \dfrac{mg \sin 30^o + \mu\, mg \cos 30^o}{m} = g\,(\sin 30^o + \mu\, mg \cos 30^o)$

$\Rightarrow \quad a = -g\,(\sin 30^o + \mu\, mg \cos 30^o) = -6.6\ m/s^2$

For the motion of block from the bottom to up the plane :

$u = +10\ m/s \qquad v^2 = u^2 + 2as$, *we get*

$0^2 = 10^2 + 2\,(-6.6)\,(s) \qquad \Rightarrow \qquad s = 7.58\ m$

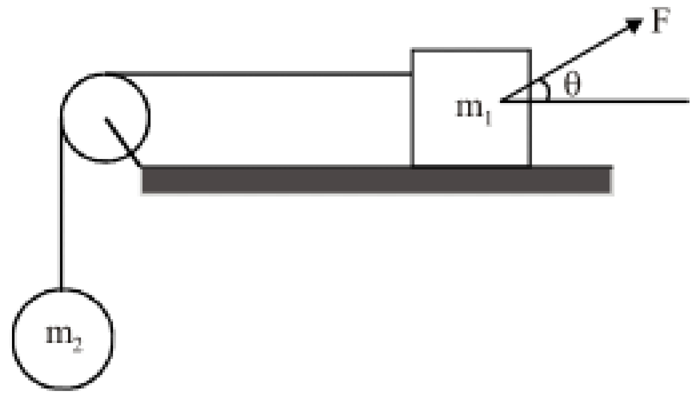
$v = u + at$

$0 = 10 - 6.6 \times t \qquad \Rightarrow \qquad t = 1.5$ *seconds*

Hence the block moves up the plane for 1.5 sec covering 7.58 m.

Illustration :

A block of mass m_1 on a rough, horizontal surface is connected to a ball of mass m_2 by a lightweight cord over a lightweight, frictionless pulley, as shown in figure. A force of magnitude F at an angle θ with the horizontal is applied to the block as shown. The coefficient of kinetic friction be tween the block and surface is μ_k. Determine the magnitude of the acceleration of the two objects.



Sol. *The block will side to the right and the ball will rise. We can identify forces and we want an acceleration, so we categorize this as a Newton's second law problem, one that includes the friction force. To analyze the problem, we begin by drawing free-body diagrams for the two objects ,as shown in figure and Next, we apply Newton's second law in component from to each object and use Equation $f_k = \mu_k n$. Then we can above for the acceleration in terms of the parameters given.*

This applied force F has x and y components F cos θ and F sin θ, respectively. Applying Newton's second law to both objects and assuming the motion of the block is to the right, we obtain.

Motion of block : $\Sigma F_x = F\cos\theta - f_k - T = m_1 a_x = m_1 a$ .....(1)

$\Sigma F_y = N + F\sin\theta - m_1 g = m_1 a_y = 0$ .....(2)

Motion of ball : $\Sigma F_x = m_2 a_x = 0$

$\Sigma F_y = T - m_2 g = m_2 a_y = m_2 a$ .....(3)

Because the two objects are connected, we can equate the magnitudes of the x component of the acceleration of the block and the y component of the acceleration of the ball. From Equation we know that $f_k = \mu_k N$, and form (2) we know that $N = m_1 g - F\sin\theta$ (in this case n is not equal to $m_1 g$) therefore,

$$f_k = \mu_k(m_1 g - F\sin\theta) \quad .....(4)$$

That is, the friction force is reduced because of the positive y component of F. Substituting (4) and the value of T from (3) into (1) gives

$$F\cos\theta - \mu_k(m_1 g - F\sin\theta) - m_2(a + g) = M_1 a$$

Solving for a, we obtain

$$a = \frac{F(\cos\theta + \mu_k \sin\theta) - g(m_2 + \mu m_1)}{m_1 + m_2} \quad .....(5)$$

The acceleration of the block can be either to the right or to the left,[5] depending on the sign of the numerator in (5). If the motion is to the left, then we must reverse the sign of f_k in (1) because the force of kinetic friction must oppose the motion of the block relative to the surface.

Illustration :

A horse pulls a sled along a level, snow-covered road, causing the sled to accelerate, as shown in figure. Newton's third law state that the sled exerts a force of equal magnitude and opposite direction on the horse. In view of this, how can the sled accelerate – don't the forces cencel ? Under what condition does the system (horse plus sled) move with constant velocity ?

Sol. *Remember that the forces described in Newton's third law act on different objects the horse exerts a force on the sled, and the sled exerts an equal magnitude and oppositely directed force on the horse. Because we are interested only in the motion of the sled, we do not consider the forces it exerts on the horse. When determining the motion of an object, you must add only the forces on that object. (This is the principle behind drawing a free-body diagram.) The horizontal forces exerted on the sled are the forward force T exerted by the horse and the backward force of friction f_{sled} between sled and snow. When the forward force on the sled exceeds the backward force, the sled accelerates to the right.*

The horizontal forces exerted on the horse are the forward force f_{horse} exerted by the earth and the backward tension force T exerted by the sled. The resultant of these two forces causes the horse to accelerate.

The force that accelerates the system (horse plus sled) is the net force $f_{horse} - f_{sled}$. When f_{horse} balances f_{sled}, the system moves with constant velocity.

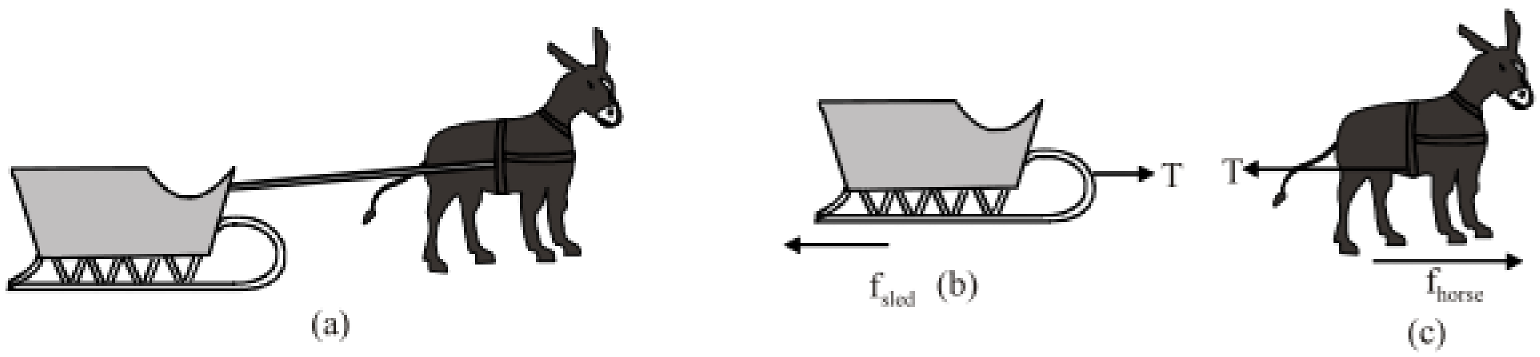
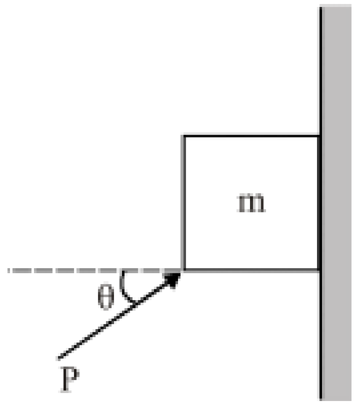


Illustration :

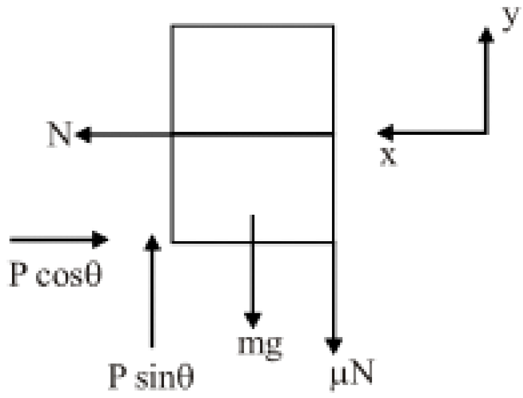
A block of mass 'm' is supported on a rough wall by applying a force P as shown in figure Coefficient of static friction between block and wall is μ_s. For what range of values of P, the block remains in static equlilibrium ?



Sol. *Impending state of motion is a critical border line between static and dynamic states of body. The block under the influence of P sin θ (Component of P) may have a tendency to move upward or it may be assumed that P sinθ just prevents downward fall of the block. Therefore there are two possibilties:*

Case (i) *Impending motion upwards : In this case force of friction is downward.*

from conditions of equilibrium



$$\Sigma F_x = N - P\cos\theta = 0$$

or $$N = P\cos\theta$$

$$\Sigma F_y = P\sin\theta - \mu N - mg = 0$$

or $$P\sin\theta - \mu P\cos\theta - mg = 0$$

or $$P_{max} = \frac{mg}{sin\theta - \mu cos\theta}$$

Case (ii) *Impending motion downward : In this case friction force acts upward.*

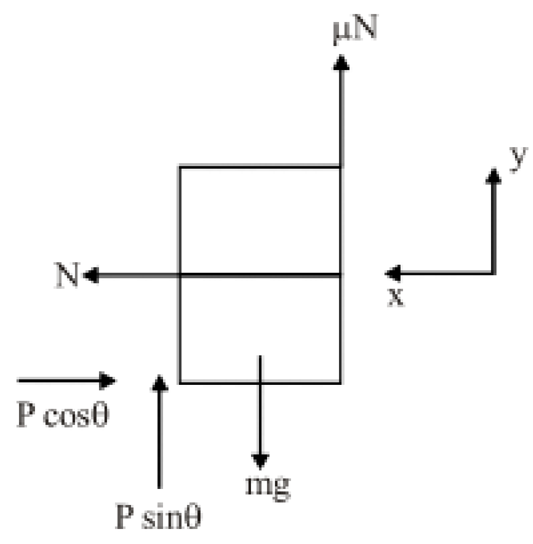
$$\Sigma F_x = N - P\cos\theta = 0$$

or $$N = P\cos\theta$$

$$\Sigma F_y = P\sin\theta + \mu N - mg = 0$$

or $$P\sin\theta + \mu P\cos\theta - mg = 0$$

or $$P_{min.} = \frac{mg}{sin\theta + \mu cos\theta}$$



Therefore the block will be in static equilibrium for

$$\frac{mg}{sin\theta + \mu cos\theta} \le P \le \frac{mg}{sin\theta - \mu cos\theta}$$

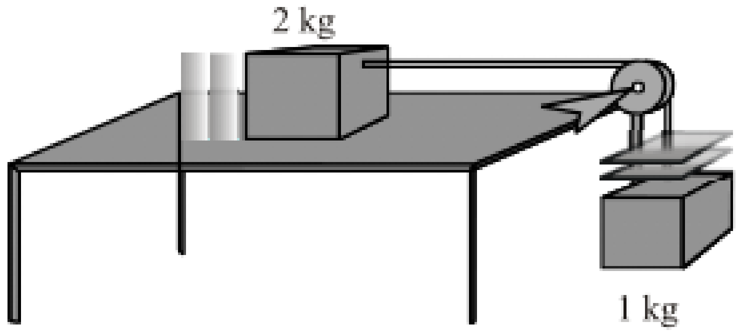
Practice Exercise

Q.1 You press your physics textbook flat against a vertical wall with your hand. What is the direction of the friction force exerted by the wall on the book?

Q.2 A crate is located in a truck. The truck accelerates to the east, and the crate moves with it, without sliding. What is the direction of the friction force exerted by the crate on the truck ?

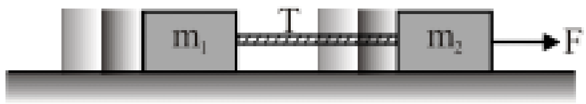
Q.3 A block lies on a floor. (a) What is the magnitude of the frictional force on it from the floor ? (b) If a horizontal force of 5 N is now applied to the block, but the block does not move, what is the magnitude of the frictional force on it ? (c) If the maximum value of the static frictional force on the block is 10 N, will the block move if the magnitude of the horizontally applied force is 8 N ? If it is 12 N ? (d) What is the magnitude of the frictional force in part (c)?

Q.4 A 1 kg hanging block is connected by a string over a pully to a 2 kg block sliding on a flat table. If the coefficient of sliding friction is 0.20, find the tension in the string.



Q.5 A 25 kg block is initially at rest on a horizontal surface. A horizontal force of 75 N is required to set the block in motion. After it is in motion, a horizontal force of 60 N is required to keep the block moving with constant speed. Find the coefficients of static and kinetic friction from this information.

Q.6 Two blocks connected by a massless rope are being dragged by a horizontal force F. Suppose that F = 60 N, m_1 = 12 kg, m_2 = 18 kg, and coefficient of kinetic friction between each block and the surface is 0.10. (a) Find the magnitude of the acceleration of the system.



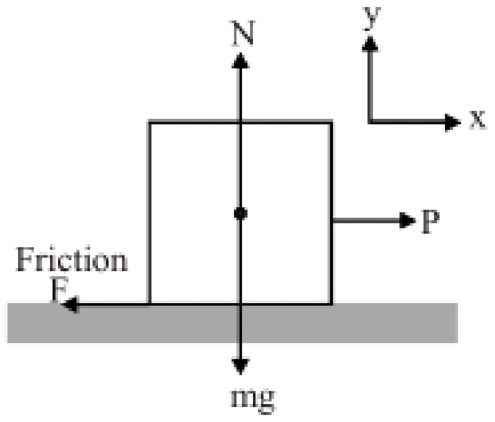
Answers

Q.1	upward	Q.2	to the west	Q.3	(a)0; (b)5; (c) No, yes (d) 8,10
Q.4	T = 8N, a = 2m/s²	Q.5	$\mu_s = 0.3$, $\mu_k = 0.24$	Q.6	a = 1 m/s²

Graph of frictional force versus applied force

Figure shows a block of weight mg resting on a rough surface. A horizontal force is applied to the block. Force P is gradually increased from zero.

When force P is very small, the block does not move form condition of equilibrium,



$$\Sigma F_x = P - F_{friction} = 0;\ F_{friction} = P$$

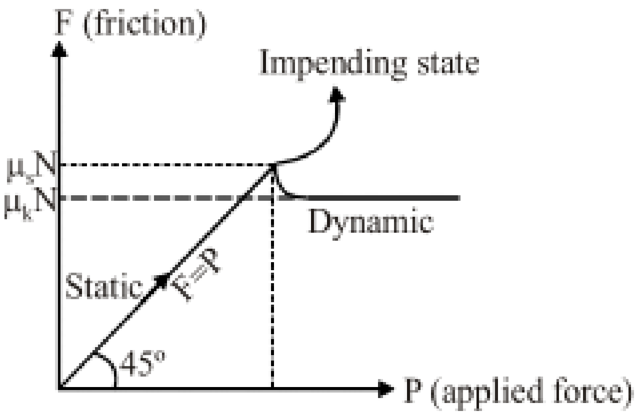
$$\Sigma F_y = N - mg = 0;\ N = mg$$

Friction force counter balance external force, till the block is static. This friction force is static friction. As external force is increased, static friction also increases.

As the force P is gradually increased, a limiting point is reached where friction force F is not sufficient to prevent onset of motion. When the block is about to move, the state of motion is called impending state of motion. At this point friction force has maximum value and given by the equation

$$F_{max} = \mu_s N$$

where μ_s is coefficient of static friction, N is normal reaction.



If force P is greater than F_{max}, the block will have a resultant froce $P - F_{max}$ on it. The block will accelerate in the direction of resultant force when sliding motion starts. At this moment friction force is given by

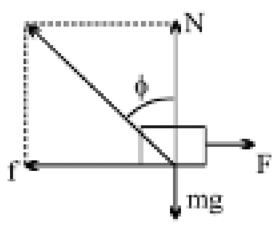
$$F = \mu_k N$$

Where μ_k is coefficient of kinetic friction. The figure shows variation of friction force versus external force graph.

Angle of friction (ϕ)

Mathematically, the angle of friction (ϕ) may be defined as the angle between the normal reaction N and the resultant of the maximum friction force f and the normal reaction.

Thus $\tan\phi = \dfrac{f}{N}$

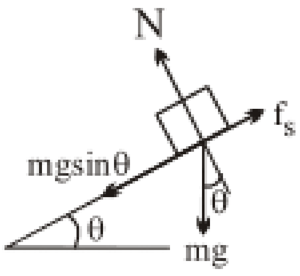


Since $f = \mu N$, therefore,

$$\tan \phi = \mu$$

Angle of Repose

Suppose a block is placed on a rough surface inclined relative to the horizontal, as shown in figure. The incline angle is increased until the block starts to move. The angle θ for which slipping just occurs is **angle of repose**.



The only forces acting on the block are the gravitational force mg, the normal force N, and the force of static friction f_s. These forces balance when the block is not moving. When we choose x-axis to be parallel to the plane and y perpendicular to it, Newton's second law applied to the block for this balanced situation gives

$$\Sigma F_x = mg\sin\theta - f_s = 0 \qquad \text{..............(1)}$$

$$\Sigma F_y = N - mg\cos\theta = 0 \qquad \text{.............(2)}$$

We can eliminate mg by substituting $mg = N\cos\theta$ from (2) into (1) to find

$$f_s = mg\sin\theta = \left(\frac{N}{cos\theta}\right)\sin\theta = N\tan\theta \qquad \text{.............(3)}$$

When the incline angle is increased until the block is on the verge of slipping, the force of static friction has reached its maximum value $\mu_s N$. The angle θ in this situation is the critical angle θ_c and (3) become

$$\mu_s N = N\tan\theta_c$$

$$\mu_s = \tan\theta_c$$

this θ_c called **angle of repose**

At this angle inclined we feel net force on the block is zero and it will move with constant velocity but, if pushed. If block is pushed slightly then it moves with acceleration because frictional force on the block will decrease. ($\mu_k < \mu_s$)

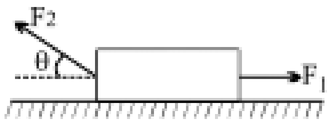
Application Automobile Antilock braking system (ABS)

(Topic for interest)

If an automobile tire is rolling and not slipping on a road surface, then the maximum friction force that the road can exert on the tire is the force of static friction μ_s. One must use static friction in this situation because at the point of contact between the tire and the road, no sliding should occur. However, if the tire starts to skid , the friction force exerted on it is reduced to the force of kinetic friction μ_k. Thus, to maximize the friction force and minize stopping distance, the wheels must maintain pure rolling motion and not skid. In directional control situations drivers typically press the brake very hard, "locking the brakes." This stops the wheels from rotating, ensuring a skid and reducing the friction force from the static to the kinetic value. To address this problem, automotive engineer have developed antilock braking systems (ABS). The purpose of the ABS is to help maintain control of automobiles and minimize stopping distance. The system briefly releases the brakes when a wheel is just about to stop turning. This maintains rolling contact between the tire and the pavement. When the brakes are released momentarily, the stopping distance is greater than it would be, if the brakes were being applied continuously.

Practice Exercise

Q.1 A block of mass m = 3 kg is experiencing two forces acting on it as shown.



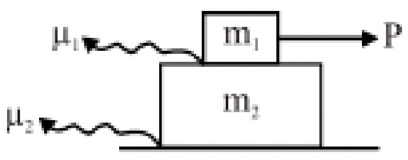
If $F_2 = 20$ N and $\theta = 60°$, determine the minimum and maximum values of F_1 so that the block remains at rest. (Take : $\mu_s = 1/\sqrt{3}$ and $\mu_k = 1/3, g = 10\,m/s^2$)

Q.2 A block, of mass m slips on a rough horizontal table under the action of a horizontal force applied to it. The coefficient of friction between the block and the table is μ. The table does not move on the floor. Find the total frictional force applied by the floor the legs of the table.

Answers

Q.1 $10\sqrt{3}$ N , $20-10\sqrt{3}$ N, 2N, left Q.2 μmg

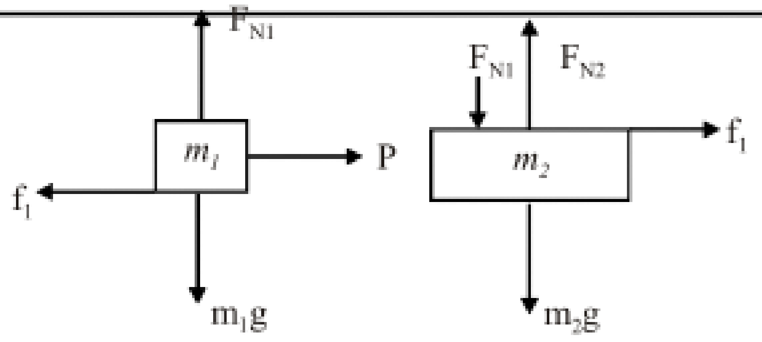
Block over block problems



Consider two block kept one over the other as shown in Figure where m_1 is kept on top of m_2. Let's say coefficient of friction between m_1 and

m_2 is μ_1 and between m_2 and ground is $\mu_2 = 0$.

To find acceleration of m_1 and m_2 we make their free-body diagrams and write Newton's second law



Along y axis

$$F_{N_1} - m_1 g = 0 \text{ and } F_{N_1} + m_2 g - F_{N_2} = 0$$

Solving the two gives,

$$F_{N_1} - m_1 g = 0 \text{ and } F_{N_2} = (m_1 + m_2)g$$

Along x axis

$$P - f_1 = m_1 a_1 \text{ and } f_1 = m_2 a_2$$

Also we know $\quad f_1 < \mu_1 F_{N_1}$

We now have two equation and three unknowns in equation along x direction. This problem cannot be solved unless we make some assumptions.

Case I : If we assume both blocks are moving together then friction between them need not be equal to maximum value. Thus, taking $a_1 = a_2$, we are down to two unknowns and we can solve the equations, but we must verify our answer by checking $f_1 \le \mu_1 F_{N_1}$. If this check fails we go to next possibility.

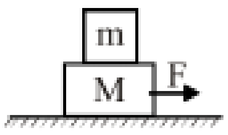
Case II : If we assume blocks are moving relative to each other, then friction between them must have reached maximum value (Kinetic friction)

$$a_1 \ne a_2 \text{ but } f_1 = \mu_1 F_{N_1}$$

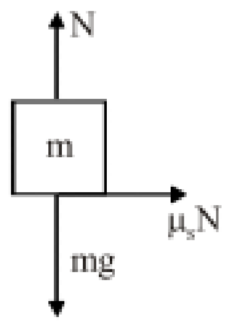
Again we get two unknowns and two equation. We can solve it, But we must verify our answer. We will check the answer by verifying that friction is opposite to the direction relative motion.

Illustration :

A block of mass m is placed on another block of mass M lying on a smooth horizontal surface. The confficient of static friction between m and M is μ_s. What is the maximum force that be applied to M so that the blocks remains at rest relative to each other ?



Sol. *Draw the force diagrams of blocks at the moment when F is at its maximum value and m is about to slide relative to it.*



Frictional force between m and M = $\mu_s N$

(N ; normal reaction between the block)

Due to the friction, M will try to drag m towrds right and hence frictional force will act on m towards right.

Let a = acceleration each block

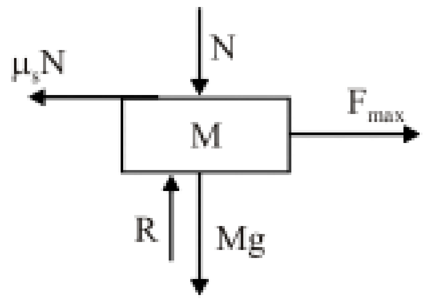
R = normal reaction between M and the surface.

Frome force diagram of m :

$$N = mg$$

$$\mu_s N = ma$$

From force diagram of M:



$$N + Mg = R$$

$$F_{max}\ \mu_s N = Ma$$

combining these two equation, we get

$$F_{max} = \mu_s\ (m + M)\ g$$

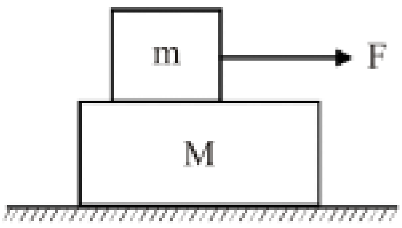
Hence μ_s (m + M) is the critical value of force F.

If F is greater than this critical value, m begins to slip relative to M and their acceleration will be different.

If F is smaller than this critical value, m and M move together without any relative motion.

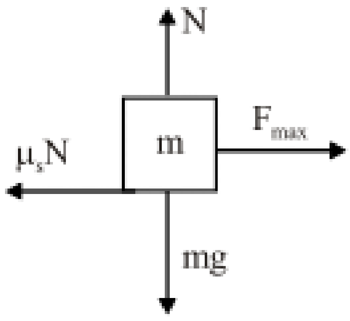
Illustration :

A block of mass m is placed on another block of mass M lying on a smooth horizontal surface. The coefficient of static friction between m and M is μ_s. What is the maximum force that can be applied to m so that blocks remains at rest relative to each other ?



Sol. *Imagine the situation when F is at its maximum value so that m is about to start slipping relative to M.*

The mass m tries to drag M toward right due to frction.



From forces on m :

$$F - \mu_s N = ma$$

$$N = mg$$

From forces on M :

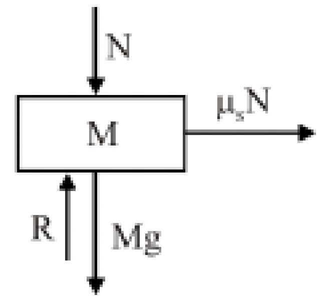
$$\mu_s N = ma$$

$$F_{max} = m_s (m + M)g$$

Solving these equations, we get :

Hence frictional force on M exerted by m will be towards right

Let a = magnitude of acceleation of blocks towards right.



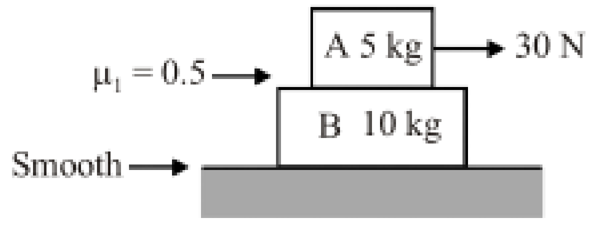
$$F_{max} = \frac{\mu_s (m+M)mg}{M}$$

If F is less than this critical value, the blocks stick together without any relative motion.

If F is greater than this critical value, the blocks slide relative to each and their acceleration are different.

Illustration :

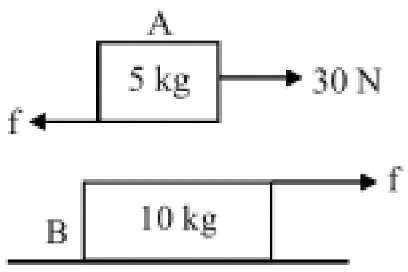
Friction coefficient between the blocks is 0.5 and ground is smooth if force of 30 N is applied on the upper block as shown. Find acceleration of the blocks



Sol. *Maximum friction force between blocks is 25 N ($f_{max} = \mu N = \mu mg = 0.5 \times 5 \times 10$) and ground is smooth. Block B must move because some force on upper surface will act on it.*
Block B can either move with same velocity and acceleration as block A or it can move relative to A.

Let's assume block B moves with block A and solve.

Making free-body diagram as shown in figure and applying Newton's second law, we get



$$30 - f = 5a \qquad \text{(for A)}$$
$$f = 10a \qquad \text{(for B)}$$

Solving we get,

$$a = 2 \text{ m/s}^2 \text{ and } f = 20 \text{ N}$$

Now check if this acceleration is possible by verifying $f \le f_{max\,A}$

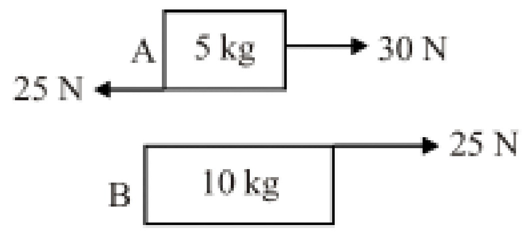
$$f = 20 < 25$$

Hence our assumption is true.
We can try by assuming they are not moving together and solve the equations to get an absurd result.

If we assume they are moving separately, friction attains maximum value.
Again making free-body diagram

$$a_A = \frac{30-25}{5} = 1\ m/s^2$$

$$a_B = \frac{25}{10} = 2.5\ m/s^2$$



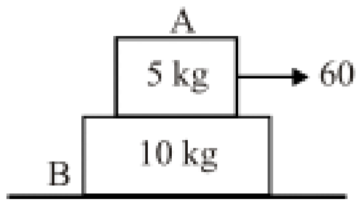
If we look at block A from reference from attached to block B we get

$$a_{A/B} = a_A - a_B = 1.5\ m/s^2$$

that is block A will be moving towards left and friction on it also acting in the same direction. This is not possible as friction must oppose relative motion.

Illustration :

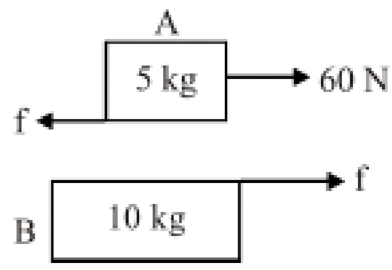
Friction coefficient between the block is 0.5 and ground is smooth. If force of 60 N is applied on the upper block as shown in figure find acceleration of the blocks.



$$f_{max} = 0.5 \times 5\ g = 25\ N$$

Let's assume they have same acceleration. The value of acceleration is $60/(10+5) = 4\ m/s^2$.

Making free-body diagram



Writing Newton's second law,

$$60 - f = 5 \times 4 \qquad \text{(for A)}$$

$$f = 10 \times 4 = 40\ N \qquad \text{(for B)}$$

Solving we get, $a = 4\ m/s^2$ and $f = 40\ N$ while limiting value of friction is 25 N, hence our assumption is wrong.

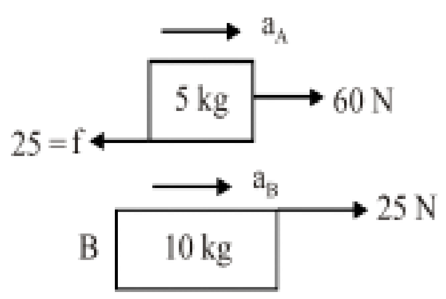
Let's say they are not moving together. Then the free-body diagram

Writing Newton's second law,

$$60 - 25 = 5\ a_A \qquad (f_K = 25)$$

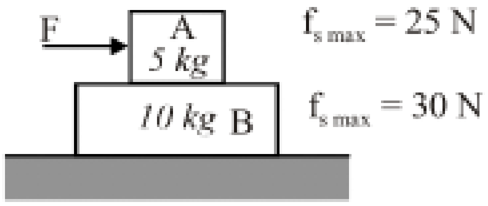
$$25 = 10\ a_B$$

$$a_A = 7\ m/s^2,\ a_B = 2.5\ m/s^2$$



If we look at block A from frame attached to block B we will find it is moving toward right and friction is acting towards left. This is satisfying friction's tendency to oppose relative motion.

Illustration :



Friction coefficient between block is 0.5 and between ground and 10 kg block is 0.2. Find acceleration of blocks if force $F = 40$ N is applied on 5 kg block as shown in figure

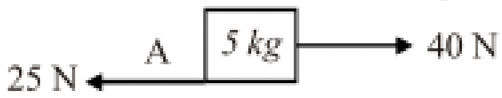
Sol. *First we find the values of limiting friction at all contact surfaces ($f_{s,max}$)*

f_{max} *between blocks* $f_1 = 0.5\times5\times10 = 25\ N$

f_{max} *between 10 kg block & ground,* $f_2 = 0.2\times15\times10 = 30\ N$

The only driving force that the block B can experience is the one applied by the lower surface of block A on block B.

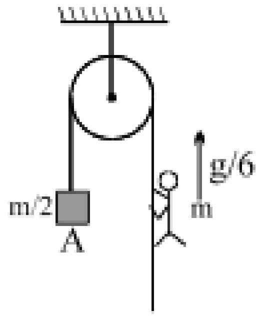
The maximum value of this force is 25N. This is lower than the minimum force required to move with respect to ground $f_{required} = 0.2 \times 15 \times 10 = 30\ N$ *Hence only the block A will move*



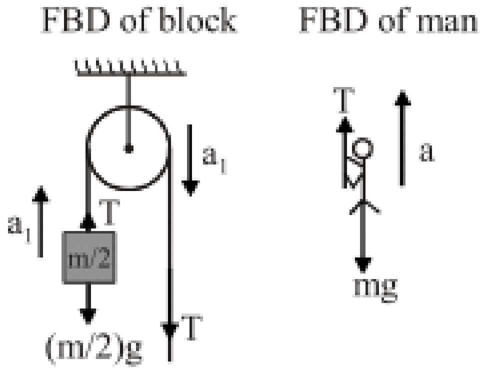
$$a_A = \frac{40-25}{5};\ a_A = 3\ m/s^2,\ a_B = 0\ m/s^2$$

Solved Example

Q.1 Block A of mass m/2 is connected to one end of light rope which passes over a pulley as shown in the Fig. Man of mass m climbs the other end of rope with a relative acceleration of g/6 with respect to rope. Find acceleration of block A and tension in the rope.



Sol.



a and a_1 are w.r.t ground.

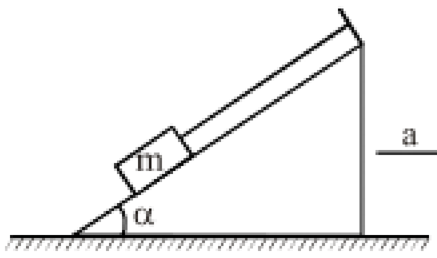
$a + a_1 = g/6$

$$T - \frac{m}{2}g = \frac{m}{2}a_1$$

$T - mg = ma$ on solving above equations we get $T = \frac{13}{18}mg$ and $a_1 = \frac{4}{9}g$

Paragraph for question nos. 2 to 4

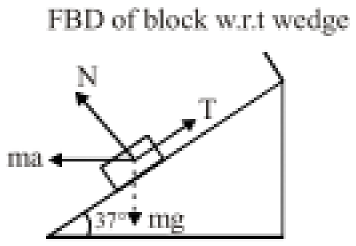
A body of mass m = 1.8 kg is placed on an inclined plane, the angle of inclination is $\alpha = 37°$, and is attached to the top end of the slope with a thread which is parallel to the slop. Then the slope is moved with a horizontal acceleration of a. Friction is negligible.



Q.2 Find the acceleration if the body pushes the slope with a force of $\frac{3}{4}$mg ?

Sol. $N = mg\cos 37° - ma\sin 37° = \frac{3}{4}\ mg$

$a = \frac{5}{6}\ m/s^2$



Q.3 Find the tension in thread ?:

Sol. $T = mg\sin 37° + ma\cos 37°$

T = 12 N

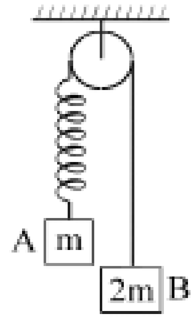
Q.4 At what acceleration will the body lose contact with plane?

Sol. $N = mg \cos 37° - ma \sin 37°$ (for lose contact $N = 0$)

$mg \cos 37° = ma \sin 37°$

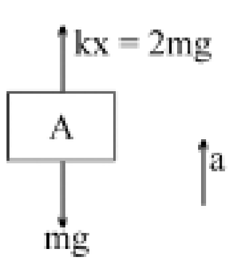
$$a = \frac{40}{3} \text{ m/s}^2$$

Q.5 In the figure, block 'A' of mass 'm' is attached to one end of a light spring and the other end of the spring is connected to another block 'B' of mass 2m through a light string. 'A' is held and B is at rest in equilibrium. Now A is released. The acceleration of A just after that instant is 'a'. The same thing is repeated for 'B'. In that case the acceleration of 'B' is 'b', then value of a/b is..........?



Sol. When A is held, B will be at rest in equilibrium, if $kx = T = 2mg$

Now, just after A is released

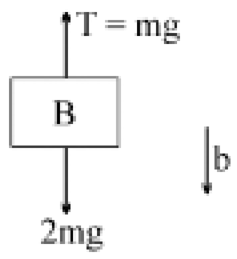


$$a = \frac{2mg - mg}{m} = g$$

When B held and A in equilibrium,

$T = kx = mg$

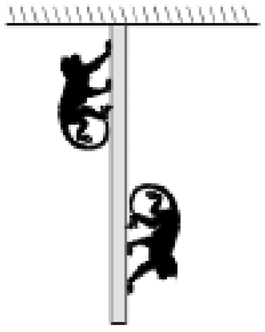
After B is released,



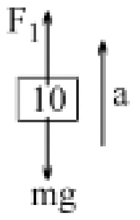
$$b = \frac{2mg - mg}{2m} = g/2$$

$a/b =$ **2**

Q.6 Two monkeys of masses 10 and 8 kg are moving along a vertical rope, the former climbing up with an acceleration of 2m/s^2 while the latter coming down with a uniform velocity of 2m/s. Find the tension in the rope at the fixed support.

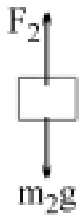


Sol. FBD of monkey (1),



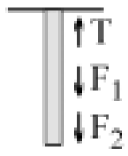
$F_1 - m_1 g = m_1 a$ $\Rightarrow$ $F_1 = m_1(g + a) = 10(10 + 2) = 120$ N

FBD of monkey (2),



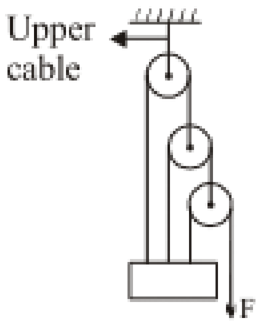
$F_2 = m_2 g = 10 \times 8 = 80$ N

FBD of rope

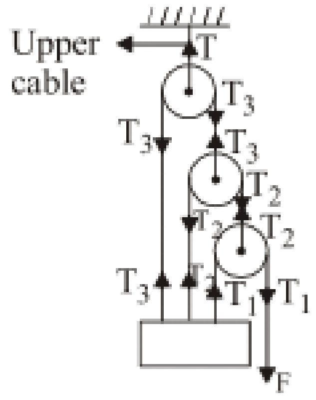


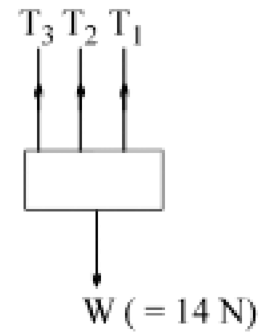
Tension $= F_1 + F_2 = 120 + 80 = 200$ N

Q.7 The pull F is just sufficient to keep the 14 N block in equilibrium as shown. Pulleys are ideal. Find the tension (in N) in the upper cable.



Sol. $T_1 = F$





$T_2 = 2T_1 = 2F$

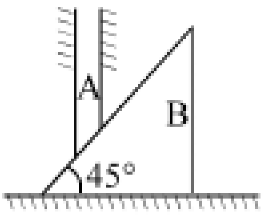
$T_3 = 2T_2 = 4F$

$T = 2T_3 = 8F$

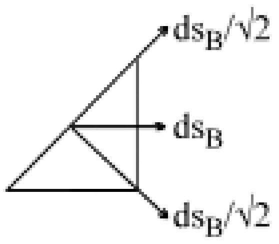
For equilibrium of block $T_1 + T_2 + T_3 = 14 \quad \Rightarrow \quad 7F = 14 \quad \Rightarrow \quad F = 2$ N

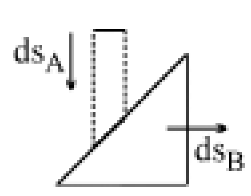
$\therefore$ T = 8F = 16 N

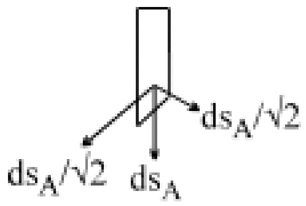
Q.9 Find the acceleration of rod A and wedge B in the arrangement shown in fig. if the mass of rod equal that of the wedge and the friction between all contact surfaces is negligible. Take angle of wedge as 45°.



Sol.





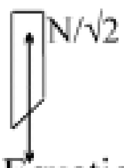


Perpendicular to the plane of contact displacement must be same.

$$\frac{ds_B}{\sqrt{2}} = \frac{ds_A}{\sqrt{2}}$$

$$ds_B = ds_A$$

Differentiating, $a_B = a_A = a$ (Let)



$mg - N/\sqrt{2} = ma$ ...(1)

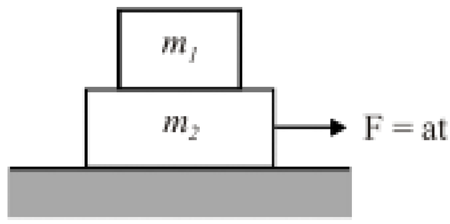
$N/\sqrt{2} = ma$ ...(2)

Equation (1) + (2)

$\Rightarrow \quad mg = 2ma$

$\Rightarrow \quad a = g/2$

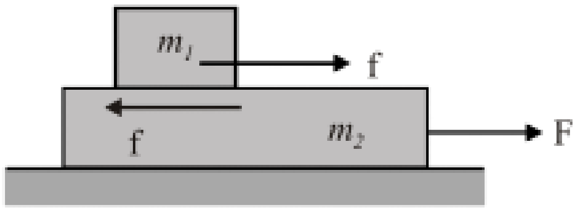
Q.10 A bar of mass m_1 is placed on a plank of mass m_2. which rests on a smooth horizontal plane. The coefficient of friction between the surfaces of the bar and the plank is equal to μ. The plank is subjected to the horizontal force F depending on time t as F = at (a is a constant).



Find (a) the moment of time t_0 at which the plank starts sliding from under the bar and (b) the acceleration of the bar a_1 and that of plank a_2 during motion.

Sol. As the force F grows, so does the static friction force f_s. However, the friction force f has the limiting value $f_{limit} = \mu m_1 g$. Unless this value is reached, both bodies move together with equal accelerations. But as soon as the force f reaches this limit, mass m_2 starts sliding under mass m_1.

Let us make free-body diagram



Writing Newton's second law for the plank and the bar, having taken the positive direction of the x axis

$f = m_1 a_1$, $F - f = m_2 a_2$

Acceleration of m_2 must be always greater than or equal to the acceleration of m_1 that is $a_2 \geq a_1$ where the equality corresponds to the moment $t = t_0$. Hence, when $f = \mu m_1 g$, then sliding begins. Putting $f = \mu m_1 g$.

$$t_0 = (m_1 + m_2)\frac{\mu g}{a}$$

When $t \leq t_0$, then

$$a_1 = a_2 = \frac{at}{(m_1 + m_2)}$$

and when $t > t_0$ then they seperate. Only force acting on m_1 is friction, whose value is constant.

Thus $\quad a_1 = \mu g = \text{constant}$

Now m_2 is experiencing force F and constant friction

thus, $\quad at - \mu m_1 g = m_2 a_2$

Solving we get

$$a_2 = \frac{(at - \mu m_1 g)}{m_2}$$

You may have been tempted to think that when external force F (=at) is equal to $\mu m_1 g$, slipping will being. You can check that at this instant t_2 they are moving with same acceleration.

$$at_2 = \mu m_1 g$$

$$t_2 = \frac{\mu m_1 g}{a}$$

Let's find force of friction between the blocks at this instant.

$$\mu m_1 g - f = m_2 a$$

$$f = m_1 a$$

Solving for friction

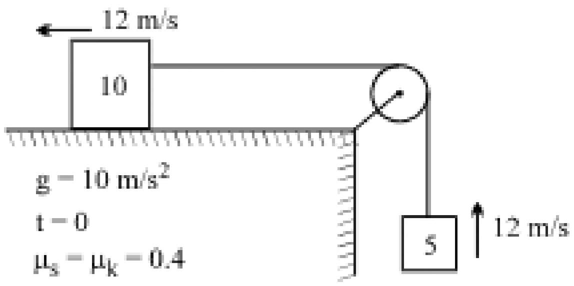
$$f = \frac{\mu m_1 g}{\left(1 + \frac{m_2}{m_1}\right)}$$

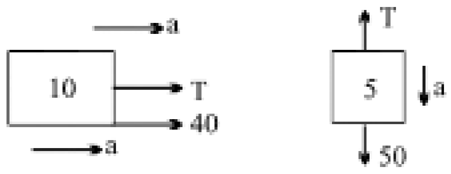
Since friction is less than the limiting value, slipping has not yet begun.

Q.11 Blocks are given velocities as shown at $t = 0$, find velocity and position of 10 kg block.

(A) at $t = 1$ sec

(B) at $t = 4$ sec.



Sol. making F.B.D. 

]

(A) $40 + T = 10a$; $50 - T = 5a$; $a = 6\ m/s^2$

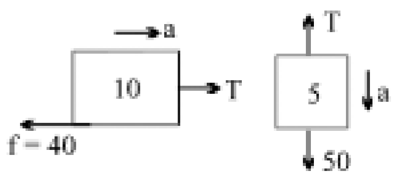
$u = 12$; $a = -6$

$v = 12 - 6 \times 1 = 6\ m/s$; $s = 12 \times 1 - 3 \times 1 = 9\ m$

(B) Let them solve it wrong. Then explain that since velocity has changed the direction during motion friction would also have changed thus direction and acceleration will change.

$u = 12$; $a = -6$ (till velocity becomes zero)

$v = 0 \Rightarrow v = 2$ sec ; $s = 2 \times 2 - 3 \times 4 = 12\ m$

now FBD 

$50 - T = 5a$

$T - 40 = 10a$

$a = 2/3\ m/s^2$

$u = 0,\ a = 2/3,\ t = 2,\ v = 4/3$

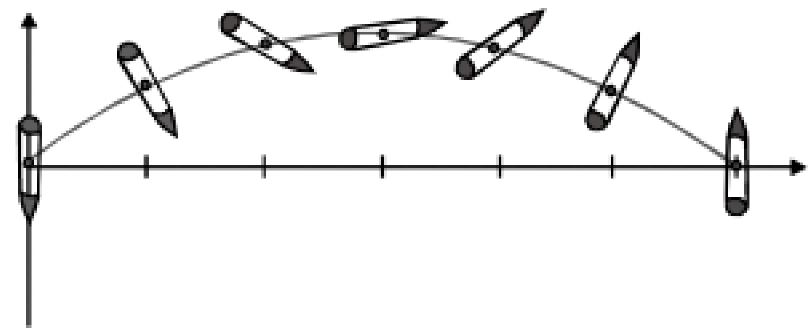
$s = \frac{1}{2} \times \frac{2}{3} \times 4 = \frac{4}{3}$; total displacement $12 - \frac{4}{3} = \frac{32}{3}$ m

Centre of Mass, Momentum & Collision

Introduction

Until now we have been mainly concerned with the motion of single particles. When we have dealt with an extended body (that is a body that has size), we assumed that it can be approximated to be a point particle or that it underwent only translational motion. Real "extended" bodies, however, can undergo rotational and other types of motion as well. For example, if you flip a pen in air, you will find that its motion is indeed very complex as every part of the pen moves in a different way. Therefore, a pen can not be represented as a particle, but as a system of particles. However, if you closely look, you will find that one of the special points of the pen moves in a simple parabolic path, as if pen's entire mass is concentrated there. That point is called the 'center of mass' of the pen. Thus, precisely speaking centre of mass is the location where the entire mass of system of particles is assumed to be concentrated.

centre of mass is an imaginary point, which *may or may not* be located on the system.

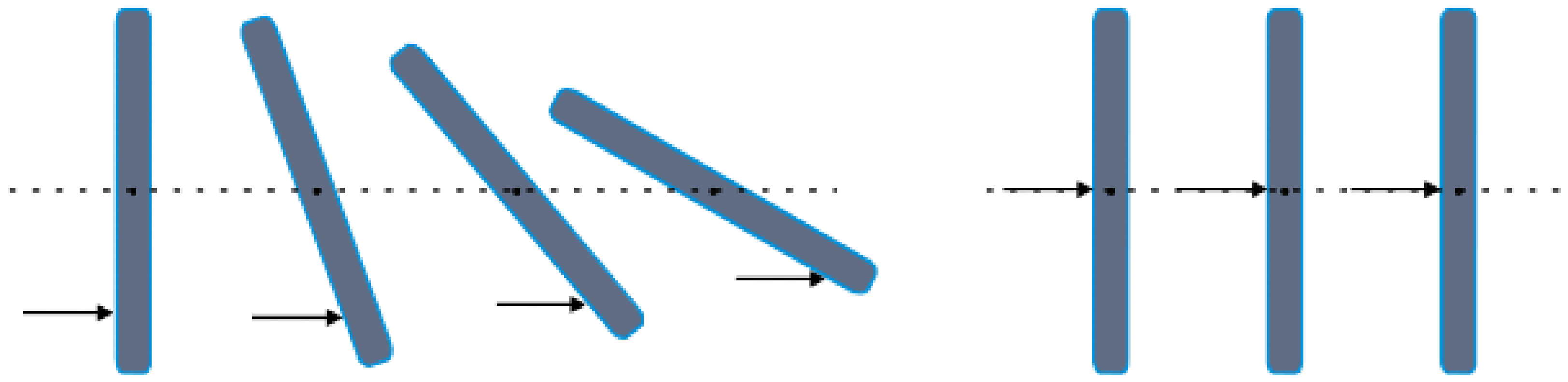


To locate, centre of mass, of a body, balance the body (let us say pen) on outstretched finger. The point on the axis, above your finger is centre of mass of the pen.

When a force is applied on a body apart from the magnitude & direction of $\vec{F}$ the motion of the body also depends upon the point of application.

Thus, $\vec{F} = m\vec{a}$ is not valid for all particles but for a special point i.e. centre of mass, $\vec{a}_{cm} = \frac{\vec{F}}{m}$.

Also, if a force is applied along a line passing through the centre of mass of the body, all the particles of the body move with same linear velocity and acceleration.



Centre of mass is useful concept and it can reduce the effort to solve many difficult problems. Let's see an example to understand this point.

Illustration:

Find the potential energy of a uniform rod of mass 'm' & length l kept vertically standing on the ground. Take potential energy at ground level to be zero.

Sol. *We can not write the potential energy of the rod directly. All the points of the rod are situated at different height from the ground. Therefore we divide it into many point masses with potential energy of a small mass being dU,*

$$dU = (dm)\, gy$$

Total P.E. would be the summation of P.E. of all the elements from y = 0 to y = l.

$$\Rightarrow \quad U = \int_0^l dm\ g\ y$$

Mass of small element, of length dy is dm :

m is mass of length l,

$$\therefore \quad \text{mass of unit length} = \frac{m}{l}$$

$$\text{mass of length } dy = \frac{m}{l}\, dy$$

$$dm = \frac{m}{l}\ dy$$

$$U = g\int_0^l \frac{m}{l}\ dy \cdot y$$

$$= \frac{mg}{l}\int_0^l y\,dy \quad = \frac{mg}{l} \times \left(\frac{l^2}{2}\right)$$

$$= mg\left(\frac{l}{2}\right)$$

In this case, rather than solving this problem, we could have said that the mass of the rod is concentrated at some height Y from the ground. And we shall replace the rod with a point mass at that height.

To determine that height y, compare U with Mg Y

$$U = Mg\ Y = mg\ \frac{l}{2}$$

$$\text{we get } Y = \frac{l}{2}$$

If we had known this position earlier, we could have solved the problem in no time without integration.

This position Y is the center of mass of the rod. Such problem would have been really cumbersome for more complicated bodies like ring, disc, sphere, cone etc. but if their centre of mass is known, we could have solved them easily without calculations.

Potential energy was an example, there are many more physical quantities of a system of particles which can be calculated by this concept of centre of mass. Now we shall explore the concept of centre of mass in detail calculation of its position and application on physical quantities.

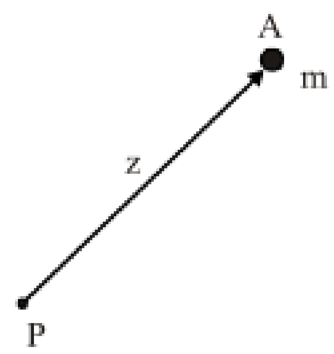
Position of centre of mass

First of all we find the position of Centre of mass of a system of particles. Just to make the subject easy we classify a system of particles in three groups :
1. System of two particles
2. System of a large number of particles and
3. Continuous bodies.

Now, lest us take them separately.

Mass Moment

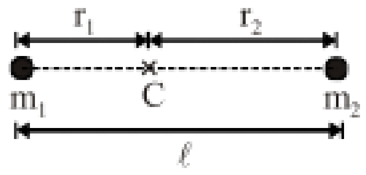
It is defined as the product of mass of the particle and distance of the particle from the point about which mass moment is taken. It is a vector quantity and its direction is directed from the point about which it is taken to the particle, as shown in figure, the mass moment of particle A (mass = m) about the point P is given by z.



There is a important property of centre of mass associated with the mass moments of the components of the system which forms the basis of analytical determination of centre of mass of a system. The property is *"The summation of mass moments of all the components of a system about its centre of mass is always equal to zero"*. This statement is an experimentally verified property which does not require any analytical proof. It can be used as a universal property in all type of system.

1. Position of Centre of mass of two particles

Consider the situation shown in figure. Two masses m_1 to m_2 are separated by a distance l, let C be the centre of mass of the system at a distance r_1 from m_1 and $(l-r_1)$ from m_2. According to the property of mass moments about centre of mass of system of two particles *The summation of mass moments of all the components of a system about its centre of mass is always equal to zero* we have



$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

in scalar form, $-m_1r_1 + m_2 r_2 = 0$ (as r_1 is towards left we consider it –ve)

$$m_1r_1 = m_2 r_2 \qquad ....(i)$$

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$$\frac{r_1}{r_1+r_2} = \frac{m_2}{m_1+m_2} \Rightarrow \frac{r_1}{\ell} = \frac{m_2}{m_1+m_2}$$

$$r_1 = \frac{m_2\ell}{m_1+m_2}$$

From equation (i). The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m).

$$r \propto \frac{1}{m}$$

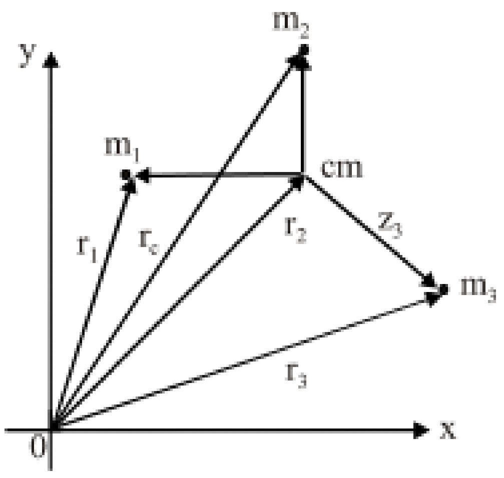
$r_1 = r_2 = \frac{d}{2}$ if $m_1 = m_2$, i.e. Centre of mass lies midway between the two particle of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_1 > m_2$ i.e. Centre of mass is nearer to the particle having larger mass.

2. Definition of Centre of mass for point particles :

Consider the situation shown in figure. There are three masses in a coordinate system with respective coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). The position vectors of these masses with respect of origin can be given as

$$r_1 = x_1 i + y_1 j + z_1 k \qquad r_2 = x_2 i + y_2 j + z_2 k \qquad r_3 = x_3 i + y_3 j + z_3 k$$



In this system, we will now locate the position of centre of mass. Let the coordinates of centre of mass be (x_c, y_c, z_c) and so the position vector will be

$$r_c = x_c i + y_c j + z_c k$$

The mass moments of the masses m_1, m_2 and m_3 about the centre of mass can be given as

$$\vec{z}_1 = m_1 \vec{r}_{1/c} = m_1 . (\vec{r}_1 - \vec{r}_c) \qquad \vec{z}_2 = m_2 \vec{r}_{2/c} = m_2 . (\vec{r}_2 - \vec{r}_c) \qquad \vec{z}_3 = m_3 \vec{r}_{3/c} = m_3 . (\vec{r}_3 - \vec{r}_c)$$

According to the property of mass moments The summation of mass moments of all the components of a system about its centre of mass is always equal to zero we have

$$\vec{z}_1 + \vec{z}_2 + \vec{z}_3 = 0$$

$$m_1 . (\vec{r}_1 - \vec{r}_c) + m_2 . (\vec{r}_2 - \vec{r}_c) + m_3 . (\vec{r}_3 - \vec{r}_c) = 0$$

On solving we get

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \qquad \text{....(i)}$$

This relation can also be generalized for n mass system. Now by substituting the vector in terms of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ and comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ we get

$$x_c\hat{i} + y_c\hat{j} + z_c\hat{k} = \frac{m_1(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + m_2(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) + m_3(x_3\hat{i} + y_3\hat{j} + z_3\hat{k})}{m_1 + m_2 + m_3}$$

$$x_c = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \qquad \text{....(ii)}$$

$$y_c = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \quad ....(iii)$$

$$z_c = \frac{m_1z_1 + m_2z_2 + m_3z_3}{m_1 + m_2 + m_3} \quad ....(iv)$$

Equation (ii), (iii), (iv) can also be extended to n - objects system.

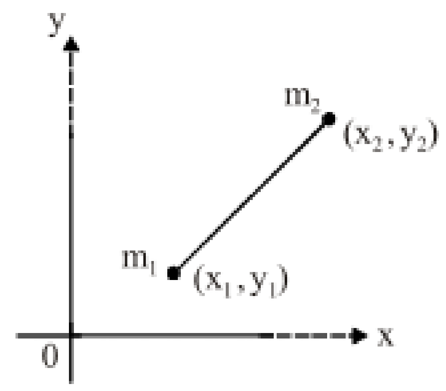
$$\vec{r}_{C.M.} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...m_n\vec{r}_n}{m_1 + m_2 + m_3 + ...m_n} = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i}$$

Thus $x_{cm} = \dfrac{m_1x_1 + m_2x_2 + m_3x_3 + ...}{m_1 + m_2 + m_3 + ...}$

for the body system this equation reduces to

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$



Note: *centre of mass divides two point masses in inverse ratio of their masses*

Illustration :

Two particles of mass 1 kg and 2 kg are located at x = 0 and x=3 m. Find the position of their centre of mass.

Sol. *Since both the particles lie on x-axis, the COM will also lie on x-axis. Let the COM is located at x = x, then*

r_1 = *distance of COM from the particle of mass 1 kg = x*

and r_2 = *distance of COM from the particle of mass 2 kg*

$= (3 - x)$

we know

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Given ; $x_1 = 0$; $x_2 = 3$

$m_1 = 1$ kg ; $m_2 = 2$kg

$$x_c = \frac{1 \times 0 + 2 \times 3}{1 + 2}$$

$= 2$ *cm*

As expected, the centre of mass is nearer to the heavier mass.

using $m_1r_{1c} + m_2r_{2c} = 0$

$m_1(0 - x) + m(3 - x) = 0$

$-x + 6 - 2x = 0$

or $x = 2$ *m*

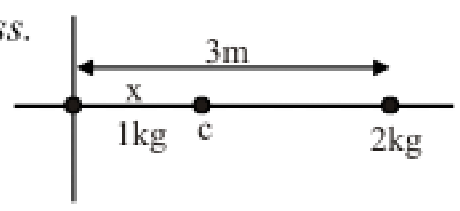
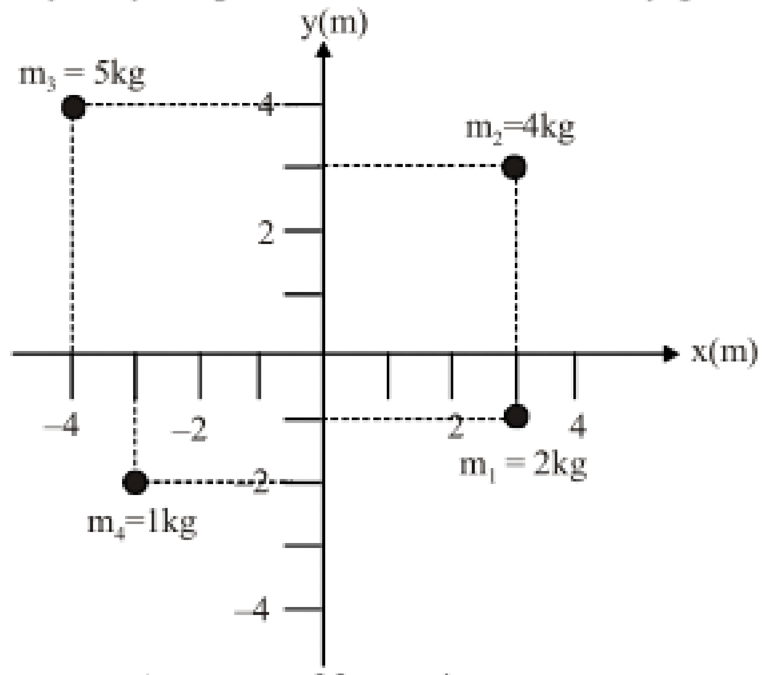


Illustration:

Find the center of mass of the four point masses as shown in figure.



A system of four point masses

Sol. *The total mass M = 12 kg,*

From equation we have, $x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i}$, $y_{cm}\ \frac{\Sigma m_i y_i}{\Sigma m_i}$

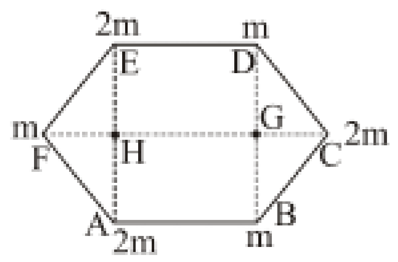
$$x_{cm} = \frac{(2kg)(3m) + (4kg)(3m) + (5kg)(-4m) + (1kg)(-3m)}{12kg} = \frac{-5}{12}m$$

$$y_{cm} = \frac{(2kg)(-1m) + (4kg)(3m) + (5kg)(4m) + (1kg)(-2m)}{12kg} = \frac{28}{12}m$$

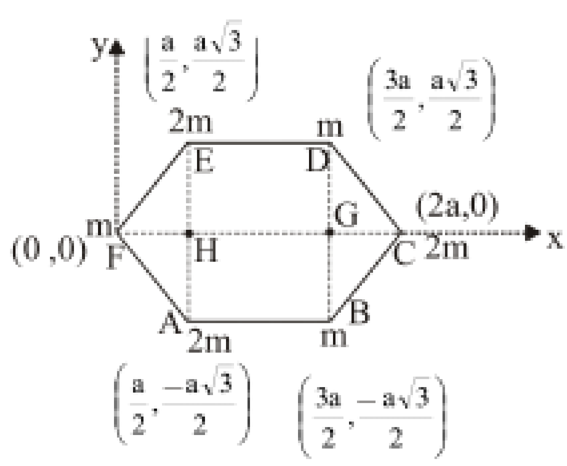
The position of the cm is $\vec{r}_{cm} = -0.42\hat{i} + 2.3\hat{j}m$

Illustration :

Find the position of centre of mass for a system of particles places at the vertices of a regular hexagon as shown in figure



Sol.



$$x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

$$= \frac{m\times 0 + 2m\times\frac{a}{2} + m\times\frac{3a}{2} + 2m\times 2a + m\times\frac{3a}{2} + 2m\times\frac{a}{2}}{m+2m+m+2m+m+2m}$$

$$x_{cm} = a$$

$$y_{cm} \frac{\Sigma m_i y_i}{\Sigma m_i}$$

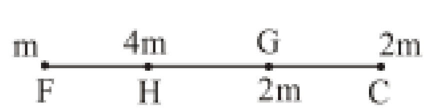
$$= \frac{m\times 0 + 2m\times\frac{a\sqrt{3}}{2} + m\times\frac{a\sqrt{3}}{2} + 2m\times 0 + m\times\left(\frac{-a\sqrt{3}}{2}\right) + 2m\times\left(\frac{-a\sqrt{3}}{2}\right)}{9m}$$

$$y_{cm} = 0$$

c.m. : (a ,0)

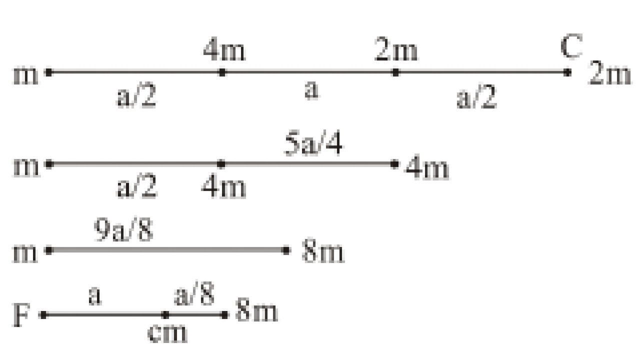
ALITER : *Masses at A & E can be placed at centre of AE, similarly masses at B & D can be placed at centre of BD.*
Objective *: While calculating centre of mass we can replace bodies by their centre of mass.*

Thus our problem reduces to :



Considering origin at F.

$y_{CM} = 0$ *(Because, centre of mass. of individual system lie on the x-axis as seen in the figure above.)*



$\therefore$ $x_{CM} = a$

$\therefore$ *centre of mass. : (a, 0)*

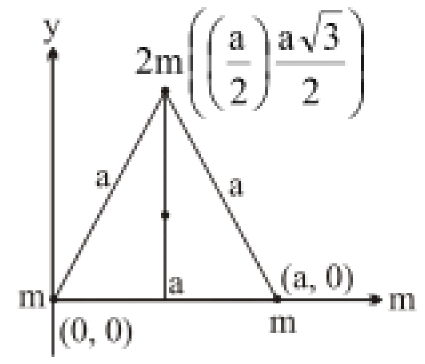
i.e. at the center of the hexagon.

Practice Exercise

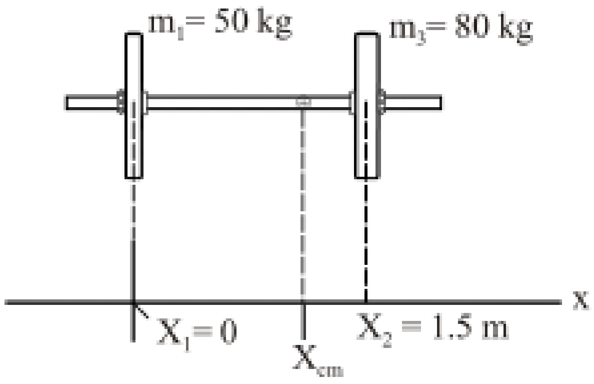
Q.1 If all the particles of a system lie in X - Y plane, is it necessary that the centre of mass be in X-Y plane ?

Q.2 If all the particle of a system lie in a cube, is it necessary that the centre of mass be in the cube ?

Q.3 Find centre of mass of a system of three particle kept at the corner equilateral triangle as shown in figure



Q.4 Find the center of mass of a barbell consisting of 50 kg and 80 kg weights at the opposite ends of a 1.5- -long bar of negligible mass.



Our sketch of the barbell

[Hint] Choosing the origin at one the masses here conveniently makes one of the terms in the sum $\Sigma m_i x_i$ zero. But, as always, the choice of origin is purely for convenience and doesn't influence the actual physical location of the center of mass.

Q.5 Consider the previous problem, at what point must the rod be picked over a knife edge, so that the barbell remains horizontal.

Answers

Q.1 Yes Q.2 Yes Q.3 Coordinates of centre of mass. : $\left(\frac{a}{2}, \frac{a\sqrt{3}}{4}\right)$ Q.4 0.92 m

Q.5 0.92 m

Centre of mass of continuous bodies

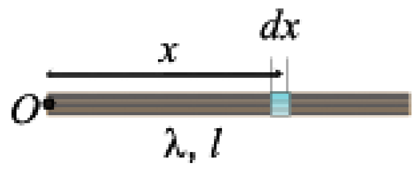
Mass Distribution in rigid bodies

Mass distribution in rigid bodies is often termed as density.

We have often heard about density as being mass per unit volume. But there are other densities as well.
Linear mass density (λ) mass/length

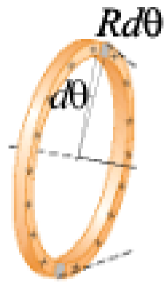
Superficial mass density (σ) mass/area
Volume mass density (r) mass/volume
Examples of linear mass density :
(a) Let λ be the linear mass density of a uniform rod of mass m and length l.

Then by definition $\lambda = \frac{m}{l}$



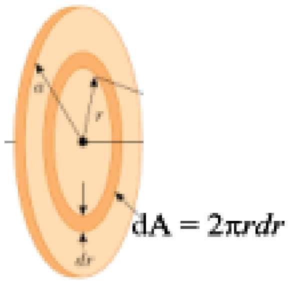
Thus, the mass of element chosen = λ dx
(b) Let λ be the linear mass density of a uniform ring of mass m & radius R

Then by definition $\lambda = \frac{m}{2\pi R}$



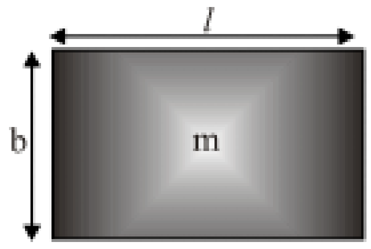
Thus, the mass of chosen arc = $\lambda R d\theta$
Examples of areal mass density :
(a) Let σ be the mass per unit area of the disc of mass m and radius R

Then by definition $\sigma = \frac{m}{\pi R^2}$



Area of ring chosen = $\pi (r + dr)^2 - \pi r^2 \approx 2\pi r\, dr$
Thus, mass of the chosen ring is = $\sigma\, 2\pi r\, dr$

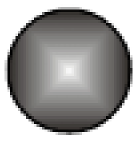
(b) Let σ be the superficial mass density of rectangular plate



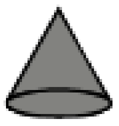
by definition , $\sigma = \frac{m}{lb}$

Examples of volume mass density (ρ)

(a) Sphere of mass m radius R



$\rho = \frac{m}{\frac{4}{3}\pi R^3}$

(b) Cone of mass m, radius R, height H,  $\rho = \dfrac{m}{\dfrac{\pi R^2 H}{3}}$

3 Centre of mass of rigid bodies $\vec{r}_{COM} = \dfrac{1}{m}\int \vec{r}\, dm$

While calculating the COM of rigid bodies, we consider small elements in the body and integrate, by replacing the element with equal mass placed at its centre of mass.

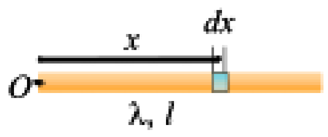
To summarize, meaning of each term

$$X_{COM} = \frac{1}{m}\int x\, dm$$

$m \rightarrow$ system's mass, $x \rightarrow$ position (co-ordinate) of COM of the element chosen

$dm \rightarrow$ mass of the chosen element

(a) Centre of Mass of Uniform Straight Rod



Let M and L be the mass and length of the rod respectively. Take the left end of the rod as the origin and the X-axis along the rod. Consider an element of the rod between the positions x and x + dx. If x = 0, the element is at its right end. x varies from 0 through L, the element covers the entire rod. As the rod is uniform, the mass per unit length is M/L. The coordinates of the element are (x, 0, 0). (The coordinates of different points of the element differ, but the difference is less than dx and that much is harmless as integration will automatically correct it.)

The x-coordinate of the centre of mass of the rod is

$$X = \frac{1}{M}\int x dm = \frac{1}{M}\int_0^L x\left(\frac{M}{L}dx\right)$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2}$$

The y-coordinate is

$$Y = \frac{1}{M}\int y dm = 0$$

and similarly Z = 0.

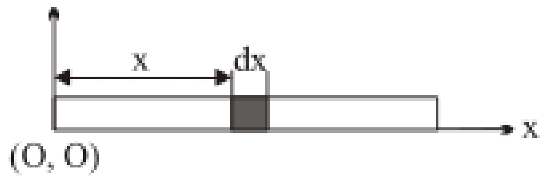
The centre of mass is at $\left(\dfrac{L}{2}, 0, 0\right)$, i.e., at the middle point of the rod.

Illustration :

Calculate Centre of mass of a non-uniform rod with linear mass density λ

$\lambda(\text{mass / length}) = kx^2$

Sol. *For a small element dx, the linear mass density can be considered constant :*



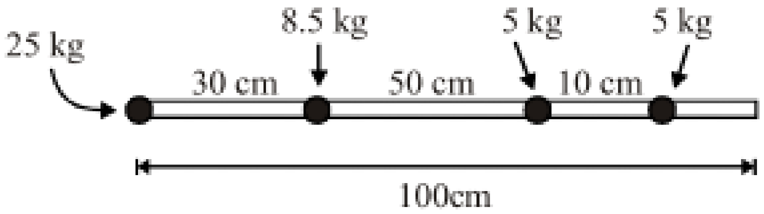
$\frac{\text{Mass}}{\text{Length}} = \lambda = kx^2$ $\quad\quad$ $\therefore$ *mass of this small element* $dm = \lambda\, dx = kx^2\, dx$

$$M = k\int_0^L x^2\, dx = \frac{KL^3}{3}$$

$$x_{cm} = \frac{\int x\, dm}{\int dm} = \frac{K\int_0^L x^3\, dx}{K\frac{L^3}{3}} = \frac{3L}{4}$$

Illustration :

Figure shows a rod of mass 10 kg of length 100cm with some masses tied to it at different positions. Find the point on the rod at which if the rod is picked over a knife edge, it will be in equilibrium about that knife edge.



Sol. *Centre of mass of the system shown in figure, will be the point, at which if we place a knife edge, system will remain in equilibrium.*

To locate the centre of mass of the system, we consider origin at the left end of the rod. With respect to this origin the position of centre of mass of the system is

$$x_C = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_{rod}x_{rod}}{m_1 + m_2 + m_3 + m_4 + m_{rod}}$$

$$x_C = \frac{25\times 0 + 8.5\times 30 + 10\times 50 + 5\times 80 + 5\times 90}{53.5} = 30cm$$

(b) Center of Mass of a Uniform Semicircular Wire

Let M be the mass and R the radius of a uniform semicircular wire. Take its centre as the origin, the line joining the ends as the X-axis, and the Y-axis in the plane of the wire. The centre of mass must be in the plane of the wire i.e. in the X-Y plane.

How do we choose a small element of the wire ?

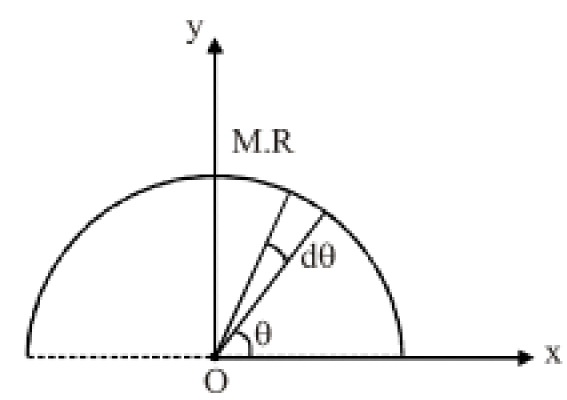
First, the element should be so defined that we can vary the element to cover the whole wire. Secondly, if we are interested in $\int x\, dm$, the x-coordinates of different parts of the element should only infinitesimally differ in range. We select the element as follows. Take a radius making an angle θ with the X-axis and

rotate it further by an angle $d\theta$. Note the points of intersection of the radius with the wire during this rotation. This gives an element of length $R\,d\theta$. When we take $\theta = 0$, the element is situated near the right edge of the wire. As θ is gradually increased to π, the element takes all positions on the wire i.e., the whole wire is covered. The coordinates of the element are $(R\cos\theta, R\sin\theta)$. Note that the coordinates of different parts of the element differ only by an infinitesimal amount.

As the wire is uniform, the mass per unit length of the wire $\lambda = \frac{M}{\pi R}$.

The mass of the element is, therefore,

$$dm = \left(\frac{M}{\pi R}\right)(R\,d\theta) = \frac{M}{\pi}\,d\theta$$



The coordinates of the centre of mass are

$$X = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^{\pi}(R\cos\theta)\left(\frac{M}{\pi}\right)d\theta = 0$$

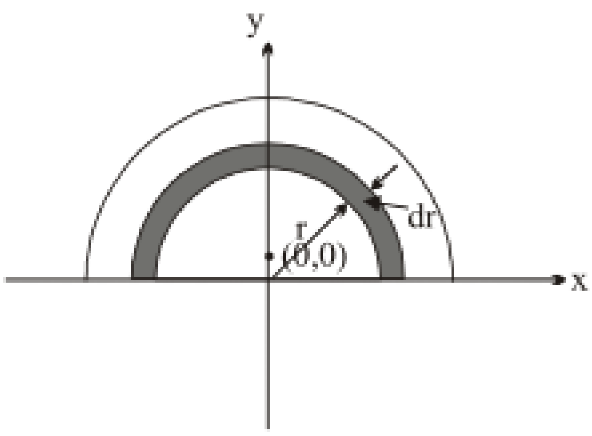
and

$$Y = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\pi}(R\sin\theta)\left(\frac{M}{\pi}\right)d\theta = \frac{2R}{\pi}$$

The centre of mass is at $\left(0, \frac{2R}{\pi}\right)$.

(c) Centre of Mass of a Uniform Semicircular Plate

This problem can be worked out using the result obtained for the semicircular wire and that any part of the system (semicircular plate) may be replaced by a point particle of the same mass placed at the centre of mass of that part.



We take the origin at the centre of the semicircular plate, the X-axis along the straight edge and the Y-axis in the plane of the plate. Let M be the mass and R be its radius. Let us draw a semicircle of radius r on the plate with the centre at the origin. We increase radius to $r + dr$ and draw another semicircle with the same centre. Consider the part of the plate between the two semicircles of figure it may be considered as a semicircular wire.

If we take $r = 0$, the part will be formed near the centre and if $r = R$, it will be formed near the edge of the plate. Thus if r is varied from '0' to R the elemental parts will cover the entire semicircular plate.

We can replace the semicircular shaded part by a point particle of the same mass at its centre of mass for the calculation of the centre of mass of the plate.

The area of he shaded part = π rdr. The area of the plate is $\pi R^2/2$. As the plate is uniform, the mass per unit area $\sigma = \dfrac{M}{\pi R^2/2}$. Hence the mass of the semicircular element

$$dm = \frac{M}{\pi R^2/2}(\pi r\, dr) = \frac{2M\, rdr}{R^2}$$

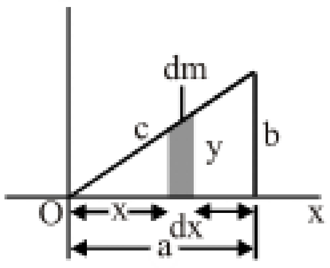
The y-coordinate of the centre of mass of this wire is $2r/\pi$. The y-coordinate of the centre of mass of the plate is, therefore,

$$Y = \frac{1}{M}\int_0^R \left(\frac{2r}{\pi}\right)\left(\frac{2Mr}{R^2}dr\right) = \frac{1}{M}\cdot\frac{4M}{\pi R^2}\frac{R^2}{3} = \frac{4R}{3\pi}$$

The x-coordinate of the centre of mass is zero by symmetry.

(d) Centre of mass of a right triangular sheet of

Dimensions are shown in figure. The object has a uniform mass per unit area.



The first step to find COM of a continuous body, is to determine the element.

We take small elements whose COM is known & then replace the element with equal mass placed at its COM, then we integrate to find COM of the body.

We divide the triangular lamina into narrow strips of width dx and height y as shown in the figure. The mass dm of each strip is

$$dm = \frac{\text{Total mass of the object}}{\text{Total area of the object}} \times \text{area of strip}$$

$$= \frac{M}{(1/2)ab}(y\,dx) = \frac{2M}{ab}y\,dx$$

Now x - coordinate of the centre of mass is

$$x_{CM} = \frac{1}{M}\int x\ dm = \frac{1}{M}\int_0^a x\left(\frac{2M}{ab}\right)y\,dx$$

$$= \frac{2}{ab}\int_0^a xy\ dx$$

To evaluate this integral we must express y in terms of x. From similar triangles in the figure we see that

$$\frac{y}{x} = \frac{b}{a} \qquad \text{or} \qquad y = \frac{b}{a}x$$

Hence,
$$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}\right)x\ dx$$

$$\frac{2}{a^2}\int_0^a x^2 dx$$

$$= \frac{2}{a^2}\left|\left(\frac{x^3}{3}\right)\right|_0^a = \frac{2}{3}a$$

On similar lines we can calculate the y-coordinate to be

$$y_{CM} = \frac{2}{3}b$$

(E) Centre of mass of a thin hemispherical shell of mass M and radius R.

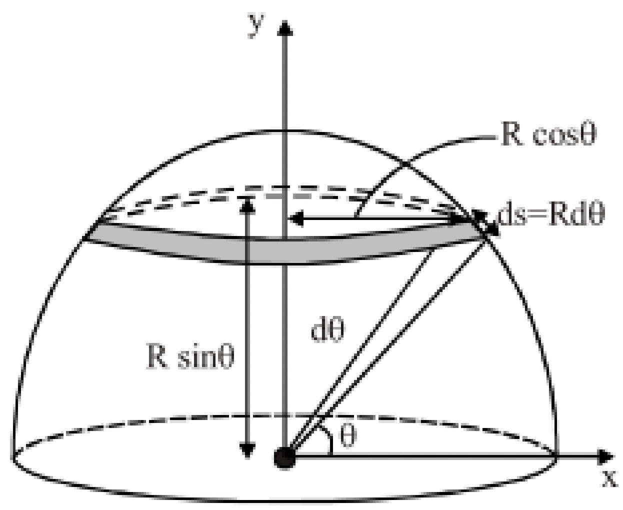
Assuming uniform mass distribution.

Sol. Here, the important part is determination of the element.

We can not take a ring element of thickness dy at a distance y from the center and integrate it from O to R.

Because the mass is distributed on the surface, by taking the above element, we do not cover the complete surface.

In this case element is a circular strip of thickness ds. The thickness of ring subtends angle $d\theta$ at the centre of the hemisphere as shown in figure. Radius of ring element is $R\cos\theta$. Mass of the element is



$$dm = \text{mass per unit area} \times \text{area of circular strip}$$

$$= \frac{M}{2\pi R^2} \times (2\pi R\cos\theta)R\ d\theta$$

Then $\quad x_{CM} = 0 \quad$ from symmetry

and

$$y_{CM} = \frac{1}{M}\int_0^{\pi/2} dm R\sin\theta$$

$$= \frac{1}{M}\int_0^{\pi/2} \frac{M}{2\pi R^2}(2\pi R\cos\theta\, R\, d\theta)R\sin\theta$$

$$= R\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \quad = \frac{R}{2}$$

(F) Center of mass of a hemispherical object of uniform density and radius R.

As shown in figure. We choose a coordinate system with the origin at the center of the flat face and we let the yz plane be the plane of the face. We now imagine slicing the hemisphere into discs parallel to the yz plane. A disk of thickness dx, located at a distance x from the plane face has a radius $\sqrt{R^2 - x^2}$. Therefore the mass of the disk $dm = \pi\rho(R^2 - x^2)dx$, where ρ is the mass density of the hemisphere. The center of mass of the hemisphere has an x coordinate x_{com} given by

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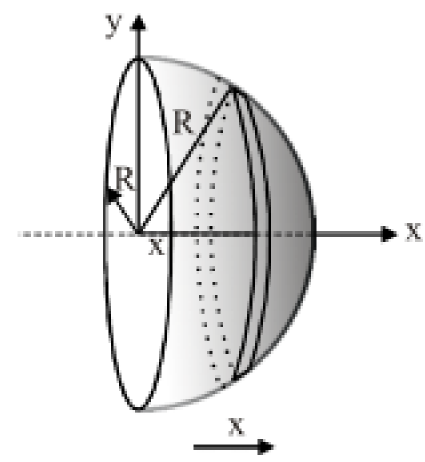
$$x_{com} = \frac{\int_0^R x\,dm}{\int_0^R dm}$$

$$= \frac{\int_0^R \pi\rho x(R^2 - x^2)dx}{\int_0^R \pi\rho(R^2 - x^2)dx}$$

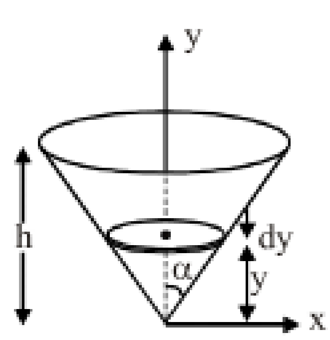


Hemispherical object of uniform density

$$x_{com} = \frac{\left(\frac{x^2R^2}{2} - \frac{x^4}{4}\right)\Big|_0^R}{\left(xR^2 - \frac{x^3}{3}\right)\Big|_0^R} = \frac{R^4/4}{2R^3/3} = \frac{3R}{8}$$

$$y_{com} = Z_{com} = 0$$

(G) Centre of mass of a uniform solid cone of height h and semi vertex angle α.



Sol. *We place the apex of the cone at the origin and axis of cone to be y axis. It is clear that the CM will lie along the y-axis. We divide the cone into disc of radius x and thickness dy. The volume of such a disc is $dV = \pi x^2\,dy = \pi\,(y \tan\alpha)^2\,dy$. The mass of the disc is $dm = \rho dV$. First we will determine the total mass of the cone.*

$$M = \int dm = \pi\rho \tan^2\alpha \int_0^h y^2 dy$$

$$= \pi\rho \tan^2\alpha\, \frac{h^3}{3} \qquad \text{......(i)}$$

The position of the CM is given by

$$y_{CM} = \frac{1}{M}\int y\,dm$$

$$= \frac{1}{M}\pi\rho\tan^2\alpha\int_0^h y^3 dy$$

$$= \frac{1}{M}\pi\rho\tan^2\alpha\,\frac{h^4}{4} \qquad \text{.....(ii)}$$

From equation (i) and(ii), we have

$$y_{CM} = \frac{3h}{4}$$

1. Centre of mass of some commonly used systems :

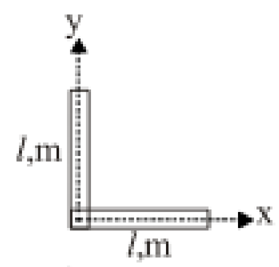
Body	Answer
a. Uniform rod of length L.	L/2
b. rod having linear mass density $\lambda = \alpha x$	2L/3
c. Quadrant of a uniform circular ring, radius R.	$2R/\pi$
d. Uniform semi circular ring of radius R.	$2R/\pi$
e. Uniform semi circular disc of radius R.	$4R/3\pi$
f. Uniform hemispherical shell of radius R.	R/2
g. Uniform solid hemisphere of radius R.	3R/8
h. Hollow cone of base radius R & height h.	h/3 from base of the cone.
i. Solid cone of base radius R and height h.	h/4 from base of the cone.

Practice Exercise

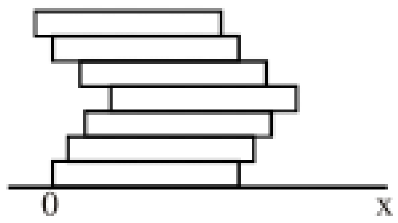
Q.1 A baseball bat of uniform density is cut at the location of its center of mass as shown in figure. The piece with the smaller mass is

(a) the piece on the right
(b) the piece on the left
(c) Both pieces have the same mass
(d) impossible to determine.

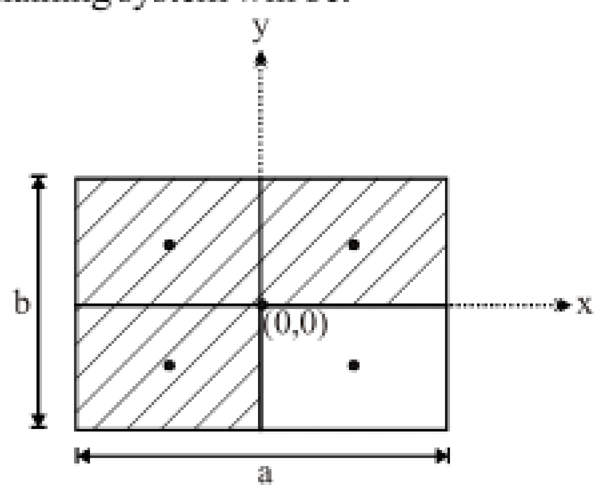
Q.2 *Calculate centre of mass of the system*



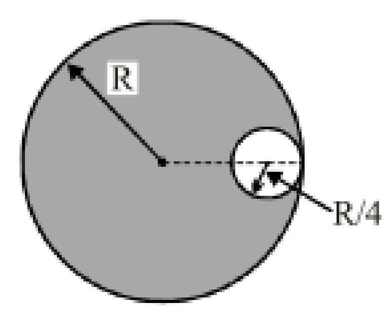
Q.3 Seven homogeneous bricks, each of length L, are arranged as shown in figure. Each brick is displaced with respect to the on in contact by L/10. Find the x-coordinate of the centre of mass relative ot the origin shown.



Q.4 Consider a rectangular plate of dimensions a ×b. If this plate is considered to be made up of four rectangles of dimensions $\frac{a}{2}\times\frac{b}{2}$ and we now remove one out of four rectangles. Find the position where the centre of mass of the remaining system will be.

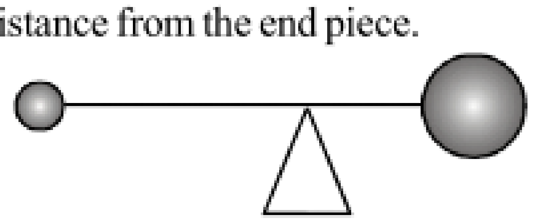


Q.5 Find the center of mass of the shaded portion of a disc



Answers

Q.1 (b) The piece with the handle will have less mass than the piece made up of the end of bat. To see why this is so, take the origin of coordinates as the center of mass before the bat was cut. Replace each cut piece by a small sphere located at the center of mass for each piece. The sphere representing the handle piece is farther from the origin, but the product of less mass and greater distance balance the product of greater mass and less distance from the end piece.



Q.2 $x_{cm}=\frac{l}{4}, y_{cm}=\frac{l}{4}$ Q.3 $\frac{22\ell}{35}$ Q.4 $x_{cm}=-\frac{a}{12}, y_{cm}=\frac{b}{12}$ Q.5 $\frac{R}{20}$ the left of disc

Motion of center of mass

Position vector of centre of mass

$$\vec{r}_{c.m.}=\frac{m_1\vec{r}_1+m_2\vec{r}_2+.....+m_n\vec{r}_n}{m_1+m_2+......m_n} \qquad ....(1)$$

Differentiating the above equation, we get the velocity of centre of mass in terms of velocity of individual particles.
Velocity vector of centre of mass

$$\vec{v}_{c.m.}=\frac{m_1\vec{v}_1+m_2\vec{v}_2+.....+m_n\vec{v}_n}{m_1+m_2+......m_n}=\frac{\sum m_i\vec{v}_i}{\sum m_i} \qquad ....(2)$$

On further differentiation, acceleration of centre of mass can be obtained.

$$\frac{d\vec{v}_{c.m.}}{dt} = \frac{m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + ... + m_n\frac{d\vec{v}_n}{dt}}{m_1 + m_2 + ... + m_n}.$$

Acceleration vector of centre of mass

$$\vec{a}_{c.m.} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_i\vec{a}_i}{\sum m_i} \quad ....(3)$$

Let net force on particle of mass m_1 be $\vec{F}_1$, m_2 be $\vec{F}_2$

$\vec{F}_1 = m_1\vec{a}_1$, $\vec{F}_2 = m_2\vec{a}_2$ ....

substituting these values in equation (3)

$$\vec{a}_{cm} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...}{m_1 + m_2 + m_3 + ...}$$

We know that summation of internal forces is zero, thus

$\vec{F}_1 + \vec{F}_2 + \vec{F}_3........\vec{F}_n$ is the net external force.

$$\vec{a}_{cm} = \frac{\left(\vec{F}_{external}\right)_{net}}{M_{total}}.$$

Displacement vector of centre of mass

$$\vec{s}_{c.m.} = \frac{m_1\vec{s}_1 + m_2\vec{s}_2 + ... + m_n\vec{s}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_i\vec{s}_i}{\sum m_i}$$

Equation (ii) and (iii) are vector equations and thus can be solved separately for the three mutually perpendicular components $\left(\hat{i},\ \hat{j} \text{ and } \hat{k}\right)$ as we did earlier in determining the position of centre of mass.

Components of velocity

$$v_{c.m.(x)} = \frac{\sum m_i v_{ix}}{\sum m_i}$$

$$v_{c.m.(y)} = \frac{\sum m_i v_{iy}}{\sum m_i}$$

Component of acceleration

$$a_{c.m.(x)} = \frac{\sum m_i a_{ix}}{\sum m_i}$$

$$a_{c.m.(y)} = \frac{\sum m_i a_{iy}}{\sum m_i}$$

Components of displacement

$$s_{c.m.(x)} = \frac{\sum m_i s_{ix}}{\sum m_i}$$

$$s_{c.m.(y)} = \frac{\sum m_i s_{iy}}{\sum m_i}$$

$$\frac{d\vec{v}_{cm}}{dt} = \vec{a}_{cm} = \frac{(\vec{F}_{ext})_{net}}{M_{total}}$$

If $\quad (\vec{F}_{ext})_{net} = 0 \Rightarrow a_{cm} = 0 \Rightarrow \vec{v}_{cm}$ is constant.

if $\quad (\vec{v}_{cm})_{initial} = 0$

During the course of motion it will remain zero, and thus displacement of the centre of mass of the system will also be zero.

Two bodies system

Initially m_1 and m_2 are at rest and they are free to move under the influence of internal forces only, then m_1 and m_2 may move with variable velocity and variable acceleration but centre of mass of system will remain at rest.

Initially

$$m_1 r_1 = m_2 r_2$$

After some time

$$m_1(r_1 - s_1) = m_2(r_2 - s_2)$$

$$m_1 r_1 - m_1 s_1 = m_2 r_2 - m_2 s_2$$

$$m_1 s_1 = m_2 s_2$$

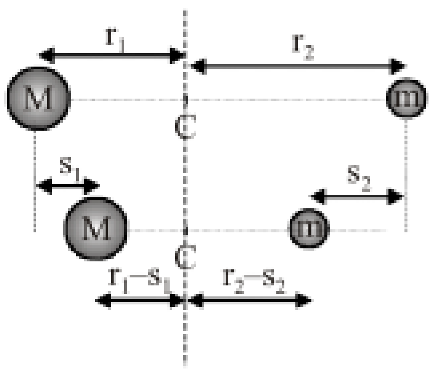
$$m_1 \frac{ds_1}{dt} = m_2 \frac{ds_2}{dt}$$

$$m_1 v_1 = m_2 v_2$$

$$m_1 \frac{dv_1}{dt} = m_2 \frac{dv_2}{dt}$$

$$m_1 a = m_2 a_2$$

$$\frac{s_1}{s_2} = \frac{v_1}{v_2} = \frac{a_1}{a_2} = \frac{m_2}{m_1} \text{ [numerically]}$$

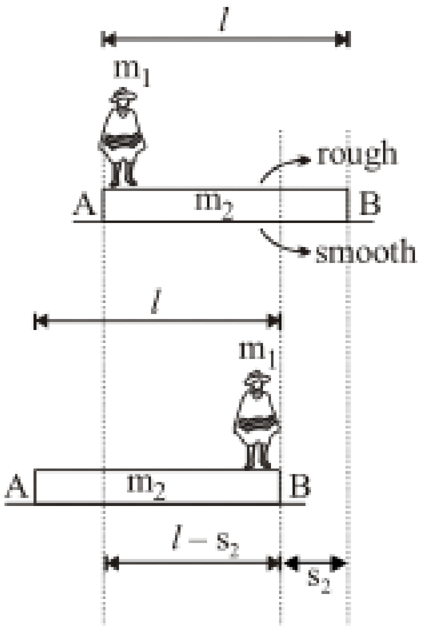


C is centre of mass of M & m

s_1 and s_2, v_1 and v_2 and a_1 and a_2 are oppositely directed.

Illustration :

A man of mass m_1 stands at an edge A of a plank of mass m_2 & length l which is kept on a smooth floor If man walks from A to the other edge B find displacement of plank.



Sol. *Let the displacement of the plank be s_2*

Initial momentum of system (man and plank) is zero.

Net external force on this system is zero.

Thus during the course of motion $\vec{P}_{sys} = 0 \Rightarrow \vec{v}_{cm} = 0 \Rightarrow \vec{s}_{cm} = 0$

$$\Rightarrow \quad m_1\vec{s}_1 + m_2\vec{s}_2 = 0$$

$$m_1\vec{s}_1 = -m_2\vec{s}_2$$

in scalar form

$$m_1 |\vec{s}_1| = m_2 |\vec{s}_2|$$

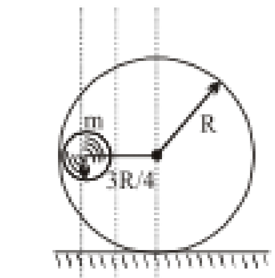
$$m_1 (l - s_2) = m_2 s_2$$

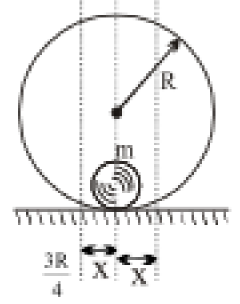
$$s_2 = \frac{m_1 \ell}{m_1 + m_2}$$

The plank moves backward because as the man moves forward, he pushes on the plank backwards.

Illustration :

Inside a smooth spherical shell of the radius R a ball of the same mass is released from the shown position (Fig.) Find the distance travelled by the shell on the horizontal floor when the ball comes to the lowest point of the shell.





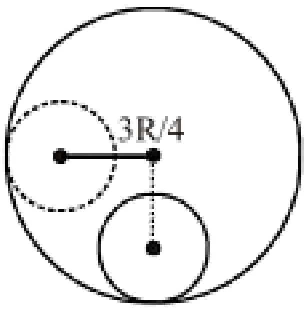
Sol. *As initial momentum of the system in x-direction is zero, and there is no net external force in x-direction the momentum of system remains zero in x-direction and thus the center of mass of the system undergoes zero displacement in x-direction*

$$m_1\vec{s}_1 + m_2\vec{s}_2 = 0$$

When the ball comes to the lowest position ; shell moves backwards say by a distance x.

Displacement of ball in x-direction = Displacement of ball w.r.t. shell + displacement of shell.

Displacement of shell = (–x)



$\therefore$ *displacement of ball is x-direction is* $\left(\frac{3R}{4} + (-x)\right)$

$$m\left(\frac{3R}{4} - x\right) - mx = 0$$

$\therefore$ $x = \frac{3R}{8}$

If we do not consider that the shell moves back ward, we can take its forward displacement to be x,

$\therefore$ *displacement of ball in x-direction* $= \frac{3R}{4} + x$

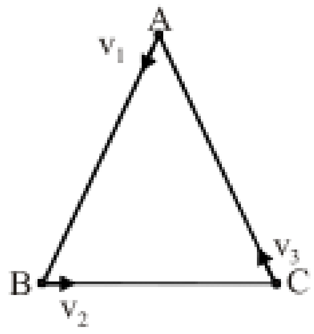
$$m\left(\frac{3R}{4} + x\right) + mx = 0$$

$x = -\frac{3R}{8}$ *(–ve sign indicates its backwards motion)*

Velocity of center of mass:

Illustration :

Let there are three equal masses situated at the vertices of an equilateral triangle, as shown in figure. Now particle-A starts with a velocity v_1 *towards particle B, particle-B starts with a velocity v_2 towards C and particle-C starts with velocity v_3 towards A. Find the displacement of the centre of mass of the three particles A, B, and C after time t. What it would be if $v_1 = v_2 = v_3$*



Sol *First we write the three velocities in vectorial form, taking right direction as positive x-axis and upwards as positive y-axis.*

$$v_1 = -\frac{1}{2} v_1 \hat{i} - \frac{\sqrt{3}}{2} v_1 \hat{j}$$

$$v_2 = v_2 \hat{i}$$

$$v_3 = -\frac{1}{2} v_3 \hat{i} + \frac{\sqrt{3}}{2} v_3 \hat{j}$$

Thus the velocity of centre of mass of the system is, $\vec{v}_{cm} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3}{m_1 + m_2 + m_3}$

$$\vec{v}_{cm} = \frac{\vec{v}_1 + \vec{v}_2 + \vec{v}_3}{3}$$

$$\vec{v}_{cm} = \frac{(v_2 - \frac{1}{2}v_1 - \frac{1}{2}v_3)\hat{i} + \frac{\sqrt{3}}{2}(v_3 - v_1)j}{3}$$

Which can be written as $\vec{v}_{cm} = v_x \hat{i} + v_y \hat{j}$

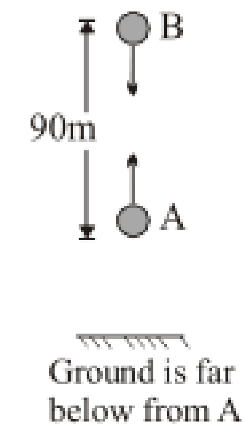
Thus displacement of the centre of mass in time t is $\Delta r = v_x t \hat{i} + v_y t \hat{j}$

If $v_1 = v_2 = v_3 = v$ *we have* $\vec{v}_{cm} = 0$

Therefore no displacement of centre of mass of the system.

Illustration :

Two particles A and B of mass 1 kg and 2kg respectively are projected in the direction shown in figure with speeds $u_A = 200$ m/s and $u_B = 50$ m/s. Initially they were 90m apart. Find the maximum height attained by the centre of mass of the particles from the initial position of A. Assume acceleration due to gravity to be constant ($g = 10$ m/s²)



Sol.

$$\vec{P}_{sys} = M_T \vec{V}_{CM}$$

$$\frac{d\vec{P}_{sys}}{dt} = M_T \vec{a}_{CM}$$

$$\vec{F}_{ext} = M_T \vec{a}_{CM}$$

Net external force is the gravitational force

$$F_{ext} = M_T \times g$$

$$\therefore \quad \vec{a}_{cm} = g \downarrow \text{ (downwards)}$$

$$(\vec{V}_{cm}) = \frac{m_A \vec{V}_A + m_B \vec{V}_B}{m_A + m_B} = \frac{1 \times 200 - 2 \times 50}{3} = \frac{100}{3} \text{ m/s} \uparrow \text{(upwards)}$$

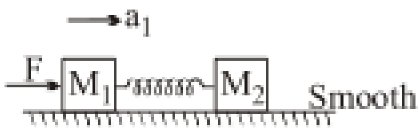
initial height of centre of mass from A , $h_0 = \frac{1 \times 0 + 2 \times 90}{1+2} = 60$ m

$$h_{max} = \text{initial height } (h_0) + \frac{v_{cm}^2}{2g} = 60 + \frac{\left(\frac{100}{3}\right)^2}{2 \times 10} = 115.55 \text{ m}$$

Acceleration of centre of mass:

Illustration :

Two blocks of masses $M_1 = 1$kg and $M_2 = 2$kg kept on smooth surface, are connected to each other through a light spring ($k = 100$ N/m) as shown in the figure. When we push mass M_1 with a force $F = 10$N find the acceleration of centre of mass of system.

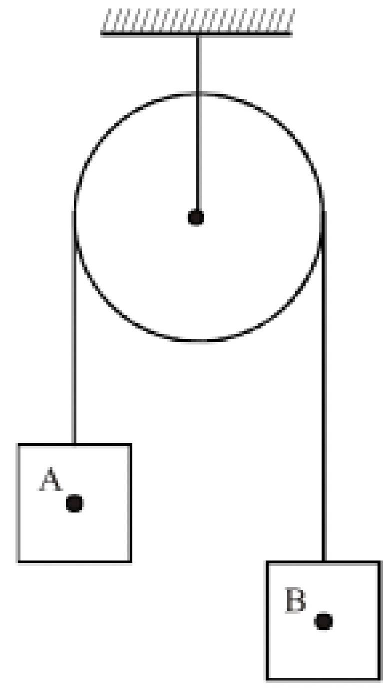


Sol. $a_{CM} = \frac{F_{ext}}{M_{Total}}$

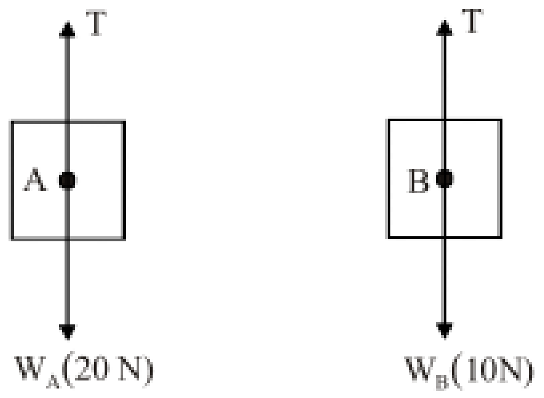
$$= \frac{10}{3} \text{ m/s}^2$$

Illustration :

In the arrangement shown in figure, $m_A = 2$ kg and $m_B = 1$ kg. The string is light and inextensible. Find the acceleration of COM of the blocks. Neglect friction everywhere.



Sol.



From Newton's II Law,

For block A, $20 - T = 2a$ .... (i)

For block B, $T - 10 = a$ .... (ii)

solving (i) and (ii), we obtain T

$$\Rightarrow T = \frac{40}{3}\text{ N}$$

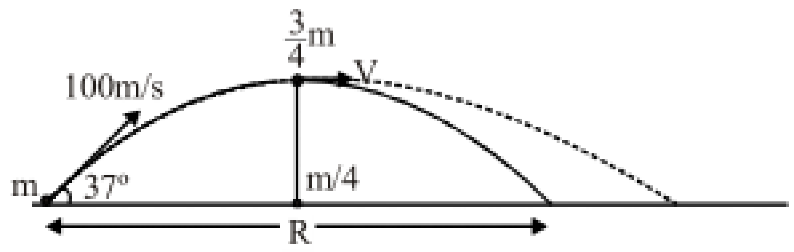
$$F_{ext} = M_{total} \times a_{cm}$$
$$= m_1 g + m_2 g - 2T = 3 \times a_{CM}$$

$$\Rightarrow \quad 30 - \frac{80}{3} = 3 \times a_{CM}, \quad a_{CM} = \frac{10}{9}\ m/s^2$$

Illustration :

A projectile is fired at a speed of 100 m/s at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1:3, the smaller coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

Sol.



As there is no external force in horizontal direction, $(P_{sys})_x$ is conserved
let the horizontal velocity of heavier mass at the highest point be v
Applying conservation of momentum in x - direction, at highest point,

$$(m \times u\cos 37°) = \left(\frac{m}{4} \times 0 + \frac{3m}{4} \times v\right)$$

$$\Rightarrow \quad v = \frac{320}{3}\ m/s$$

for projectile time to reach highest point and time to reach ground are equal to $\frac{T}{2}$

for time of flight T, using concept of projectile motion,

$$T = \frac{2u\sin\theta}{g}$$

$$T = 12\ sec\ ;$$

$$\therefore \quad T/2 = 6\ sec$$

$\therefore$ *horizontal distance travelled in first 6 sec. & next 6 sec is* $(u\cos 37° \times 6)$ & $(v \times 6)$ *respectively*

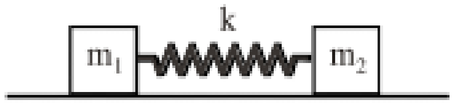
$$x = 480 + 640$$
$$= 1120\ m$$

Practice Exercise

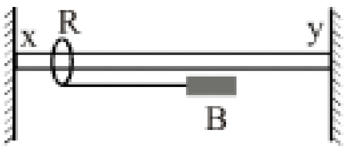
Q.1 A projectile is fired from a gun at an angle of 45° with the horizontal and with a speed of 20 m/s relative to ground. At the highest point in its flight the projectile explodes into two fragments of equal mass. One fragment, whose horizontal speed is zero falls vertically. How far from the gun does the other fragment land , assuming a level terrain ? Take $g = 10\ m/s^2$?

Q.2 Two particles of mass 2 kg and 4kg are approaching towards each other with acceleration $1\ m/sec^2$ and $2\ m/sec^2$ respectively on a smooth horizontal surface. Find the acceleration of centre of mass of the system.

Q.3 Figure shows two blocks of masses $m_1 = 5$ kg and $m_2 = 2$ kg placed on a frictionless surface and connected with a spring. An external kick gives a velocity 14 m/s to the heavier block in the direction of lighter one. Find (a) the velocity gained by the center of mass and (b) the velocities of the two blocks in the center of mass co-ordinate system just after the kick.



Q.4 The ring R of mass m in the arrangement shown can slide along a smooth fixed, horizontal rod XY. It is attached to the block B of mass m by a light string of length L. The block is released from rest, with the string horizontal. Find the displacement of ring when the string becomes vertical.



Answers

Q.1 60 m Q.2 1 m/sec^2 (towards 2 kg block)

Q.3 (a) $V_{CM} = \dfrac{m_1v_1 + m_2v_2}{m_1 + m_2} = \dfrac{5\times142\times0}{7} = 10$ m/s

(b) $v_{m_1/CM} = 14 - 10 = 4\,m/s$; $v_{m_2/CM} = 0 - 10 = -10\,m/s$

(c) 30 m

Q.4 $\dfrac{ML}{m+M}$

Linear momentum and its conservation principle

The (linear) momentum of a particle is defined as $\vec{p} = m\vec{v}$. The momentum of an N-particle-system is the (vector) sum of the momenta of the N particles i.e.,

$$\vec{P}_{sys} = \sum_i \vec{p}_i = \sum_i m_i\vec{v}_i\ .$$

But $$\sum_i m_i\vec{v}_i = \frac{d}{dt}\sum_i m_i\vec{r}_i = \frac{d}{dt}M\vec{R}_{CM} = M\vec{V}_{CM}$$

Thus, $\vec{P}_{sys} = M\,\vec{V}_{CM}$

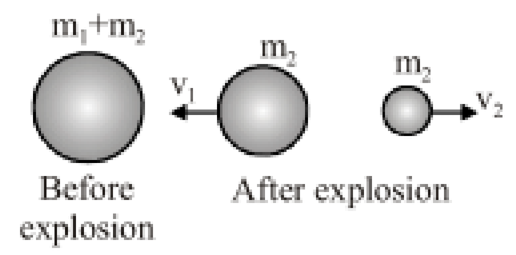
As we have seen, if the external forces acting on the system add up to zero, the centre of mass moves with constant velocity, which means $\vec{p}$ = constant. Thus the linear momentum of a system remains constant (in magnitude and direction), if the external forces acting on the system add up to zero. This is known as the principle of conservation of linear momentum.

Let as see a simple example of a bomb explosion.

Consider a bomb placed on a horizontal surface which suddenly explodes into two parts of masses m_1 and m_2. The forces that are responsible for the explosion are internal. As there is no external force on the

system, momentum of system remains conserved. The initial momentum of the system is zero, Thus the final momentum of the system must also be zero.

i.e. After explosion, if mass m_1 moves with velocity $\vec{v}_1$, and mass m_2 moves with velocity $\vec{v}_2$, then by conservation of linear momentum.



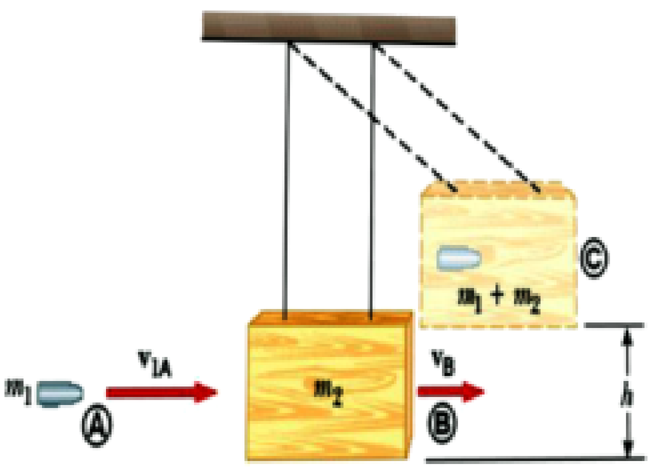
$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

in scalar form,

$$m_1v_1 + m_2(-v_2) = 0$$
$$m_1v_1 = m_2v_2.$$

Illustration :

The ballistic pendulum is an apparatus used to measure the speed of a fast moving projectile, such as a bullet. A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from light wires. The bullet embeds in the block and the entire system swings through a height h. Determine the initial speed of the bullet in terms of h?



Sol. *As there is no external force in x-direction, applying conservation of momentum during collision*

$$m_1v_1 = (m_1 + m_2)\, V_B$$

$$\therefore \quad V_B = \frac{m_1v_1}{m_1 + m_2} \qquad .....(i)$$

Applying conservation of mechanical energy after collision

$$\frac{1}{2}\,(m_1 + m_2)\, V_B^2 = (m_1 + m_2)\, g\, h$$

here h is the displacement of center of mass :

$$V_B = \sqrt{2gh} \qquad ....(ii)$$

From (i) & (ii)

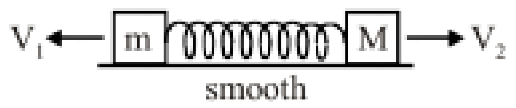
$$\therefore \quad V_1 = \left(1 + \frac{m_2}{m_1}\right)\sqrt{2gh}$$

Illustration :

A light spring of constant k is kept compressed between two blocks of masses m & M on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions. The spring looses contact with the blocks when it acquires natural length. If the spring was initially compressed through distance x, find the speed of two blocks.

Sol. *As net external force acting on the system is zero (net $F_{ext} = 0$)*

∴ Applying conservation of momentum in horizontal direction



$$P_f = P_i$$

$$MV_2 - mv_1 = 0 \qquad \text{....(i)}$$

applying conservation of mechanical energy (as no work is done)

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}MV_2^2 \qquad \text{.....(ii)}$$

$$v_1 = \sqrt{\frac{MK}{m(m+M)}} \;;\; v_2 = \sqrt{\frac{mK}{M(m+M)}}$$

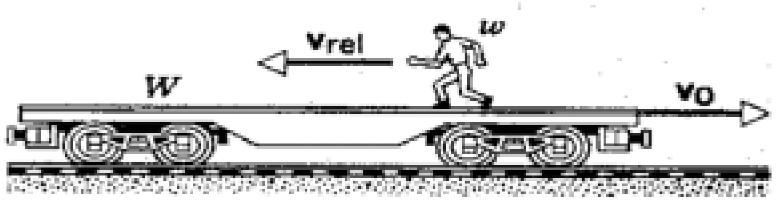
Illustration :

A railroad flatcar of mass M can roll without friction along a straight horizontal track. Initially, a man of mass 'm' is standing on the car, which is at rest. What is the velocity of the car if the man runs to the left so that his speed relative to the car is v_{rel}.

Sol. *Let velocity of man w.r.t. ground be V_m and the velocity attained by car be v_0 in backward direction.*

$V_{rel} = v_m - (-v_0)$ *(considering left to be positive)*

$v_m = v_{rel} - v_0$



Applying conservation of momentum in x-direction,

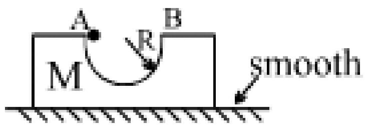
$$mv_m + M(-v_0) = 0$$

$$m(v_{rel} - v_0) - Mv_0 = 0$$

$$\therefore \qquad v_0 = \frac{mv_{rel}}{m+M}$$

Illustration :

In the figure shown the wedge of mass M has a semicircular groove. A particle of mass $m = \frac{M}{2}$ is released from A. It slides on the smooth circular track and starts climbing on the right face.



Find the maximum velocity of wedge during process of motion.

Sol. *Maximum velocity of wedge will be when the ball is at the lowest point in the wedge as till this point the horizontal component of normal on the wedge will be speeding the wedge but after this it will be opposite to the direction of motion of wedge, thereby slowing it down.*

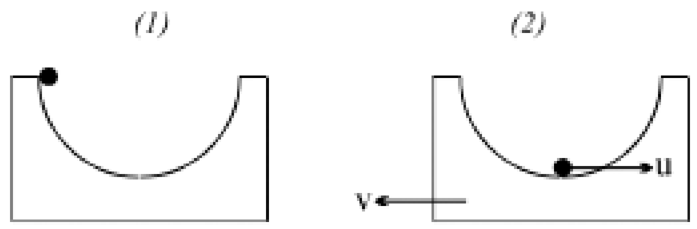
Applying conservation of linear momentum in horizontal direction at positions 1 & 2.

Initial momentum, $p_i = 0$

Fianl momentum, $p_f = -Mv + mu$

$$p_i = p_f$$

$$u = \frac{Mv}{m} = 2v$$



Applying conservation of mechanical energy

$$U_i + K_i = U_f + K_f$$

$$mgR + 0 = 0 + \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$$

$$2mgR = m(2v)^2 + Mv^2$$

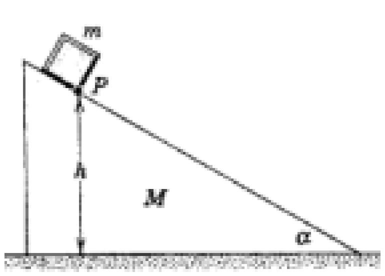
$$2 \times \frac{M}{2} \times gR = 4\frac{M}{2}v^2 + Mv^2$$

$$MgR = 3Mv^2$$

$$v = \sqrt{\frac{gR}{3}}$$

Illustration :

A block of mass m rests on a wedge of mass M which, in turn, rests on a horizontal table, as shown in figure. All surfaces are frictionless. If the system starts at rest with point P of the block a distance h above the table, find the speed of the wedge the instant point P touches the table.



Sol. As net external force in horizontal direction is zero, (F_{ext}) horizontal $= 0$

applying momentum conservation in horizontal direction,

$$mu_x = Mv \qquad \text{.........(i)}$$

from constraint we obtain(refere figure - (2)),

$$u_y = (u_x + v)\tan\alpha \qquad \text{.........(ii)}$$

applying conservation of mechanical energy,

$$mgh = \frac{1}{2}m(u_x^2 + u_y^2) + \frac{1}{2}Mv^2 \qquad \text{.........(iii)}$$

Solving equation (i), (ii) and (iii), we get

$$v = \sqrt{\frac{2mgh}{\frac{M^2}{m} + \frac{M}{2} + \frac{\tan^2\alpha(m+M)^2}{m}}}$$

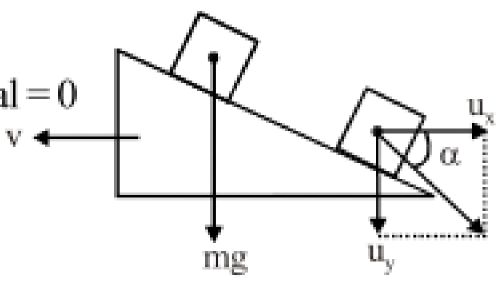
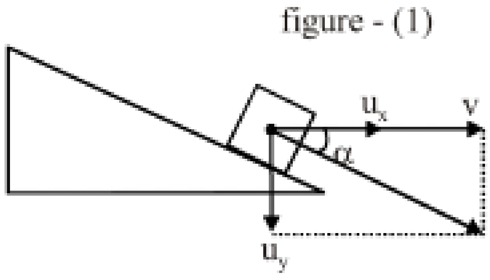


figure - (1)

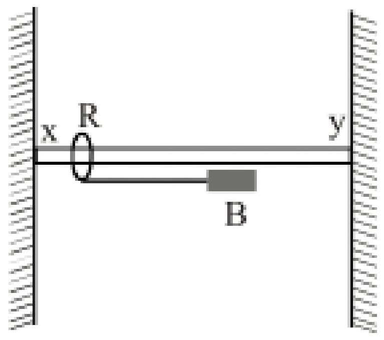


motion of block w.r.t. wedge

figure - (2)

Practice Exercise

Q.1 The ring R in the arrangement shown can slide along a smooth fixed, horizontal rod XY. It is attached to the block B by a light string. The block is released from rest, with the string horizontal. Then which of the following are true.



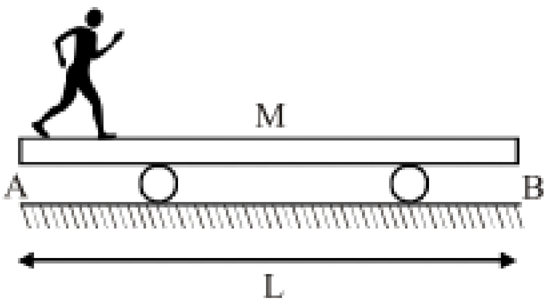
(a) One point in the string will have only vertical motion

(b) R and B will always have momentum of same magnitude

(c) When the string becomes vertical, the speeds of R and B will be inversely proportional to their masses

(d) R will lose contact with the rod at some point

Q.2 The figure shows a man of mass m standing at the end A to a trolley of mass M placed at rest on a smooth horizontal surface. The man starts moving towards the end B with a velocity u_{rel} with respect to the trolley. The length of the trolley is L.



(a) Find the time taken by the man to reach the other end.

(b) As the man walk on the trolley, find the velocity centre of mass of the system (man + trolley).

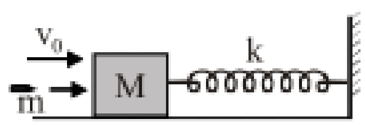
(c) When the man reaches the end B, find the distance moves by the trolley with respect to ground.

(d) Find the distance moved by the man with respect to ground.

Q.3 A man of mass 60 kg jumps from a trolley of mass 20 kg standing on smooth surface with absolute velocity 3 m/s. Find the velocity of trolley and total energy produced by man.

Q.4 Three particles of mass 20g, 30 g, and40 g are initially moving along the positive direction of the three coordinate axes respectively with the same velocity of 20 cm/s, when due to their mutual interactional the first particle comes to rest, the second acquires a velocity $10\hat{i}+20\hat{k}$. What is then the velocity of the third particle ?

Q.5 A bullet of mass m strikes a block of mass M connected to a light spring of stiffness k, with a speed v_0 and gets into it. Find the loss of K.E. of the bullet.



Answers

Q.1 (a), (c)

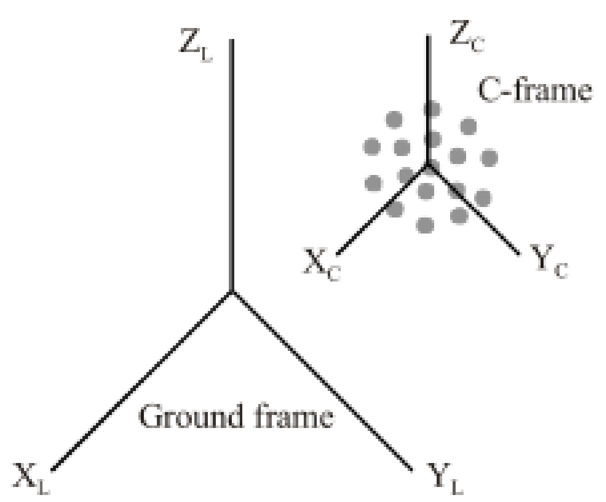
Q.2 (a) $\frac{L}{u_{rel}}$ (b) zero (c) $\frac{mL}{m+M}$ (d) $\frac{ML}{m+M}$

Q.3 9 m/s, 1.08 kJ Q.4 $2.5\hat{i}+15\hat{j}+5\hat{k}$ Q.5 $\frac{Mmv_0^2}{2(M+m)}$

C-Frame :

The total momentum of a system of particles in the C-frame of reference is always zero. We can attach a frame of reference to the center of mass of a system, this is called the center of mass or C-frame of reference (figure). Relative to this frame, the center of mass is at rest ($v_{com}=0$) and according to equation $P = M\vec{v}_{com}$, the total momentum of a system of particles in the C-frame of reference is always zero.

$$\vec{P}=\sum_i \vec{P}_i = 0 \text{ in the C-frame of reference}$$



C-frame of reference

The C-frame is important because many situations can be more simply analyzed in this frame. It is clear that the C-frame moves with a velocity v_{com} relative to the ground frame. When no external forces act on a system, the C-frame becomes an inertial frame.

Illustration :

The velocities of two particles of masses m_1 and m_2 relative to an observer in an inertial frame are v_1 and v_2. Determine the velocity of the center of mass relative to the observer and the velocity of each particle relative to the center of mass.

Sol. *From definition*

$$v_{com} = \frac{dr_{com}}{dt} = \frac{1}{M}\sum_i m_i \frac{dr_i}{dt} = \frac{\sum_i m_i v_i}{M}$$

The velocity of the center of mass relative to the observer is

$$v_{com} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

The velocity of each particle relative to the center of mass (figure) using the relative motion equations for velocities is

$$v_{1c}' = v_1 - v_{com} = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Answers

Q.1 (a), (c)

Q.2 (a) $\dfrac{L}{u_{rel}}$ (b) zero (c) $\dfrac{mL}{m+M}$ (d) $\dfrac{ML}{m+M}$

Q.3 9 m/s, 1.08 kJ Q.4 $2.5\hat{i}+15\hat{j}+5\hat{k}$ Q.5 $\dfrac{Mmv_0^2}{2(M+m)}$

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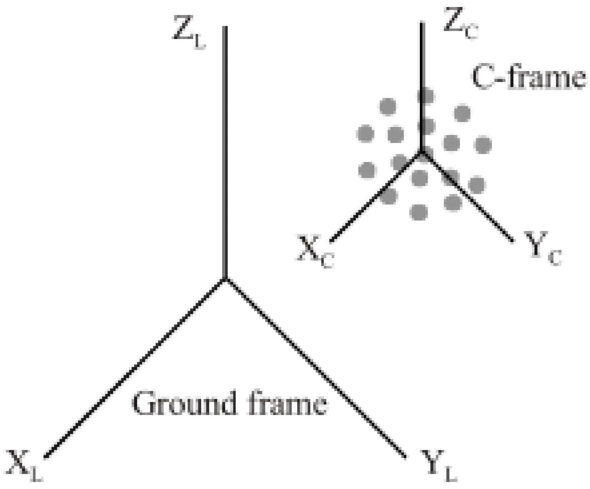
$$\vec{P} = \sum_i \vec{P}_i = 0 \text{ in the C-frame of reference}$$



C-frame of reference

The C-frame is important because many situations can be more simply analyzed in this frame. It is clear that the C-frame moves with a velocity v_{com} relative to the ground frame. When no external forces act on a system, the C-frame becomes an inertial frame.

Illustration :

The velocities of two particles of masses m_1 and m_2 relative to an observer in an inertial frame are v_1 and v_2. Determine the velocity of the center of mass relative to the observer and the velocity of each particle relative to the center of mass.

Sol. *From definition*

$$v_{com} = \frac{dr_{com}}{dt} = \frac{1}{M}\sum_i m_i \frac{dr_i}{dt} = \frac{\sum_i m_i v_i}{M}$$

The velocity of the center of mass relative to the observer is

$$v_{com} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

The velocity of each particle relative to the center of mass (figure) using the relative motion equations for velocities is

$$v_{1c}' = v_1 - v_{com} = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \frac{m_2(v_1 - v_2)}{m_1 + m_2} = \frac{m_2 v_{12}}{m_1 + m_2}$$

$$v_{2c}' = v_2 - v_{com} = \frac{m_1(v_2 + v_1)}{m_1 + m_2} = -\frac{m_1 v_{12}}{m_1 + m_2}$$

where $v_{12} = v_1 - v_2$ is the relative velocity of the two particles.
Thus, in the C-frame, the two particles appear to be moving in opposite directions with velocities inversely proportional to their masses.

Also relative to the center of mass, the two particles move with equal but opposite momentum since

$$p_1' = m_1 v_{1c}' = \frac{m_1 m_2 v_{12}}{(m_1 + m_2)} = p_2'$$

The expressions for two particle problems are much simpler when they are related to the C-frame of reference.

Kinetic energy of system of particles

Let us find relation between kinetic energy of a system from ground frame and C-frame. We have a system consisting of many particles, let's say speed of the i[th] particle is v_i. Then kinetic energy of system, K, in ground frame will be summation of individual kinetic energies.

$$K_{sys} = \Sigma\left(\frac{1}{2} m_i v_i^2\right)$$

now $\qquad v_i = v_{i/c} + v_c$

where v_i is velocity of the i[th] particle in ground frame, $v_{i/c}$ is velocity of the i[th] particle in reference to frame attached to the center of mass and v_c is velocity of center of mass in ground frame.

$$K_{sys} = \frac{1}{2} \Sigma m_i \left(\vec{v}_{i/c} + \vec{v}_c\right)^2$$

$$K_{sys} = \frac{1}{2} \Sigma m_i v_{i/c}{}^2 + \frac{1}{2} \Sigma m_i \vec{v}_c^2 + \frac{1}{2} \times 2\left(\Sigma m_i \cdot \vec{v}_{i/c} \cdot \vec{v}_c\right)$$

$$K_{sys} = \frac{1}{2} \Sigma m_i v_{i/c}{}^2 + \frac{1}{2} (\Sigma m_i) \vec{v}_c^2 + \vec{v}_c^2 \left(\Sigma m_i \cdot \vec{v}_{i/c}\right) \vec{v}_c$$

We can take v_c out of summation in second and third term as it is constant. Now third term becomes zero, as $\Sigma m_i \vec{v}_{i/c} = M\vec{v}_{c/c} = 0$ (total momentum of a system of particles in the C-frame of reference is always zero.)

$v_{c/c}$ is velocity of center of mass in frame of com. Which is zero. Also it represents momentum of system in C-frame which is zero.

$$\left(\frac{1}{2} \Sigma m_i \vec{v}_{i/c}^2\right) = K_{sys/c}$$

Thus, we get $\qquad K_{sys} = K_{sys/c} + \frac{1}{2} m v_c^2$

Where $K_{sys/c}$ means kinetic energy of system in C-frame. This important conclusion will be useful again in rotational dynamics; we can do little manipulation to write the equation as

$$K_{sys} = K_{sys/c} + \frac{p_c^2}{2M}$$

A system of two Particles

Suppose the masses of the particles are equal to m_1 and m_2 and their velocities in the K reference frame be $\vec{v}_1$ and $\vec{v}_2$. respectively. Let us find the expressions defining their moment and the total kinetic energy in the C-frame.

The momentum of the first particle in the C-frame is

$$\vec{P}_{1/c} = m\vec{v}_{1/c} = m_1(\vec{v}_1 - \vec{v}_c)$$

Where v_c is the velocity of the center of mass of the system in the ground frame. Substituting in this formula expression.

$$\vec{v}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$\vec{P}_{1/c} = \mu(\vec{v}_1 - \vec{v}_2)$$

Where μ is the reduced mass of the system, given by

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

Similarly, the momentum of the second particle in the C-frame is

$$\vec{P}_{2/c} = \mu(\vec{v}_2 - \vec{v}_1)$$

Thus, the momenta of the two particles in the C-frame are equal in magnitude and opposite in direction; the modulus of the momentum of each particle is

$$\vec{P}_{1/c} = \mu v_{rel}$$

Where $v_{rel} = |\vec{v}_1 - \vec{v}_2|$ is the velocity of one particle relative to another.

Finally, let us consider kinetic energy. The total kinetic energy of the two particles in the C-frame is

$$K_{sys/c} = K_1 + K_2 = \frac{\vec{P}^2}{2m_2} + \frac{\vec{P}^2}{2m_2}$$

we know $\quad \mu = \dfrac{m_1 m_2}{m_1 + m_2} \quad$ or $\quad \dfrac{1}{m_2} + \dfrac{1}{m_2} = \dfrac{1}{\mu}$

Then $\quad K_{sys/c} = \dfrac{\vec{P}^2}{2\mu} = \dfrac{\mu v_{rel}^2}{2}$

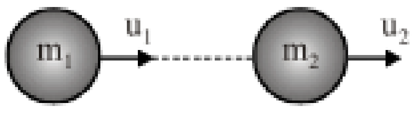
$$K_{sys} = K_{sys/c} + K_c, \text{ we get}$$

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{m v_c^2}{2} \qquad (\text{where } m = m_1 + m_2)$$

Read this Illustration after collision

Illustration :

Two particles of mass m_1, m_2 moving with initial velocity u_1 and u_2 collide head-on. Find minimum kinetic energy that system has during collision. Thus. Prove that maximum kinetic energy is lost in perfectly inelastic collision



Sol. *Particles moving with velocity u_1 and u_2 in the same direction.*
In C-frame initial kinetic energy of system is

$$K_{sys} = K_{sys/c} + K_c, \text{ we get}$$

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{mv_c^2}{2}$$

$$\frac{1}{2}\mu(u_2 - u_1)^2 + \frac{mv_c^2}{2}$$

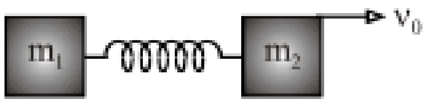
$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

During collision, at the instant of maximum deformation. we get minimum kinetic energy in C-frame as particles attain same. velocity, thus relative velocity becomes zero.
When an isolated system has minimum kinetic energy in C-frame, it will also have minimum kinetic energy in ground frame, as velocity of center of mass is constant.
Thus, minimum kinetic energy during collision is

$$\frac{1}{2}(m_1 + m_2)v_c^2$$

Where $$v_c = \frac{(m_1 u_1 + m_2 u_2)}{m_1 + m_2}$$

In perfectly inelastic collision, since both the particles move together, the relative velocity becomes zero. Thus, final kinetic energy is $\frac{1}{2}(m_1 + m_2)v_c^2 (m_s = m_1 + m_2)$, *as velocity of center of mass is constant. This is the minimum possible kinetic energy that a system will have because in all other case there will be one more term adding in the kinetic energy of system because of particles having relative velocity.*
Two block of mass m_1 and m_2 connected by an ideal spring of spring constant k are kept on a smooth horizontal surface. Find maximum extension of the spring when the block m_2 is given an initial velocity of v_o toward right as shown in figure.



Blocks of masses m_1 and m_2 connected by an ideal spring

When a block of mass m_2 is given an initial velocity of v_0 toward right, the spring extends and pulls the block toward left and the same extended spring will pull the block m1 toward right. Initially the force acting on m_2 will reduce its speed and the force acting on m_1 will increase its speed. Thus, we can see that initially the extension be increasing.

If we consider the two blocks and spring as one system, then total mechanical energy must be conserved as there is no dissipative force present. Also, momentum will be conserved as there is no external force present.

Now there will be an instant when the block will have same velocity, that is, velocity of m_1 has increased sufficiently to become equal to the velocity of m_2 which has been decreasing continuously. At this moment. The spring will have the maximum extension x_{max}, as till this point distance between the blocks was continuously increasing because m_2 had larger velocity. Now it will start decreasing as m_1 will be moving faster than m_2 and it will reduce the distance between the two blocks. Thus when $v_1 = v_2$, extension is maximum. This can also be understood alternatively by looking at m_1 from reference frame attached to m_2. To an observer sitting on m_1 the block m_2 will be closest or farthest when it is relative at rest.

Since no external force is present, velocity of canter of mass is given as

$$v_{com} = \frac{(m_2 v_0)}{m_1 + m_2}$$

From the reference frame of center of mass, the initial kinetic energy is given by

$$K = \frac{1}{2}\mu(v_{2f} - v_{1f})^2 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(u)^2$$

From the reference frame of center of mass, the final kinetic energy is given by

$$K = \frac{1}{2}\mu(v_{2f} - v_{1f})^2 = 0$$

Thus, equating initial and final energies in C-frame, we get

$$\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_0)^2 = 0 + \frac{1}{2}kx_{max}^2$$

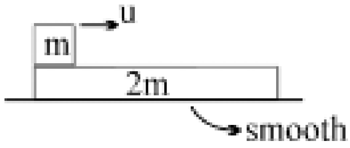
Thus, maximum extension

$$x_{max} = v_0\sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

This problem can be thought exactly as the opposite of the previous Illustration as here the maximum extension is occurring the relative velocity is zero.

Illustration:

Find total work done by friction assuming plank is sufficiently long.



Solve this question in both ground frame and C-frame.

Sol. ***In ground frame*** *:*

let v be the final velocity of block & plank when relative motion ceases between block and plank; applying conservation of linear momentum, mu = (2m + m)v

$\therefore \quad v = u/3$

w_f = change in K.E. = $K.E._f - K.E._i$

$$w_f = \frac{1}{2}3m\left(\frac{u}{3}\right)^2 - \frac{1}{2}mu^2 = -\frac{mu^2}{3}$$

In C-frame :

Considering block and plank as a system

Work done by friction is change in kinetic energy

w_f = change in K.E. = $K.E._f - K.E._i$

$$K.E._i = \frac{1}{2}\mu(u-0)^2 + \frac{(2m+m)v_c^2}{2} \qquad \left(\mu = \frac{2m\times m}{2m+m}\right)$$

$$K.E._f = 0 + \frac{(2m+m)v_c^2}{2}$$ (v_c is constant as external force is zero)

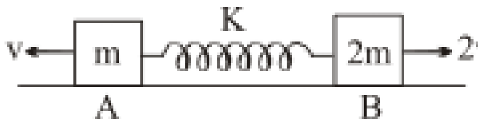
$$w_f = K.E._f - K.E._i = -\frac{mu^2}{3}$$

Note: P_{sys} = conserved if $f_{ext.} = 0$ although internal friction are doing work.

Illustration :

. Two blocks A and B of masses m & 2m placed on smooth horizontal surface are connected with a light spring. The two blocks are given velocities as shown when spring is at natural length.

(i) Find velocity of centre of mass (b) maximum extension in the spring



Sol. (a) $V_{CM} = \dfrac{2m\times 2v - m_x v}{3m} = v$

(b) There will be maximum extension in the spring when $v_{rel} = 0$

$\therefore$ applying conservation of mechanical energy,

$$\frac{1}{2}MV_{CM}^2 + 0 + \frac{1}{2}kx^2 = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\mu v_{rel}^2$$

$$x = \sqrt{\frac{\mu}{k}}v_{rel} = \sqrt{\frac{2m}{3k}}\times 3v = \sqrt{\frac{6m}{k}}v$$

Impulse

The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval. The Impulse is a vector quantity.

For any arbitrary force. The impulse $\vec{J}$ is defined as

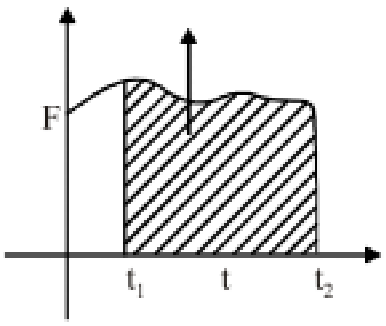
$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{ext} dt$$

$$\vec{F}_{ext} dt = d\vec{P}_{sys}$$

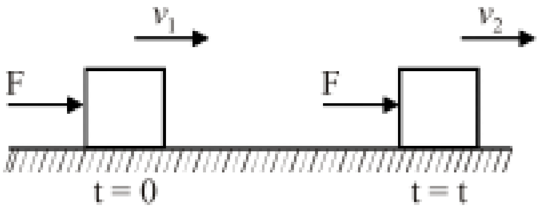
$$\vec{J} = \int_{P_i}^{P_f} \overrightarrow{dP}_{sys}$$

$$\vec{J} = \vec{P}_f - \vec{P}_i$$

ared under the curve is impulse



The concept of impulse can be better explained by an example shown in figure



A is a block of mass m moving with a velocity v_1, at time t = 0, a constant force F is applied on it in the direction of velocity for a time t. Due to this force the velocity of the body increases hence momentum increase. If after time t the velocity of the body becomes v_2, then according to momentum conservation we have

Initial momentum + momentum imparted = final momentum

$$mv_1 + F\,t = mv_2 \qquad \text{.......(i)}$$

If applied force is opposite to the direction of v_1 then we'll have

$$mv_1 - F\,t = mv_2 \qquad \text{.......(ii)}$$

Equation (i) and (ii) are similar to the equations written for work - energy theorem as work done by the system or on the system are subtracted or added to the initial kinetic energy, gives the final kinetic energy of the system. Similar to that in initial momentum impulse due the forces acting on the system are added or subtracted, gives the final momentum of the system. If force is in the direction of the initial velocity of the particle, impulse is added to the initial momentum and if it is against the velocity, impulse is subtracted from the initial momentum

Impulsive Force

When a force of high magnitude acts for a time that is short compared with the time of observation of the system, it is referred as an impulsive force. An impulsive force can change the momentum of a body by a finite magnitude in a very short time interval.

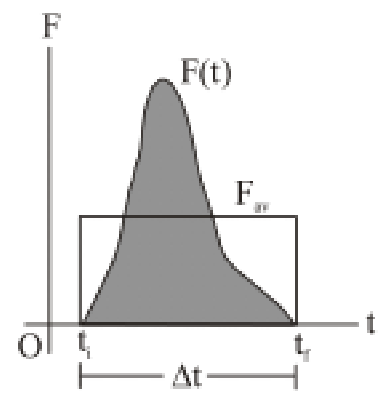
An impulsive force can only be balanced by another impulsive force.

We defined the impulse in terms of a single force, but the impulse-momentum theorem deals with the change in momentum due to the impulse of the net force –the is, the combined effect of all the forces that act on the particle. In the case of a collision involving two particles, there is often no distinction because each particle is acted upon by only one force, which is due to the other particle. In this case, the change in momentum of one particle is equal to the impulse of the force exerted by the other particle.

Average Force

The impulsive force whose magnitude is represented in figure is assumed to have a constant direction. The magnitude of the impulse of this force is represented by the area under the F(t) curve. We can represent that same area by the rectangle in figure of width Δt and height F_{av} where F_{av} is the magnitude of the average force that acts during the interval Δt. Thus

$$J = F_{av}\,\Delta t.$$



Impulsive force is a relative term. There is no clear differentiation between an impulsive and non-impulsive force.

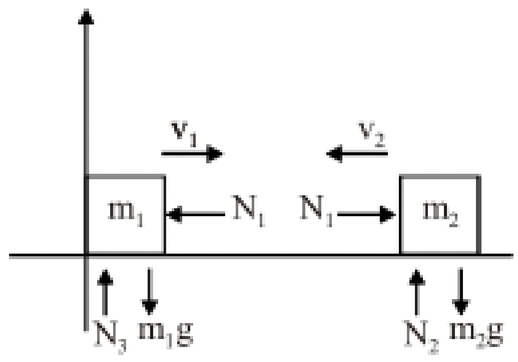
1. Gravitational force and spring force are always non-impulsive.
2. Normal, tension and friction are case dependent.

1. Impulsive Normal :

In case of collision, normal forces at the surface of collision are always impulsive.

eg. (i) N_1 is Impulsive ;

Normal reaction due to ground N_2 & N_3 are Non-impulsive

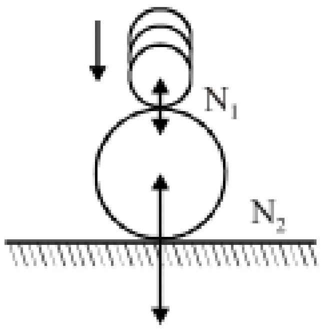


(a)

(ii) Consider a ball dropped on a large ball.

Both normal forces N_1 and N_2 are impulsive

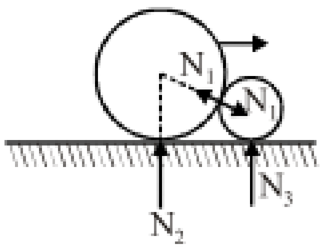
N_2 is impulsive, as it balance N_1 for the large ball.



(a)

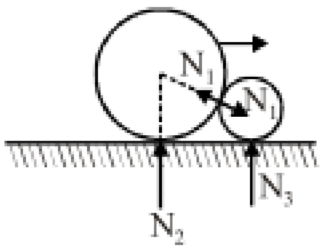
(iii) Consider a large ball colliding with small ball N_1, N_3 are impulsive ; N_2 is non-impulsive

N_1 can be easily seen to be impulsive, as it is the normal force during collision here N_3 balance a component of N_1, therefore it is also impulsive.

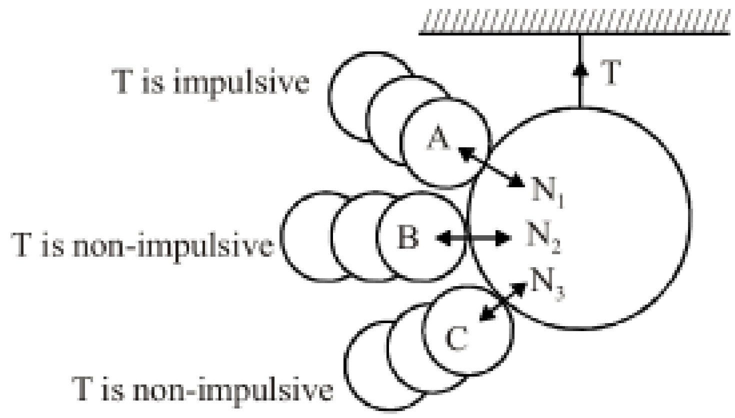


(b)

2. **Impulsive Friction :** If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.



3. **Impulsive Tensions In a string :** When a string is jerked out equal and opposite tension acts suddenly at each end and impulses act on the bodies attached with the string in the direction of the string.



All normal are impulsive but tension
T is impulsive only for the ball A

One end of the string is fixed : The impulsive force which acts at the fixed end of the string cannot change the momentum of the fixed object attached at the other end. The object attached to the free end however will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string. In this direction string cannot exert impulsive forces.

Both ends of the string attached to movable objects : In this case equal and opposite impulses act on the two object, producing equal and opposite changes in momentum. The total momentum of the system therefore remains constant, although the momentum of each individual object is changed in the direction of the string. Perpendicular to the string however, no impulse acts and the momentum of each particle in this direction is unchanged.

In case of rod : Tension is always impulsive.

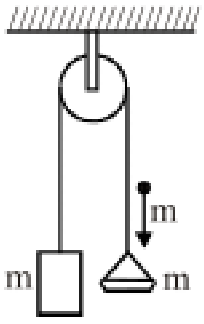
In case of spring : Tensions always non-impulsive.

Illustration :

A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed u, find the speed with which the system moves just after the collision.

Sol. *Let the required speed be V.*

As there is sudden change in the speed of the block, the tension must change by a large amount during the collision.



Let N = magnitude of the contact force between the particle and the pan

T = tension in the string

Consider the impulse imparted to the particle. The force is N is in upward direction and the impulse is $\int N\ dt$. *This should be equal to the change in its momentum.*

Thus, $\int N\ dt = mu - mV.$ ...(i)

Similarly considering the impulse imparted to the pan,

$$\int (N - T)\, dt = mV \quad ....(ii)$$

and that to the block,

$$\int T\, dt = mV \quad ....(iii)$$

Adding (ii) and (iii),

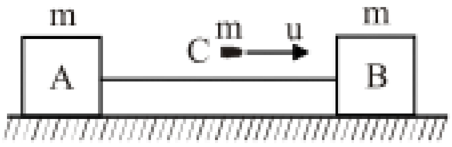
$$\int N\, dt = 2mV$$

comparing with (i)

$$mu - mV = 2\ mV \qquad or \qquad V = u/3.$$

Illustration :

Two identical block A and B connected by a massless string are placed on a frictionless horizontal plane. A bullet having same mass, moving with speed u strikes block B from behind as shown. If the bullet gets embedded into the block B then find :



(a) The velocity of A,B,C after collision.

(b) Impulse on A due to tension in the string

(c) Impulse on C due to normal force of collision.

(d) Impulse on B due to normal force of collision.

Sol. (a) After collision, the blocks & the bullet will move together with same velocity, say v
By conservation of linear momentum $mu = 3mv$

$$v = \frac{u}{3}$$

(b) Net impulse on A is due to tension force;
Impulse on $A = P_f - P_i$

$$\int T\,dt = \frac{mu}{3} - 0$$

(c) On the bullet C, net impulse is due to N

$$-\int N\,dt = P_f - P_i$$

$$= \frac{mu}{3} - mu$$

$$= \frac{-2mu}{3}$$

(d) On B two impulsive forces act i.e. Normal & Tension.

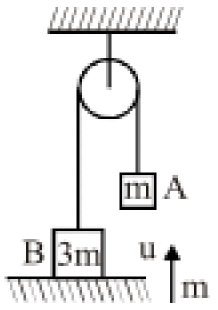
$$\vec{J} = \int (N - T)\,dt = \int N\,dt - \int T\,dt = \frac{mu}{3}$$

$$\Rightarrow \quad \int N\,dt = \frac{2mu}{3}$$

The impulse due to normal force on both the colliding bodies is equal. Thus we can directly say impulse on B due to normal is same as impulse on C due to normal.

Illustration :

Two blocks of masses m and 3m are connected by an inextensible string and the string passes over a fixed pulley which is massless and frictionless. A bullet of mass m moving with a velocity 'u' hits the hanging block of mass 'm' and gets embedded in it. Find the height through which block A rises after the collision.



Sol As soon as the collision occurs, the string becomes slack, tension becomes zero. Gravitational force is acting vertically downwards. But as gravitational force is a weaker force than impulse force, therefore a possible change of momentum due to gravitational force during collision can be neglected.

Hence, conserving momentum of the system during collision along vertical:
If v = velocity of the combined mass block A & bullet just after collision, then

$$mu = 2mv \quad or \quad v = \frac{u}{2}$$

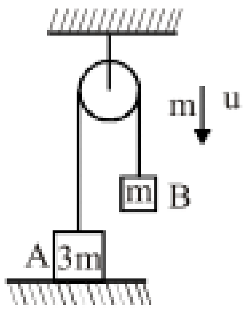
Block A & bullet start moving with velocity $\frac{u}{2}$

Now, maximum height through which the combined mass rises,

$$h_{max} = \frac{v^2}{2g} = \frac{u^2}{8g}$$

Illustration :

A system of two blocks A and B are connected by an inextensible massless strings as shown. The pulley in massless and frictionless. A bullet of mass 'm' moving with a velocity 'u' (as shown) hits the block 'B' and gets embedded into it. Find the impulse imparted by tension force to the system to block A



Sol. *Let velocities of B and A after collision have magnitude v.*

At the time of collision, tension is T and normal force between bullet and block B is F.

Impulse provided by tension = $\int T\ dt$

For bullet: considering downward direction to be positive

$$-\int F\ dt = mv - mu \qquad \ldots (i)$$

For block B:

$$\int (F - T)\ dt = mv \qquad \ldots (ii)$$

For block A

$$\int T\ dt = 3mv \qquad \ldots (iii)$$

Adding (i), (ii) & (iii)

$$mu = 5\ mv$$

or $\quad v = \dfrac{u}{5}$

Hence, impulse imparted by tension force, $\int T\ dt = 3mv$

$$= 3m\left(\frac{u}{5}\right) = \frac{3mu}{5}$$

Collisions

Introduction

Newton's laws are useful for solving a wide range of problems in dynamics. However, there is one class of problem in which, even though Newton's laws still apply as we have defined them, we may have insufficient knowledge of the forces to permit us to analyze the motion. These problems involve collisions between two or more objects

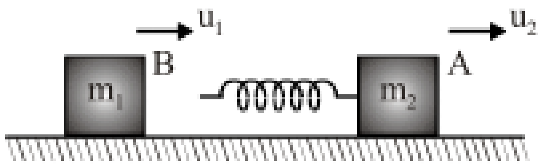
In this section we will learn how to analyze collisions between two objects. In doing so, we will find that we need a new dynamic variable apart from velocity, acceleration, force and energy, called linear momentum. We will see that the law of conservation of linear momentum, one of the fundamental conservation laws of physics, can be used to study the collisions of objects from the scale of subatomic particles to the scale of galaxies.

In a collision, two object exert forces on each other for an identifiable time interval, so that we can separate the motion into three parts : before, during and after the collision. Before and after the collision, we assume that the objects are far enough apart that they do not exert any force on each other. During the collision, the objects exert forces on each other; these forces are equal in magnitude and opposite in direction, according to Newton' third law. We assume that these forces are much larger than any forces exerted on the two objects by other objects in their environment. The motion of the objects (or at least one of them) changes rather abruptly during the collision, so that we can make a relatively clear separation of the situation before the collision from the situation after the collision.

When a bat strikes a baseball, for example, the bat is in contact with the ball for an interval that is quite short compared with the time during for which we are watching the ball. During the collision the bat exerts a large force on the ball. This force varies with time in a complex way that we can measure only with difficulty.

When an alpha particle (^{4}He nucleus) collides with another nucleus, the force exerted on each by the other may be the repulsive electrostatic force associated with the charges on the particles. The particles may not actually come into direct contact with each other, but we still may speak of this interaction as collision because a relatively strong force, acting for a time that is short compared with the time that the alpha particle is under observation, has a substantial effect on the motion of the alpha particle.

Collision



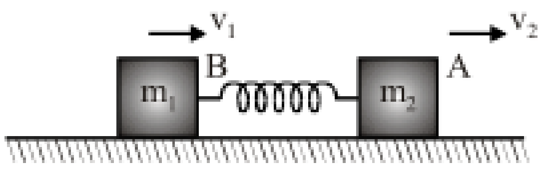
Consider the situation shown in figure . Two block of masses m_1 and m_2 are moving on the same straight line on a frictionless horizontal table. The block m_2, which is ahead of m_1 is going with a speed u_2 smaller than the speed u_1 of m_1 . A spring is attached to the rear end of m_2. Since $u_1 > u_2$, the block m_1 will touch the rear end the spring at some instant.

Since m_1 moves faster then m_2, the length of the spring will decrease. The spring will compress, it pushes back both the blocks with force kx. This force is in the direction of the velocity of m_2, hence m_2 will accelerate. However, this is opposite to the velocity of m_1 and so m_1 will decelerate. The velocity of the front block A (Which was slower initially will gradually increase, and the velocity of the rear block B (which was faster initially) will gradually decrease. The spring will continue to become more and more compressed as long as the rear block B is faster than the front block A. There will be an instant when the

two blocks will have equal velocities.
This corresponds to the maximum compression of the spring. Thus ***"the spring compression is maximum when the two blocks attain equal velocities"***
Now, the spring being already compressed, continues to push back the two blocks. Thus, the front block A will still be accelerated and the rear block B will still be decelerated. At the instant of maximum compression velocities were equal and hence after this the front block will move faster than the rear block. And so do the ends of the spring as they are in contact with the blocks. The spring will thus increase its length. This process will continue till the spring acquires its natural length. Once the spring regains its natural length, it stops exerting any force on the blocks. As the two blocks are moving with different velocities by this time, the rear one slower, the rear block will leave contact with the spring and the blocks will move with constant velocities. Their separation will go on increasing.

During the whole process, the momentum of the two - blocks system remains constant.
This is because there is no resultant external force acting on the system. Note that the spring being massless, exerts equal and opposite forces on the blocks.
Next consider the energy of the system. As there is no friction anywhere, the sum of the kinetic energy and the elastic potential energy remains constant. The elastic potential energy is $\frac{1}{2}kx^2$ when the spring is compressed by x. If v_1 and v_2 are the speeds at this time, we have,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}kx^2 = E$$



Compression in the spring is x

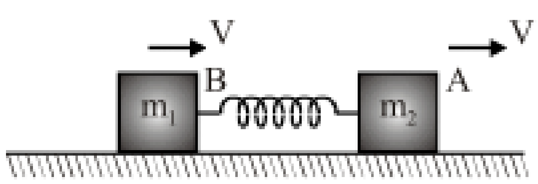
Where E is the total energy of the system.
Initially the spring is at its natural length so that,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = E \qquad \text{....(i)}$$

At the time when the compression of the spring is maximum.

$$v_1 = v_2 = v$$

$$\frac{1}{2}(m_1 + m_2)V^2 + \frac{1}{2}kx_{max}^2 = E$$



During maximum compression $v_1 = v_2 = V$

when the spring acquires its natural length, so that,

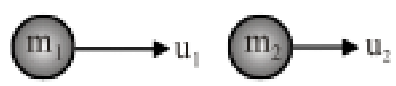
$$\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 = E \qquad \text{....(ii)}$$

From (i) and (ii),

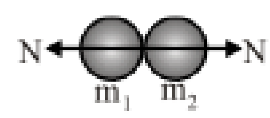
$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2$$

The kinetic energy before the collision is the same as the kinetic energy after the collision. However, we can not say that the kinetic energy remains constant because it changes as a function of time, during the process.

Now consider a very similar situation as shown figure. The two bodies of masses m_1 and m_2 are moving along a line with velocities u_1 and u_2 respectively, u_1 should be greater then u_2 for two bodies to collide.

Fig. I 

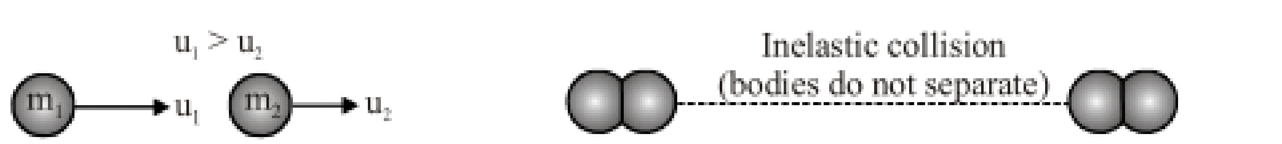
After some time the two bodies will come in contact as in the previous situation.

Fig. II 

As the velocity of m_1 (u_1) is greater than that of m_2 (u_2), there will be a deformation in the bodies (as we saw in the previous example of spring mass system). This deformation continues till they acquire same velocity. Now a question arises, that why will they acquire same velocity.

As shown figure II both m_1 and m_2 exert force on each other equal in magnitude (Newton's IIIrd Law) This force is in the direction of the velocity of m_2, hence m_2 will accelerate. However, this is opposite to the velocity of m_1 and so m_1 will decelerate. The velocity of the front body m_2 (which was slower initially) will gradually increase, and the velocity of the rear body m_1 (which was faster initially) will gradually decrease.

When they have attained a common velocity ; if bodies do not possess elastic properties, the two bodies will continue to move together and the process of collision ends here at maximum deformation. This is known as perfectly inelastic collision.



Just before collision | Just after collision | Long time after collision

As net external force is zero therefore by conservation momentum, we can find the common velocity.

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\,V$$

$$V = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

If bodies possess elastic property they will try to regain their shape that is the potential energy stored in deformation period will get converted into kinetic energy. If bodies are perfectly elastic then total potential energy will get converted into kinetic energy that is kinetic energy will be same before and after the collision, this type of collision in known as perfectly elastic collision.

Just before collision | Just after collision

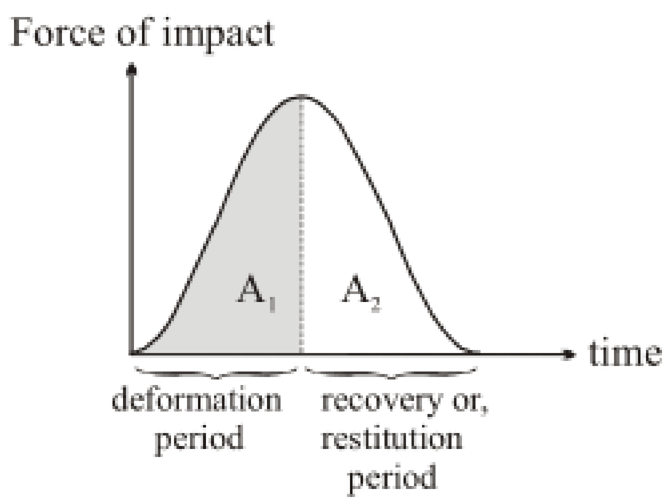
By conservation momentum.

$$m_1u_1 + m_2u_2 = m_1\,v_1 + m_2\,v_2$$

During the whole process, the momentum of the two -block system remains constant. Before and after collision kinetic energy may be same but not during the whole process.

Force of impact

The total time period of impact Δt is divided into period of deformation and period of restitution (recovery). Impulse of deformation and restitution (recovery) are finite and appreciable although the time interval of impact is extremely small.



Area A_1 represents impulse of deformation

Area A_2 represent impulse of reformation (restitution).

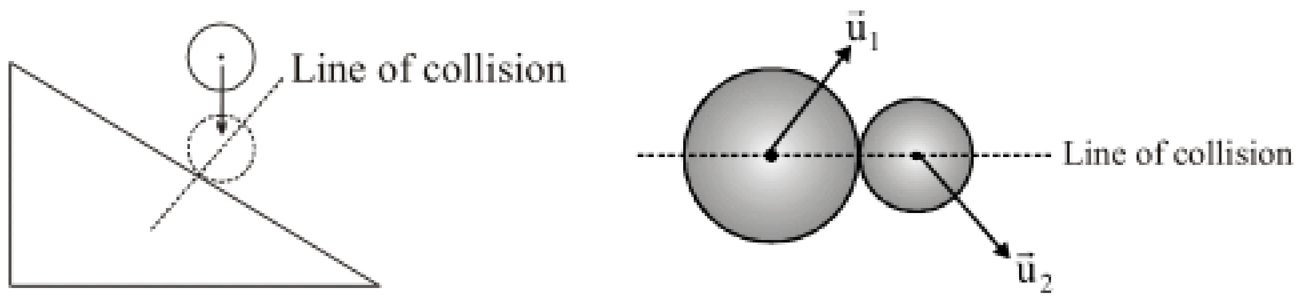
Deformation is maximum at the end of deformation period or at the start of recovery period.

In collision (impact) we consider the situation just before and just after collision. Just before collision is the moment when deformation period starts and just after collision is the moment when recovery period ends. At these moments one may say that force of impact becomes effectively zero.

Since time interval of impact is very small so impulse due to weight of body or weight dependent force is neglected during collision.

Line of collision (LOC)

When two bodies collide, they exert force on each other through point of contact, perpendicular to the plane of contact. The direction of force of interaction is line of collision. Line of collision is independent of the direction of velocities of colliding bodies.



After collision, only the components of velocity along line of collision changes, the perpendicular components of velocity remain unaffected.

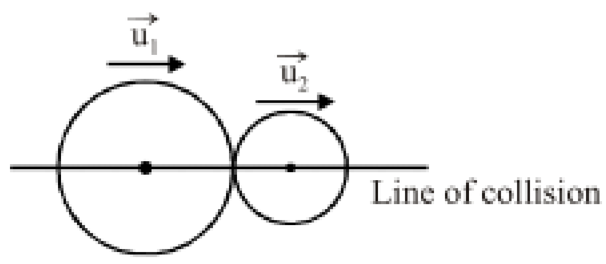
According to initial velocities and line of collision, collision is of two types :

(A) Head on

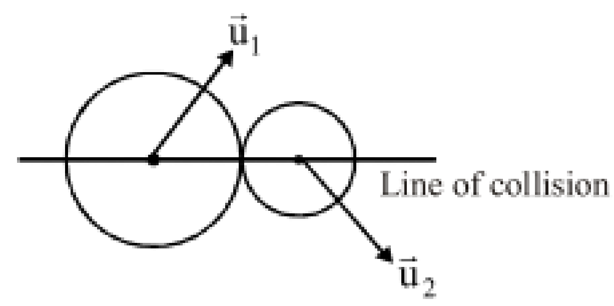
Line of collision is same as the line of motion (velocities) of the two colliding bodies.

(B) Oblique

When the lines of motion of two bodies are different, then the collision is oblique.



Head on collision



Oblique collision

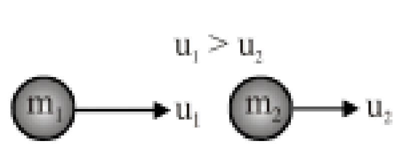
Laws of collision

(i) Conservation of momentum

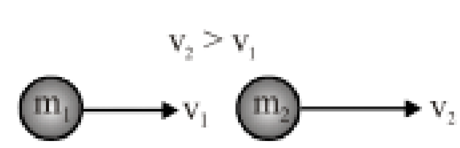
If no external impulsive forces act in a particular direction, the total momentum of system in that direction remains conserved.

Here we talk about external impulsive force because the force of impact is impulsive and changes the momentum of the individual bodies in a very short interval of time. The effect of non-impulsive forces such as gravity, spring force in such a small interval of time is negligible. Thus momentum remains conserved

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{u}_1 + m_1\vec{u}_1$$



Just before collision

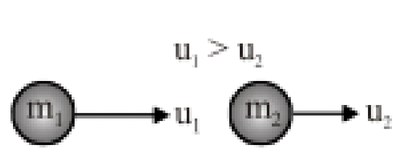


Just after collision

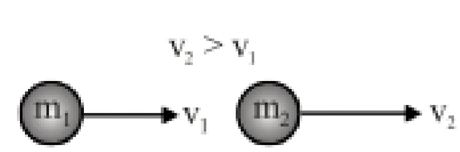
(ii) Newton's Experimental Law.

This is an experimental law which relates the velocity of approach of the bodies before collision and the velocity of separation after the impact.

When the collision is neither perfectly elastic nor perfectly inelastic, for such cases, Newton did certain experiments and gave another law which is now called as Newton's Experimental Law.



Just before collision



Just after collision

$$\frac{v_2 - v_1}{u_2 - u_1} = -e \text{ (coefficient of restitution) or } \frac{v_2 - v_1}{u_1 - u_2} = e$$

$v_2 - v_1$ is velocity of separation and $u_1 - u_2$ is velocity of approach.

Relative velocity of impact = e [Relative velocity of approach before impact]

v_1 and v_2 : Components of velocities of masses colliding, along the line of contact, after collision (with sign).

u_1 and u_2 : Components of velocities of colliding masses, along the line of contact, before collision (with sign)

Newton's Experimental Law can be stated **"velocity of separation is e times velocity of approach"**

Coefficient of Restitution is the property of colliding bodies which depends on their elastic behaviour.

Note : This Law is valid even when momentum is not conserved i.e. external impulsive forces act.

(iii) Type of collision depends on e

For e = 1, collision is perfectly elastic

for 0 < e < 1, collision is inelastic

for e = 0, collision is perfectly inelastic (Bodies will move together)

Illustration:

A ball of mass 4 kg moving with a velocity of 12 m/s impinges directly on another ball of mass 8 kg moving with a velocity of 4 m/s in the same direction. Find their velocities after impact if e = 0.5.

Sol.

$u_1 = 12\ m/s \qquad m_1 = 4\ kg$

$u_2 = 4 m/s \qquad m_2\ 8\ kg$

Let v_1 *and* v_2 *be the velocity after impact.*

By conservation of momentum :

$$m_1v_1 + m_2v_2 = m_1u_1 + m_1u_2$$

$$\Rightarrow \quad 4v_1 + 8v_2 = 80 \qquad ....(i)$$

By Newton's experimental Law :

$$v_2 - v_1 = e\ (u_1 - u_2)$$

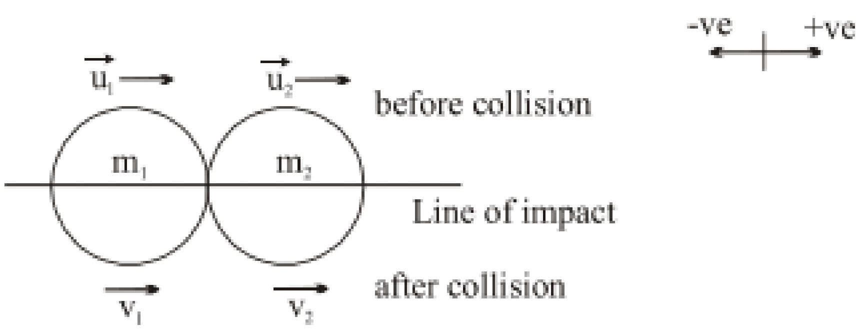
$$v_2 - v_1 = 0.5\ (12 - 4) = 4 \qquad ....(ii)$$

Solving (i) and (ii), we get :

$$v_1 = 4\ m/s \text{ and } v_2 = 8\ m/s$$

Head-on collision

Two bodies m_1 & m_2 are moving with velocities $\vec{u}_1$ and $\vec{u}_2$ along the same line as shown.



1. By conservation of momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad ... (i)$$

2. By Newton's experimental law

$$\frac{v_2 - v_1}{u_1 - u_2} = e \qquad \text{... (ii)}$$

From (i) & (ii)

$$v_1 = \frac{(m_1 - em_2)u_1}{m_1 + m_2} + \frac{(1+e)m_2 u_2}{m_1 + m_2}$$

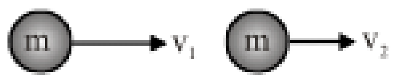
$$v_2 = \frac{(1+e)m_1 u_1}{m_1 + m_2} + \frac{(m_2 - em_1)u_2}{m_1 + m_2}$$

for perfectly elastic collision velocities can be obtained by substituting e = 1.

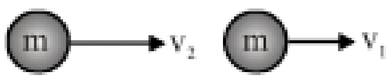
$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}u_1 + \frac{2m_2}{m_1 + m_2}u_2 \qquad \& \qquad v_2 = \frac{2m_1}{m_1 + m_2}u_1 + \frac{m_2 - m_1}{m_1 + m_2}u_2$$

Special cases:

Case 1: $m_1 = m_2$: In this case the velocities of particles are exchanged.



Just before collision



Just after collision

Case 2: $m_1 >> m_2$, $e = 1$: $v_1 \approx u_1$ & $v_2 = 2u_1 - u_2$

Loss in kinetic energy

The loss in kinetic energy is equal to initial kinetic energy (k_i) minus final kinetic energy (k_f)

$$\Delta KE = k_i - k_f$$

$$k_i = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$k_f = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Substituting the values of v_1 & v_2 we obtain

$$\Delta KE = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (u_1 - u_2)^2 (1 - e^2)$$

This result can also be obtained by using concept of C-frame.

For e = 1, i.e. perfectly elastic collision

$$\Delta KE = 0$$

As we discussed earlier there is no loss of kinetic energy

However, in elastic collision, KE is same before & after the collision,

i.e. $\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$

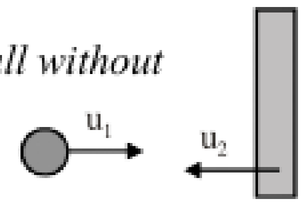
This equation is not independent, using Newton's experimental law & momentum conservation we can solve for final velocities more easily.
For $e=0$, i.e. perfectly in elastic collision

$$\Delta KE=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2$$

in this case, there is maximum loss in kinetic energy.

Illustration :

A particle of mass m moves with velocity $u_1 = 20$ m/s towards a wall that is moving with velocity $u_2 = 5$ m/s. If the particle collides with the wall without losing its energy, find the speed of the particle just after the collision.



Sol. *Velocity of approach*

$$u_{approch}=(u_1+u_2)$$

Let the velocity of the particle just after the collision be $\vec{v}_1$.

The velocity of separation,

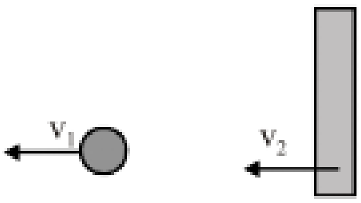
$$v_{seperation}=(v_1-v_2)$$

Since, according to Newton's experimental law

$$e=\frac{v_1-v_2}{u_1+u_2}$$

As there is no loss of energy, collision is elastic
Putting the value of e = 1, we obtain

$$\frac{v_1-v_2}{(u_1-u_2)}=1 \text{ or } u_1+u_2=v_1-v_2$$



or $v_1 = u_1 + u_2 + v_2$
Putting $u_1 = 20$ m/s, $u_2 = 5$ m/s and $v_2 = u_2$, since the wall being very heavy (infinite mass), moves with constant velocity, we obtain

$$v_1 = v_0 + 2v = 20 + 2(5) = 30 \text{ m/sec.}$$

Solving (i) & (ii), we obtain $v_1=\frac{12}{7}$ *m/s,* $v_2=\frac{26}{7}$ *m/s*

Illustration :

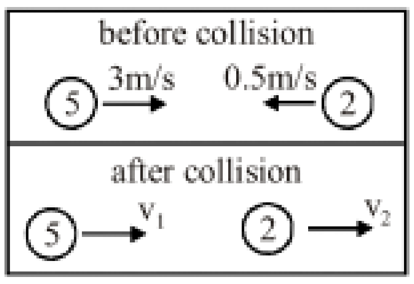
A ball of mass 5 kg moving velocity 3 m/s impinges direction on another ball of mass 2 kg moving with velocity 0.5 m/s towards the first ball. Find the velocity after impact, if $e=\frac{4}{7}$

Sol. *By conservation of momentum*

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$5\times3 + 2\,(-0.5) = 5v_1 + 2v_2$

$14 = 5v_1 + 2v_2$ ....(i)



By Newton's Law of collision $e = \dfrac{v_2 - v_1}{u_1 + u_2} = \dfrac{v_2 - v_1}{3 - 5}$

$v_2 - v_1 = \dfrac{4}{7} \times 3.5 = 2\ m/s$ ....(ii)

Practice Exercise

Q.1 Find the final velocities of the masses after in collision the given situations.

(i) 

$e = 1$

(ii) 

$e = 5$

(iii) 

$e = 5$

(iv) 

$e = 3/5$

(v) 

$e = 1$

Answers

Q.1 (i) $v_1 = 0$, $v_2 = 3$ (Rightwards)

(ii) $v_1 = .3$ (Rightwards), v_2 1.8(Rightwards)

(iii) $v_1 = 2.1$ (Rightwards), $v_2 = 2.6$(Rightwards)

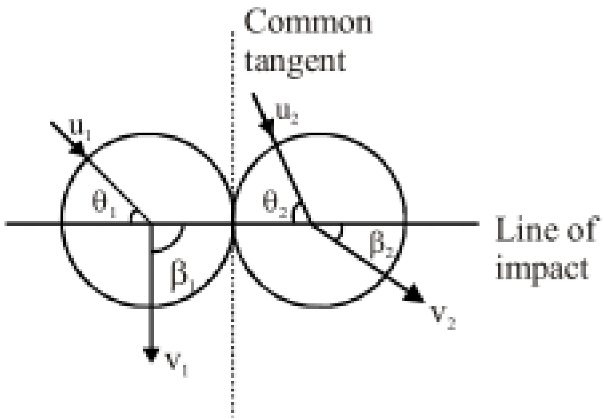
(iv) $v_1 = 1.8$ (Leftwards), $v_2 = 1.2$ (Rightwards)

(v) v_2 (Leftwards) , v_1 (Rightwards)

Oblique collision

In, oblique impact, the relative velocity of approach of the bodies doesn't coincide with the line of impact.

Conserving the momentum of the system along and perpendicular to the line of impact (due to absence of any other external impulsive force) we obtain



$$m_1 u_1 \cos\theta_1 + m_2 u_2 \cos\theta_2 = m_1 v_1 \cos\beta_1 + m_2 v_2 \cos\beta_2 \qquad \ldots(1)$$

and, $m_1 u_1 \sin\theta_1 + m2\ u2 \sin\theta_2 = m_1 v_1 \sin\beta_1 + m_2 v_2 \sin\beta_2$

Since no force is acting on m_1 and m_2 along the tangent, the individual momentum of m_1 and m_2 remains conserved.

$\Rightarrow \quad m_1 u_1 \sin\theta_1 = m_1 v_1 \sin\beta_1 \qquad \ldots(2)$

and $\quad m_2 u_2 \sin\theta_2 = m_2 v_2 \sin\beta_1 \qquad \ldots(3)$

Newton's experimental Law:

(We consider velocities along the line of collision)

$$e = \frac{v_2 \cos\beta_2 - v_1 \cos\beta_1}{u_1 \cos\theta_1 - u_2 \cos\theta_2} \qquad \ldots(4)$$

Now we have four equations and four unknown's v_1, v_2, β_1 and β_2. Solving four equations for four unknown we obtain.

$$v_1 \cos\beta_1 = \frac{(m_1 - em_2)u_1 \cos\theta_1 + m_2(1+e)u_2 \cos\theta_2}{m_1 + m_2} \qquad \ldots(5)$$

and $$v_2 \cos\beta_2 = \frac{m_1(1+e)\cos\theta_1 + (m_2 - em_2)u_2 \cos\theta_2}{m_1 + m_2} \qquad \ldots(6)$$

$\therefore \quad v_1 = \sqrt{(v_1 \sin\beta_1)^2 + (v_1 \cos\beta_1)^2}$

and $\quad \tan\beta_1 = \dfrac{v_1 \sin\beta_1}{v_1 \cos\beta_1}$

$\Rightarrow \quad \beta_1 = \tan^{-1}\left(\dfrac{v_1 \sin\beta_1}{v_1 \cos\beta_1}\right)$

[Put $v_1 \sin\beta_1$ from (2) and $v_1 \cos\beta_1$ from (5)]

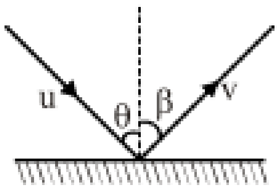
Similarly, find v_2 and β_2

$$\text{Impulse} = \frac{m_1 m_2}{m_1 + m_2}(1+e)(u_1 \cos\theta_1 - u_2 \cos\theta_2)$$

Energy loss $= \dfrac{m_1 m_2}{2(m_1 + m_2)}(1 - e^2)(u_1 \cos\theta_1 - u_2 \cos\theta_2)^2$

Oblique collision on a Fixed Plane

Let a small ball collides with a smooth horizontal floor with a speed u at an angle θ to the vertical as shown in the figure. Just after the collision, let the ball leaves the floor with a speed v at an angle β to vertical.



It is quite clear that the line of action is perpendicular to the floor. Therefore, the impact takes place along the (normal) vertical. Now we can use Newton's experimental law as

$$e = \frac{\text{velocity of seperation}}{\text{velocity of approach}}$$

$\Rightarrow$ $e[\text{velocity of approach}] = \text{velocity of seperation}$

$\Rightarrow$ $e(u\cos\theta)(-\hat{j}) = -(v\cos\beta)(+\hat{j})$

or $v\cos\beta = e\,u\cos\theta$ ...(i)

Since impulsive force N acts on the body along the normal, we cannot conserve its momentum. Since along horizontal the component of N is zero, therefore we can conserve the horizontal momentum of the body.

Velocity

$\Rightarrow$ (P_x) body = Constant

$\Rightarrow$ $(P_x)_{\text{initial}} = (P_x)_{\text{final}}$

$\Rightarrow$ $m\,u\sin\theta = m\,v\sin\beta$

$\Rightarrow$ $v\sin\beta = u\sin\theta$ ...(ii)

Squaring equations (i) and (ii) and adding,

$v^2\cos^2\beta + v^2\sin^2\beta = e^2u^2\cos^2\theta + u^2\sin^2\theta$

$\Rightarrow$ $v^2 = u^2[e^2\cos^2\theta + \sin^2\theta]$

$\Rightarrow$ $v = u\sqrt{\sin^2\theta + e^2\cos^2\theta}$

Dividing equation (i) by (ii)

$$= \frac{v\cos\beta}{v\sin\beta} = \frac{eu\cos\theta}{u\sin\theta}$$

$\Rightarrow$ $\cot\beta = e\cot\theta$

$\Rightarrow$ $= \cot^{-1}(e\cot\theta)$.

Impulse

Impulse of the blow = change of momentum of the body

$= \{mv\sin\beta\,\hat{i} + (mv\cos\beta)\}\,\hat{j}\} - (mu\sin\theta\,\hat{i} - mu\cos\theta\,\hat{j})$

$\Rightarrow$ Impulse $= m(v\sin\beta - u\sin\theta)\,\hat{i} + m(v\cos\beta + u\cos\theta)\,\hat{j}$

Since $v\sin\beta = u\sin\theta$

$\Rightarrow \quad \text{Impulse} = m\,(v\cos\beta + u\cos\theta)\,\hat{j}$

Putting $v\cos\beta = eu\cos\theta$ from eq (i), we obtain

$\text{Impulse} = m(1+e)u\cos\theta\,\hat{j}$

$\therefore$ Magnitude of the impulse $= m(1+e)\,u\cos\theta$ (in the direction of line of collision)

Change in kinetic energy

$$\Delta K.E. = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Putting the value of v we obtain

$$= \frac{1}{2}m\left[\left[u\sqrt{\sin^2\theta + e^2\sin^2\theta}\right]^2 - u^2\right]$$

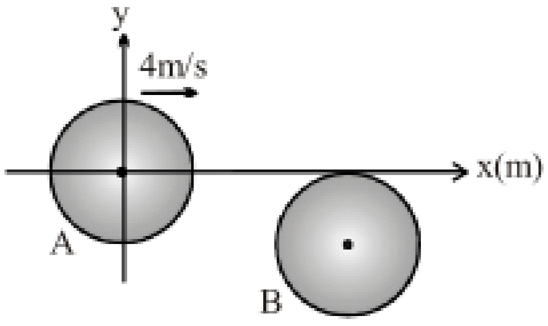
$$= \frac{1}{2}mu^2\,[\sin^2\theta + e^2\cos^2\theta - 1]$$

$$\Delta\,K.E. = \frac{1}{2}(1-e^2)\,mu^2\cos^2\theta$$

Negative sign indicates the loss of K.E. i.e. final K.E. is less than initial K.E.

Illustration :

Two smooth balls A and B each of mass m and radius R, have their centre at (0, 0) and at (5R, –R) respectively, in a coordinate system as shown. Ball A, moving along positive x-axis, collides with ball B. Just before the collision, speed of ball A is 4m/s and ball B is stationary. The collision between the balls is elastic.

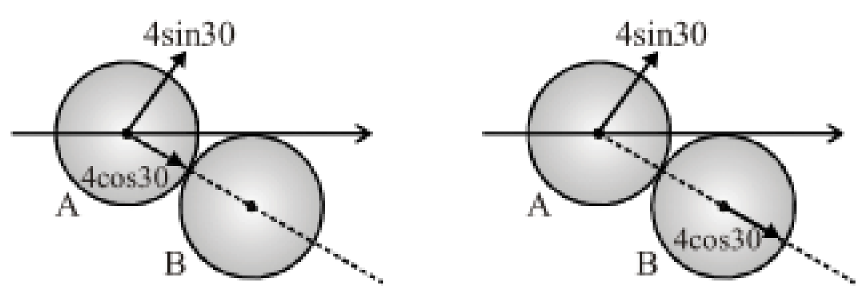


(a) *Find speed of the ball A just after the collision.*

(b) *Find impulse of the force exerted by A on B during the collision.*

Sol.

(a) As the collision is elastic & bodies are of equal mass, the velocity along the line of collision will interchange. As no force acts perpendicular to line of impact, therefore velocity of A perpendicular to the line of impact (4 sin 30) will remain same.



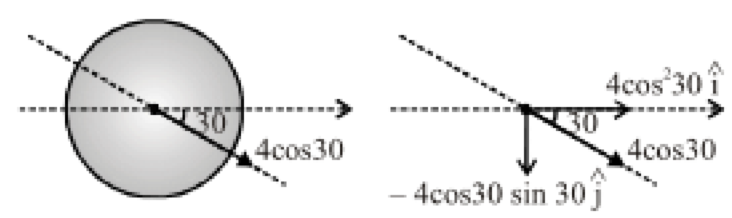
just before collision   just after collision

from the figure it is clear that velocity of A is 4 sin 30 and velocity of B is 4 cos 30 and both move at 90° to each other.

(b) $\vec{J}_{A\,on\,B} = m(\vec{v}_{Bf} - \vec{v}_{Bi})$

$m[4\cos 30^o(\cos 30^o\,\hat{i} - \sin 30^o\,\hat{j}] - 0]$

$(3m\hat{i} - \sqrt{3}m\,\hat{j})$ kg – m/s



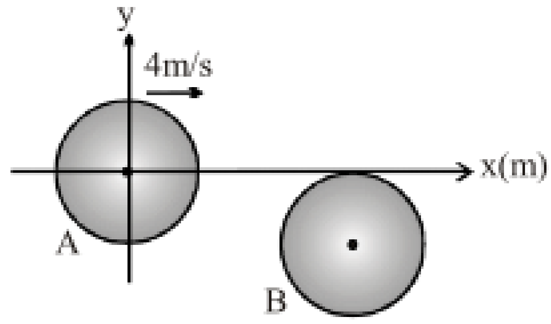
Final velocity of B (v_{bf})

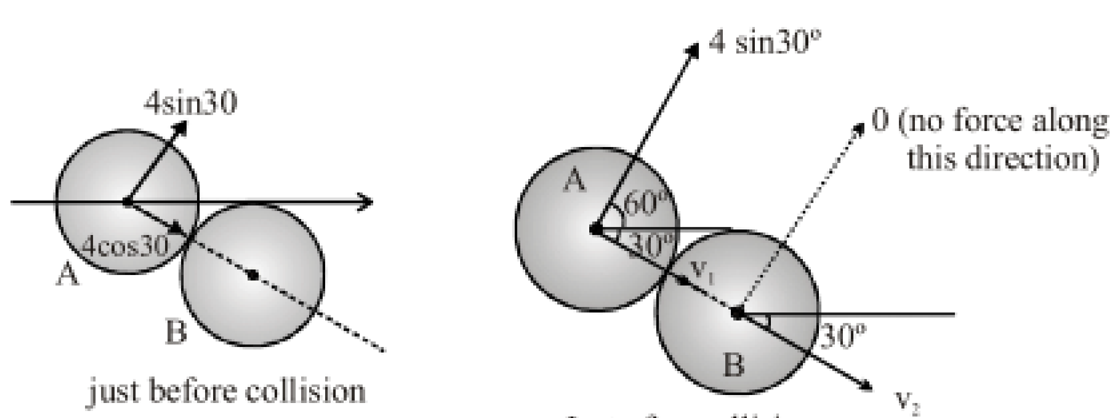
Impulse of collision is always along the line of collision.

Illustration :

In the previous illustration if coefficient of restitution during the collision is changed to 1/2, keeping all other parameters unchanged. What is the velocity of the ball B after the collision ?

Sol.





just before collision   Just after collision

By Newton's experimental law

$$\frac{1}{2} = \frac{(v_2 - v_1)}{(4\cos 30^o)}$$

$$v_2 - v_1 = \sqrt{3} \qquad \text{........(i)}$$

By conservation of momentum along line of collision

$$m\frac{4\sqrt{3}}{2} = mv_1 + mv_2$$

$$v_1 + v_2 = 2\sqrt{3} \qquad \text{........(ii)}$$

From equation (i) & (ii),

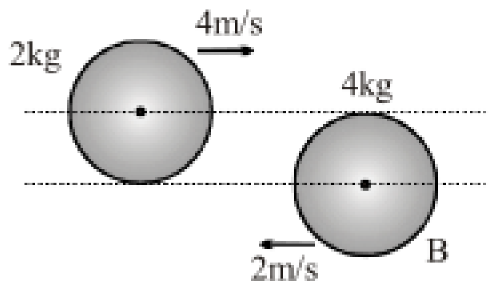
$$v_1 = \frac{\sqrt{3}}{2}\,m/s, \quad v_2 = \frac{3\sqrt{3}}{2}\,m/s$$

$$\vec{v}_2 = \frac{3\sqrt{3}}{2}[\cos 30^o\,\hat{i} + \sin 30^o(-\hat{j})]\ m/s$$

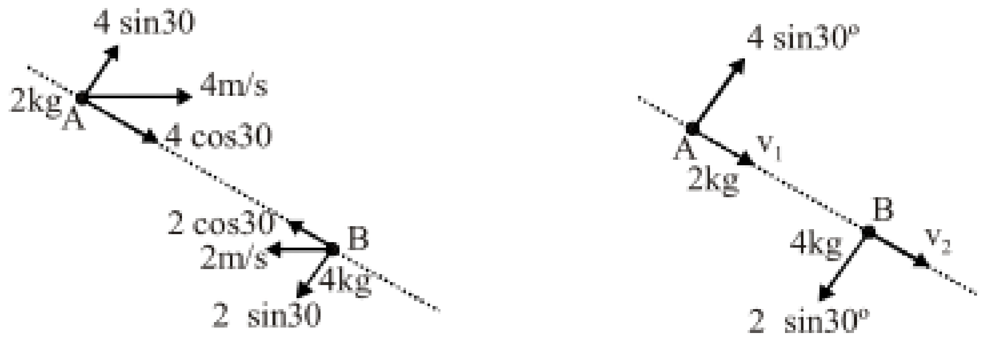
$$= \left(\frac{9}{4}\hat{i} - \frac{3\sqrt{3}}{4}\hat{j}\right)$$

Illustration :

Two spheres are moving towards each other. Both have same radius but their masses are 2kg and 4kg. If the velocities are 4 m/s and 2m/s respectively and coefficient of restitution is e = 1/3, Find final velocities along line of impact.



Sol. *Let v_1 and v_2 be the final velocities of A and B respectively then by conservation of momentum along line of impact.*



Before collision

Just After collision

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2 v_2$$

$$2\,(4\cos 30^o) - 4\,(2\cos 30^o) = 2(v_1) + 4\,(v_2)$$

or $$0 = v_1 + 2v_2 \quad ....(1)$$

By Newton's Experimental Law,

$$e = \frac{\text{velocity of separation along LOC}}{\text{velocity of approach along LOC}}$$

or $$\frac{1}{3} = \frac{v_2 - v_1}{4\cos 30^o + 2\cos 30^o}$$

or $$v_2 - v_1 = \sqrt{3} \quad ....(2)$$

From the above two equations,

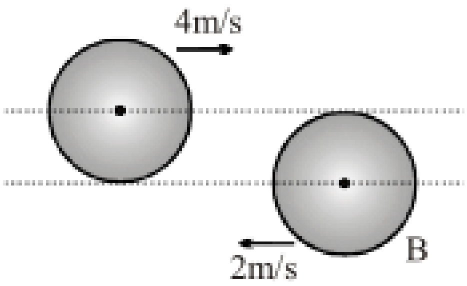
$$v_1 = -\frac{2}{\sqrt{3}}\,m/s$$

Negative answer denotes that we have chosen the wrong direction, actual direction of final velocity will be opposite to the direction we assumed in figure

$$v_2 = \frac{1}{\sqrt{3}}\,m/s$$

Practice Exercise

Q.1 Two spheres are moving towards each other. Both have same radius but their masses are 4kg and 2kg. If the velocities are 4 m/s and 2m/s respectively and coefficient of restitution is e = 1. Find, final velocity along line of impact.



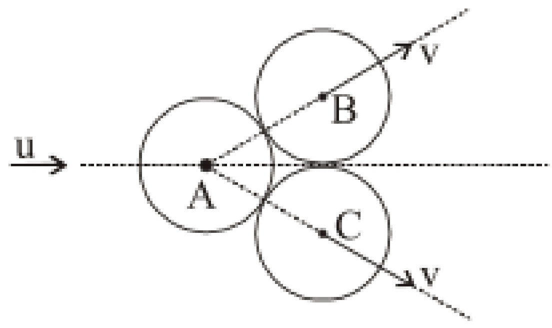
Q.2 Two spheres are moving towards each other. Both have same radius but their masses are 4kg and 4kg. If the velocities are 4 m/s and 2m/s respectively and coefficient of restitution is e = 1, find. Final velocity along line of impact.

Q.3 A ball of mass m moving with a speed u_1 collides elasticity with another identical ball moving with velocity u_2.

(a) Find the velocities of the balls after collision if the impact is direct.

(b) Find the angle between velocities after collision if they collide obliquely and $u_2 = 0$.

Q.4 Two equal sphere of mass m are in contact on a smooth horizontal table. A third identical sphere impinges symmetrically on them and is reduced to rest. Find e and the loss of KE.



Answers

Q.1 $v_1 = 0, \quad v_2 = 3\sqrt{3}\ m/s$

Q.2 $v_1 = -\sqrt{3}\ m/s, \quad v_2 = 2\sqrt{3}\ m/s$

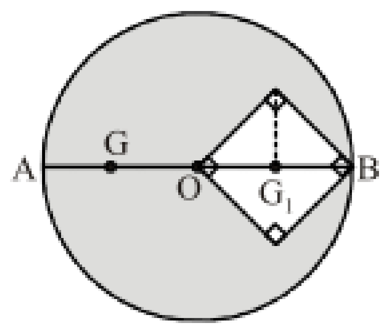
Q.3 (a) $v_1 = u_2$, $v_2 = u_1$; (b) $\frac{\pi}{2}$

Q.4 $\frac{2}{3}, \frac{mu^2 6}{6}$

Solved Examples

Q.1 A square hole is punched out of a circular lamina, the diagonal of the square being the radius of the circle. If 'a' be the diameter of the circle, find the distance of centre of mass of the remainder from the centre of the circle.

Sol. Consider the figure shown below. Let AB be the diameter passing through the diagonal OB of the square portion where O is the centre of the circle.



As mass is proportional to area of a uniform laninar body,
Mass of the portions can be replaced by their respective areas at their centre of mass

Area of circular potion $= \dfrac{\pi a^2}{4}$

Area of square portion $= \dfrac{a^2}{8}$

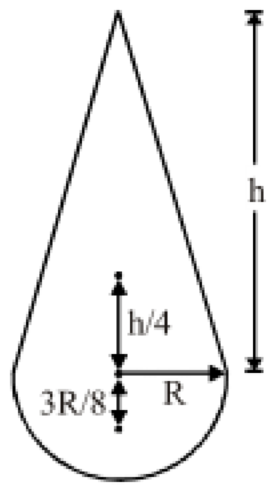
If G_1 and G the positions of centre of mass of the cut square portion and remaining portion.

Then $$OG = \frac{-\dfrac{\pi a^2}{4}(0) - \dfrac{a^2}{8}\left(\dfrac{a}{4}\right)}{\dfrac{\pi a^2}{4} - \dfrac{a^2}{8}} = \frac{\dfrac{a}{32}}{\left(\dfrac{2\pi - 1}{8}\right)}$$

$$= \frac{a}{4(2\pi - 1)}$$

$\therefore$ The centre of mass of the remaining parting is at a distance of $\dfrac{a}{4(2\pi - 1)}$ from the centre.

Q.2 Find out the centre of mass of a composite object shown in figure. Object consists of a cone with its base joint with the base of a hemisphere. The dimensions of the object are shown in figure. Assume uniform density of the system.



Sol. The shown object is made up of joining a solid cone and a hemisphere. We already know the location of the centre of mass of a cone and that of a hemisphere. The masses of the two are in proportion of their volume. The masses of cone and hemisphere are

Mass of cone is $\quad m_1 = \rho \frac{1}{3} \pi R^2 h$

and that of hemisphere is $\quad m_2 = \rho \frac{2}{3} \pi R^3$

Now we apply the result of two body system to find the centre of mass of the composite body. Let l be the distance between the independent centre of mass of the bodies cone and hemisphere, then

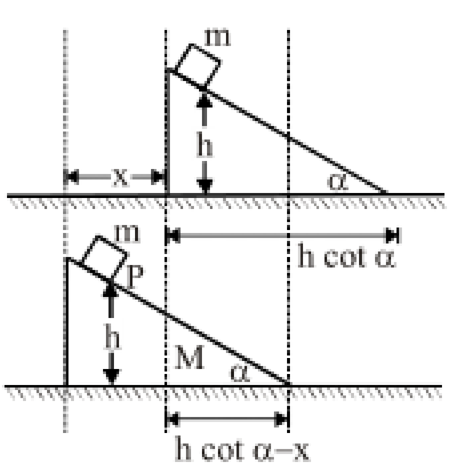
$$l = \frac{3R}{8} + \frac{h}{4}$$

The position of centre of mass from m_2 is

$$x = \frac{m_1 l}{m_1 + m_2} = \frac{\rho \frac{1}{3} \pi R^2 h \left(\frac{3R}{8} + \frac{h}{4} \right)}{\rho \frac{1}{3} \pi R^2 h + \rho \frac{2}{3} \pi R^3}$$

$$x = \frac{h(3R + 2h)}{8(h + 2R)}$$

Q.3 For the figure shown, block of mass m is released from the rest. Find the distance of the wedge from initial position, when block m arrives at the bottom of the wedge. All surfaces are frictionless.



Sol. As there is no net external force in the x direction, thus the momentum of system in x direction $(P_{sys})_x$ is conserved.

$Mv_1 = mv_2$, Initially $(P_{sys})_x = 0$

$\therefore$ Displacement of centre of mass in x-direction = 0

i.e. $\quad Mx_1 = mx_2 \quad$ ....(i)

Let the displacement of wedge M be x backwards

$\therefore \quad$ displacement of block $= h \cot \alpha - x$

Using equation (i)

$$m (h \cot\alpha - x) = Mx$$

$$x = \frac{m}{m + M} h \cot \alpha$$

Q.4 Two bodies A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a light spring of stiffness k. A third body C of mass m moves with velocity v_0 along the line joining A and B and collides elastically with A. If ℓ_0 be the natural length of the spring then find the minimum separation between the blocks.

Sol. Initially there will be collision between C and A which is elastic, therefore, by using conservation of momentum we obtain,

$$mv_0 = mv_A + mv_C \quad ; \quad v_0 = v_A + v_C$$

Since $e = 1$, $v_0 = v_A - v_C$

Solving the above two equation, $v_A = v_0$ and $v_C = 0$

Now A will move and compress the spring which in turn acceleration B and retard A and finally both A and B will move with same velocity v.

(a) Since net external force is zero, therefore momentum of the system (A and B) is conserved.

Hence $mv_0 = (m + 2m)v$

$\Rightarrow \quad v = v_0/3$

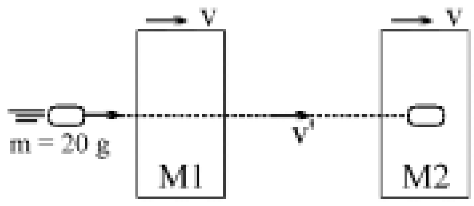
(b) If x_0 is the maximum compression, then using energy conservation

$$\frac{1}{2}mv_0^2 = \frac{1}{2}(m+2m)v^2 + \frac{1}{2}kx_0^2$$

$$\Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}(3m)\frac{v_0^2}{9} + \frac{1}{2}kx_0^2 \qquad \Rightarrow \quad x_0 = v_0\sqrt{\frac{2m}{3k}}$$

Hence minimum distance $D = \ell_0 - x_0 = \ell_0 - v_0\sqrt{\frac{2m}{3k}}$

Q.5 A 20 g bullet pierces through a plate of mass $M_1 = 1$ kg and then comes to rest inside a second plate of mass $M_2 = 2.98$ kg (refer figure). It is found that the two plates, initially at rest, now move with equal velocity v. Find the velocity of the bullet (in m/s) when it is between M_1 and M_2. Given that it entered M_1 with 100 m/s.



Sol. From the principle of conservation of linear momentum we have

$$mu = M_1v + mv'$$

and $$mv' = (m + M_2)v$$

or $$20u = 1000v + 20v'$$

and $$20v' = (20 + 2980)v$$

or $$u = 50v + v' \quad ...(i)$$

and $$v' = 150\ v \quad ...(ii)$$

From (i) and (ii), we get $\quad 3u = v' + 3v' = 4v' \quad$ or $\quad v' = 3u/4 = 75$

Q.6 A block of mass 4 kg is moving with a velocity of 7 m/s on a surface. It collides with another block of mass 3 kg elastically. The surface is smooth for 4 kg block but rough for 3 kg block ($\mu = 0.4$). Find the time (in sec) after which next collision will occur.

Sol. 

Let after collision their velocities are v_1 & v_2

applying conservation of momentum

$$4 \times 7 + 0 = 4v_1 + 3v_2$$

$$4v_1 + 3v_2 = 28 \qquad \text{... (1)}$$

also $\quad e = -\left(\dfrac{v_2 - v_1}{u_2 - u_1}\right)$

$$1 = -\left(\frac{v_2 - v_1}{0 - 7}\right)$$

$$v_2 - v_1 = 7 \qquad \text{....(2)}$$

solving (1) & (2) $\quad \Rightarrow \quad v_2 = 8 \text{ m/s}$

$\qquad v_1 = 1\text{m/s}$

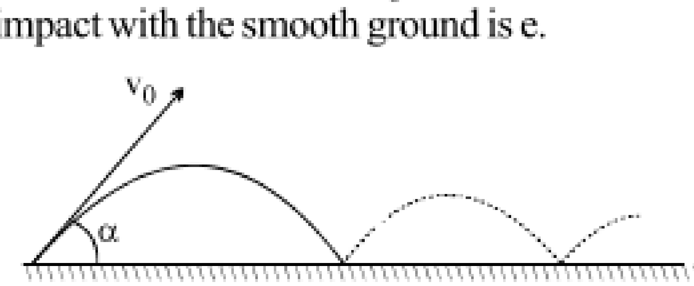
time after 2^{nd} block stops $= \dfrac{u}{a} = \dfrac{8}{4} = 2$ sec

distance travelled by 2^{nd} block till this moment $s = 8t + \dfrac{1}{2}\,at^2$

$$s = 8 \times 2 - \frac{1}{2}4 \times 2^2 = 8\text{m}$$

so time elasped till 2^{nd} collision $= \dfrac{8}{1} = 8$ sec.

Q.7 A particle of mass 'm' is projected with velocity v_0 at an angle 'α' with the horizontal. The coefficient of restitution for any of its impact with the smooth ground is e.



Find total time taken by the particle before it stops moving vertically ?

Sol. Total time taken by the particle to stop

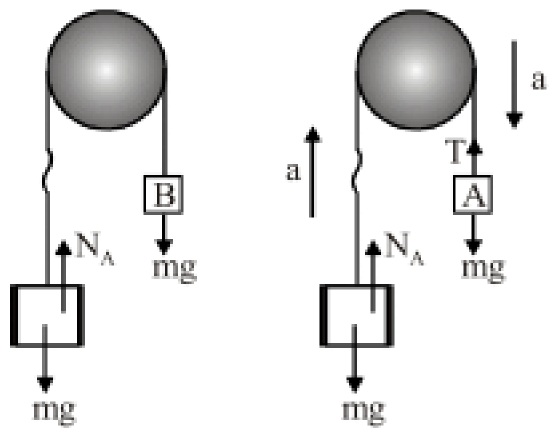
$$T = \frac{2v_0 \sin\alpha}{g} + \frac{2ev_0 \sin\alpha}{g} + \frac{2e^2v_0 \sin\alpha}{g} + \ldots$$

$$= \frac{2v_0 \sin\alpha}{g}(1 + e + e^2 + \ldots) \qquad = \frac{2v_0 \sin\alpha}{g(1-e)}$$

Q.8 After falling from rest through a height h a body of mass m begins to raise a body of mass M (M > m) connected to it through a pulley.

(a) Determine the time it will take for the body of mass M to return to its original position.

(b) Find the fraction of kinetic energy lost when the body of mass M is jerked into motion.



(a) Before the string tautens

(b) after the string tautens

Sol. (a) The speed of the body B just before the string becomes taut is $v = \sqrt{2gh}$. When the string is jerked, large impulsive reactions are generated in the string. At this moment effect of gravity is negligible. So momentum of the system is conserved at this instant. Let v' be the common speed of the two bodies after they are jerked into motion. From conservation of momentum, we have

$$mv = (M + m)\,v' \qquad \text{or} \qquad v' = \frac{m}{M+m}v$$

The acceleration of the system is

$$\Sigma F = Mg - mg = (M + m)\,a \qquad \text{or} \qquad a = -\frac{M-m}{M+m}g$$

The acceleration is negative, (opposite to v')

Let the system return to original position at time t.

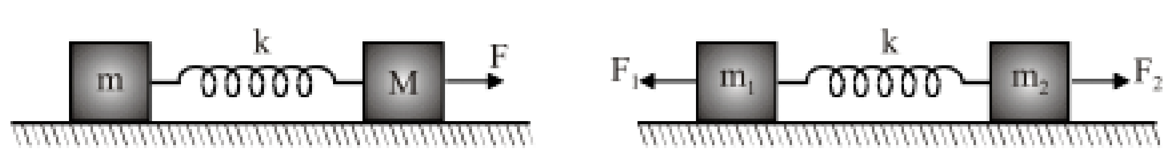
$$0 = v't + \frac{1}{2}at^2$$

$$\text{or} \qquad t = -\frac{2v'}{a} = \frac{2m}{M-m}\sqrt{\frac{2h}{g}}$$

(b) The fractional loss of kinetic energy is

$$\frac{\frac{1}{2}mv^2 - \frac{1}{2}(M+m)v'^2}{\frac{1}{2}mv^2} = \frac{M}{M+m}$$

Q.9 A block of mass m is connected to another block of mass M by massless spring constant k The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.



(a) (b)

Sol. Let us take the two blocks plus the spring as the system. The centre of mass of the system moves with an acceleration $a = \frac{F}{m+M}$. Let us work from a reference frame with its origin at the centre of mass. As this frame is acceleration with respect to the ground we have to apply a pseudo force ma towards left on the block of mass m and Ma towards left on the block of mass M. The net external force on m is

$$F_1 = ma = \frac{mF}{m+M} \text{ towards left}$$

and the net external force on M is

$$F_2 = F - Ma = F = \frac{MF}{m+M} = \frac{mF}{m+F} \text{ towards right}$$

The situation from this frame is shown in figure. As the centre of mass is at rest in this frame, the block move in opposite direction and come to instantaneous rest at some instant. The extension of the spring will be maximum of this instant. Suppose the left block through a distance x_2 from the initial positions. The total work done by the external forces F_1 and F_2 in this period are

$$W = F_1x_1 + F_2x_2 = \frac{mF}{m+F}(x_1 + x_2).$$

This should be equal to the increase in the potential energy of the spring as there is no change in the kinetic energy. Thus,

$$\frac{mF}{m+F}(x_1 + x_2) = \frac{1}{2}k(x_1 + x_2)^2$$

or, $$x_1 + x_2 = \frac{2mF}{k(m+M)}$$

This is the maximum extension of the spring.

Q.10 A glass ball collides with a smooth horizontal surface with a velocity $\vec{v} = a\hat{i} - b\hat{j}$. If the coefficient of restitution of collision be e, find the velocity of the ball just after the collision. (Take y-axis along vertical)

Sol. Collision takes place along the normal. Therefore the magnitude normal component (v_y) of the velocity of the glass ball is changed to $v_{y'} = e\,v_y$ just after the collision whereas the horizontal component (v_x) of its velocity remains constant due to the absence of any horizontal force.

$\Rightarrow$ The velocity of the ball just after the impact

$$= \vec{v} = \vec{v}_x + \vec{v}_y$$

$\Rightarrow$ $$\vec{v}' = v'_x\,\hat{i} + v'_y\,\hat{j}$$

where, $v'_x = a$ & $v'_y = eb$

$\Rightarrow$ $$\vec{v}' = a\,\hat{i} + eb\hat{j}$$

Therefore the magnitude of the velocity $\vec{v}' = |\vec{v}'| = \sqrt{a^2 + e^2b^2}$ and the direction is given as

$$\theta = \tan^{-1}\left(\frac{v_{x'}}{v_{y'}}\right) = \tan^{-1}\left(\frac{a}{eb}\right) \text{ to the normal (vertical)}$$

Q.11 A stationary body explodes into four identical fragments such that three of them fly off mutually perpendicular to each other each with same kinetic energy (E_0). Find the energy of explosion.

Sol. Let three of the fragments move along X,Y & Z axes. Therefore their velocities can be given as

$\vec{v}_1 = v\hat{i}, \vec{v}_2 = v\hat{j}$ & $\vec{v}_3 = v\hat{k}$ where v = speed of each of the three fragments. Let the velocity of the fourth fragment be $\vec{v}$. Since, in explosion no net external force is involved, the net momentum of the system remains conserved just before and after the explosion.

$\Rightarrow \quad (\vec{p})_f = (\vec{p})_i$

$\Rightarrow \quad m\vec{v}_1 + m\vec{v}_2 + m\vec{v}_3 + m\vec{v}_4 = 0$ ($P_i = 0$ because the body was stationary), putting the values of $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$, we obtain,

$$\vec{v}_4 = -v(\hat{i} + \hat{j} + \hat{k})$$

Therefore, $v_4 = \sqrt{3}v$

The energy of explosion (ΔKE) system

$\Rightarrow \quad E = KE_f - KE_i$

$$= \left(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2 + \frac{1}{2}mv_4^2\right) - (0)$$

Putting $v_1 = v_2 = v_3 = v, v_4 = \sqrt{3}v$

$$E = \left(\frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}m(\sqrt{3}v)^2\right) - (0)$$

$$E = 3mv^2 \qquad \text{.........(i)}$$

As we know from question that kinetic energy of three are equal and equal to E_0

$\therefore \quad \frac{1}{2}mv^2 = E_0$, Putting this value in equation (i)

we obtain, $E = 6E_0$.

Q.12 A man of mass m climbs a rope of length L suspended below a balloon of mass M. The balloon is stationary with respect to ground. If the man begins to climb up the rope at a speed v (relative to rope) in what direction and with speed (relative to ground) will the balloon move ?

Sol. Balloon is stationary

$\Rightarrow$ No net external force acts on it.

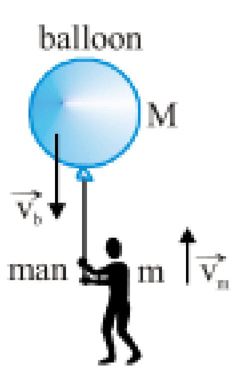
$\Rightarrow$ The conservation of linear momentum of the system (balloon + man) is valid

$\Rightarrow \quad M\vec{v}_b + m\vec{v}_{mb} = 0$

where $\vec{v}_m = \vec{v}_{mb} + \vec{v}_b$

$\Rightarrow \quad M\vec{v}_b + m[\vec{v}_{mb} + \vec{v}_b] = 0$

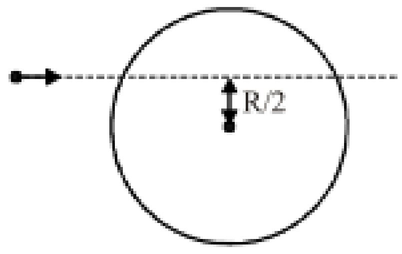
where v_{mb} = velocity of man relative to the balloon (rope)



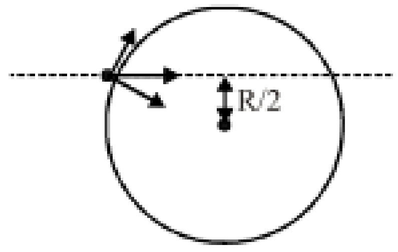
$\Rightarrow \quad \vec{v}_b = -\dfrac{m\vec{v}_{mb}}{M+m}$

Where $v_{mb} = v \Rightarrow v_b = \dfrac{mv}{M+m}$ and directed opposite to that of the man.

Q.13 A particle of mass m strikes elastically with a disc of radius R, with a velocity $\vec{v}$ as shown in the figure. If the mass of the disc is equal to that of the particle and the surface of the contact is smooth, find the velocity of the disc just after the collision.



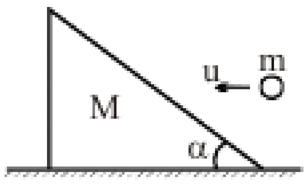
Sol. We see that impact takes place along the normal. Therefore, the particle and the disc change their momentum along that line. However, no external force acts on the system along the normal line. Hence we can conserve the linear momentum of the system (disc + particle) along the normal. Since the masses of the disc and particle are equal, so the exchange of momentum takes place along the normal. That means, the particle completely delivers the part (component) of its momentum (m v cos θ) along the normal



$\Rightarrow$ Velocity of the disc, $\vec{v}_1 = (v\cos\theta)\,\hat{j}$ where, $\cos\theta = \cos 30° = \dfrac{\sqrt{3}}{2}$

$\Rightarrow$ $\vec{v}_1 = \dfrac{\sqrt{3}v}{2}\,\hat{j}$

Q.14 A wedge of mass m rest on horizontal surface. The inclination of the wedge is α. A ball of mass m moving horizontal with speed u hits the inclined face of the wedge inelastically and after hitting slides up the inclined face of the wedge. Find the velocity of the wedge just after collision. Neglect any friction.



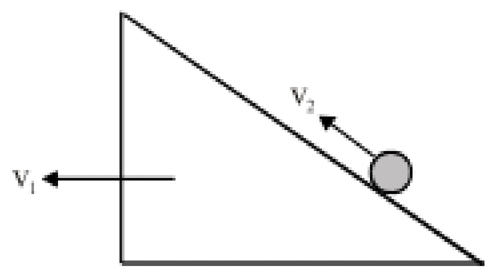
Sol. Let velocity of the ball after collision is $\vec{v}_2$ (w.r. to wedge) in directions as shown in the figure. Conserving momentum along horizontal, we get

$mu = m[v_2 \cos\alpha + v_1] + Mv_1$

$\Rightarrow mu = mv_2 \cos\alpha + (M + m)v_1$ ......(i)

Since common normal is along y', therefore momentum of ball remains constant along the incline (along x' $\because \vec{F}_{x'} = 0$)

$\Rightarrow \quad u\cos\alpha = v_2 + v_1 \cos\alpha$ .....(ii)



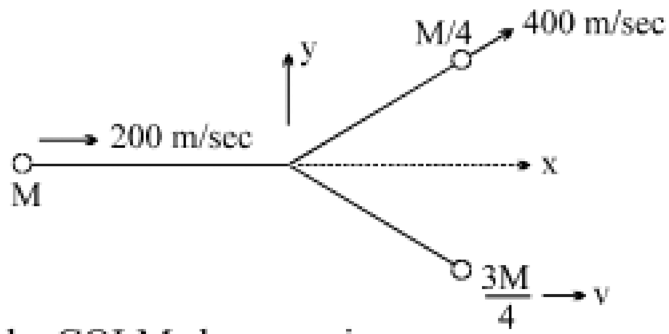
from equation (i) and (ii), we get

$mu = mu\cos^2 a - mv_1 \cos^2 a + (M+m)v_1$

$$\Rightarrow \quad v_1 = \frac{mu\sin^2\alpha}{M + m\sin^2\alpha}$$

Q.15 A missile of mass M moving with velocity v = 200 m/s explodes in midair breaking into two parts of mass M/4 & 3M/4. If the smaller piece flies off at an angle of 60° with respect to the original direction of motion with an initial speed of 400 m/s, what is the magnitude and direction of the initial velocity of the other piece.

Sol.



by COLM along x axis :

$$200M = 400 \times \frac{M}{4}\cos 60° + \frac{3M}{4} V\cos\theta$$

$$\frac{3}{4} v\cos\theta = 150 \qquad \Rightarrow v\cos\theta = 200 \qquad ... (i)$$

by COLM along y axis :

$$0 = 400\frac{M}{4}\sin 60° - \frac{3M}{4} V\sin\theta$$

$$3v\sin\theta = \frac{400\sqrt{3}}{2}$$

$$v\sin\theta = \frac{200}{\sqrt{3}} \qquad ... (ii)$$

(ii) / (i) gives

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = 30°$$

$$\Rightarrow \quad v = \frac{400}{\sqrt{3}} \text{ m/s}$$

Circular Motion

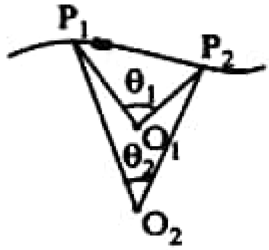
Introduction

A car rounds a curve. A staellite circles Earth. Electrons revolve around the nucleus. Since they are not in straight lines, their velocities are changing either with direction or magnitude or with both i.e they are accelerated. Newton's Laws tell us that force acts on each. Which is this force and how it does so, will be discussed in this chapter.

Angular Variables

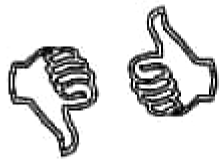
Angular displacement

Angle substended by a moving particle on a fixed point is called angular displacement about the fixed point. Thus in above discussion angular displacement about O_1 is θ_1 & about O_2 is θ_2. It is dimensionless quantity and its unit is radian. It should not be used in degree.

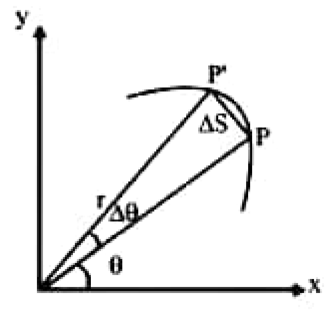


Angular displacement depends on reference frame , but angular displacement is different for different observers in the same frame). (The linear displacement is same for two observers at different positions in same frame) e.g. O_1 & O_2 will observe same linear displacement but different angular displacement although both points are in the same ground frame.

It is a scalar quantity. For small angles it canbe treated as a vector.
Although angular displacement is a scalar quantity, but if a body is rotating in a plane, then we treat it as a vector. Generally, anticlockwise is taken as positive and clockwise is taken as negative
Direction of angular displacement vector is decided by right hand rule.i.e. move your right hand fingers in sense of motion and direction of your thumb will be the direction of angular displacement.



Angular velocity



Rate of change of angular displacement is called angular velocity. Its unit is rad / sec.

Suppose a particle moving in circular path of radius "r" moves from P to P' so that its angular position changes from θ to (θ + Δθ) as shown.

$$\Rightarrow \quad \omega_{av} = \frac{\Delta\theta}{\Delta T}$$

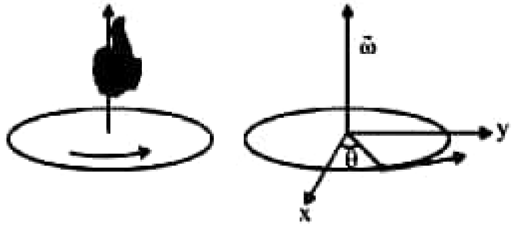
$$\Rightarrow \quad \omega = \lim_{\Delta t \to 0}\left(\frac{\Delta\theta}{\Delta t}\right) = \frac{d\theta}{dt}$$

It is the measure of how fast a line joining origin and particle is rotating. For example faster rotating fan means it has greater angular velocity.

Angular velocity vector

Angular speed ω is the magnitude of vector called the angular velocity ω of the particle. Direction of $\vec{\omega}$

can be determined from circular motion right hand rule. Curl your fingers of right hand in the sense of revolution of particle,then the extended thumb points in the dirction of $\vec{\omega}$.



Here the angle θ is measured from the x-axis.

Suppose a particle is completing " f" revolutions per second (called frequency). In each revolution, 2π radians are covered. So number of radians covered per second, $\omega = 2\pi f$ rad/s

Time period i.e. time taken to complete one revolution is $T = \frac{1}{f}$

$\Rightarrow \quad \omega = \frac{2\pi}{T}$

If particle revolutes with n r.p.m., $f = \frac{n}{60}$ cycles/sec $\quad \Rightarrow \omega = \frac{2\pi n}{60}$ rad/s

Relation between linear velocity (v) and angular velocity (ω)

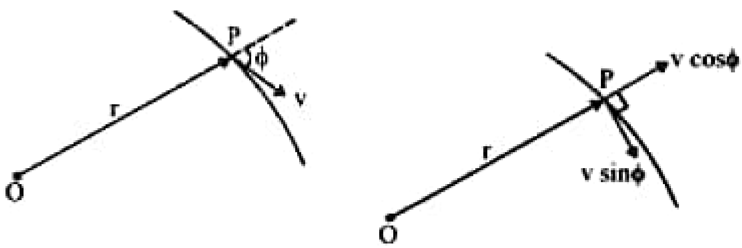
For a particle undergoing circular motion,

$$v = \lim_{\Delta t \to 0}\left(\frac{\Delta S}{\Delta t}\right) = r \lim_{\Delta t \to 0}\left(\frac{\Delta \theta}{\Delta t}\right)$$

$\Rightarrow \quad v = r\,\omega$

Thus different points of second's hand of a clock are rotating with same angular velocity but with different speeds and its tip has greatest speed.

For any curvilinear motion (like the motion of particle P as shown below)



If the particle has only velocity componant v cosϕ (along $\vec{r}$) If the particle has only velocity componant v cosϕ (along $\vec{r}$), the observer O need not turn his head to always look at the particle i.e. this componant does not contribute in angular motion. Thus only the componant v sinϕ is responsible for changing the angular displacment.

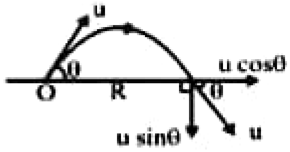
$\therefore \quad \omega = \frac{v \sin \phi}{r}$

In general, $\omega = \dfrac{\text{velocity componant perpendicular to the line joning the particle and the observer}}{\text{distance between the particle and observer}}$

$$= \frac{v_\perp}{r}$$

Illustration :

A particle is launched from horizontal plane with speed u and angle of projection θ. Find angnlar velocity as observed from the point of projection of the particle at the time of landing.

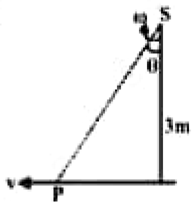


Sol. w.r.t. O, $\omega = \dfrac{u\sin\theta}{R}$

$$\omega = \frac{u\sin\theta}{\left(\dfrac{u^2\sin 2\theta}{g}\right)} = \frac{g}{2u\cos\theta}$$

Illustration :

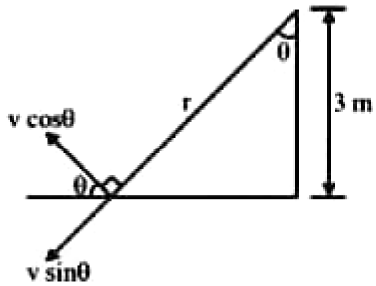
A spotlight S rotates in a horizontal plane with a constant angular velocity of 0.1 rad/sec. The spot of light P moves along the floor at a distance of 3m. Find the velocity of the spot P when θ = 45°



Sol. $\omega = \dfrac{v\cos\theta}{r}$

$$v = \frac{r\omega}{\cos\theta}$$

where $r = \dfrac{3}{\cos\theta}$



$\therefore \quad v = \dfrac{3\omega}{\cos^2\theta}$

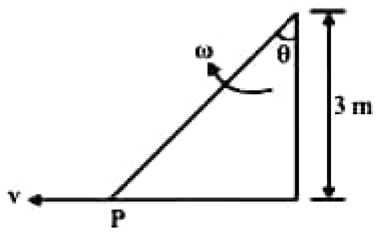
At the instant shown, θ = 45°

$\therefore \quad v = 0.6$ m/s

Alternatively

$x = 3\tan\theta$

Also $v_P = \dfrac{dx}{dt}$



$\therefore \quad v = 3\sec^2\theta \cdot \dfrac{d\theta}{dt}$

$= 3\,\omega\sec^2\theta$

$\therefore$ *at the instant shown*

$$v = 3 \times 0.1 \times \left(\sqrt{2}\right)^2 = 0.6 \text{ m/s}$$

Practice Exercise

Q.1 Find the linear speed of tip of second's hand of length 10 cm of a clock

Q.2 A particle is moving along a stright line $y = d$ with speed u. Find its angular speed w.r.t. origin at the instant it makes angle ϕ with y-axis as shown.

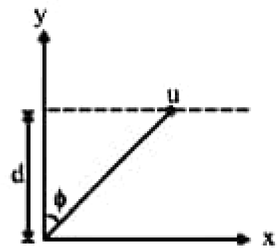


Illustration :

A particle is revoluting in a circular path of radius 0.5 m complating 1200 r.p.m (a) Find its linear speed (b) It now retards at the contant rate of 5π rad/s^2. Find the number of revolutions completed by it from the moment retardation begins till it stops.

Sol. (a) $\omega_0 = \dfrac{2\pi}{60} \times 1200 = 40\,\pi = 20\,\pi$ rad/s

$v = r\,\omega_0 = 0.5 \times 40\pi = 20$ m/s

(b) when it stops $\omega = 0$

$\therefore \quad \omega^2 = \omega_0^2 + 2\,\alpha\,(\Delta\theta) = 0$

$\therefore \quad (20\pi)^2 + 2\,(-5\pi)\Delta\theta = 0$

$\Rightarrow \quad \Delta\theta = 40\pi$ radians

Also number of revolutions is N then

$\therefore \quad N = \dfrac{\Delta\theta}{2\pi} = 20$

Unit vectors along Radial direction ($\hat{r}$) and tangential direction ($\hat{v}$)

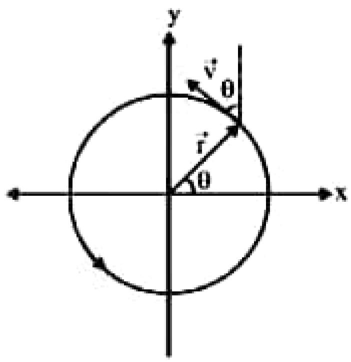
Suppose a particle moving in a cirular path of radius r in x y plane with origin as centre and makes angle θ with x-axis as shown, at any instant. Its position vector at this instant can be given by

$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$\Rightarrow \quad \hat{r} = \dfrac{\vec{r}}{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Also $\quad \vec{v} = -v\sin\theta\,\hat{i} + v\cos\theta\,\hat{j}$

$\therefore \quad \hat{v} = \dfrac{\vec{v}}{v} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$



Uniform circular motion

When a particle is moving with constant speed in a circular path, its motion is called unifrom circular motion. Although magnitude of velocity is constant but its direction changes continusly. It means it is continuonsly having some acceleration. This acceleration is always directing towards the centre. This is called radial acceleration (a_r) or centripetal acceleration (a_c).

As we have discussed in the previous topic

$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

and $\quad \vec{v} = -v\sin\theta\,\hat{i} + v\cos\,\hat{j} = v(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$$\vec{a} = \frac{d\vec{v}}{dt} = v\left(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}\right)\left(\frac{d\theta}{dt}\right) + v(\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{dv}{dt}$$

$|\vec{v}|$ is constant $\Rightarrow \quad \frac{d|\vec{v}|}{dt} = 0 \quad$ i.e. $\frac{dv}{dt} = 0$

Also $\quad \frac{d\theta}{dt} = \omega$

$\therefore \quad \vec{a} = -v\,\omega\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$\vec{a} = v\,\omega\,\hat{a}$

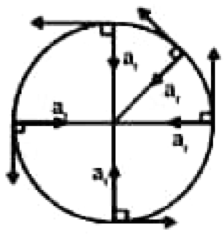
here $\quad \hat{a} = -(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = -\hat{r}$

Thus direction of this acceleration is opposite to $\hat{r}$ i.e. radially inwards.

Also $\quad |\vec{a}| = v\,\omega$

$\Rightarrow \quad a_r = v\,\omega$ or $\frac{v^2}{r}$ or $\omega^2 r \qquad [\because\ v = r\,\omega]$

Magnitude of this acceleration is constant but direction changes continuounsly, always being normal to velcoity as shown in the figure above.



Non-uniform circular motion

For a particle moving in non-uniform cicular motion both direction as well as magnitude of velocity change.

Now, $\quad \vec{v} = v\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$

$\therefore \quad \vec{a} = \frac{d\vec{v}}{dt} = -v(\sin\theta\,\hat{i} + \cos\,\hat{j})\cdot\frac{d\theta}{dt} + (-\sin\theta\,\hat{i} + \cos\,\hat{j})\cdot\frac{dv}{dt}$

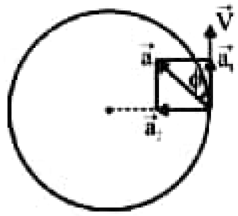
$$= -v\,\omega\,\hat{r} + \frac{dv}{dt}\hat{v}$$

we can write $\vec{a} = \vec{a}_r + \vec{a}_t$

where $\vec{a}_r = v\omega(-\hat{r})$ i.e. radial acceleration.

and $\vec{a}_t = \left(\frac{dv}{dt}\right)\hat{v}$ having unit vector equal to $\hat{v}$ i.e. it is also in tangential direction and is called tangential acceleration.

Thus in this case acceleration has two componants one along velocity i.e. tangential acceleration and another normal to velocity i.e. radial acceleration as shown.



If net acceleration ($\vec{a}$) makes angle ϕ with tangential direction, we may write

$$\tan\phi = \frac{a_r}{a_t}$$

also $\quad |\vec{a}| = \sqrt{a_r^2 + a_t^2}$

Some points to remember

- $$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\,\alpha$$

$\therefore \quad |\vec{a}| = \sqrt{(\omega^2 r)^2 + (r\alpha)^2}$

- Students need not to be confused with $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d}{dt}|\vec{v}|$. $\vec{v}$ contains both magnitude and driection. Thus

$\frac{d\vec{v}}{dt}$ means $\vec{a}_{net}$ i.e. $\left|\frac{d\vec{v}}{dt}\right| = |\vec{a}_{net}|$.

Also $|\vec{v}|$ means magnitude of velocity only which changes because of tangential acceleration.

$$\therefore \quad \frac{d|\vec{v}|}{dt} = a_t$$

Illustration :

A particle is revoluting a circular path of radius 0.2 m with angular velocity $\omega = 20\,t^2$ rad/s, where t is in seconds. Find its acceleration at t = 0.5 sec.

Sol. $\quad a_r = \omega^2 r = (20t^2)^2\,(0.2) = 80\,t^4$ m/s²

$\therefore$ at $\quad t = 0.5$ sec; $\quad a_r = 80\,(0.5)^4 = 5$ m/s²

Also $\quad \alpha = \frac{d\omega}{dt} = 40\,t$ rad/s²

$\Rightarrow \quad a_t = r\alpha = 8t$ m/s²

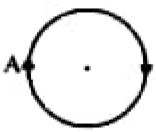
at $\quad t = 0.5$ sec; $\quad a_t = 8\,(0.5) = 4$ m/s²

$\therefore \quad |\vec{a}| = \sqrt{a_t^2 + a_r^2} = \sqrt{(4)^2 + (5)^2} = \sqrt{41}$ m/s²

$\therefore \quad a = 6.4$ m/s²

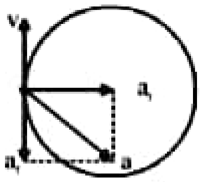
Illustration :

A particle is moving in circular path clockwise as shown with decreating speed. When it is at point A, the direction its acceleration may be given as



(A)  *(B)*  *(C)*  *(D)* 

Sol. *Since its speed is decreasing so its tangental acceleation is opposite to v. When it is is at A*



so (D) is correct

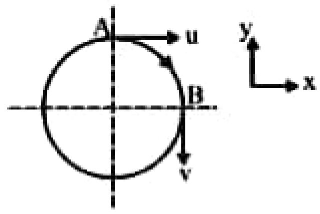
Illustration :

A particle is moving in a circular path of radius R with speed u, when it begins to speed up at a constant rate. After that when it completes one fourth revolution, change in its velocity vector has magnitude 2u. At that moment, find

(i) its radial acceleration

(ii) angle it makes with its velocity vector.

Sol. *(i) Suppose initially it was at point A with speed u and now it is at point B as shown with speed v.*



Now $\Delta\vec{v} = \vec{v}_f - \vec{v}_{in}$

$= (-v\,\hat{j}) - (u\,\hat{i})$

$\therefore \quad |\Delta\vec{v}| = \sqrt{v^2 + u^2}$

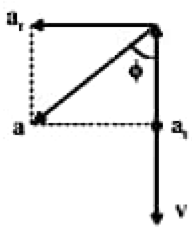
$\Rightarrow \quad \sqrt{v^2 + u^2} = 2u$

$\Rightarrow \quad v^2 = 3u^2$

$a_r = \dfrac{v^2}{R} \qquad \Rightarrow \qquad a_r = \dfrac{3u^2}{R}$

(ii) Also $\omega^2 = \omega_0^2 + 2\,\alpha\,(\Delta\theta)$

$\Rightarrow \quad \left(\dfrac{v}{R}\right)^2 = \left(\dfrac{u}{R}\right)^2 + 2\,\alpha\left(\dfrac{\pi}{2}\right)$



$\Rightarrow \quad \alpha = \dfrac{v^2 - u^2}{\pi R^2} = \dfrac{3u^2 - u^2}{\pi R^2} = \dfrac{2u^2}{\pi R^2}$

$\Rightarrow \quad a_t = R\alpha = \dfrac{2u^2}{\pi R}$

Now $\tan\phi = \dfrac{a_r}{a_t} \qquad \Rightarrow \qquad \phi = \tan^{-1}\left(\dfrac{3\pi}{2}\right)$

Radius of curvature

If a body is moving in any curvilinear path, then at different locations, the curvature would be different, thus the radius would be different.

For general curvilinear motion, when the particle crosses a point A, it is satisfying condition of moving on an imaginary circle. At this instant, if $a_{\perp} = \frac{V^2}{R_c}$ (where R_c is radius of curvature at this instant.)

$$R_c = \frac{V^2}{a_{\perp}}$$

$$R_c = \frac{(speed)^2}{comp.\ of\ acceleration\ perpendicular\ to\ velocity}$$

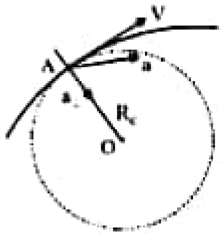
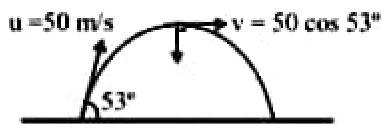


Illustration :

An object is projected with speed 50 m/s at an angle 53° with the horizontal from ground. Find radius of its trajectory (i) at the instant it is at heighest point (ii) at t = 1 sec. after projection.

Sol. *(i) At any instant acceleration of the projectile is 'g' downward. At the heighest point velcoity has magnitude = 50 cos 53° = 30 m/s and is in horizontal direction. Thus acceleration perpendicular to velocity is 'g' itself.*

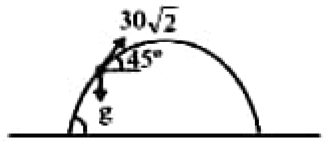


$$\therefore \quad a_r = \frac{v^2}{R} = g$$

$$\Rightarrow \quad R = \frac{v^2}{a_r} = \frac{(30)^2}{10} = 90m$$

(ii) at $t = 1sec$ $\quad V_x = 50 \cos 53° = 30\ m/s$

and $\quad V_y = 50 \sin 53° - g\,(1) = 30\ m/s$



$$\therefore \quad V = \sqrt{V_x^2 + V_y^2} = 30\sqrt{2}\ m/s$$

i.e $\vec{v}$ *is at angle 45° with the horizontal*

$$\therefore \quad a_{\perp} = \text{componant of g perpendicular to velocity} = \frac{g}{\sqrt{2}}$$

$$\therefore \quad \frac{(30\sqrt{2})^2}{R} = \frac{10}{\sqrt{2}}$$

$$\Rightarrow \quad R = 180\sqrt{2}\ m$$

Dynamics of circular motion

Here we deal with the forces which are responsible for keeping on object in circular path.

Centripetal force

Force or combination of forces which provide contrepetal acceleration necessary for the revolution of a particle is called centripetal force or radial force. For example

(a) For object tied to a string and is revolving on a smooth horizontal surface, tension is centripetal force.

(b) To revolve satellite around the Earth, gravitational force provides centripetal acceleration, so gravitational force is centripetal force.

(c) For motion of electron around nucleus, the electrostatic force on electron is centripetal force on it.

(d) For an object placed on a rough rotating table, friction on the object due to table is centripetal force.

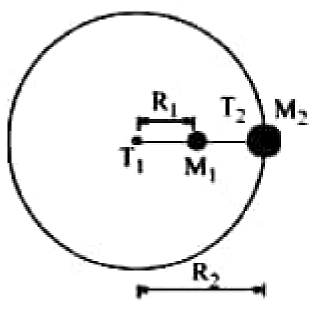
Problem solving strategy

1. Identify the plane of circular motion.
2. Locate the centre of ratation and calculate the radius.
3. Make F.B.D.
4. Resolve force along the radial direction and along the direction perpendicular to it.
5. The net force along radial direction is mass times the radial acceleration i.e. $m\left(\frac{V^2}{R}\right)$ or $m(\omega^2 R)$

Centripetal force $\left(\frac{mV^2}{R}\right)$ is no saperate force like Tension, Weight, Spring force, Normal reaction, Friction ect. In fact anyone of these or their combination may play a role of centripetal force.

Illustration :

*Two different masses are connected to two **light and inextensible** strings as shown in the figure. Both masses revolve about a central fixed point with constant angular speed of 10 rad s^{-1} on a smooth horizontal plane. Find the ratio of tensions $\frac{T_1}{T_2}$ in the strings.*



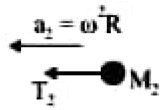
$M_1 = 0.25$ kg
$M_2 = 1.0$ kg
$R_1 = 5$cm
$R_2 = 10$cm

Sol. *Both the masses are moving in horizontal plane with same angular speed 10 rad/s. Here forces in radial directin can be tensions only.*

For M_2

$$F_{net} = T_2 = M_2 a_2$$

F.B.D. of M_2

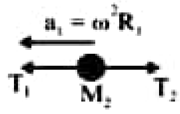


$$\Rightarrow \quad T_2 = M_2 \omega^2 R_2 \qquad ....(i)$$

For M_1

$$T_1 - T_2 = m_1 a_1$$

F.B.D. of M_1



$$T_1 = M_1 a_1 + T_2$$

$$T_1 = M_1 \omega^2 R_1 + M_2 \omega^2 R_2 \qquad \text{..........(ii)}$$

Dividing equation (ii) from (i), we get

$$\therefore \quad \frac{T_1}{T_2} = \frac{M_1R_1 + M_2R_2}{M_2R_2} = \frac{M_1}{M_2} \times \frac{R_1}{R_2} + 1$$

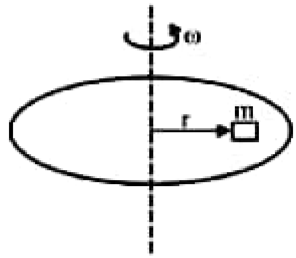
$$= \frac{0.25}{1} \times \frac{5}{10} + 1$$

$$\Rightarrow \quad \frac{T_1}{T_2} = \frac{9}{8}$$

Here centripetal force on M_2 is "T_2" and on M_1 is "$T_1 - T_2$".

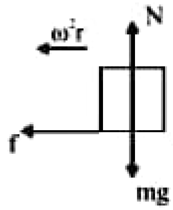
Illustration :

A rough horizontal table can rotate about its axes as shown. A small block is placed at r = 20 cm from its axis. The coefficient of friction between them is 0.5. Find the maximum angular speed that can be given to the block table system so that the block does not slip on the table.



Sol. *The force acting on the block are as shown below.*

N and mg are in vertical direction, so the only force that can provide necessary centrepetal acceleration in horizontal plane is friction. It makes the block to revolve with the table without slipping.



$$(F_{net})_y = 0 \Rightarrow N = mg$$

& $\quad (F_{net})_x = ma.$

$\Rightarrow \quad f = m\omega^2 r$

but $\quad f \le \mu N$

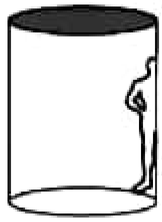
$$\Rightarrow \quad m\omega^2 r \le \mu mg \qquad \Rightarrow \qquad \omega \le \sqrt{\frac{\mu g}{r}}$$

$$\therefore \quad \omega_{max} = \sqrt{\frac{\mu g}{r}} \qquad = \sqrt{\frac{0.5 \times 10}{0.2}}$$

$$\Rightarrow \quad \omega_{max} = 5 \text{ rad/s}^2$$

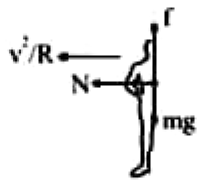
Illustration :

In a rotor, a hollow vertical cylindrical structure rotates about its axis and a person rests against the inner wall. At a particular speed of the rotor, the floor below the person is removed and the person hangs resting against the wall without any floor. If the radius of the rotor is 2m and the coefficient of static friction between the wall and the person is 0.2, find the minimum speed at which the floor may be removed. Take g = 10 m/s².



Sol. *Here friction (f) is upward and opposes tendency to move down. Also normal force is radially towards the centre to provide centripetal acceleration.*

$$(F_{net})_x = \frac{mV^2}{R}$$



$$\Rightarrow \quad N = \frac{mV^2}{R}$$

Also as man doesnot fall down,

$$f = mg \le \mu N$$

$$\Rightarrow \quad mg \le \mu \frac{mV^2}{R}$$

$$\Rightarrow \quad \sqrt{\frac{gR}{\mu}} \le V$$

$$\Rightarrow \quad V_{min} = \sqrt{\frac{gR}{\mu}} = \sqrt{\frac{10\times 2}{0.2}} = 10 m/s$$

Illustration :

A block of mass 25 kg rests on a horizontal floor (μ = 0.2). It is attached by a 5m long horizontal rope to a peg fixed on floor. The block is pushed along the ground with an initial velocity of 10 m/s so that it moves in a circle around the peg. Find

(a) *Tangential acceleration of the block*

(b) *Speed of the block at time t.*

(c) *Time when tension in rope becomes zero.*

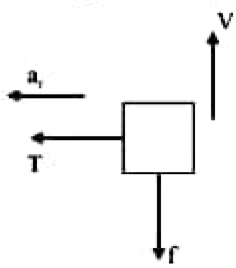
Sol. *The block is pushed on a horizontal stationary rough surface, the friction here is kinetic and is always opposite to velocity i.e. it is tangential force and is in horizontal plane.*

Also Normal force is in vertical direction to balance mg

i.e. N = mg = 250 N

$$\Rightarrow \quad f = \mu N = 50 N$$

F.B.D. of M_2
(Top view)



(a) $f = m a_t$

$$\Rightarrow \quad a_t = \frac{f}{m} = -\frac{50}{25} = -2 m/s^2$$

(Here –ve sign is taken to indicate that it is always opposite to velocity)

(b) $a_t = \frac{dV}{dt} = -2 \Rightarrow \int_{10}^{V} dV = -2\int_0^t dt$

$$\Rightarrow \quad V - 10 = -2t$$

$$\Rightarrow \quad V = 10 - 2t$$

(c) Tension $\quad T = \dfrac{mV^2}{R}$

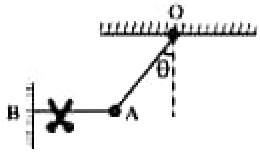
$\therefore \quad T = 0$ *when* $V = 0$

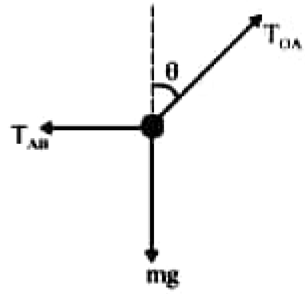
i.e. $10 - 2t = 0$

$\Rightarrow \quad t = 5$ *sec*

Illustration :

Find tension in OA before and just after AB is cut. The mass of the particle is m.





Sol. ***Before AB is cut****, the particule is in static equilibrium i.e.* $a = 0$.

Resolving in vertical and horizontal direction,

$$T_{OA} \cos \theta = mg$$

$$\Rightarrow \quad T_{OA} = \frac{mg}{\cos\theta}$$

After AB is cut, *the particles moves in a circular path around O, with radius equal to the length of the string OA.*

Now the acceleration cannot be directly taken zero, rather we have to consider acceleration in radial and tangential direction.

Resolving the force in radial and tangential direction.

$\therefore \quad T - mg \cos \theta = m\, a_r$

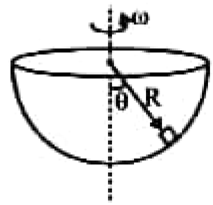
but just after the string AB is cut, speed of the particle is zero $\Rightarrow a_r = \dfrac{V^2}{\ell} = 0$

$\therefore \quad T - mg \cos\theta = 0$

$\Rightarrow \quad T = mg \cos \theta$

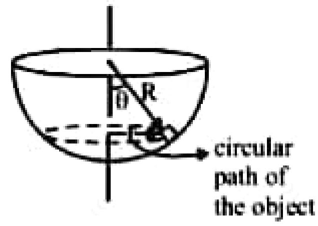
Illustration :

A hallow hemispherical bowl having radius of inner smooth surface $R = 80$ *cm is rotated with angular velocity* $\omega = 5$ *rad/s. A small object is placed at rest w.r.t. the bowl at position as shown. Find angle* θ.



Sol. *While the bowl is rotating, the plane of circular path of the particle is horizontal and its radius is r as shown.*

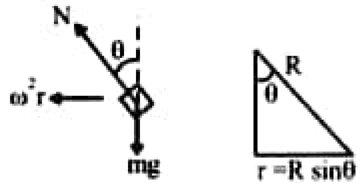
$$N \cos\theta = mg.$$



$$\Rightarrow \qquad N = \frac{mg}{\cos\theta}$$

Also $\qquad N\sin\theta = m\,\omega^2 r$

$$\Rightarrow \qquad \left(\frac{mg}{\cos\theta}\right)\sin\theta = m\omega^2 (R\sin\theta)$$



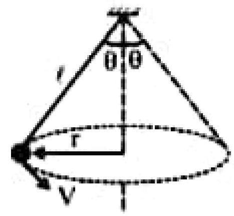
$$\Rightarrow \qquad \frac{g}{\cos\theta} = \omega^2 R$$

$$\Rightarrow \qquad \cos\theta = \frac{g}{\omega^2 R} = \frac{10}{(5)^2(0.8)}$$

$$\Rightarrow \qquad \cos\theta = \frac{1}{2}$$

$$\Rightarrow \qquad \theta = 60^o$$

Conical Pendulum

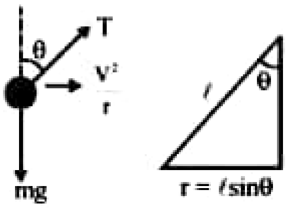


It consists of a small sphere (bob) connected to a light string. The bob is given some speed such that it revolve in a horizontal circular path while the string makes constant angle (say θ) with the vertical as shown.

Resolving in horizontal (radial) direction and vertical direction,

$(F_{net})_y = 0 \Rightarrow \quad T\cos\theta = mg$

$$\Rightarrow \qquad T = \frac{mg}{\cos\theta}$$



Also $(F_{net})_x = ma \quad \Rightarrow \quad T\sin\theta = \dfrac{mV^2}{r}$

$$\Rightarrow \qquad \left(\frac{mg}{\cos\theta}\right)\sin\theta = \frac{mV^2}{\ell\sin\theta}$$

$$\Rightarrow \qquad V = \sin\theta\sqrt{\frac{g\ell}{\cos\theta}}$$

Time period of revolution (t) $= \dfrac{2\pi r}{V} = \dfrac{2\pi\ell\sin\theta}{\sin\theta\sqrt{\dfrac{g\ell}{\cos\theta}}} = 2\pi\sqrt{\dfrac{\ell\cos\theta}{g}}$

Results obtained

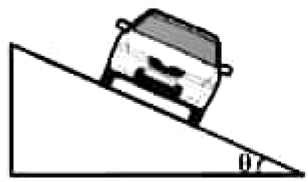
Tension (T) $= \dfrac{mg}{\cos\theta}$

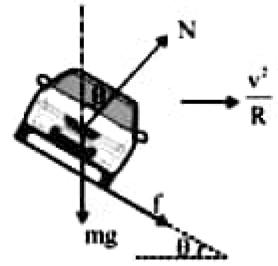
Speed (V) $= \sin\theta\sqrt{\dfrac{g\ell}{\cos\theta}}$

Time period (t) $= 2\pi\sqrt{\dfrac{\ell\cos\theta}{g}}$

Banking of road

When a vehicle is taking on horizontal circular turn, friction on tyres due to the road provides centripetal acceleration. Many times the friction is not sufficient enough, for example when speed is too large or the turn is too narrow. In such cases, instant of keeping the road horizontal, it is made tilted with the outer-side as higher side. This is called banking of road and the angle with which the road is kept inclined is called angle of banking. This enables the normal force to provide a horizontal component to help friction to provide centripetal acceleration. This can be explained below mathematically.



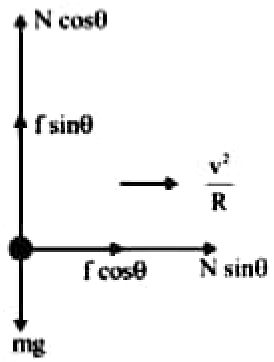


$(F_y) = 0 \Rightarrow N\cos\theta + f\sin\theta = mg$

Under limithing condition, i.e. when the car is about to skid, $f = \mu N$

$\Rightarrow \quad N\cos\theta + \mu N\sin\theta = mg$

$$N = \frac{mg}{\cos\theta + \mu\sin\theta} \qquad \text{..........(i)}$$



Also $\quad (F_x)_{net} = m a_r$

$$\Rightarrow \quad N\sin\theta + f\cos\theta = m\left(\frac{V^2}{R}\right)$$

$$\Rightarrow \quad N(\sin\theta + \mu\cos\theta) = \frac{mV^2}{R}$$

substituting value of N found in equation (i), we get

$$\frac{mg(\sin\theta + \mu\cos\theta)}{\cos\theta + \mu\sin\theta} = \frac{mV^2}{R}$$

$$\Rightarrow \quad V = \sqrt{\frac{gR(\sin\theta + \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)}}$$

This is the speed at which friction achieves its maximum value i.e. it is the maximum speed with which the vehicle can take turn safely on banked road.

Special cases :

I If friction is neglected i.e. μ is taken zero.

$$V = \sqrt{gR\tan\theta}$$

II If banking is not provided i.e. $\theta = 0°$

$$V = \sqrt{\mu gR}$$

Centrifugal force

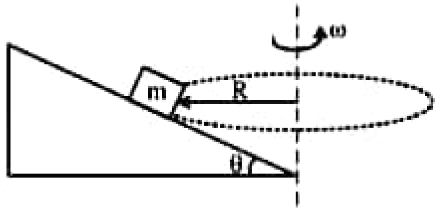
This the pseudo force which has to be taken under account when the frame of reference is rotating

object itself, since the revolving object is accelerated i.e. non-inertial frame of refernce. The magnitude of this force is $\frac{mV^2}{r}$ or $m\omega^2 r$ (where r - radius of circular path of the object) and its direction is always radially outwards.

Solving problems with the concept of centrifugal force

Illustration :

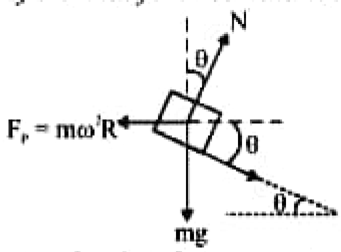
A small block is placed on a rough traingular shaped wedge which is revolving as shown such that the block undergoes circular path of radius R. The coefficient of friction between the block and the wedge is μ. Find the range of angular speed ω so that the block does not slip with respect to the wedge.



Sol. *If we select the frame of reference as the point at which the small block is placed. We take centrifugal force $m\omega^2R$ outward.*

(i) ***Maximum angular speed (ω_{max})***

More is ω, more is centrifugal force to balance which more horizontal force is required radially inwards. The friction force acts down the incline as shown so that its horizontal componant helps horizontal componant of Normal force to balance centrifugal force.



$(F_{net})_y = 0 \Rightarrow N\cos\theta - f\sin\theta - mg = 0$

Under limiting condition i.e. when the block is about to slip, $f = \mu N$

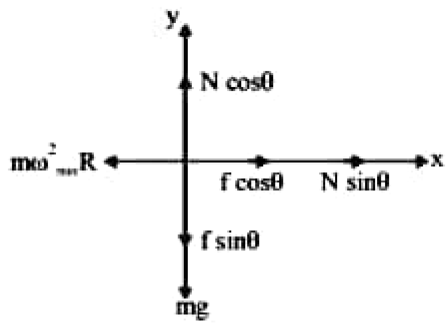
$\therefore \quad N(\cos\theta - \mu\sin\theta) = mg$

$$\Rightarrow \quad N = \frac{mg}{\cos\theta - \mu\sin\theta}$$

$(F_{net})_x = 0 \quad \Rightarrow N\sin\theta + (\mu N)\cos\theta - m\omega_{max}^2 R = 0$

$\Rightarrow N(\sin\theta + \mu\cos\theta) = m\omega_{max}^2 R$

$$\Rightarrow mg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right) = m\omega_{max}^2 R$$



$$\therefore \quad \omega_{max} = \sqrt{\frac{g}{R}\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)}$$

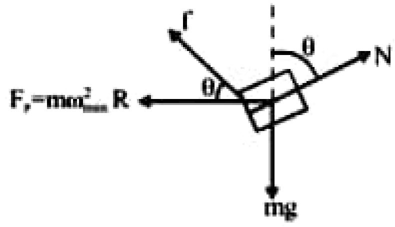
(ii) *For minimum angular speed (ω_{min})*

Lesser is ω, lesser is centrifugal force thus friction now acts up the incline so as to provide horizontal componant along centifugal force to balance horizontal componant of Normal force.

$(F_{net})_y = 0 \Rightarrow N\cos\theta + f\sin\theta - mg = 0$

$\Rightarrow \quad N(\cos\theta + \mu\sin\theta) = mg$

$$N = \frac{mg}{\cos\theta + \mu\sin\theta}$$

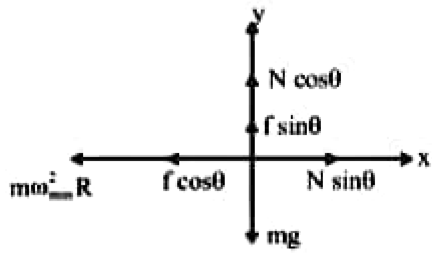


$(F_{net})_x = 0 \quad \Rightarrow N\sin\theta - f\cos\theta - m\omega_{min}^2 R = 0$

$\Rightarrow N(\sin\theta - \mu\cos\theta) = m\omega_{min}^2 R$

$$\Rightarrow \frac{mg(\sin\theta - \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)} = m\omega_{min}^2 R$$

$$\Rightarrow \quad \omega_{min} = \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)}}$$

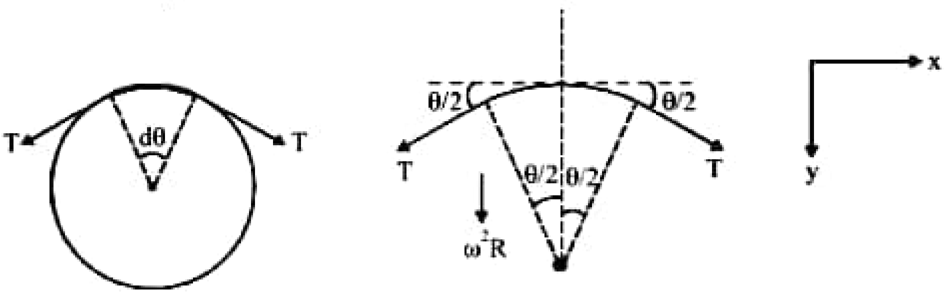


$$\therefore \quad \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)}} \le \omega \le \sqrt{\frac{g(\sin\theta + \mu\cos\theta)}{R(\cos\theta - \mu\sin\theta)}}$$

Tension in rotating ring

Diagram shown top view of a uniforming of mass m roating in horizontal plane.

Here tension inside the ring is an internal force for the ring. So to find it, we take a very small part of the ring which subtends angle dθ as shown.



we take y- axis towards the centre and x-axis in tangential direction and origin as the mid-point of the element

$\therefore \quad (F_{net})_y = (dm)\,\omega^2 R$

$\therefore \quad 2T\sin\left(\frac{d\theta}{2}\right) = (dm)\,\omega^2 R \qquad$ ....(i)

but $\quad \sin\left(\frac{d\theta}{2}\right) \simeq \frac{d\theta}{2} \qquad$ ($\because$ $d\theta$ is very small)

Also dm = (Mass per unit length of ring)× length of element

$\therefore \quad dm = \left(\frac{M}{2\pi R}\right)(Rd\theta) = \left(\frac{M}{2\pi}\right)d\theta$

Putting these values in equation (i), we get

$$2T\left(\frac{d\theta}{2}\right) = \left(\frac{M}{2\pi}\right)d\theta \times \omega^2 R$$

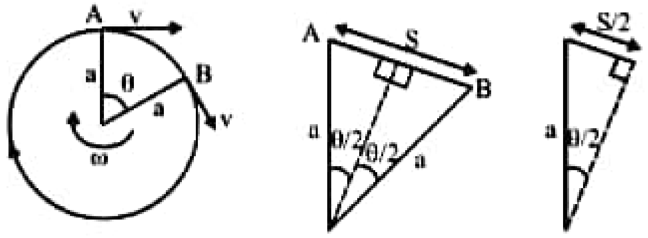
$\therefore \quad T = \frac{M\omega^2 R}{2\pi}$

Solved Example

Q.1 The magnitude of displacement of a particle moving in a circle of radius a with constant angular speed ω varies with time 't' as.

(A) $2a \sin\omega t$ (B) $2a \sin\frac{\omega t}{2}$ (C) $2a \cos \omega t$ (D) $2a \cos\frac{\omega t}{2}$

Sol. In time t, let the particle moves from A to B and rotates by angle $\theta = \omega t$



Magnitude of displacement

$$S = |\vec{S}| = AB$$

$\Rightarrow$ $S/2 = a \sin(\theta/2)$

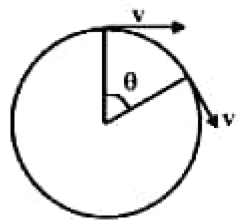
$\therefore$ $S = 2a \sin(\theta/2) = 2a \sin\left(\frac{\omega t}{2}\right)$

Q.2 The speed of an object undergoing uniform circular motion is 4 m/s. Find the minimum possible centripetal acceleration (in m/s^2) of the object. [Take $\pi = 25/8$]

Sol. Let the particle rotates by angle θ (= ωt) in the given time interval

$\therefore$ $|\Delta\vec{V}| = |\vec{V}_f - \vec{V}_{in}| = \sqrt{|\vec{V}_f|^2 + |\vec{V}_{in}|^2 - 2|\vec{V}_f| \times |\vec{V}_{in}| \times \cos\theta}$

but $|\vec{V}_f| = |\vec{V}_{in}|$ = V and also $|\Delta\vec{V}| = V$



$\therefore$ $V = \sqrt{V^2 + V^2 - 2V^2\cos\theta}$

$\Rightarrow$ $\cos\theta = \frac{1}{2}$

$\therefore$ θ i.e. $\omega t = \frac{\pi}{3}, \frac{5\pi}{3}$ etc.

$\Rightarrow$ $\omega_{min} t = \frac{\pi}{3}$

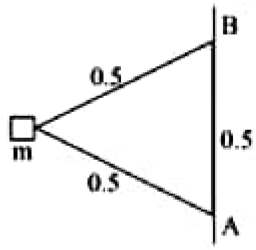
$$\therefore \quad \omega_{max} = \frac{\pi}{3t} = \frac{\pi}{3(0.5)} = \frac{2\pi}{3}$$

Also $a_r = \omega V$

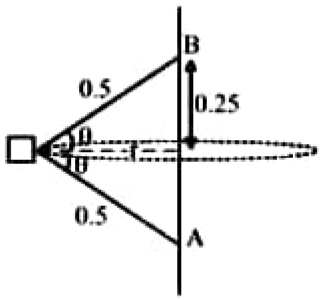
$$\Rightarrow \quad a_r = \frac{2\pi}{3} \times 4 = \frac{8\pi}{3} \text{ m/s}^2$$

$$\therefore \quad a_r = \frac{25}{3} \text{ m/s}^2 = 8.33 \text{ m/s}^2$$

Q.3 Two strings of length $l = 0.5$ m each are connected to a block of mass $m = 2$ kg at one end and their ends are attached to the point A and B 0.5 m apart on a vertical pole which rotates with a constant angular velocity $\omega = 7$ rad/sec. Find the ratio $\frac{T_1}{T_2}$ of tension in the upper string (T_1) and the lower string (T_2). [Use $g = 9.8$ m/s²]



Sol. $\text{Sin}\theta = \frac{0.25}{0.50} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$



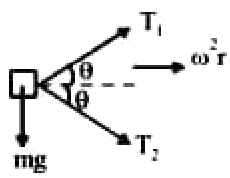
$$r = 0.5\,\text{Cos}\theta = 0.5\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$

$(F_{net})_y = 0 \Rightarrow T_1 \sin\theta - T_2 \sin\theta - mg = 0$

$\therefore \quad (T_1 - T_2)\sin\theta = mg \qquad$ .....(i)

$\quad (F_{net})_x = m\,\omega^2 r$

$\therefore \quad (T_1 + T_2)\cos\theta = m\,\omega^2 r \qquad$ ......(ii)



Dividing equation (ii) by (i)

$$\left(\frac{T_1 + T_1}{T_1 - T_2}\right)\cot\theta = \frac{\omega^2 r}{g}$$

$$\therefore \quad \frac{T_1 + T_1}{T_1 - T_2} = \frac{\omega^2 r \tan\theta}{g} = \frac{(7)^2 \times \left(\sqrt{3}/4\right) \times 1/\sqrt{3}}{9.8}$$

$$\therefore \quad \frac{T_1 + T_1}{T_1 - T_2} = \frac{5}{4}$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{5+4}{5-4} = 9$$

Q.4 A long horizontal rod has a bead which can slide along its length, and initially placed at a distance L from one end of A of the rod. The rod is set in angular motion about A with constant angular acceleration α. If the coefficient of friction between the rod and the bead is μ and gravity is neglected, then the time after which the bead starts slipping is

(A) $\sqrt{\frac{\mu}{\alpha}}$ (B) $\frac{\mu}{\sqrt{\alpha}}$ (C) $\frac{1}{\sqrt{\mu\alpha}}$ (D) infinitesimal

Sol. $N = ma_t = m\alpha L$

and $f = ma_r = m\omega^2 L$

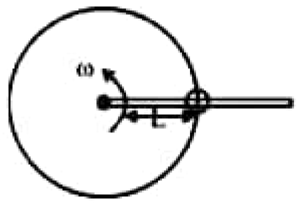
but $\omega = 0 + \alpha t = \alpha t$

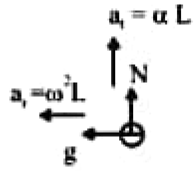
$\therefore$ $f = \mu\alpha^2 t^2 L$

Also at the instant of slipping, $f = \mu N$

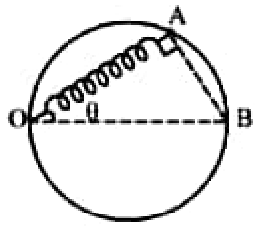
$\therefore$ $m\alpha^2 t^2 L = \mu(m\alpha L)$

$\therefore$ $t = \sqrt{\frac{\mu}{\alpha}}$





Q.5 A bead of mass m = 300gm moves in gravity free region along a smooth fixed ring of radius R = 2m. The bead is attached to a spring having natural length R and spring constant k = 10 N/m. The other end of spring is connected to a fixed point O on the ring.

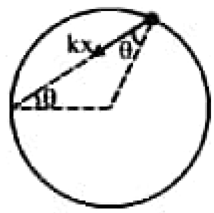


$AB = \frac{6R}{5}$. Line OB is diameter of ring.

Find (a) Speed of bead at A if normal reaction on bead due to ring at A is zero.

(b) The rate of change in speed at this instant.

Sol. The length of the spring, when bead is at A, is



$$OA = \sqrt{(OB)^2 - (AB)^2} = \sqrt{(2R)^2 - \left(\frac{6R}{5}\right)^2} = \frac{8R}{5}$$

Elongation in the spring = OA – natural length

$$= \frac{8R}{5} - R = \frac{3R}{5}$$

Also $\cos\theta = \frac{(8R/5)}{2R} = \frac{4}{5}$ and $\sin\theta = \left(\frac{6R/5}{2R}\right) = \frac{3}{5}$

Radial component of spring force $= kx\cos\theta = \frac{mv^2}{R}$

$$\Rightarrow \quad k\left(\frac{3R}{5}\right)\left(\frac{4}{5}\right)=\frac{mv^2}{R}$$

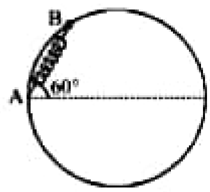
$$\Rightarrow \quad v=\sqrt{\left(\frac{12kR^2}{25m}\right)}=8 \text{ m/s}$$

Tangential component of force $= k\,x\,\sin\theta = k\left(\frac{3R}{5}\right)\left(\frac{3}{5}\right)=\frac{9kR}{25}$

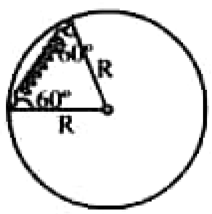
$$\Rightarrow \quad m\frac{dv}{dt}=\frac{9kR}{25}$$

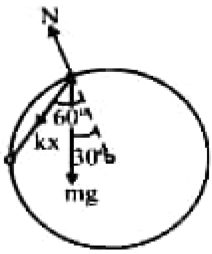
Rate of change in speed $\quad \frac{dv}{dt}=\frac{9kR}{25m} \quad = 24 \text{ m/s}^2$

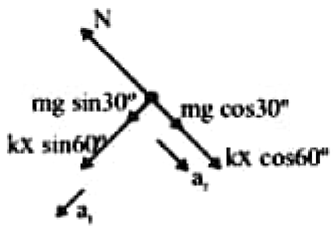
Q.6 A bead of mass m is attached to one end of a spring of natural length $\sqrt{3}$ R and spring constant $k=\frac{(\sqrt{3}+1)\,mg}{R}$. The other end of the spring is fixed at point A on a smooth fixed vertical ring of radius R as shown in the figure. What is the normal reaction at B just after the bead is released?



Sol.







At the instant of release, compression in the spring is

$$x=\sqrt{3}R-(\sqrt{3}-1)R$$

$\therefore \quad$ spring force, $kx=\frac{(\sqrt{3}+1)mg}{R}\times(\sqrt{3}-1)R=2\,mg$

Also at the instant of release, speed = 0

$\therefore \quad$ radial acceleration, $a_r=\frac{v^2}{R}=0$

$\therefore \quad$ Along radial direction, $F_{net}=0$

$\therefore \quad N - mg\cos 30° - kx\cos 60° = 0$

$$N = mg\,\frac{\sqrt{3}}{2}+(2\,mg)\times\frac{1}{2}=\frac{mg(\sqrt{3}+2)}{2}$$

Work Energy and Power

Introduction

Figure (a) shows a skier starting from rest at the top of a uniform slope. What's the skier's speed at the bottom ? You can solve this problem by applying Newton's second law to find the skier's constant acceleration and then the speed. But what about the skier in figure (b) ? Here the slope is continuously changing and so is the acceleration. Constant-acceleration equations are not applicable here, so solving for the details of the skier's motion is difficult.

Figure (a)

Figure (b)

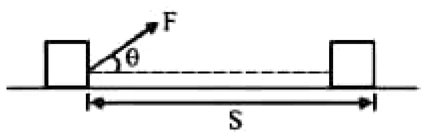
There are many cases where motion involves changing forces and accelerations. In this chapter, we introduce the important physical concepts of work and energy. These powerful concepts enable us to "shortcut" the detailed application of Newton's law to analyze these more complex situations. We begin with the concept of work.

Work

In day-today life, me oftenely use the word "Work". In physics, we define the work in quite different manner than we usually use the word "Work" in daily life. Work done by a constant force $\vec{F}$ on an object when displaced by $\vec{S}$ is given by

$$W = F'S$$

Where F' is the component of force which is in the direction of displacement



In above figure, $F' = F\cos\theta$

$$\therefore \quad W = (F\cos\theta)\,S = F\,S\cos\theta$$

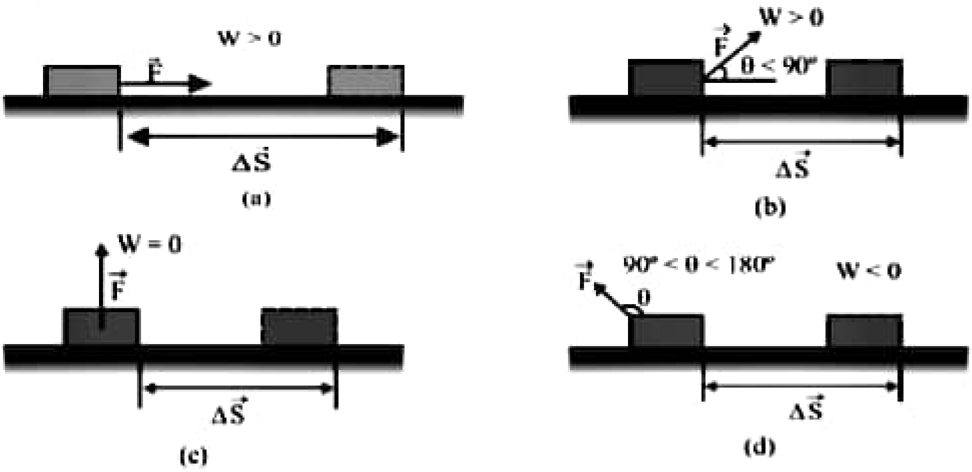
$$\therefore \quad \mathbf{W} = \vec{\mathbf{F}}\cdot\vec{\mathbf{S}}$$

So work may also be given as dot product of force and displacement.

• Its unit is Newton meter (N.m) or Joule (J)

• If $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and $\vec{S} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$, then $W = \vec{F}\cdot\vec{S} = F_x(\Delta x) + F_y(\Delta y) + F_z(\Delta z)$

- Work done force is frame dependent as displacement is frame dependent
- Work can be positive or negative or zero. When a force speed up the praticle, it does positive work. A force acting at 90° to the motion does no work. And when a force slow down the motion, it does negative work.



e.g. According to equation the person pushing the car in figure (a) does work equal to the force he applies times the distance the car moves. But person pulling the suitcase in figure (b) does work equal to only the horizontal component of the force times the distance the suitcase moves. Furthermore, by our definition, the waiter of figure (c) does no work on the tray. Why not ? Because the force on the tray is vertical while the tray's displacement is horizontal; there's no component of force in the direction of the tray's motion.

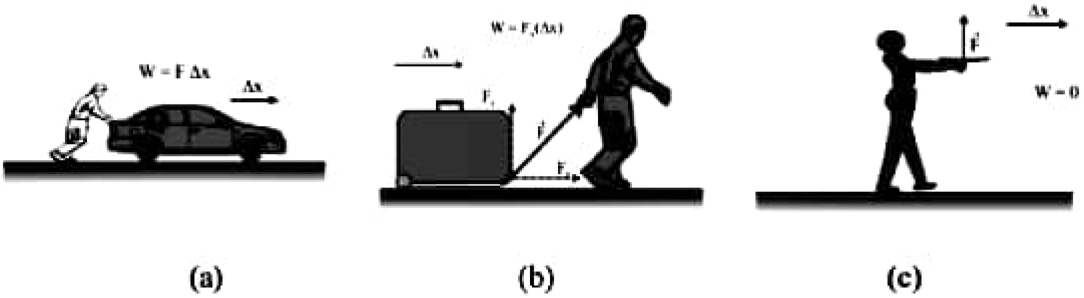


Illustration :

The airline passenger in above figure (b) exerts a 80 N force on his suitcase pulling at an angle 60° with the horizontal. What work he does on suitcase while pulling it 50 m on the floor ?

Sol. $W = (80\ N)\ (50\ m) \cos 60°$

$= 2000\ J = 2\ KJ$

Illustration :

A particle is moving along a straight line from point A to point B. The position vectors for points A and B are $(2\hat{i}+7\hat{j}-3\hat{k})$m and $(5\hat{i}-3\hat{j}-6\hat{k})$m respectively. One of the force acting on the particle is $\vec{F}=20\hat{i}-30\hat{j}+15\hat{k}$ N. Find the work done by this force.

Sol. $\vec{S} = (5\hat{i} - 3\hat{j} - 6\hat{k}) - (2\hat{i} + 7\hat{j} - 3\hat{k})$

$= 3\hat{i} - 10\hat{j} + 3\hat{k}$ m

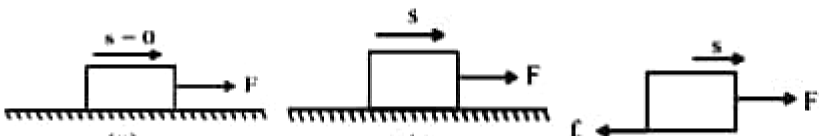
$\vec{F} = 20\hat{i} - 30\hat{j} + 15\hat{k}$

Now $W = \vec{F}\cdot\vec{S}$

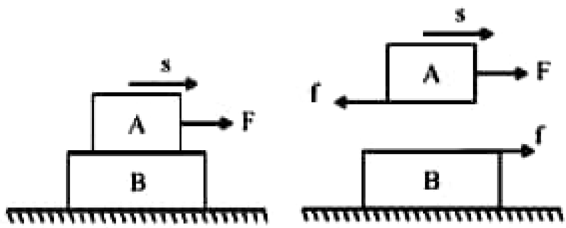
$= 60 + 300 - 45 = 315\ J$

Work Done by friction

There is a misconception that the force of friction always does negative work. In reality, the work done by fricition may be zero, postive or negative depending upon the situation as shown in the figure.



(a) (b)

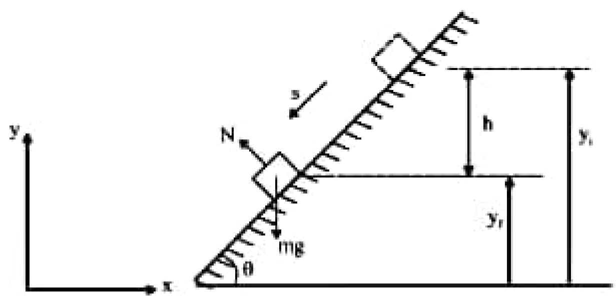


(c)

(a) When a block is pulled by a force F and the block does not move, the work done by friction is zero.
(b) When a block is pulled by a force F on a stationary surface, the work done by the kinetic friction is negative.
(c) Block A is placed on the block B. When the block A is pulled with force F, the friction force does negative work on block A and positive work on block B, which is being accelerated by a force F. the displacement of A relative to the table is in the forward direction. The work done by kinetic friction on block B is positive.

Work Done by Gravity

Consider a block of mass m which slides down a smooth inclined plane of angle θ as shown in figure.



Let us assume the coordinate axes as shown in the figure to specify the components of the two vector although the value of work will not depend onthe orientation of the axes.

Now, the force of gravity, $\vec{F}_g = -mg\,\hat{j}$

and the displacement is given by

$$\vec{s} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$$

The work done by gravity is

$$W_g = \vec{F}_g.\vec{s} = -mg\,\hat{j}.(\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k})$$

or $\quad W_g = -mg(\Delta y)$

Since $\quad \Delta y = y_f - y_i = -h$

$\therefore \quad W_g = +mgh$

If the block moves in the upward direction, then the work done by gravity is negative and is given by

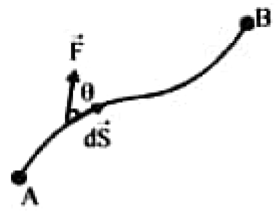
$$W_g = -mgh$$

Important

1. The work done by the force of gravity depends only on the initial and final vertical coordinates, not on the path taken.
2. The work done by gravity is zero for path that returns to its initial point.

Work done by a variable force

Often the force applied to an object varies with position. Important examples include electric and gravitational force, which vary with the distance between interacting objects. The force of a spring that we encountered in previous chapter provides another example: as the spring stretches, the force increases.



In this case we have difficulty to apply $W = \vec{F}.\vec{S}$, since $\vec{F}$ is not same for complete $\vec{S}$.

Thus, we take a very small part $d\vec{S}$ of its path. This displacement $d\vec{S}$ is so small that in variation force may be neglected during it. So we may write, for the work done during this displacement as

$$dW = \vec{F}.d\vec{S}$$
$$= F\,dS\cos\theta$$

The total work done in going from A to B as shown may be calculated by summing up i.e. integrating the work done during its small fractions.

i.e. $$W_{A\to B} = \int_A^B \vec{F}.d\vec{S} = \int_A^B (F\cos\theta)dS$$

In terms of rectangular components,

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

and $\quad d\vec{S} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

therefore,

$$W_{A\to B} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

Illustration :

A force $\vec{F} = x\hat{i} + y^2\hat{j}$ N acts on a particle and the particle moves from (1,2) m to (–3,4) m. Find work done by the force $\vec{F}$.

Sol. $dW = \vec{F}\cdot d\vec{S}$

where $d\vec{S} = dx\hat{i} + dy\hat{j}$

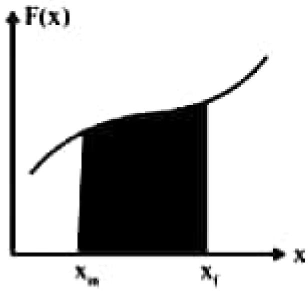
$\therefore \quad dW = xdx + y^2dy$

and $W = \int dW = \int_1^{-3} xdx + \int_2^4 y^2dy = \left.\frac{x^2}{2}\right|_1^{-3} + \left.\frac{y^3}{3}\right|_2^4 = \frac{68}{3} J$

Work done as Area under the force displacement graph

Suppose of particle moving along a straight line and a force acting on it varies with its displacement x as shown.

$W = \int_{x_{in}}^{x_f} F.dx$



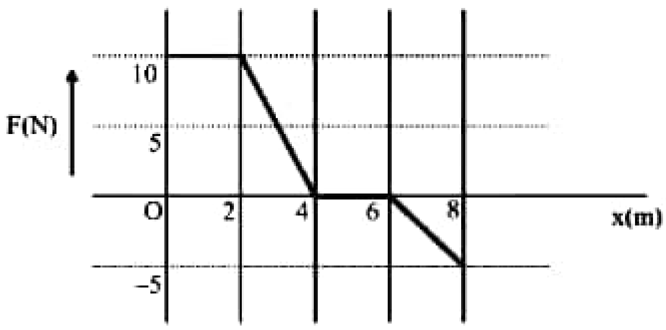
= Area under F vs x graph from $x = x_{in}$ to x_f

In general, the work done by a point x_{in} to final point x_f is given by the area under the force - displacement curve as shown in the figure.

Area (work) above the x-axis is taken as positive, and below x-axis as negative.

Illustration :

A 5 kg block moves in a straight line on a horizontal frictionless surface under the influence of a force that varies with position as shown in the figure. Find the work done by this force as the block moves from the origin to x = 8m.

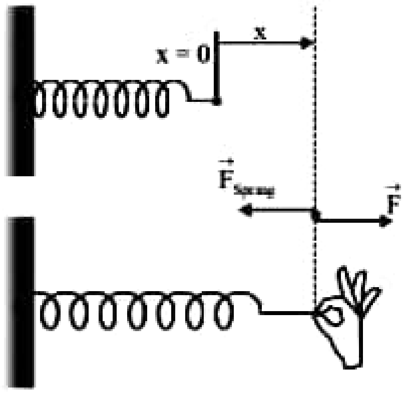


Sol. *The work from x = 0 to x = 8 m is the area under the curve.*

$$W = 10\times 2 + \frac{1}{2}(10)(4-2) + 0 + \frac{1}{2}(-5)(8-6) = 25 J$$

Work done by spring force

A spring provides an important example of a force that varies with position. We've seen that an ideal spring exerts a force proportional to its displacement from equilibrium : $F = -kx$, where k is the spring constant and the minus sign shows that the spring force is opposite the direction of the displacement.



$\therefore$ Work done (W_s) by spring force when its deformation changes from x_{in} to x_f is

$$W_s = -k\int_{x_{in}}^{x_f} x.dx$$

$\Rightarrow$ $$W_s = -\frac{1}{2}\, k\left(x_f^2 - x_{in}^2\right)$$

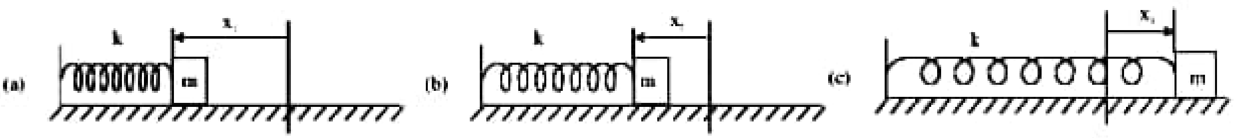
Note : Work done by a spring force to stretch it from its undeformed length to deform it upto x is

$$W = -\frac{1}{2}\, k\,(x^2 - 0) = -\frac{1}{2}\, kx^2$$

- Work done by a spring force may be negative ($x_f > x_{in}$) or may be positve (if $x_f < x_{in}$) or may be zero (if $x_f = x_{in}$).
- Work done by the spring force depends only on initial and final deformation.
- In the equation, $W = -\frac{1}{2}k(x_f^2 - x_{in}^2)$

x_{in} & x_f are magnitudes of deformations no matter if these are compressions or extensions.

e.g.



Work done by spring force from position (a) to (b) and that from (a) to (c) are same and equal to $-\frac{1}{2}k\left(x_2^2 - x_1^2\right)$

Work Energy Theorem

Closely related to work is energy – One of the most important concepts in all of physics. Here we introduce the energy associated with motion i.e. kinetic energy. Our goal is to relate kinetic energy and work.

The net work is done by all the forces acting on an object, so we use the net force in our expression for work. We'll consider one-dimensional motion with force and displacement along the same line. In that case, the Equation below gives the net work :

$$W_{net} = \int F_{net} dx$$

But net force can also be written as

$$F_{net} = ma = mv\frac{dv}{dx}$$

$$\Rightarrow \quad F_{net}\, dx = mvdv$$

$$W_{net} = \int_{v_{in}}^{v_f} mvdv$$

$$\therefore \quad W_{net} = \frac{1}{2}m\left[v_f^2 - v_{in}^2\right]$$

$$\therefore \quad W_{net} = \Delta\ K.E.$$

Thus change in an object's kinetic energy is equal to the net work done on the object. This called Work Energy Theorm.

Important

(i) The kinetic energy of an object is a measure of the amount of work needed to increase it speed from zero to a given value.

(ii) The kinetic energy of a particle is the work it can do on its surroundings in coming to rest.

(iii) Since the velocity and displacement of a particle depend on the frome of reference, the numerical values of the work and the kinetic energy also depend on the frame.

(iv) If work done by net force is positive, kinetic energy of the system increases. If net work done is negative K.E. decreases and if net work is zero, K.E. remains contact

Illustration :

A 60 gm tennis ball thrown vertically up at 24 m/s rises to a maximum height of 26 m. What was the work done by resistive forces?

Sol. *By Work - Energy theorem,* $W_{net} = \Delta\ K.E$

$$W_g + W_{res} = (0 - \frac{1}{2}mu^2)$$

$$-mgh + W_{res} = -\frac{1}{2}mu^2$$

$$W_{res} = 0.06 \times 10 \times 26 - \frac{1}{2} \times 0.06 \times (24)^2$$

$$= -1.68\ J$$

Illustration :

A force of $(3\hat{i} - 1.5\hat{j})$ N acts on a 5 kg body. The body is at a position of $(2\hat{i} - 3\hat{j})$ m and is travelling at 4 ms^{-1}. The force acts on the body until it is at the position $(\hat{i} + 5\hat{j})$ m. Assuming no other force does work on the body, the final speed of the body.

Sol. Given; Mass of the body = 5 kg

Force $\vec{F} = 3\hat{i} - 1.5\hat{j}$

Displacement $\overrightarrow{\Delta s} = \{(\hat{i} + 5\hat{j}) - (2\hat{i} - 3\hat{j})\}\ m = (-\hat{i} + 8\hat{j})\ m$

From Work Energy theorem

$$W = \vec{F}\cdot\Delta\vec{s} = \frac{1}{2}m(v^2 - u^2)$$

$$-3 - 12 = \frac{1}{2}\times 5\left[v^2 - (4)^2\right]$$

$\Rightarrow \quad v = \sqrt{10}$ m/s

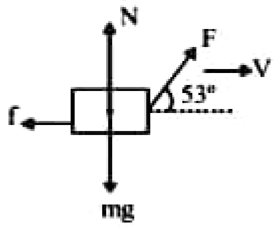
Illustration :

A block of mass m = 4 kg is dragged 2 m along a horizontal surface by a force F = 30 N acting at 53° to the horizontal. The initial speed is 3 m/s and $\mu_k = 1/8$.

(a) Find the change in kinetic energy of the block

(b) Find its final speed

Sol.



(a) The forces acting on the block are shown in the figure.

Clearly, $W_N = 0$ and $W_g = 0$, whereas $W_F = FS\cos\theta$

$W_f = -fs = -(\mu_k N)S$, where $N = mg - F\sin\theta$

The work-energy theorem,

$$\Delta K = W_{net} = W_F + W_f$$

therefore, $\Delta K = F\ S\cos\theta - \mu_k(mg - F\sin\theta)\cdot S$

$$= (30)(2)(0.6) - \frac{1}{8}(40 - 24)(2) = 32\ J$$

(b) Now $\Delta K{\cdot}E{\cdot} = \frac{1}{2}m\left[v_f^2 - v_i^2\right] = 32\ J$

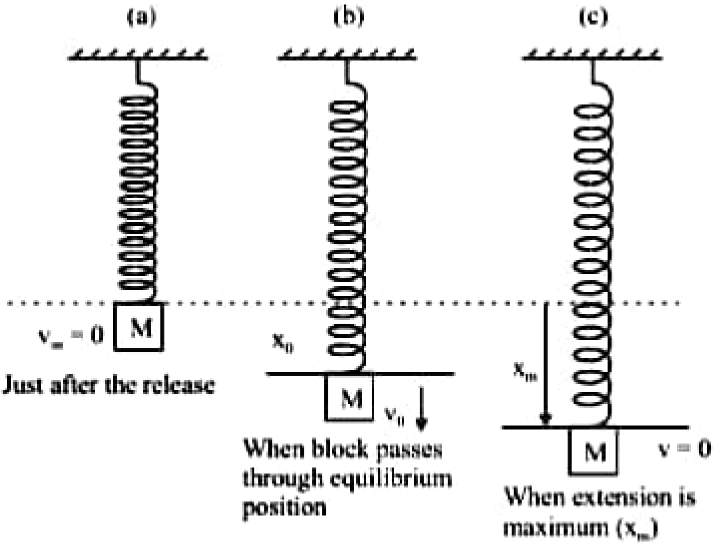
$\Rightarrow \quad \frac{1}{2}\times 4\left[v_f^2 - (3)^2\right] = 32$

$\Rightarrow \quad v_f = 5$ m/s

Illustration :

A sping of spring constant k is attached to the ceiling. A block of mass M is attached to its lower end and is released suddenly. Find (i) maximum extension in the spring (ii) Find its speed at the instant it passes throught the equilibrium position.

Sol.

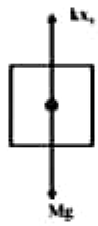


Applying work energy theorem for motion from (a) to (c)

$$W_g + W_{spring} = \Delta KE$$

$$Mg\,x_m - \frac{1}{2}kx_m^2 = \frac{1}{2}M(0-0) = 0$$

$\therefore$ *maximum extension,* $x_m = \dfrac{2Mg}{k}$



(ii) When the block is equilibrium (i.e. $F_{net} = 0$*)*

$$kx_0 = Mg$$

$$\Rightarrow \quad x_0 = \frac{Mg}{k}$$

$\therefore$ *Applying work energy therom for motion from (a) to (b)*

$$Mg\,x_0 - \frac{1}{2}kx_0^2 = \frac{1}{2}M(V_0^2 - 0)$$

$$\Rightarrow \quad Mg\left(\frac{Mg}{k}\right) - \frac{1}{2}k\left(\frac{Mg}{k}\right)^2 = \frac{1}{2}MV_0^2$$

$$\Rightarrow \quad v_0 = g\sqrt{\frac{M}{k}}$$

Illustration :

A spring is fixed at the bottom end of an incline of inclination 37°. A small block is released from rest on an incline from a point 4.8 m away from the spring. The block compresses the spring by 20 cm, stops momentarily and then rebounds through a distance of 1 m up the incline. Find (a) the friction coefficient between the plane and the block and (b) the spring constant of the spring. Take g = 10 m/s^2.

Sol.

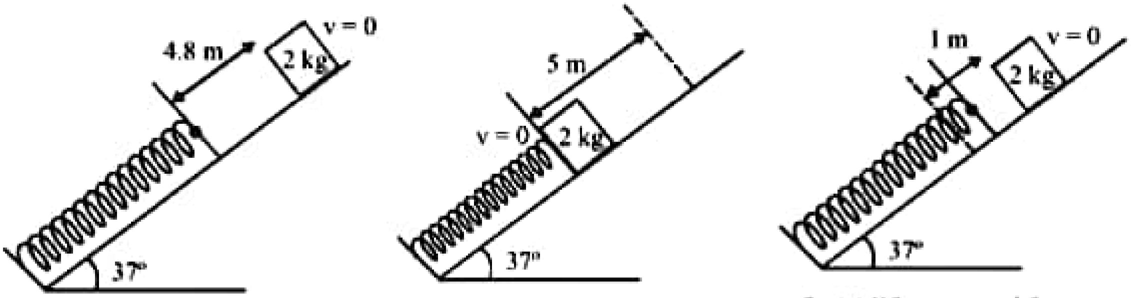


fig (a) Just after release

fig (b) When stopped for the first time

fig (c) When stopped for the second time

Applying work energy theorem for motion from (a) to (b)

$$W_{gravity} + W_{friction} + W_{spring} = \Delta K{\cdot}E = \frac{1}{2} m\,(0 - 0) = 0$$

$$\therefore \quad 20 \times 5 \sin 37^\circ - \mu\,(20 \cos 37^\circ)5 - \frac{1}{2}\, k\,[(0.2)^2 - 0] = 0 \qquad \text{..........(i)}$$

Applying work energy for motion form (b) to (c)

$$-\,20 \times 1 \times \sin 37^\circ - \mu\,(20 \cos 37^\circ) \times 1 - \frac{1}{2} k\left[0 - (0.2)^2\right] = 0 \qquad \text{..........(ii)}$$

Adding equation (i) and (ii)

$$\therefore \quad -\,20\,(5 - 1) \times \frac{3}{5} - \mu\,(20 \times \frac{4}{5})\,(5 + 1) = 0$$

$$\Rightarrow \quad \mu = 0.5$$

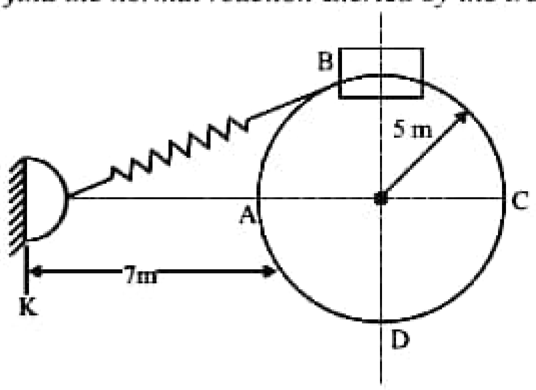
Putting this value in equation (i), we get

$$k = 1000 \text{ N/m}$$

Here velocity is maximum at equilibrium since before this, spring force was less than the weight of the block and the block was accelerating and after this , the spring force is greater than the weight thus retarding the block to zero velocity upto the lowest position.

Illustration :

A collar B of mass 2 kg is constrained to move along a horizontal smooth and fixed circular track of radius 5 m as shown in figure. The spring lying in the plane of the circular track and having spring constant 200 N/m is undeformed when the collar is at A. If the collar starts from rest from B, then find the normal reaction exerted by the track on the collar when it passes through A.



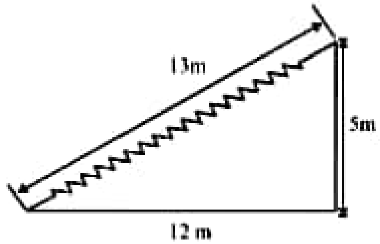
Sol. *Initially ,*

Length of spring $= 13\ m$

undeformed lingth $= 7\ m$

Initial extention $(x_{in}) = 13 - 7 = 6$

final extension $(x_f) = 0$

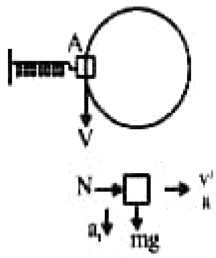


Applying work energy theorem for motion from B to A

$$\frac{1}{2}k\left(x_{in}^2 - x_f^2\right) = \frac{1}{2}mv^2$$

$mv^2 = 200\ (y^2 - 0^2) = 200 \times 49$

$mv^2 = 9800$



At point A, along radial direction, $F_{net} = N = m\frac{v^2}{R}$

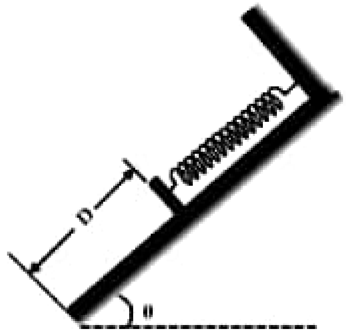
$$\therefore \quad N = \frac{mv^2}{R} = \frac{9800}{5}$$

$$= \frac{200}{5}\ [0 - (6)^2]$$

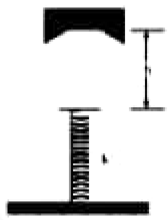
$= 1960$ *Newton*

Q.4 In figure a spring with k = 168 N/m is in its relaxed length and is at the top of a frictionless incline of angle $\theta = 37^\circ$. The lower end of the incline is at distance D = 1 m from the end of the spring. A small blcok of mass 2 kg is pushed against the spring until the spring is compressed by 0.2 m and released from rest.

(a) What is the speed of the block at the instant the spring returns to its relaxed length (which is when the block loses contact with the spring)? (b) What is the speed of the block when it reaches the lower end of the incline ?



Q.5 A block of mass m = 2.0 kg is dropped from height h = 40 cm onto a spring of spring constant k = 1960 N/s. Find the maximum distance through which the spring is compressed. Take $g = 9.8\ m/s^2$



Answers

Q.1 35 m/s Q.2 (i) 6 m/s (ii) 3 m/s Q.3 (i) $\frac{2F}{k}$ (ii) $\frac{F}{\sqrt{mk}}$

Q.4 (a) 2.4 m/s (b) 4.2 m/s Q.5 10 cm

Consevative and Non-conservertive force

If we throw a body up along smooth incline plane with some speed v_0, then it moves along the incline till it becomes stationary for a moment and then moves down the incline. It is observed the, when it reaches the point of projection, its speed is v_0 again, which proves that during the journey the net work done on the block is zero. Two forces act on the block during its motion. One is Normal force (N) which is continuously perpendicular to the block's motion, so its work for any part of the path is zero. Another one is weight (mg) which does negative work while upward motion and positive work of same magnitude during downward motion, does zero net work when the body reaches the initially position again.

Now we throw the same body on a rough incline plane. When it reaches the initial position, its speed is lesser than the speed of projection since friction does negative work for motion during up and as well as down the incline.

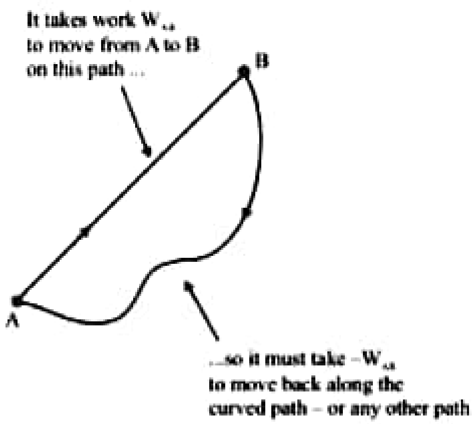
Thus, here we find two categories of force .

Conservative force

When the total work done by a force F acting as an object moves over any closed path is zero, the force is conservative. Mathematically.

$$\oint \vec{F}\cdot d\vec{r} = 0 \qquad \text{(conservative force)}$$

Suppose we move an object along the straight path between point A and B shown in figure, along which a conservative force acts; let the work done by the conservative force be W_{AB}. Since the work done over any closed path is zero, the work W_{BA} done in moving back from B to A must be $-W_{AB}$, whether we return along the straight path or the curved path or any other path. So, going from A to B involves work W_{AB}, regardless of the path taken.



In other words : *The work done by a conservative force in moving between two points is independent of the path taken; mathematically,* $\int_A^B \vec{F}\cdot d\vec{r}$ *depends only on the endpoints A and B, not on the path between them.*

These include force due to gravity (mg), spring force, electrostatic force etc.

Non-conservative force

Work done by non-conservative forces depends on path also. Least work is done for straight line path and any curved path it involves different work. These include frictional force, viscous force etc.

Illustration :

A particle moves in x-y plane from (0,0) to (a,a) and is acted upon by a force $\vec{F} = k(y^2\,\hat{i} + x^2\,\hat{j})$ *N, where k is a constant and x and y are coordinates in meter. Find work done by this force, if the particle mones along*

(i) Two straight lines first from (0,0) to (a,0) and then from (a,0) to (a,a)

(ii) A single straight line.

Sol. $dW = \vec{F}\cdot d\vec{s} = k(y^2\,\hat{i} + x^2\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$

$= ky^2\,dx + k\,x^2\,dy.$

(i) when it moves from (0,0) to (a,0)

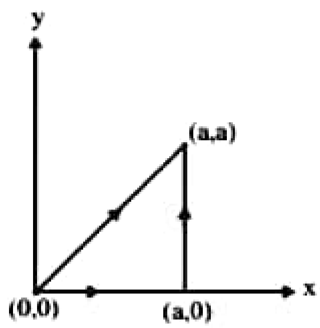
$y = constant = 0$

$\Rightarrow \quad dy = 0$

$\therefore \quad dW_A = k\,(0)\,dx + kx^2\,(0)$

$= 0$

$\Rightarrow \quad W_A = \int dw_A = 0$



When it moves from (a,0) to (a,a)

$x = constant = a \quad \Rightarrow \quad dx = 0$

and y changes from 0 to a

$\therefore \quad dW_B = ky^2\,(0) + ka^2\,dy = ka^2\,dy$

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$$\therefore \quad W_B = ka^2\int_0^a dy = ka^3$$

$$\therefore \quad W = W_A + W_B = ka^3$$

(ii) When moves from (0,0) to (a,a) as shown in above figure, along path C which is a straight line for which y = x

$$\Rightarrow \quad dy = dx$$

$$\therefore \quad dW = kx^2\,dx + k\,x^2\,dx = 2k\,x^2\,dx$$

$$\therefore \quad W = \int dW = 2k\int_0^a x^2 dx = \frac{2ka^3}{3}$$

In above illustration, the work done by the force is different for different paths for different paths taken, so it provides an example of non-conservative force.

Potential Energy

When we move an object upto some height against gravity and release, it gains kinetic energy while moving down. It means work done against a conservative force like gravity is somehow stored, in the sense that we can get it back again in the form of kinetic energy. We can consider the "stored work" as potential energy U.

The change ΔU_{AB} in potential energy associated with a conservative force is the negative of the work done by that force as it acts over any path from point A to point B:

$$\Delta U_{AB} = -\int_A^B \vec{F}\cdot d\,\vec{r}$$

If a conservative force does positive work (as does gravity on a falling object), then potential energy must decrease – and that means ΔU must be negative. Conversely, if a conservative force does negative work (as does gravity on a weight being lifted), then energy is stored and ΔU must be positive.

Change in potential energy are all that ever matter physically; the actual value of potential energy is meaningless. All though optenly it is convinient to establish a refernce point at which the potential energy is defined to be zero. When we say "the potential energy U,"we really mean the potential-energy difference ΔU between the point we're considering and the reference point.

Gravitational Potential Energy (Near the Earth's Surface)

The work done by gravity on a particle of mass m whose vertical coordinate changes from y_A to y_B is

$$W_g = -\,mg\,(y_B - y_A)$$

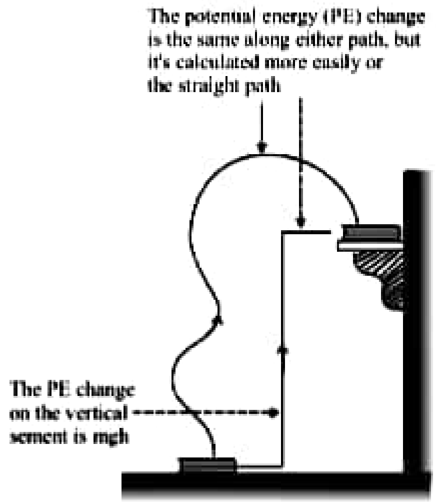
From equation, we have $W_g = -\,\Delta U_g = -\,(U_B - U_A)$

Thus gravitational potential energy at the point B near the surface of the Earth is given by

$$U_B = U_A + mgh$$

If we assume potential energy at the point A to be zero, then potential energy at the point B is given by

$$U_B = 0 + mgh = mgh$$



Elastic Potential Energy

When you stretch or compress a spring, you do work against the spring force, and that work gets stored as elastic potential energy. For an ideal spring, the force is $F = -kx$, where x is the distance the spring is stretched from equilibrium, and the minus sign shows that the force opposes the stretching or compression.

$$\Delta U = -\int_{x_1}^{x_2} F(x)\,dx = -\int_{x_1}^{x_2} (-kx)\,dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

where x_1 and x_2 are the initial and final values of the stretch. If we take $U = 0$ when $x = 0$ –that is, when the spring is neither stretched nor compressed –then we can use this result to write the potential energy $U_2 = U$ at an arbitrary stretch (or compression) $x_2 = x$, $U = \frac{1}{2}kx^2$

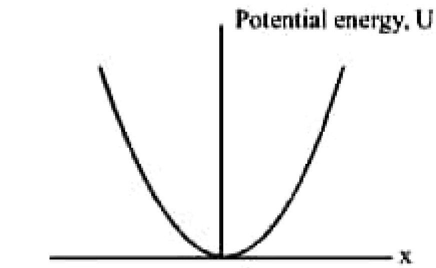
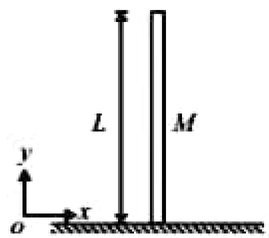


Illustration :

A uniform rod of mass M and length L is held vertically upright on a horizontal surface as shown in the figure. Find the potential energy of the rod if the zero potential energy level is assumed at the horizontal energy level is assumed at the horizontal surface.



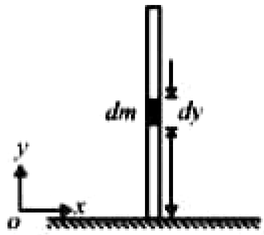
Sol. *Since the parts of the rod are at different level with respect to the horizontal surface, therefore, we have to use the integration to find its potential energy. Consider a small element of length dy at a height y from the horizontal.*

Mass of the element is

$$dm = \frac{M}{L}dy$$

Its potential energy is given by

$$dU = (dm)gy$$



or $$dU = \frac{M}{L}gydy$$

On integration, we get

$$U = \frac{Mg}{L}\int_0^L y\,dy$$

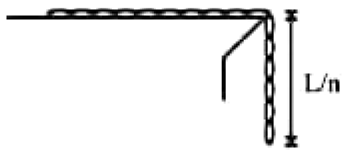
or $$U = \frac{Mg}{L}\left|\frac{y^2}{2}\right|_0^L$$

or $$U = \frac{1}{2}MgL$$

Note that the potential energy of the rod is equal to the product of Mg and height of the center of mass $\left(\frac{L}{2}\right)$ *from the surface.*

Illustration :

A unifrom chain of mass M and length L lies on a table with n^{th} part of it hanging off the table. Find the work required to slowly pull the hanging part up to the table.



Sol. *Here work is done agains the granity.*

$\therefore$ $$W = \Delta U = U_f - U_{in}$$

Taking table as reference level, i.e. $U_f = 0$

$\therefore$ $$W = -U_{in} = -[-mgh_{C.M.}]$$
$$= Mgh_{C.M.}$$

here mass of hanging part is $m = \frac{M}{n}$

and its centre of mass is $h_{C.M.}$ hight below the table

where $h_{C.M.} = \frac{(L/n)}{2} = \frac{L}{2n}$

$\therefore$ $$W = \left(\frac{M}{n}\right)g\left(\frac{L}{2n}\right) = \frac{MgL}{2n^2}$$

Conservation of Mechanical Energy

The work-energy theorem, shows that the change ΔKE in a body's kinetic energy is equal to the net work done on it :

$$\Delta KE = W_{net}$$

Consider separately the work W_c done by conservative force and the work W_{nc} done by nonconservative forces. Then

$$\Delta KE = W_c + W_{nc}$$

We've defined the change in potential energy ΔU as the negative of the work done by conservative forces. So we can write

$$\Delta KE = -\Delta U + W_{nc}$$

or $\quad \Delta KE + \Delta U = W_{nc}$

We define the sum of the kinetic and potential energy as the mechanical energy. Then Equation shown that the change in mechanical energy is equal to the work downe by non-conservative forces.

i.e. $\quad \Delta E = W_{nc}$

$\therefore \quad \Delta E = 0 \quad \text{if } W_{nc} = 0$

Thus if work done by non-conservative forces is zero the mechanical energy of the system is unchanged. This called law of conservation of mechanical energy. It may also be written as

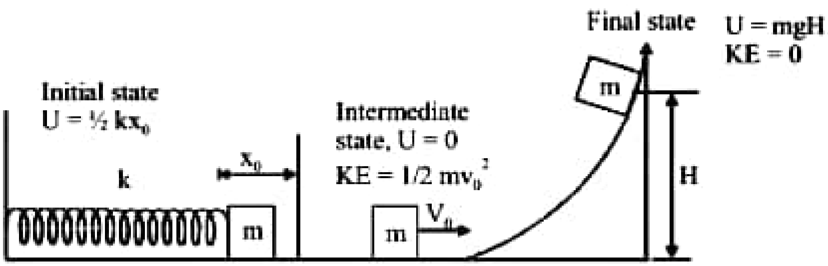
$$\Delta U + \Delta K\,E = 0$$

or $\quad \Delta U = -\Delta KE$

or $\quad U + K\,E = \text{constant}$

or $\quad U_{in} + K\,E_{in} = U_f + K\,E_f$

The surfaces shown in the figure are frictionless and horizontal surface is taken as reference level.

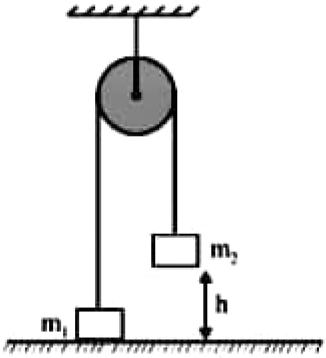


By conservation of mechanical energy

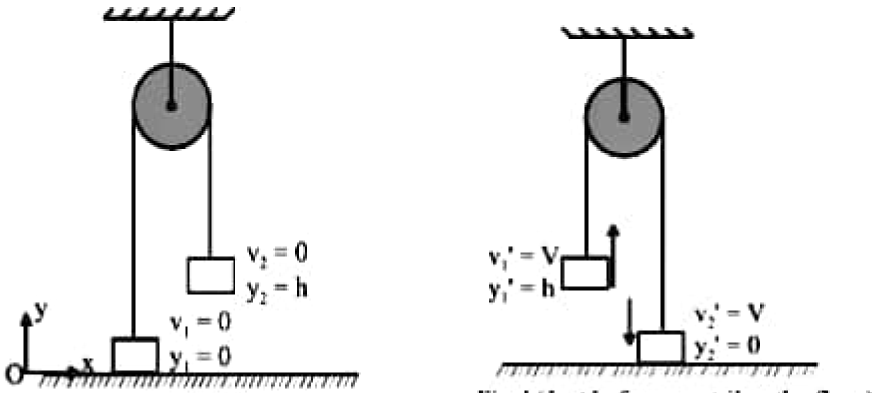
$$\frac{1}{2}kx_0^2 + 0 = 0 + \frac{1}{2}mv_0^2 = mgH + 0$$

Illustration :

Two block with masses m_1 = 3kg and m_2 = 5 kg are connected by a light string that slides over a frictionless pulley as shown in figure. Initially, m_2 is held 5 m off the floor while m_1 is on the floor. The system is then released. At what speed does m_2 hit the floor ?



Sol. *The inital and final configurations are shown in the figure. It is convenient to set U_g = 0 at the floor. Initially, only m_2 has potential energy. As it falls, it loses potential energy and gains kinetic nergy. At the same time, m_1 gains potential energy and kinetic energy. Just before m_2 lands, it has only kinetic energy. Let v the final speed of each mass. Then, using the law of conservation of mechanical energy.*



Initial

Final (Just before m_2 strikes the floor)

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}(m_1 + m_2)v^2 + m_1gh = 0 + m_2gh$$

$$v^2 = \frac{2(m_2 - m_1)gh}{m_1 + m_2}$$

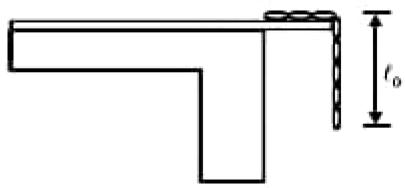
Putting $m_1 = 3$ kg; $m_2 = 5$ kg; $h = 5$ m and $g = 10\ m/s^2$

we get $$v^2 = \frac{2(5-3)(10)(5)}{5+3}$$

or $v = 5$ m/s.

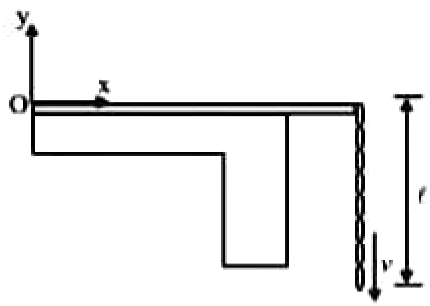
Illustration :

A chain of length ℓ = 80 cm and mass m = 2 kg is hanging from the end of plane so that the length ℓ_0 of the vertical segment is 50 cm as shown in the figure. The other end of the chain is fixed by a nail. At a certain instant, the nail is pushed out, what is the velocity of the chain at the moment it completely slides off the plane ? Neglect the friction.



Sol. *We assume the zero potential energy level at the horizontal plane. The initial and final configuration of the chain are shown in the figure. Initially,* $KE_{in} = 0$

$$U_{in} = 0 + \left(\frac{m}{l}l_0\right)g\left(-\frac{l_0}{2}\right)$$



or $$U_{in} = -\frac{ml_0^2}{2l}g$$

Note *that the part of chain lying over the table has zero potential energy.*

Finally, $$KE_f = \frac{1}{2}mv^2$$

Where v is the final velocity of the chain.

and $$U_f = -mg\frac{l}{2}$$

Using the law of energy conservation

$$KE_f + U_f = KE_{in} + U_{in}$$

$$\frac{1}{2}mv^2 - mg\frac{\ell}{2} = 0 - \frac{ml_0^2 g}{2l}$$

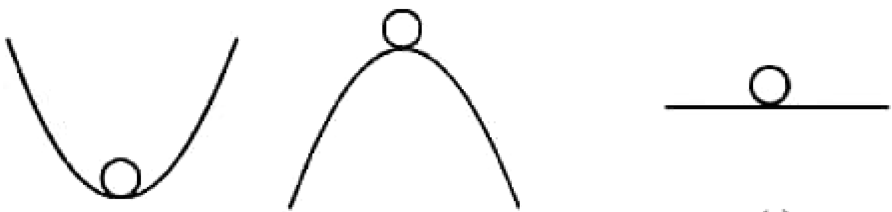
or $$v = \sqrt{\frac{g}{l}\left(l^2 - l_0^2\right)}$$

Putting $l = 0.8m$; $l_0 = 0.5m$; $g = 10\ m/s^2$, *we get*

$$v = \sqrt{5.1}\ m/s$$

or $$v = 2.23\ m/s$$

Types of equilibrium on the basis of stability



(a) (b) (c)

Suppose a small ball is placed on a smooth track under three different situations as shown in figure (a), (b) & (c). In all the three situation, the ball, is in equilibrium

(A) In figure (a), when the ball is slightly displaced from its equilibrium position, it tries to attain the shown position again, such type of equilibrium is called **stable equilibrium**.

Here potential energy in equilibrium position is minimum as compares to its neighbouring points i.e. under stable equilibrium, potential energy is minimum

$$\text{i.e. } \frac{dU}{dr} = 0 \text{ and } \frac{d^2U}{dr^2} > 0$$

(B) In figure (b), when the ball is slightly displaced from its equilibrium position it tends to move farther form the shown equilibrium position. Such type of equilibrium is called **unstable equilibrium.**

Here potential energy in equilibrium is maximum as compared to its near by points

$$\text{i.e. } \frac{dU}{dr} = 0 \text{ and } \frac{d^2U}{dr^2} < 0$$

(C) In figure (c), when the ball is displaced, it accepts the new position as equilibrium position, such type of equilibrium is called **neutral equilibrium.**

Here potential energy remains uniform for the equilibrium position

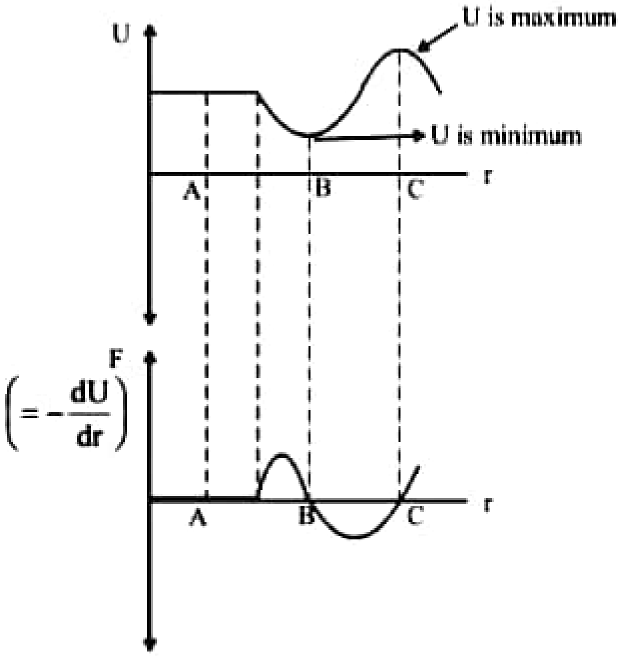
i.e. $\frac{dU}{dr}=0$ and $\frac{d^2U}{dr^2}=0$

Although, the above discussion is under the effect of gravity but the results observed is applicable in other situations also where a particle can move under the effect of the conservative force only.

The above result can also be studied with the help of the following graphs.

For a particle whose position (r) varies along a straight line, the graphs below

Show variation of U vs r and F vs r.



At Point A $F=0$; $\frac{dU}{dr}=0$, but $F=0$ at its nearly points also. So when slightly displaced from A, the new position is also equilibrium. Thus point, A shown is the position of neutral equilibrium.

At Point B $F=0$; $\frac{dU}{dr}=0$, Now when it is slightly displaced towards left of B, force is positive i.e. towards right and when it is slightly displaced towards right of B, force is negative i.e. towards right. Thus force tries to bring the particle towards B again. This type of force is called restoring force and the point B is the position of stable equilibrium.

At Point C $F=0$, $\frac{dU}{dr}=0$ but when particle is displaced slightly form it towards any direction, force acts in that direction only i.e. to move the particle away from C. Thus point C is the position of unstable equilibrium.

Illustration :

The potential energy of a conservative system is given by

$$U = ax^2 - bx$$

Where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable unstable or neutral.

Sol. *In a conservative field*

$$F = -\frac{dU}{dx} = -\frac{d}{dx}(ax^2 - bx) = -(2ax - b)$$

$$\therefore \quad F = b - 2ax$$

For equilibrium F = 0

or $$b - 2ax = 0 \quad \therefore \quad x = \frac{b}{2a}$$

From the given equation we can see that $\frac{d^2U}{dx^2} = 2a$ *(positive), i.e., U minimum.*

Therefore, $x = \frac{b}{2a}$ *is the stable equilibrium position.*

Power

Power is the work done per unit time. If ΔW a work done in time Δt, the average power is $P_{av} = \frac{\Delta W}{\Delta t}$

Average power becomes intantaneous power as Δt approches to zero.

$\therefore$ Instantaneous Power $P = \lim_{\Delta t \to 0}\left(\frac{\Delta W}{\Delta t}\right)$

$$\Rightarrow \quad P = \frac{dW}{dt}$$

Due to a particular force on a particle

$$\Delta W = \vec{F}\cdot\Delta\vec{S}$$

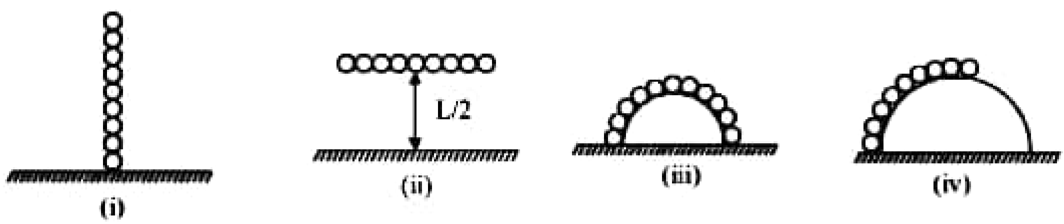
$$\therefore \quad P = \lim_{\Delta t \to 0} \vec{F}\cdot\frac{\Delta\vec{S}}{\Delta t} = \vec{F}\cdot\frac{d\vec{s}}{dt}$$

$$\Rightarrow \quad P = \vec{F}\cdot\vec{V}$$ (where $\vec{V}$ is velocity)

Centripetal force is always perpendicular to velocity, so power due to it is zero.

Practice Exercise

Q.1 A chain of length L and mass M is arranged as shown in following four cases. Arrange the potential energy (assumed zero at horizontal surface) in decreasing order.



Q.2 A particle is constrained to move in the region $x \geq 0$ and its potential energy is given by

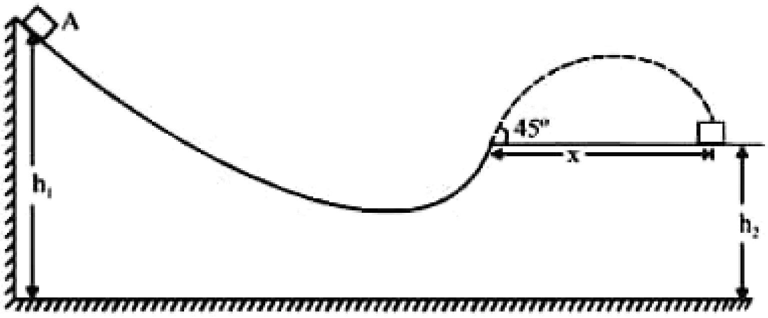
$U = 2x^4 - 3x^2$ J (where x is in m)

For what values of x, the particle is under

(i) Stable equilibrium (ii) Unstable equilibrium

Q.3 A force $\vec{F} = -k(y\hat{i} + x\hat{j})$ (where K is a positive constant) acts on a particle moving in the X–Y plane.Starting from the origin, the particle is taken along the positive x-axis to the point (a, 0) and then parallel to the y-axis to the point (a,a). Find the total work done by the force F on the particle.

Q.4 A block starts from rest at point A, slides down a frictionless incline that terminates in a ramp pointing up at 45° angle, as shown in figure. Find an expression for the horizontanl range x shown in the figure as a function of the heights h_1 and h_2.



Q.5 A horse pulls a cart upto 200m in 5 minutes, exerting a force of 750N along the displacement of the cart. Find its power output measured in watts and in horsepower.

Answers

Where work done by tension, $W_T = 0$ (since tension is continuously perpendicular to motion) and work done by gravity $W_g = -mgh$

$$= -mgR(1-\cos\theta)$$

$\therefore \quad -mgR(1-\cos\theta) = \frac{1}{2}m(v^2-u^2)$

$\therefore \quad v^2 = u^2 - 2gR(1-\cos\theta) \qquad$ ......(ii)

$\therefore$ from equation (i) and (ii)

$$T = mg\cos\theta + \frac{m}{R}[u^2 - 2gR(1-\cos\theta)]$$

$\therefore \quad T = \frac{mu^2}{R} + 3mg\cos\theta - 2mg \qquad$ ....(iii)

Thus T is maximum when $\theta = 0^\circ$ i.e. at the lower most position A

$$T_{max} = T_A = mg + \frac{mu^2}{R}$$

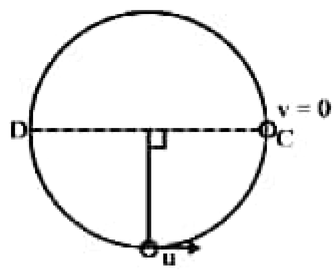
Also T is minimum when $\theta = 180^\circ$ i.e. at the top most point B

$$T_{min} = T_B = \frac{mu^2}{R} - 5mg$$

Thus : $T_{max} - T_{min} = 6\,mg$ (independent of initial speed provided the particle is able to complete the circular path)

From equations (i) and (iii), following results are obtained

Case I



If $u = \sqrt{2gR}$, V becomes zero exactly at point C

Also when it is at point C tension, $T = m\frac{v^2}{R} = 0$

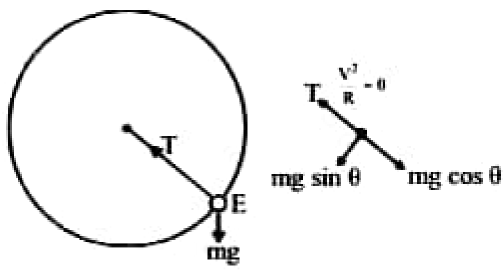
Now it oscillates between points C and D.

Case II

If $u < \sqrt{2gR}$, V becomes zero

even before $\theta = \frac{\pi}{2}$, let at point E

At this point



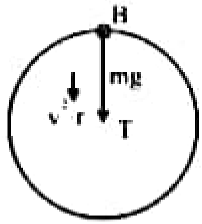
$$T - mg\cos\theta = m\left(\frac{v^2}{R}\right) = 0$$

$\Rightarrow \quad T = mg\cos\theta \neq 0$

Case III

If $u = \sqrt{5gR}$ then at the top most point B where $\theta = 180°$, the speed of the particle is

$$v = \sqrt{5gR - 2gR(1-(-1))} = \sqrt{5gR - 4gR} = \sqrt{gR}$$



At this point, $mg + T = \dfrac{m(\sqrt{gR})^2}{R}$

$\Rightarrow \quad T = 0$

i.e. string is just slacked, but it still continues to move in circuler path, since it has got speed.

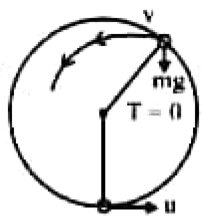
Case IV

If $u > \sqrt{5gR}$ and $T > 0$ even at the topmost point (B) and particle completes circular path

Case V

If $\sqrt{5gR} > u > \sqrt{2gR}$

T becomes zero at some point between C and B i.e. for $\frac{\pi}{2} < \theta < \pi$. And that instant v is non zero [see equation (ii)] and thus particle move in parabolic path after that due to gravity only.

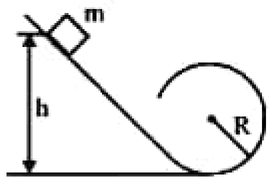


Motion of a particle at the inner surface of a vertical circular track

Same results as above are obtained when a body is given some speed u at the bottom most point inside a fixed circular loop as shown. Here instead of tension; normal force due to inner surface of the loop comes into action.

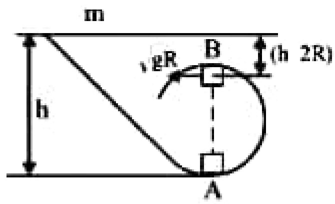
Illustration :

Figure shows a an incline which ends into a circular track of radius R. What should be the minimum value of height (h), so that the small object shown after release, is able to complete the loop. Neglect friction.



Sol. *It completes the loop, if its speed is atleast $\sqrt{gR}$ at B and $\sqrt{5gR}$ at A. Applying work energy theorem, for motion from starting point to the point B,*

$$mg\,(h - 2R) = \frac{1}{2} m\,[(\sqrt{gR})^2 - 0]$$



$$\Rightarrow \quad h = \frac{5R}{2}$$

Alternatively : Applying work - energy theorem for motion from starting point to A.

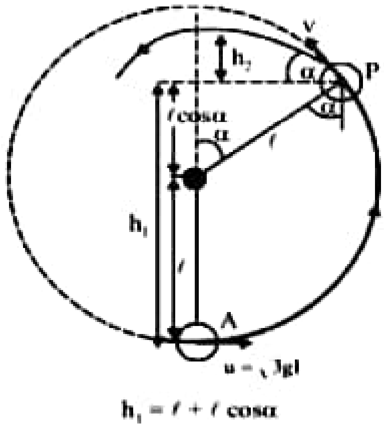
$$mgh = \frac{1}{2} m \left[(\sqrt{5gR})^2 - 0\right]$$

$$\Rightarrow \quad h = \frac{5R}{2}$$

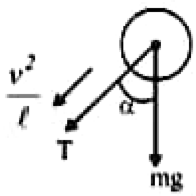
Illustration :

The bob of a simple pendulum of length ℓ is given a sharp hit to impart it a horizontal speed of $\sqrt{3g\ell}$. When it was at its lowermost position. Find (i) angle α shwon of the string from upside of vertical and speed of the particle when the string becomes slack. (ii) maximum height (from the bottom)

Sol. *(i) Since $\sqrt{2gl} < u < \sqrt{5gl}$, the string slacks somewhere between horizontal point and the topmost point.*
Let string slock at P, where speed is say v.



At point P,



$$\Rightarrow \quad T + mg\cos\alpha = \frac{mv^2}{\ell}$$

As the string slacks, T = 0

$$\Rightarrow \quad mg\cos\alpha = \frac{mv^2}{\ell}$$

$$\Rightarrow \quad v = \sqrt{g\ell\cos\alpha} \qquad \text{........(i)}$$

Applying work energy theorem for motion from A to P

$$-mg\,h_1 = \frac{1}{2}m(v^2 - u^2)$$

$\therefore$ *from equation (i)*

$$-mg\ell\,(1+\cos\alpha) = \frac{1}{2}\,m\,[g\,(\ell\cos\alpha) - 3g\ell]$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{3}$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \quad \text{equation (i), } u = \sqrt{\frac{g\ell}{3}}$$

(ii) Now, after slackening of the string, the motion of the bob is under gravity only, for which the maximum height from P is given by

$$h_2 = \frac{v^2\sin^2\alpha}{2g}$$

where $v^2 = \dfrac{g\ell}{3}$ *and* $\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\dfrac{1}{3}\right)^2 = \dfrac{8}{9}$

$$\therefore \quad h_2 = \frac{\left(\frac{g\ell}{3}\right)\left(\frac{8}{9}\right)}{2g} = \frac{4\ell}{27}$$

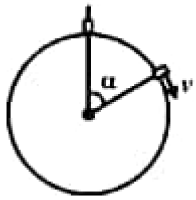
$$\therefore \quad \text{maximum height from } A \text{ is} = h_1 + h_2 = \ell\left(1+\frac{1}{3}\right) + \frac{4\ell}{27}$$

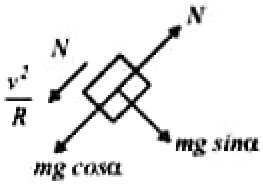
$$= \frac{40\ell}{27}$$

Illustration :

A particle slides on the surface of a fixed smooth sphese starting from the topmost point. Find the angle rotated by radius through the particle, where it leaves contact with the sphere. Also find speed at that instant.

Sol. *Let it rotates by angle* α





$$mg\cos\alpha - N = \frac{mv^2}{R}$$

It looses contact i.e. N = 0

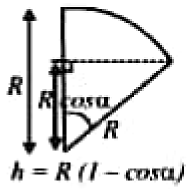
$$\Rightarrow \quad mg\cos\alpha = \frac{mv^2}{R}$$

$$\Rightarrow \quad v^2 = gR\cos\alpha \qquad \text{.........(i)}$$

Also applying work energy theorem

$$mgh = \frac{1}{2}m(v^2 - 0)$$

$$\Rightarrow \quad mgR(1-\cos\alpha) = \frac{1}{2}m(gR\cos\alpha - 0)$$



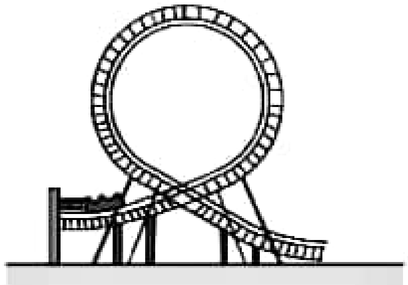
$$\Rightarrow \quad \cos\alpha = \frac{1}{3}$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{1}{3}\right)$$

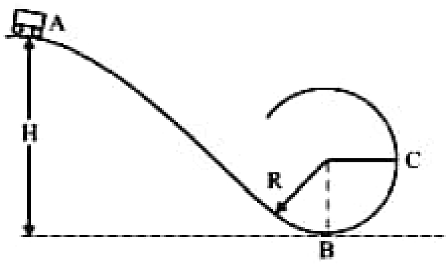
Also from equation (i), $\quad v = \sqrt{\frac{gR}{3}}$

Practice Exercise

Q.1 An 900-kg roller-coaster car is launched from a giant spring of constant k = 31 kN/m into a frictionless loop-the-loop track of radius 6.2 m as shown in Figure. What is the minimum amount that the spring must be compressed if the car is to stay on the track ?



Q.2 A small toy car of mass m slides with negligible friction on 'loop' the loop track as shown in figure. The toy car starts from rest at point A which is height H = 2 R above level of the lowest point of the track :



(i) What normal force is exerted by the track on the toy car at point B ?

(ii) What are the speed and normal force at point C ?

(iii) At what height will the ball leave the track and to what maximum height will it rise afterwards ?

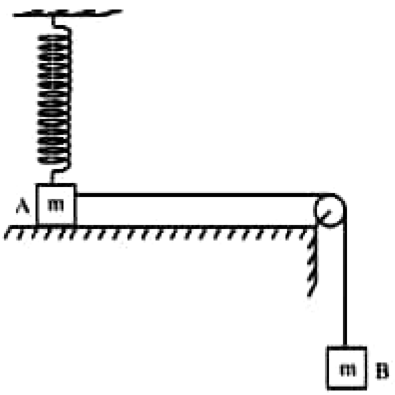
Q.3 A particle attached to a vertical string of length 1 m is projected horizontally with a velocity of $5\sqrt{2}$ m/s.

(a) What is maximum height reached by the particle from the lowermost point of its trajectory.

(b) If the string breaks when it makes an angle of 60° with downward vertical, find maximum height reached by the particle from the lowermost point of its trajectory.

Solved Example

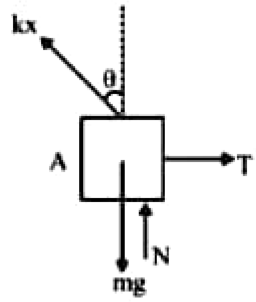
Q.1 Two blocks A and B, each having a mass of 320 g connected by a light string passing over a smooth light pulley. The block A is attached to a spring of spring constant 40 N/m whose other end is fixed to a support 40 cm above the horizontal surface as shown. Initially, the spring is vertical and ustretched when the system is released to move. Find the velocity of the block A at the instant it breaks off
the surface below it . Take $g = 10\ m/s^2$.



Sol. At the instant of break-off ,N = 0

$\Rightarrow$ $kx \cos\theta = mg$

Where extension in the spring



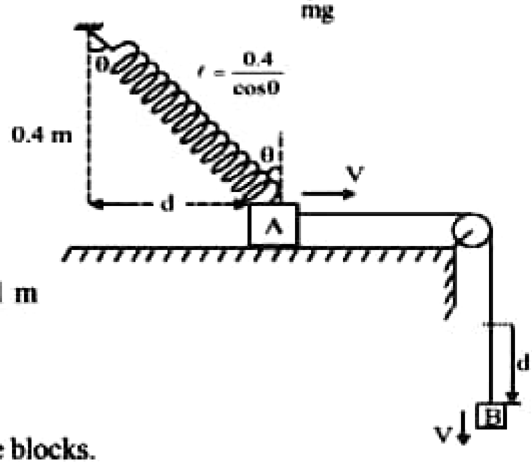
$$x = \ell - 0.4 = \frac{0.4}{\cos\theta} - 0.4$$

$\therefore$ $$k\left(\frac{0.4}{\cos\theta} - 0.4\right)\cos\theta = mg$$

$$k(0.4)(1-\cos\theta) = mg$$

$\Rightarrow$ $$\cos\theta = 1 - \frac{mg}{0.4k}$$

$$= 1 - \frac{0.320\times 10}{0.4\times 40}$$



$$\cos\theta = \frac{4}{5} \quad \Rightarrow \quad x = \frac{0.4\times 5}{4} - 0.4 = 0.1\ m$$

Also $$\tan\theta = \frac{3}{4} \quad \Rightarrow \quad d = 0.4\tan\theta = 0.3\ m$$

$\therefore$ By work energy theorem for motion of both the blocks.

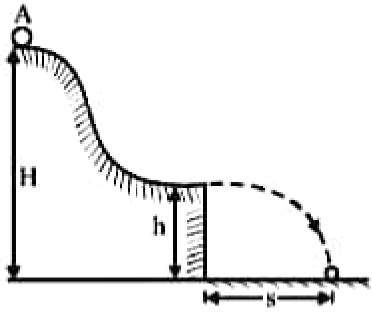
$$W_g + W_s = \Delta KE$$

$\Rightarrow$ $$mgd - \frac{1}{2}kx^2 = \left(\frac{1}{2}mv^2\right)\times 2$$

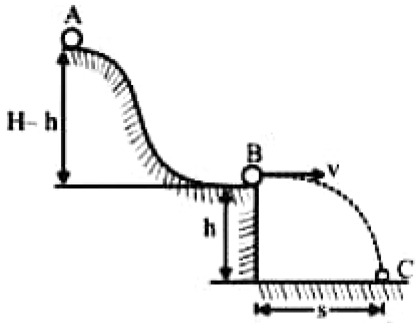
$\Rightarrow$ $$0.32\times 10\times 0.3 - \frac{1}{2}(40)(0.1)^2 = (0.32)\,v^2$$

$\Rightarrow$ $v = 1.54\ m/s$

Q.2 A small body starts sliding from the height H with zero velocity down a smooth hill which has horizontal portion as shown. What must be the height of the horizontal portion 'h' to ensure the maximum distance 's' covered by the body ? What is the maximum value of 's' ?



Sol.



Applying work energy theorem for motion from A to B

$$mg(H-h) = \frac{1}{2}m(V^2 - 0)$$

$\therefore \quad V = \sqrt{2g(H-h)}$

If time taken by the disc to move from point B to the point C on ground is t, then

$$\Delta y = u_y t - \frac{1}{2}gt^2 = h$$

$\therefore \quad 0 - \frac{1}{2}gt^2 = -h \Rightarrow t = \sqrt{\frac{2h}{g}}$

$S = \Delta x = vt$

$\therefore \quad S = \sqrt{2g(H-h)\times\frac{2h}{g}} = 2\sqrt{Hh - h^2}$ .....(i)

for s to be maximum, $(Hh - h^2)$ is maximum

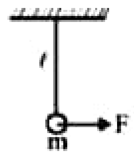
i.e. $\frac{d}{dh}\left[Hh - h^2\right] = 0$

$\therefore \quad H - 2h = 0 \quad \Rightarrow \quad h = \frac{H}{2}$

Putting this value of h in equation (i)

$\therefore \quad S_{max} = 2\sqrt{H\left(\frac{H}{2}\right) - \left(\frac{H}{2}\right)^2} = H$

Q.3 A constant horizontal force F of magnitude equal to $\frac{mg}{2}$ begins to act on the bob of pendulum shown when the bob was at rest. What maximum angle the string makes with the vertical ?



Sol. Let the string makes angle θ with the vertical when it comes to rest (momentarily) as shown

$\therefore \quad x = \ell \sin\theta$

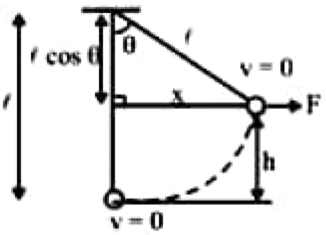
and $\quad h = \ell - \ell\cos\theta = \ell(1 - \cos\theta)$

Here work done by gravity $W_g = -mgh = -mg\ell(1 - \cos\theta)$

Work done by tension, $W_T = 0$ [$\vec{T} \perp \vec{v}$]

Work done by force F, $W_F = F \times$ (displacement in the direction of F) = Fx

$$= \frac{mg}{2}\ell\sin\theta$$



$\therefore$ Applying work-energy theorem

$$W_g + W_F + W_T = \Delta K \cdot E = 0$$

$$\therefore \quad \frac{1}{2}\sin\theta = 1 - \cos\theta$$

$$\therefore \quad \frac{1}{2}\left[2\times\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right] = 2\sin\left(\frac{\theta}{2}\right)$$

$$\therefore \quad \tan\left(\frac{\theta}{2}\right) = \frac{1}{2}$$

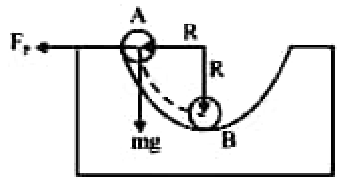
$$\therefore \quad \theta = 2\tan^{-1}\left(\frac{1}{2}\right)$$

Q.4 A small body is placed at rest at the bottom B of a smooth hemispherical surface of wedge as shown. If the wedge is shifted horizontally towards right with acceleration $a_0 = 3g$, find speed of the body w.r.t the wedge at the instant the body reaches points A.

Sol. Here we need calculation w.r.t the wedge which is accelerated i.e. non-inertial frame of reference, So we have to consider Pseudo froce (F_P) and work done by the Pseudo force (W_P) also.

Where $F_P = ma_0 = 3mg$ and is towards left i.e. apposite to a_0

By work-energy theorem for motion from B to A

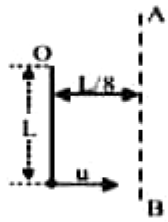


$$F_P R - mgR = \frac{1}{2}m(v^2 - 0)$$

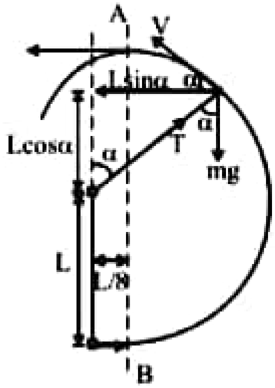
$$\Rightarrow \quad 3mgR - mgR = \frac{1}{2}mv^2$$

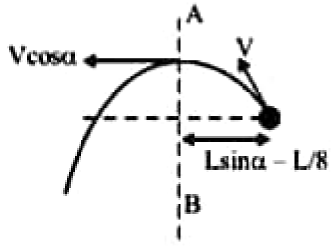
$$\therefore \quad v = 2\sqrt{gR}$$

Q.5 A particle is suspended vertically from a point O by an inextensible massless string of length L. A vertical line AB is at a distance L/8 from O as shown. The object given a horizontal velocity u. At some point, its motion ceases to be circular and eventually the object passes through the line AB. At the instant of crossing AB, its velocity is horizontal. Find u.



Sol Let the string slacks when the particle is at point P as shown





At point P, $T + mg\cos\alpha = \dfrac{mV^2}{L}$

where $T = 0$ (as string slacks)

$\therefore \quad mg\cos\alpha = \dfrac{mV^2}{L}$

$\Rightarrow \quad V^2 = gL\cos\alpha$ ....(i)

After this it undergoes parabolic path. When it passes through line AB its velocity is horizontal which implies that $(L\sin\alpha - L/8)$ is half of horizontal range.

i.e. $L\sin\alpha - \dfrac{L}{8} = \dfrac{1}{2}\left[\dfrac{V^2\sin(2\alpha)}{g}\right]$

$\therefore$ from equation (i)

$$L\sin\alpha - \frac{L}{8} = \frac{1}{2}\left[\frac{gL\cos\alpha(2\sin\alpha\ \cos\alpha)}{g}\right]$$

$\therefore \quad \sin\alpha - \dfrac{1}{8} = \sin\alpha\cos^2\alpha$

$\Rightarrow \quad \sin\alpha\,(1 - \cos^2\alpha) = \dfrac{1}{8}$

$\Rightarrow \quad \sin^3\alpha = \dfrac{1}{8} \quad \Rightarrow \quad \sin\alpha = \dfrac{1}{2} \quad \Rightarrow \quad \alpha = 30^\circ$

$\therefore \quad V^2 = \dfrac{gL\sqrt{3}}{2}$

Also by applying work energy theorem

$$-mgL(1 + \cos\alpha) = \frac{1}{2}m(V^2 - u^2)$$

$= \quad -gL\left(1 + \dfrac{\sqrt{3}}{2}\right) = \dfrac{1}{2}\left(\dfrac{gL\sqrt{3}}{2} - u^2\right)$

$\therefore$ On solving, we get $\quad u = \sqrt{\dfrac{gL}{2}\left(4 + 3\sqrt{3}\right)}$

Rotational Mechanics

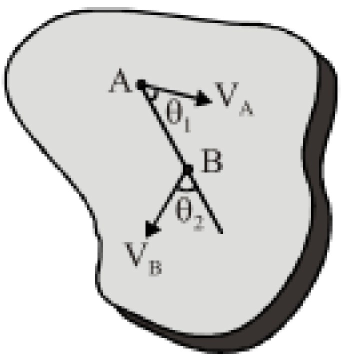
Introduction

The previous chapter we are studied only the translational motion of objects. The most general motion of a rigid body includes rotational as well as translational motions. Thus to study this general motion appropriate kinematics as well as dynamics must be studied in detail. Any body in general interacts with its surroundings through four basic interactions strong, electromagnetic, weak and gravitational. Not all forces produce rotational motion in the body. It is only certain conditions on force and its position and orientation that can produce rotational effects. The rotational variables have a definite relation with corresponding linear variables. This is studied under rotational kinematics. The causes of rotational motion and the factors governing changes in rotational state of motion are subject matter of rotational dynamics. The subject matter of this chapter includes kinematics and dynamics of rotational motion.
There can be pure rotation of a body or can be rotation with translation. We begin with pure rotational cases, go on developing the basics of rotational motion and subsequently deal with rolling motion which is a combination of rotational as well as translational motion.

Rotational kinematics

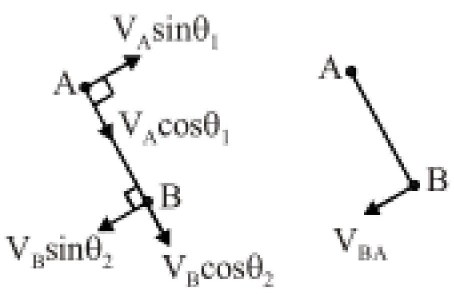
Rigid body :

Rigid body is defined as system of particles in which distance between each pair of partices remain constant (with respect to time) that means the shape and size do not change, during the motion Eg : Fan, Pen, Table, stone and so on.



Our body is not a rigid body, two blocks with a spring attached between them is also not a rigid body.

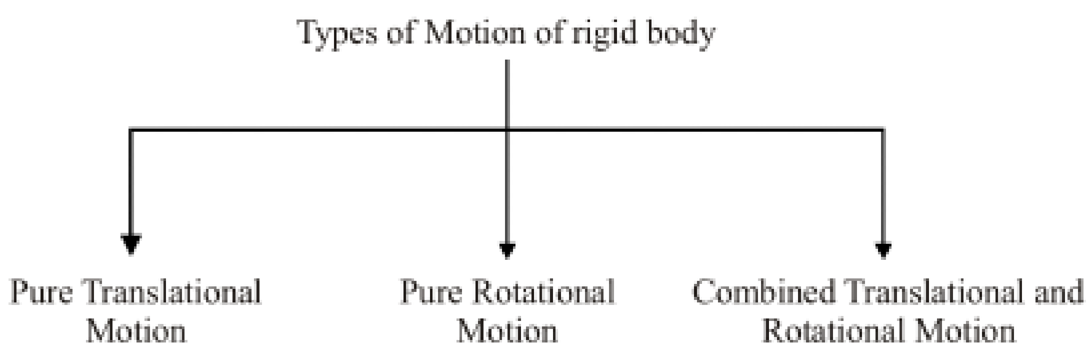
For every pair of particles in a rigid body, there is no velocity of seperation or approach between the particles. In the figure shown velocities of A and B with respect ground re V_A and V_B respectively.



If the above body is rigid

$$V_A \cos\theta_1 = V_B \cos\theta_2$$

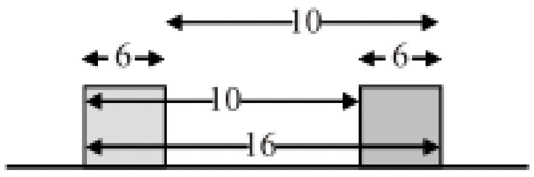
V_{BA} = relative velcoity of B with respect to A.



Pure Translation Motion :

A body is said to be in pure traslational motion if the displacment of each particle is same during any time interval howsoever small or large. In this motion all the particle have same $\vec{S}$, $\vec{V}$ & $\vec{a}$ at an instant.

Ex : A box is being pushed on a horizontal surface.



$$\vec{V}_{CM} = \vec{V} \text{ of any particle}$$

$$\vec{a}_{CM} = \vec{a} \text{ of any particle}$$

$$\Delta\vec{S}_{CM} = \Delta\vec{S} \text{ of any particle}$$

Pure Rotational Motion :

A body is said to be in pure rotational motion if the perpendicular distance of each particle remains constat from a fixed line or point and do not move parallel to the line, and that line is known as axis of rotation. In this motion all the particles have same $\vec{\theta}, \vec{\omega}$ & $\vec{a}$ at an instant. Eg : - a rotating ceiling fan, arms of a clock.

For pure rotation motion -

$\theta = \dfrac{s}{r}$ Where θ = angle rotated by the particle

s = length of arc traced by the particle.

r = distance of particle from the axis of rotation

$\omega = \dfrac{d\theta}{dt}$ Where ω = angular speed of the body

$\alpha = \dfrac{d\omega}{dt}$ Where α = angular acceleration of the body.

All the paramenters θ, ω and α are same for all the particles. Axis of rotation is perpendicular to the plane of rotation of particles.

Specila case : If α = constant,

$\omega = \omega_0 + \alpha t$ Where ω_0 = initial angular speed

$\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$ t = time interval

$\omega^2 = \omega_0{}^2 + 2\alpha\theta$

Practice Exercise

Q.1 The motor of an engine is rotation about its axis with an angular velocity of 100 rev/minute. It comes to rest in 15s, after being switched off. Assuming cosntant angular decelaration, calculate the number of revolutions made by it before coming to rest.

Q.2 Starting from rest, a fan takes five seconds to attain the maximum speed of 400 rpm (revolutions per minute). Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.

Answers

Q.1 12.5 rev   Q.2 2.5 sec.

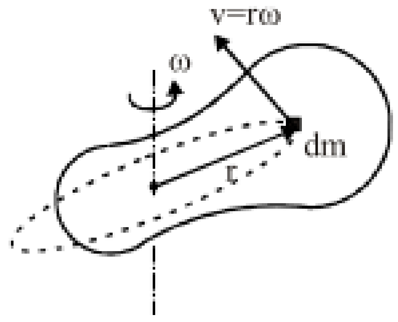
Kinetic Energy of Rotating Body

The rotating blade of a fan has some kinetic energy due to rotational motion which can not be expressed directly as $K.E = \frac{1}{2}mv^2$ since all the points do not have same speed.

To find rotational K.E. we take the fan's blade as a collection of different very small particles called elements. One such element has mass dm and is at distance r from the rotational axis as shown. Its kinetic energy can be given as

$$d(K.E.) = \frac{1}{2}(dm)v^2 = \frac{1}{2}(dm)(r\omega)^2$$

$$= \frac{1}{2}(dm)r^2\omega^2$$



The rotational kinetic energy of the body is given by summing i.e. intergrating the kinetic energy of all the elements of the body.

$$K.E. = \int d(K.E.) = \int \frac{1}{2}(dm)r^2\omega^2$$

Since ω is same for every element of a rigid body so we take ω^2 outside the integral.

$$\therefore \quad K.E. = \frac{1}{2}\omega^2 \int r^2 dm$$

we may write $\mathbf{K.E. = \frac{1}{2}I\omega^2}$

where $I = \int r^2 dm$ called Moment of Inertia.

The above equation is analogous to the $K.E. = \frac{1}{2}mv^2$ i.e. Kinetic energy of a body having translational motion. Here ω is analogous to v. Also I is analogous to mass m i.e. I plays the same role in rotational motion as that of mass in translational motion. In other words as the inertia to the translational motion is due to the mass, inertia to the rotational motion is due to the quantity **Moment of Inertia**.

Moment of Inertia

Definition

It is the property of a rigid body by virtue of which it opposes change in its rotational motion.

* This is always taken w.r.t. a axis of rotation.

* This plays same role in rotational motion as mass plays in translational motion

* Difference between mass & M.I. (Moment of Inertia) is that mass is property of body & is independent of any reference axis choosen but MI depends on the mass as well as its distribution about the given axis of rotation. In other words it depends on

(i) axis of rotation

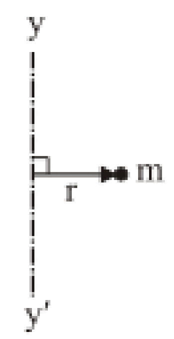
(ii) shape of the body

(iii) size of the body

(iv) density of the material of the body] mass depends only on these two things.

MI of a point mass :

$I_{yy'} = mr^2$

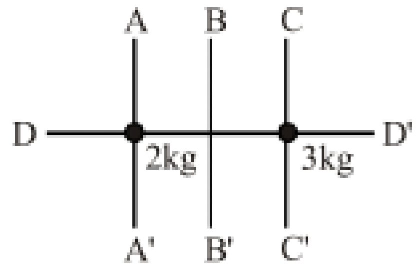


r is perpendicular distance from mass to axis of rotation yy′

Illustration :

Two particles of masses 2 kg and 3 kg are separated by 4 m as shown. Find M.I. of the system of particle about axis.

(A) AA' *(B) BB' (C) CC'* *(D) DD'*



Sol. (A) $I_{AA'} = 2(0)^2 + 3(4)^2 = 48\ kgm^2$ *(2 kg lies on the axis only)*

(B) $I_{BB'} = 2(2)^2 + 3(2)^2 = 20\ kgm^2$

(C) $I_{CC'} = 2(3)^2 + 3(0)^2 = 18\ kgm^2$

(D) $I_{DD'} = 2\,(0)^2 + 3(0)^2 = 0$ *(as both lies on the axis)*

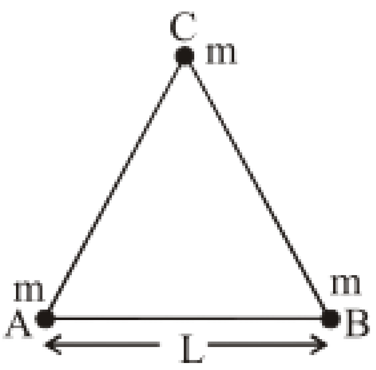
Illustration:

Find M.I. of a system of three particles lying on an equilateral triangle as shown about the axis

(A) Which is $\perp$ to AB and passes through its centre.

(B) Passing through the side AC

(C) Passing through centroid and $\perp$ to the plane of ABC.

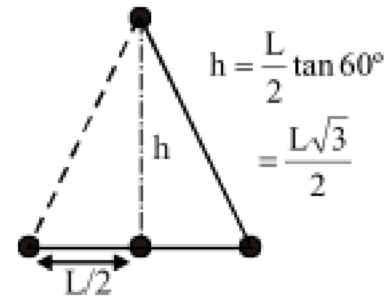


Sol. (A) $I = I_! + I_B + I_C$

$= m\,(L/2)^2 + m\,(L/2)^2 + m\times 0$

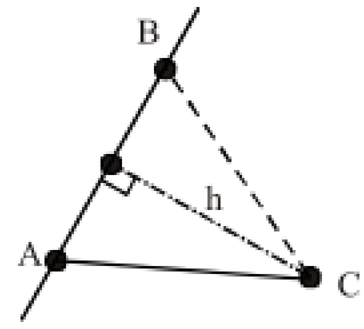
$= mL^2$

(B) $I_{BB'}.\ 0+0+mh^2$ $\qquad [\because I_A = I_B = 0]$

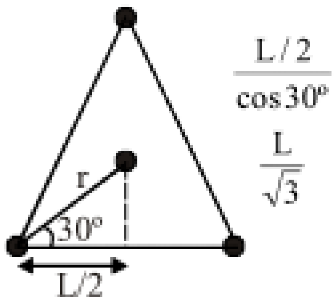


$= \dfrac{3mL^2}{4}$

(C) $I = 3 \times mr^2$



$= 3\,m\left(\dfrac{L}{\sqrt{3}}\right)^2$



$= mL^2$

Illustration:

Find the moment of inertia about C.O.M. system of two particles of mass m & M separated by distance l.

Sol. *Position C.O.M. from m is*

$$r_1 = \frac{m(0)+M(l)}{m+M} = \frac{Ml}{m+M}$$

from M is, $r_2 = l - r_1 = \dfrac{ml}{m+M}$

$$\therefore \quad I = mr_1^2 + Mr_2^2 = m\left(\frac{Ml}{m+M}\right)^2 + M\left(\frac{ml}{M+m}\right)^2$$

$$= \left(\frac{mM}{M+m}\right)l^2$$

M.I of continuous rigid body

As we have discussed in previous topic for a continuous rigid body, $I = \int r^2 dm$

To solve for above equation, we take dm in terms of variable r/dr or both r and dm in terms of another variable.

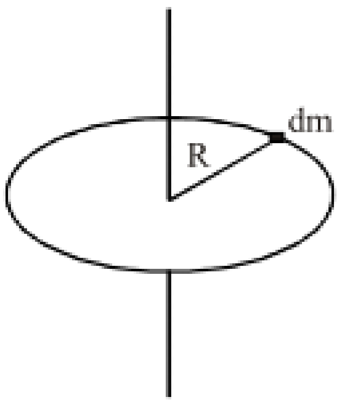
To substitute dm, we take linear, areal and volume mass densities as we had taken in previous chapter.

M.I. of a ring

Let mass of the ring is M and radius is R.

$\therefore$ M.I. of small element of mass dm is $dI = (dm)R^2$

$\therefore \quad I = \int R^2 (dm)$



but here distance of each element from the axis is same (= R) and so can be taken out of integral

$\therefore \quad I = R^2 \int dm$

$\therefore \quad I = MR^2$

M.I. of uniform rod

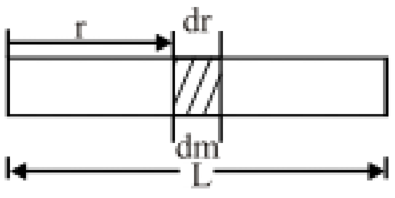
(i) About one of its end

M.I. of the element which is at distance r from the end is

$I = \int dI = \int r^2 (dm)$

Where, mass of the dement, dm = (mass per unit length)×(length of the element)

$$dm = \left(\frac{M}{L}\right) dr$$

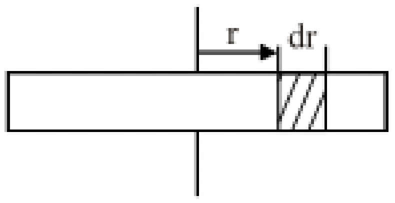


$$\therefore \quad I = \int_0^L r^2 \left(\frac{M}{L}\right) dm = \frac{M}{L}\int_0^L r^2 dr = \frac{M}{L}\left[\frac{r^3}{3}\right]_0^L$$

$$\therefore \quad I = \frac{ML^2}{3}$$

(ii) About its C.O.M

$$dI = (r^2)(dm) = \frac{M}{L}\, r^2\, dr$$



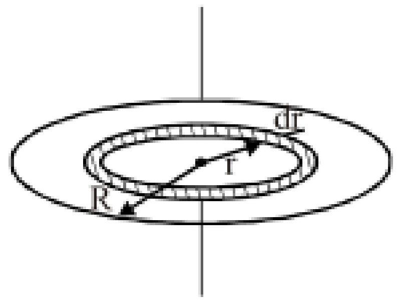
here M.I. can be fround out by integrating the above from left end to rigth end

$$\therefore \qquad I = \frac{M}{L}\int_{-L/2}^{L/2} r^2 dr = \left[\frac{r^2}{3}\right]_{-L/2}^{L/2} = \frac{M}{L}\times\frac{1}{3}\left[\left(\frac{L^3}{8}\right)=\left(\frac{-L^3}{8}\right)\right]$$

$$\therefore \qquad I = \frac{ML^2}{12}$$

Note : Here we can observe M.I. about the axis passing through C.O.M is less then that about end. In fact for various axis which are parallel to each other M.I. is minimum about the axis which passes through C.O.M. This is also proved later on in this chapter only.

M.I. of a uniform disc



Mass – M, Radius R. Lets taken an elementary ring of radius r and thickness dr.

$\therefore$ $\qquad dI = (dm)r^2$ (for ring)

Also $\qquad dm$ = (mass per unit area) × (area of alimentary ring)

Where $\qquad$ area of elementary ring, $dA = \pi\,[(r + dr)^2 - r^2]$

$$= \pi\,[2\,r\,dr + (dr)^2]$$

$\therefore$ $\qquad dA \simeq 2\,\pi\,r\,dr$ $\qquad$ [neglecting $(dr)^2$ being a ring small quantity]

$$\therefore \qquad dm = \frac{M}{\pi R^2} \times 2\,\pi\,r\,dr = \frac{2M}{R^2}\,r\,dr$$

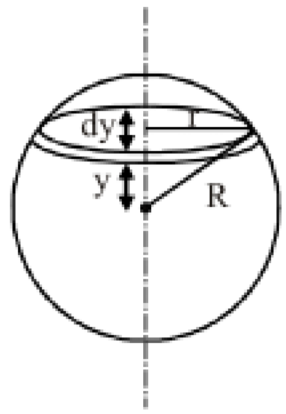
$$\therefore \qquad dI = (dm)\,r^2 = \frac{2M}{R^2}\,r^3\,dr$$

$$\therefore \qquad I = \frac{2M}{R^2}\int_0^R r^3\,dr$$

$$\therefore \qquad I = \frac{MR^2}{2}$$

M.I. of solid sphere (Mass-M, Radius - R)

Let taken n elemntary disc of radius r and at distance from the centre. Let its thikness is dy.



$$dm = \frac{M}{\frac{4}{3}\pi R^3} \times \pi r^2 dy$$

$$= \frac{3M}{4R^3} r^2 dy$$

$\therefore$ M.I. of elementary disc is $dI = \frac{1}{2}(dm)r^2 = \frac{3M}{8R^2} r^4 dy$

Also $r^2 = R^2 - y^2 \Rightarrow r^4 = (R^2 - y^2)^2 = R^4 + y^4 - 2R^2 y^2$

$\therefore$ $dI = \frac{3M}{8R^3}(R^4 + y^4 - 2R^2 y^2)\ dy$

$\therefore$ M.I. of the sphere is

$$I = \int dI = \frac{3M}{8R^3}\left[\int_{-R/2}^{R/2} dy + \int_{-R/2}^{R/2} y^4.dy - 2R^2 \int_{-R/2}^{R/2} y^2.dy\right]$$

$$\therefore \quad I = \frac{3M}{8R^3}\left[R^4[y]_{-R/2}^{R/2} + \left[\frac{y^5}{5}\right]_{-R/2}^{R/2} - 2R^2\left[\frac{y^3}{3}\right]_{-R/2}^{R/2}\right]$$

$$\therefore \quad I = \frac{2}{5}MR^2$$

M.I. of thin hollow sphere (mass - M; Radius - R)

Here we take an elementary ring of radius r and thickness = $Rd\theta$

$\therefore$ Mass of elementary ring, $dm = \sigma\, dA$

$$\therefore \quad dm = \frac{M}{4\pi R^2} \times (2\pi r)\, R\, d\theta$$

$$= \frac{M}{2R}\, r\, d\theta$$

$$\therefore \quad dI = (dm)r^2 = \frac{M}{2R}\, r^3\, d\theta$$

Aslo $r = \sin\theta \Rightarrow dI = \dfrac{M}{2R}\sin^3\theta\, d\theta$

or $dI = \dfrac{M}{2R}(1-\cos^2\theta)\sin\theta\, d\theta$ [Taking $\sin^2\theta = 1-\cos^2\theta$]

Let $\cos\theta = x \Rightarrow \dfrac{dx}{d\theta} = -\sin\theta \Rightarrow \sin\theta\, d\theta = -dx$

$$\therefore \quad dI = \frac{M}{2R}(1-x^2)(-dx) = \frac{M}{2R}(x^2-1)\,dx$$

Also When θ varies from 0 to π ; x i.e. $\cos\theta$ varies from 1 to -1

$$\therefore \quad I = \frac{M}{2R}\int_1^{-1}(x^2-1)dx = \frac{M}{2R}\left[\frac{x^3}{3} - x\right]_1^{-1}$$

$$\therefore \quad I = \frac{2}{3}MR^2$$

M.I. of some common bodies

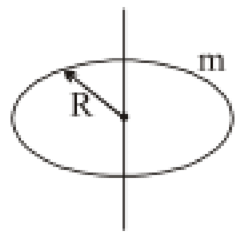
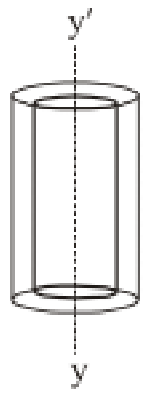
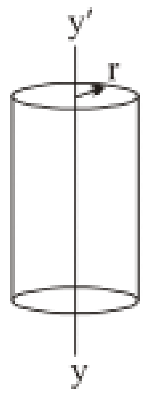
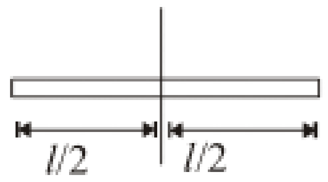
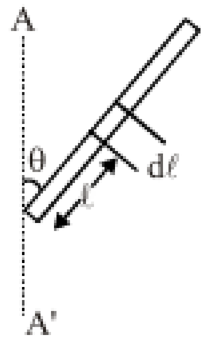
Uniform ring (about an axis passing through centre and perpendicular to the plane.)	$I = Mr^2$
Disc (about axis passing through centre and perpendicular to plane of disc). 	$= \dfrac{MR^2}{2}$
Hollow cylinder (about yy′ axis) Note : Independent of length of cylinder of same mass. 	$= Mr^2$
 **Solid cylinder** (about yy′ axis)	$= \dfrac{MR^2}{2}$
Hollow sphere (about a diameter)	$= \dfrac{2}{3}\ MR^2$
Solid sphere (about a diameter)	$= \dfrac{2M}{5}\ R^2$
Uniform rod mass M length (about axis passing through one end & perpendicular).	$= \dfrac{ml^2}{3}$
uniform rod (about an axis passing at $\dfrac{L}{4}$ from one end and perpendicular) 	$= \dfrac{ml^2}{12}$
 uniform rod (about axis passing through rod)	$I = 0$

Illustration :

Find M.I. of a unifrom rod of mass M and langth L about an axis which is at angle θ to it as shown .

Sol. *M.I. of the element*



$$dI = (dm)\, r^2 \left(\frac{M}{L} dl\right) r^2$$

Also, $r = l \sin\theta$

$$\therefore \quad dI = \left(\frac{M}{L}\sin^2\theta\right) l^2\, dl$$

$$\therefore \quad I = \frac{M}{L}\sin^2\theta \int_0^L l^2\, dl$$ *[since θ is common to all points, $\sin\theta$ comes out of the integral]*

$$\therefore \quad I = \frac{M}{L}\sin^2\theta\left[\frac{L^3}{3}\right] = \frac{ML^2}{3}\sin^2\theta$$

Illustration :

A rod AB of length L has linear mass density (λ) varying with distance (x) from the end A as $l = \alpha x^2$ where α is a constant. Find M.I. of the rod about the end A.

Sol. $dm = l\, dx$

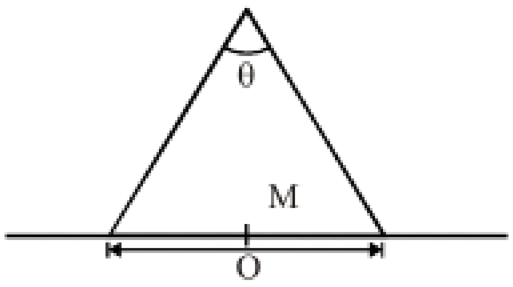
$= \alpha\, x^2\, dx$

$\therefore \quad dI = dmx^2 = \alpha\, x^4\, dx.$

$$\therefore \quad I = \int dI = a\int_0^L x^4 dx \quad \Rightarrow \quad I = \frac{\alpha L^2}{5}$$

Illustration :

Find M.I. of a uniform traingular plate of mass M about its base as shown.



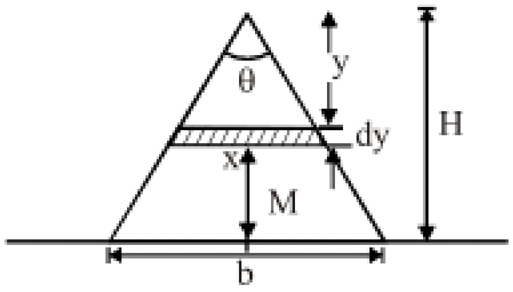
Sol. σ *(mass per unit Area) × (Area of the element)*

$= (\sigma)\,(x\, dy)$

Since each point of the dementary strip is at distance

$(H - y)$ from the axis

$\therefore \quad dI = (dm)\,(H - y)^2 = (\sigma\ xdy)\,(H^2 + y^2 - 2Hy)$



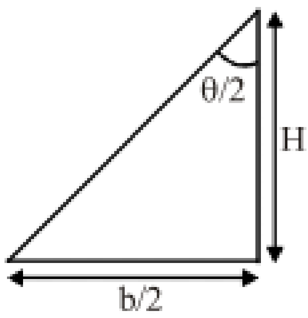
Also $$\frac{x}{b} = \frac{y}{H} \Rightarrow x = \left(\frac{b}{H}\right) y$$

$$\therefore \qquad dI = \sigma \frac{b}{H}\left(H^2 y + y^3 - 2Hy^2\right) dy$$

$$\therefore \qquad I = \sigma \frac{b}{H}\left[H^2 \int_0^H y.dy + \int_0^H y^3.dy - 2H\int_0^H y^2.dy\right]$$

$$= \sigma \frac{b}{H}\left[\frac{H^4}{2} + \frac{H^4}{4} - \frac{2H^4}{3}\right]$$

$$= \sigma \frac{bH^3}{12}$$



H = b/2 cot (θ/2)

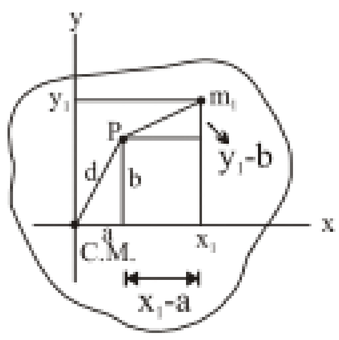
Also $$\sigma = \frac{Total\ mass}{Total\ area} = \frac{M}{\left(\frac{1}{2}bH\right)} = \frac{2M}{bH}$$

$$\therefore \qquad I = \frac{MH^2}{6} = \frac{M}{6}\left[\frac{b}{2}cot\left(\frac{\theta}{2}\right)\right]^2$$

$$= \frac{Mb^2}{24} cot\left(\frac{\theta}{2}\right)$$

Parallel axis Theorem

Used to find moment of inertia about an axis which parallel to the axis passing through C.M.



C.M. is at origin

$I_{CM} \rightarrow$ MI of the rigid body about an axis through CM

$I_P \rightarrow$ MI of the rigid body about an axis which is parallel to the above axis through CM & is at distance d from the axis through CM

$$I_{CM} = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

$$I_P = \sum_i m_i \left[(x_i - a)^2 + (y_i - b)^2\right]$$

z coordinates are not involved so m_i can be replaced by sum of mass of all particles placed on z-axis with co-ordinate (x_i, y_i)

$$= \Sigma m_i \left(x_i^2 + y_i^2\right) + \Sigma m_i (a^2 + b^2) - \Sigma\, 2am_i x_i - \Sigma 2bm_i y_i$$

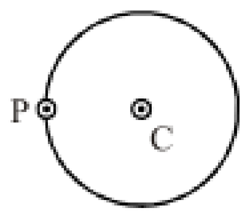
$$= I_{CM} + (a^2 + b^2)\, \Sigma m_i - 2a\, \Sigma m_i x_i - 2b\Sigma m_i y_i$$

$$I_P = I_{CM} + Mh^2$$

If M.I. of a body of mass M about an axis passing through its C.O.M. is I_C them M.I. of another axis which is parallel to the above central axis and is parallel to it is given by $I_{AA'} = I_C + Mh^2$

Illustration :

Find M.I. of a uniform body of mass M and radius R about an axis passing through a point p on its periphery and is perpendicular to its plane. Take the body as

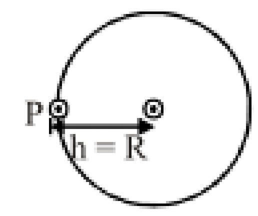


(a) Ring *(b) Disc*

Sol. $I_P = I_C + M(R)^2$

(a) for ring, $I_P = MR^2 + M(R)^2$

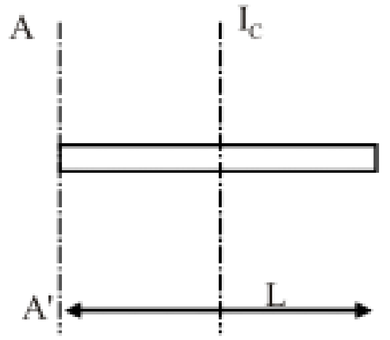
$= 2\,MR^2$



(b) for disc, $I_P = \dfrac{MR^2}{2} + M(R)^2 = \dfrac{3}{2}MR^2$

Illustration :

Find I_C *of the uniform rod of mass M shown, knowing that M.I. about an end is* $\dfrac{ML^2}{3}$



Sol. $I_{AA'} = I_C + Mh^2$

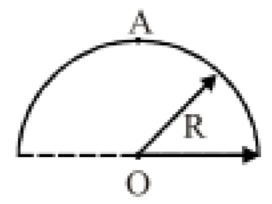
$\therefore$ $I_C = I_{AA'} - Mh^2$

where $h = \dfrac{1}{2}$ and $I_{AA'} = \dfrac{ML^2}{12}$

$\therefore$ $I_C = \dfrac{ML^2}{3} - M\left(\dfrac{L}{2}\right)^2 = \dfrac{ML^2}{12}$

Illustration :

Find M.I. of thin semicircular wire of mass m about axis passing through point A and is perpendicular to its plane as shown.

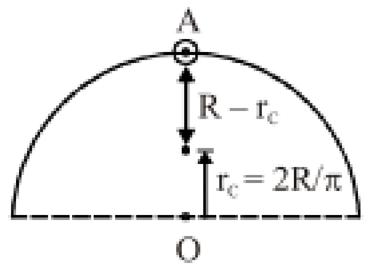


Sol. $I_0 = mR^2$

By parallel axis theorem

$I_0 = I_C + mr^2_C \quad \Rightarrow \quad I_C = I_0 - mr^2_C$

Where r_C *is distance of C.O.M. of the helf ring from its base ;*



$r_C = \dfrac{2R}{\pi}$

$$\therefore \qquad I_C = I_0 - m\left(\frac{2R}{\pi}\right)^2$$

Again applying parallel axis theorem, now for axes through A and C.O.M.

$$\therefore \qquad I_e = I_C + m\,(R - r_C)^2$$

$$= \left[mR^2 + m\left(\frac{2R}{\pi}\right)^2\right] + m\left(R - \frac{2R}{\pi}\right)^2$$

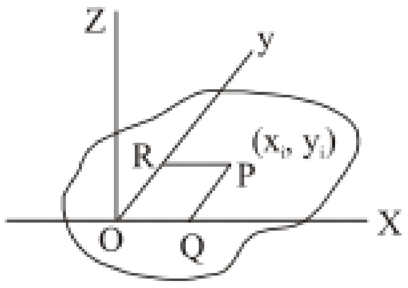
$$\therefore \qquad I_A = 2mR^2\left(1 - \frac{2}{\pi}\right)$$

Perpendicular axis theorem

This theorem is applicable only for the laminar bodies (i.e. plane bodies). (e.g. ring, disc, not sphere)

I_x & I_y are MI of body about a common pt. O in two mutually $\perp$ directions in the plane of body

I_z is MI of body about an axis $\perp$ to X & Y axis & passing through pt. O



$I_x = \Sigma m_i\, y_i^2$

$I_y = \Sigma m_i x_i^2$

$I_z = \Sigma m_i\,(x_i^2 + y_i^2)$ $\qquad\qquad I_z = I_x + I_y$

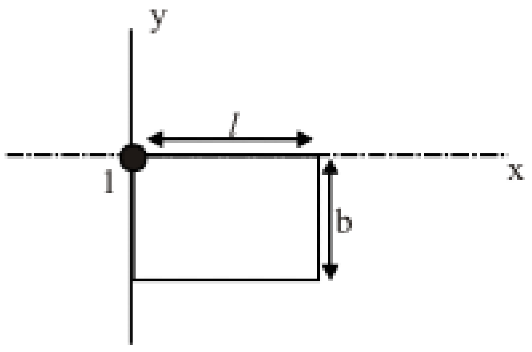
The point of intersection of the three axis need not be centre of mass, it can be any point in the plane of body which lie on the body or even outside it. For relation from perpendicular axis theorem, $I_3 = I_1 + I_2$ axis (3) must be perpendicular the plane of the body and axis (1) and axis (2) must be in the plane of the body.

Illustration :

Find M.I. of a uniform rectangular plate of sides l and b shown, obout the axes passing through

(i) Point 1 i.e. corener

(ii) Point 2 i.e. centre



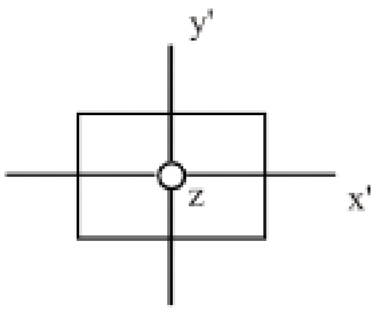
Sol. *(i)* $\qquad I_X = \dfrac{Mb^2}{3}\ ;\ I_y = \dfrac{Ml^2}{3}$

By perpendicular axis theorem

$$I_z = I_x + I_y$$

$$= \frac{M}{3}\,[l^2 + b^2]$$

(ii) $I_{x'} = \frac{Mb^2}{12}, I_y = \frac{Ml^2}{12}$



$\therefore \quad I_Z = I_{x'} + I_y = \frac{M}{12}(l^2 + b^2)$

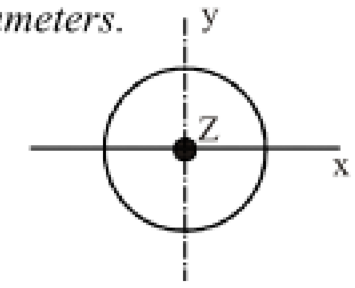
In this question, for squese plate of side b = l

$$I_1 = \frac{M}{3}\ (l^2 + l^2) = \frac{2Ml^2}{3}$$

& $\quad I_Z = \frac{M}{12}(l^2 + l^2) = \frac{Ml^2}{6}$

Illustation :

Find M.I. of the ring and the disc about the axis passing through their diameters.



Sol. *By perpendicular axis theorem*

$$I_X + I_Y + I_Z$$

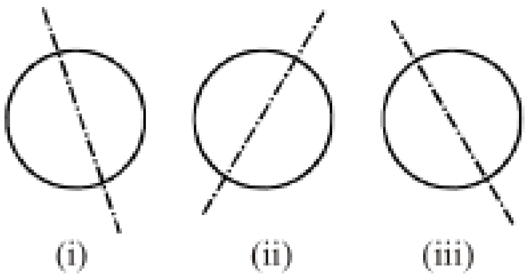
Also, since mass distrioution of the body about x and y axis are similar

$\therefore \quad I_x = I_y \quad \Rightarrow \quad I_x = I_y = I_z/2$

(i) For ring, $I_Z = MR^2 \Rightarrow \quad I_X = I_y = \frac{MR^2}{2}$

(ii) For disc, $I_Z = \frac{MR^2}{2} \Rightarrow \quad I_X + I_y = \frac{MR^2}{4}$

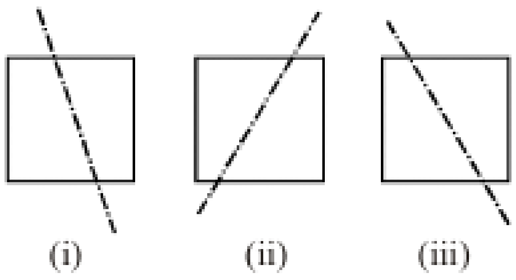
Note :



We can observe mass distrbution about the diamenteral axis shown in (i), (ii) and (iii) are similar

$\therefore \quad I_{(i)} = I_{(ii)} = I_{(iii)}$

Similarly for square plate



$$I_{(i)} = I_{(ii)} = I_{(iii)}\ \frac{Ml^2}{6}$$

Illustration :

A hole of radius $\frac{R}{2}$ is made in a uniform ciruclar plate of radius R. The mass of remaining portion shaded is m. Find M-I. of this body about point O.

Sol. *To solve this, we use the concept. M.I. of the remaining postion*

= M.I. of the complete body before removed – M.I. of the removed part.

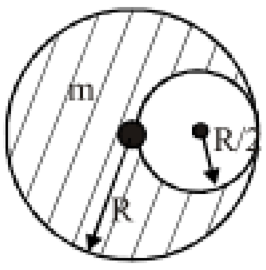
(A) Let 'm' be the mass of reamining disc.

Area of original disc $= \pi r^2$,

Area of removed part $= \pi(r/2)^2$,

Therefore area of the remaining disc $= \pi r^2 - \pi(r/2)^2 = (3/4)\pi r^2$

Mass per unit area $= \sigma = \dfrac{m}{(3/4)\pi 3^2} = \dfrac{4}{3}\cdot\dfrac{m}{\pi r^2}$,



Mass of the removed part $= \dfrac{1}{4}\pi r^2 \sigma = \dfrac{1}{3}m$,

Moment of inertia of the complete disc $(I_{com}) = \dfrac{1}{2}\times\dfrac{4}{3}mr^2 = \dfrac{2}{3}mr^2$

Moment of inertia of the removed part $(I_{removed}) = \dfrac{1}{2}\times\dfrac{m}{3}\left(\dfrac{r}{2}\right)^2 + \dfrac{m}{3}\left(\dfrac{r}{2}\right)^2$

$$= \frac{1}{8}mr^2$$

Therefore the moment of inertia of the remaining disc

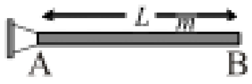
$$I_{remaining} = \left(\frac{2}{3}\right)mr^2 - \left(\frac{1}{8}\right)mr^2$$

$$I_{remaining} = \left(\frac{13}{24}\right)mr^2$$

Illustration :

A non-uniform bar AB of mass m has linear mass density $\lambda = \lambda_0 \dfrac{x}{L}$ (x is calculated from one end).

Find the (a) Mass of the rod (b) Moment of inertia of the rod aobut the end A.



(c) Momemt of inertia about centre of mass

Sol. (a) $dm = \lambda\, dx = \lambda_0 \frac{x}{L} dx$

$$\therefore \quad M = \int dm = \frac{\lambda_0}{L} \int_0^L x\, dx = \frac{\lambda_0 L}{2}$$

(b) About A, $dI = (dm)x^2 = \frac{\lambda_0 x^3}{L} dx$

$$\therefore \quad I_A = \int dI = \frac{\lambda_0}{L} \int_0^L x^3 dx = \frac{\lambda_0 L^3}{4}$$

(c) Position of C.O.M from end A is

$$x_C = \frac{\int (dm)x}{\int dm} = \frac{\frac{\lambda_0}{L}\int_0^L x^2 dx}{\left(\frac{\lambda_0 L}{2}\right)}$$

$$\Rightarrow \quad x_C = \frac{2}{L^2}\left[\frac{L^3}{3}\right] = \frac{2L}{3}$$

By paraell axis theorem, $I_A = I_C + M x_C^2$

$$\therefore \quad I_C = I_A - Mx_c^2 = \frac{\lambda_0 L^3}{4} - \left(\frac{\lambda_0 L}{2}\right)\left(\frac{2L}{3}\right)^2$$

$$= \frac{\lambda_0 L^3}{36}$$

Radius of gyration (K)

If M.I. aobut an axis of a system of mass M is I, then we may write, $I = MK^2$ where k is called radius of gyration of the body about that axis $= \sqrt{\frac{I}{M}}$

In other words M.I. aobut an axis of system is same as M.I. of a particle of same mass placed t the distance equal to K from the axis of a uniform disc of radius R, radius of gyration about an axis passing through its centre & is to its plane is

$K = \sqrt{\frac{\left(\frac{MR^2}{2}\right)}{M}} = \frac{R}{\sqrt{2}}$ i.e. the disc will same rotational inertia as that of a particle of same mass placed at distance $\frac{R}{\sqrt{2}}$ from the axis

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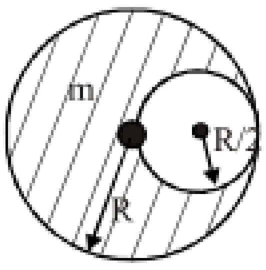

$$\text{Mass of the removed part} = \frac{1}{4}\pi r^2 \sigma = \frac{1}{3}m,$$

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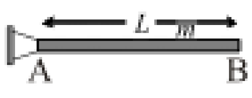

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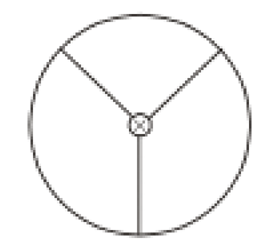
$K = \sqrt{\frac{\left(\frac{MR^2}{2}\right)}{M}} = \frac{R}{\sqrt{2}}$ i.e. the disc will same rotational inertia as that of a particle of same mass placed at distance $\frac{R}{\sqrt{2}}$ from the axis

Note : In case of system of combitionation of various bodies $k = \sqrt{\dfrac{I_{total}}{M_{total}}}$

Illustration

Each wheels has an outer ring having radius R and mass m. Other than outer ring the wheels comprise of some uniform rods (each of mass m and length R).

Calculate radius of gyration about centre and perpendicular to plane.



Sol. $I_{total} = m(R^2) + 3 \times \left(\dfrac{mR^2}{3}\right) = 2mR^2$

Also $M_{total} = 4\,m$

$\therefore \quad K = \sqrt{\dfrac{2mR^2}{4m}} = \dfrac{R}{\sqrt{2}}$

Illustration :

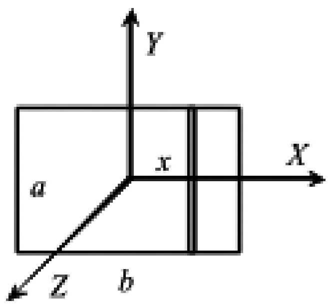
Find the MI of a ring about the chord which is parallel to the diameter of the ring at a distance R/2 from the diameter

MI of Plate:

$I_x = \int \dfrac{1}{12} dm\, a^2 = \dfrac{1}{12} m a^2$

$\therefore I_y = \dfrac{1}{12} m b^2$

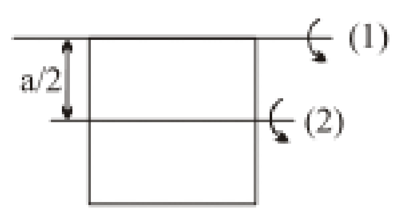
$\therefore I_z = I_x + I_y = \dfrac{1}{12} m \left(a^2 + b^2\right)$



1. What would have happened if it was a square plate?
2. Prove that MI of square plate of mass m and side a about any axis passing through COM along the surface of plate is $ma^2/12$

Illustration:

Find the moment of inertia of a Square plate about an edge in the plane



$I_{CM} = I_2 = \dfrac{ma^2}{12} \qquad \therefore I_1 = \dfrac{ma^2}{12} + m\left(\dfrac{a}{2}\right)^2 = \dfrac{ma^2}{12} + \dfrac{ma^2}{4} = \dfrac{ma^2}{3}$

Practice Exercise

Q.1. Find the moment of inertia of a pair of spheres, each having a mass m and radius r, kept in contact about the tangent passing through the point of contact.

Q.2 The moment of inertia of a uniform rod of mass $m = 0.50$ kg and length $l = 1$ m is $I = 0.10\ kg-m^2$ about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.

Q.3 Find the moment of inertia of a uniform square plate of mass m and edge a about one of its diagonals.

Q.4 The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius. Find the distance of the line form the centre.

Q.5 Calculate the moment of inertia of a uniform rod of mass m & length l about an axis passing through one end & making angle $\theta = 45^\circ$ with its length.

Q.6 The surface density (mass/area) of a circular disc of radius a depends on the distance from the centre of $\rho(r) = A + Br$. Find its moment of inertia about the line perpendicular to the plane of the disc through its centre.

Answers

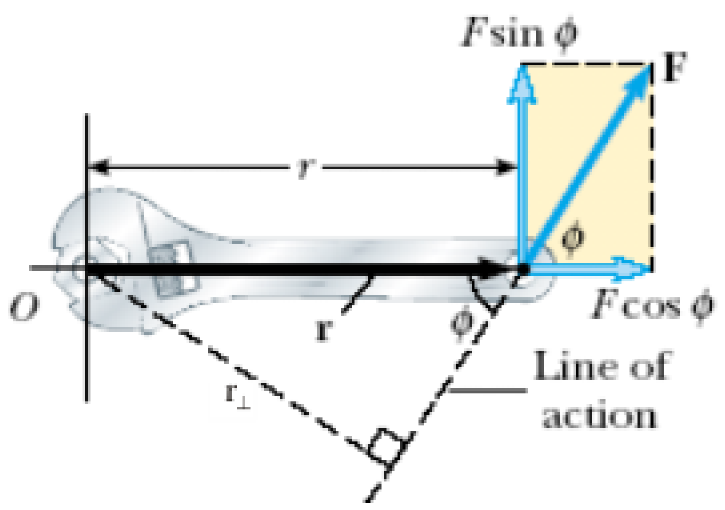
Q.1. $14\ mr^2$ Q.2 $\sqrt{\frac{1}{2}\left(1-\frac{ml^2}{12}\right)} = \sqrt{\frac{7}{60}} = 0.34m$ Q.3 $ma^2/12$ Q.4 $r/\sqrt{2}$

Q.5 $\frac{ml^2}{6}$ Q.6 $2\pi\left(\frac{Aa^4}{4}+\frac{Ba^5}{5}\right)$

4. Torque

The quantitative measure of the tendency of a force to cause or change the rotational motion of a body is called torque. Consider an example to understand this.

In the figure below, the wrench is trying to open the nut. Now the ability of wrernch to open the nut will depend not only on the applied force, but the distance at which force is applied. This gives birth to **new physical quantity** called torque.



If only radial force F_r were present, the nut could not be turned. Thus the force causing the rotation is tangential force F_T only. The magnitude of the torque about an axis due to a force is given by

τ = (Force causing the rotation)×(distance of point of application of force from the axis)

i.e., $= (F \sin \phi)\, r$

we may also write $\tau = F\,(r \sin\phi) = F\,(r_\perp)$

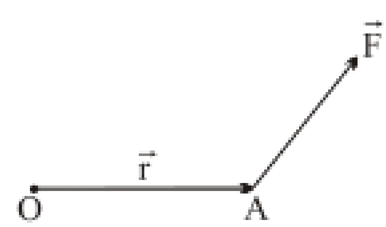
$$\therefore \tau = r F \sin\phi = (r \sin\phi) F$$
$$= r F_\tau = r_\perp F$$

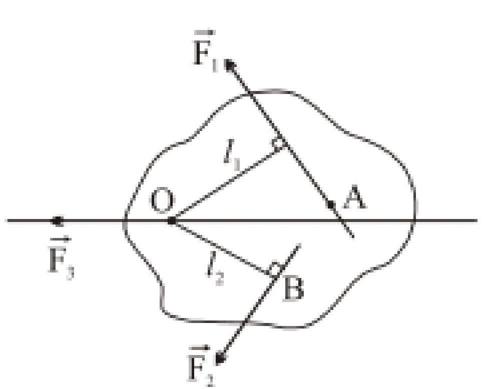
$r_\perp(d)$: moment arm, lever arm

$\vec{\tau} = \vec{r} \times \vec{F}$ Direction of torque is found by sliding the force vector at the axis of rotation and using right hand thumb rule.

Torque also follows superposition principle. $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots + \vec{\tau}_n$

$\overrightarrow{OA} = \vec{r}$ = P.V. of pt. of application of force wrt fixed axis.(centre of rotation)





Note :

1. Torque & force are entirely diff. quantities. As torque is always defined with reference to point about which body is rotating, while force does not depend on it. Like torque of F_3 about O is zero, while about A or B is not zero.
2. When equal & opposite force acts on a body having different line of action is called **couple**.

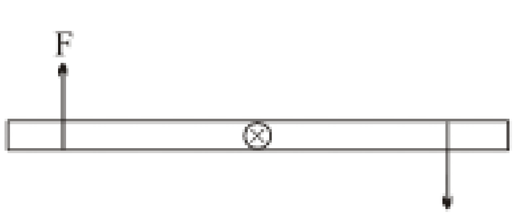
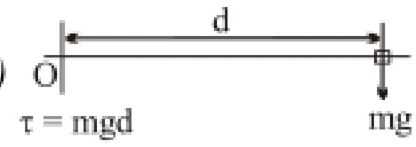


Illustration :

A particle is falling freely along line y = d. Find torque on this particle due to gravity, about origin when it

(i) Crosses x axis *(ii) is at y = d*

Sol. *(i)*



(ii)

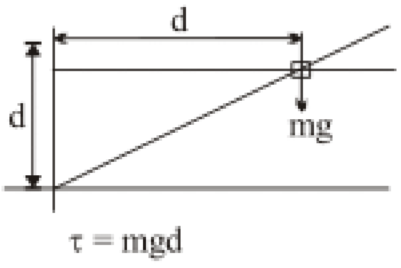


Illustration :

Find torque due to gravity at any time t about pt. of projection, if a body is projected with velocity u at an angle θ.

Sol.

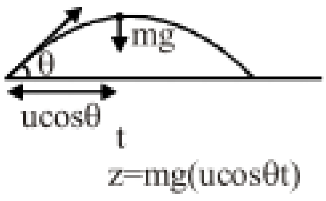
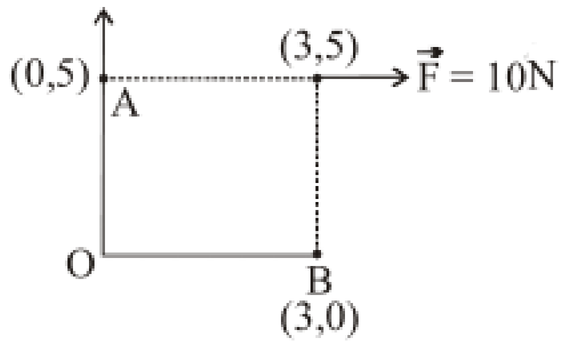
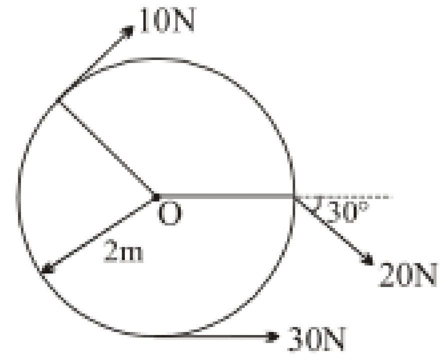


Illustration :

Find out the torque about point A, O and B for fig (i) & about 'O' for fig. (ii).



(i)



(ii)

Sol. $\tau_A = 0$

$\tau_B = 10 \times 5 = 50$ *N-m*

$\tau_0 = 10 \times 5 = 50$*N-m*

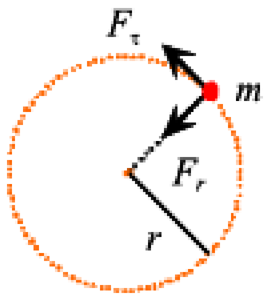
Torque about O

$\tau = -10 \times 2 - 20 \sin 30^o \times 2 + 30 \times 2 = 20$ *N-m*

Relationship between torque and angular acceleration

Rotational analog of Newton's second law

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force F_τ and a radial force F_r, as shown in figure.



$$F_t = m a_t$$

Magnitude of torque about the center of circle is:

$$\tau = F_t\, r = m a_t\, r = m(\alpha r) r = \alpha (m r^2) = I\alpha$$

$$\therefore \tau = I\alpha$$

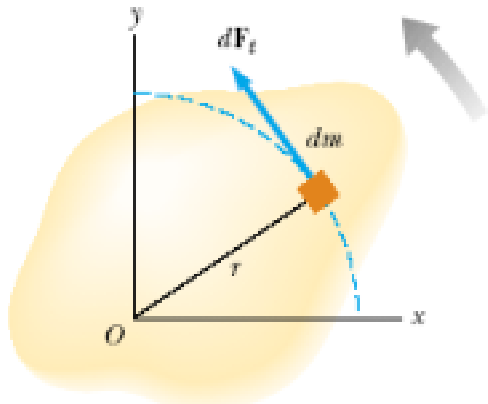
That is, the torque acting on the particle is proportional to its angular acceleration.

We can also understand that since torque is written about O, we should write '*I*' also about O.

Torque on rigid body:

Proof: Consider a rigid body shown.

Torque acting on *i* th particle will be:



$$\vec{\tau}_i = \left(m_i r_i^2\right)\vec{\alpha} \quad \Rightarrow \vec{\tau}_{net} = \sum \vec{\tau}_i = \sum \left(m_i r_i^2\right)\vec{\alpha}$$

$$\therefore \vec{\tau}_{net} = I\vec{\alpha}$$

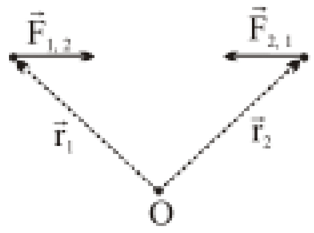
$\vec{\tau} = I\vec{\alpha} = \vec{r} \times \vec{F}$: This is the general relation that we are going to use in this chapter.

Note:

a) Torque due to internal forces is alwys zero.

b) Torque due to gravity is found by showing the gravitational force at the COM of the rigid body.

Proof (a): Consider two particles as shown in figure. From Newton's third law, the forces exerted by these particles are equal and opposite.



$$\vec{F}_{2,1} = \vec{F}_{1,2}$$

The sum of torques of these forces about origin O is

$$\vec{\tau}_1 + \vec{\tau}_2 = \vec{r}_1 \times \vec{F}_{2,1} + \vec{r}_2 \times \vec{F}_{1,2}$$
$$= \vec{r}_1 \times \vec{F}_{2,1} + \vec{r}_2 \times (-\vec{F}_{2,1})$$
$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{2,1}$$

The vector $\vec{r}_1 - \vec{r}_2$ is along the line joining the two particles, so $\vec{F}_{2,1}$ is either parallel or antiparallel to $(\vec{r}_1 - \vec{r}_2)$ thus

$$(\vec{r}_1 - \vec{r}_2) \times \vec{F}_{2,1} = 0$$

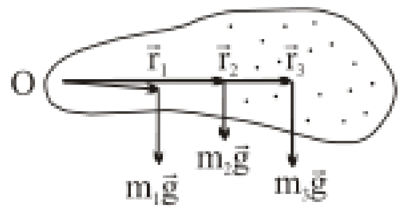
So the internal forces (torques) cancel in pairs.

Proof (b):

τ = torque of gravity about O

$$\vec{\tau} = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g} + ...\vec{r}_n \times m_n\vec{g} = (m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...+ m_n\vec{r}_n) \times \vec{g}$$

$$\vec{\tau} = \frac{(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...+ m_n\vec{r}_n)}{(m_1 + m_2 + m_3 + ...+ m_n)} \times (M\vec{g})$$



$$\vec{\tau} = \vec{r}_{cm} \times (M\vec{g})$$

Where r_{cm} is PV of C.M. wrt. point O.

Practice Exercise

Q.1 A force $F = A\hat{i} + B\hat{j}$ is applied to a point whose radius vector relative to the origin of coordinates O is equal to $r = a\hat{i} + b\hat{j}$, where a, b & A, B are constants, and $\hat{i}$, $\hat{j}$ are the unit vectors of the x and y axes. Find the Torque due to force.

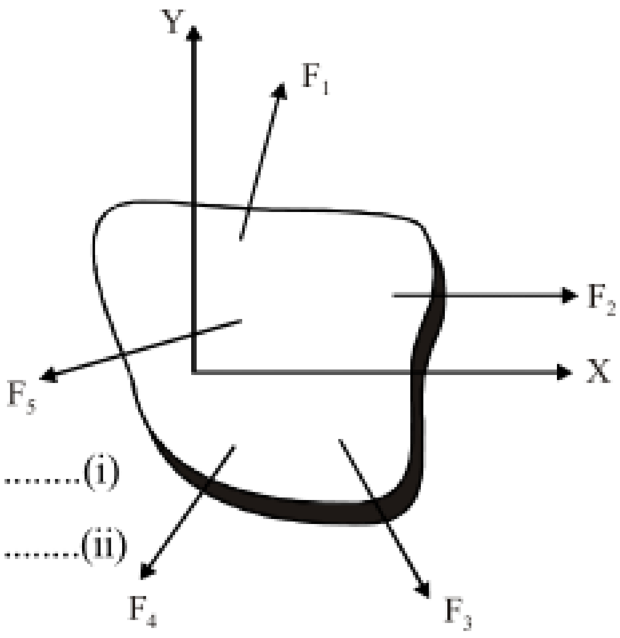
Answers

Q.1 $Z = (aB - bA)\ \hat{k}$

Rotational Equilibrium

If net external torque acting on the body is zero, then the body is said to be in rotational equilibrium.

The centre of mass of a body remains in equilibrium if the total external force acting on the body is zero. Similarly, a body remains in rotational equilibrium if the total external torque acting on the body is zero.



For translational equilibrium.

$$\Sigma F_x = 0 \quad ....(i)$$

and $$\Sigma F_y = 0 \quad ....(ii)$$

$$\Sigma F_z = 0$$

The condition of rotational equilibrium is

$$\Sigma Z_{ext} = 0$$

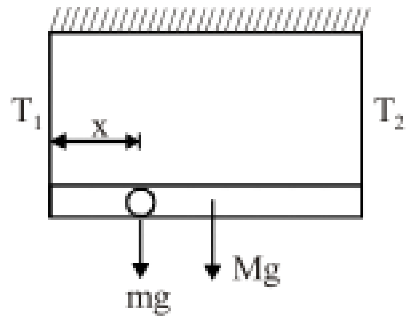
The equilibrium of a body is called stable if the body tries to regain its equilibrium position after being slightly displaced and released. It is called unstable if it gets further displaced after being slightly dis-

placed and released. If it can say in equilibrium even after being slightly displaced and released, it is said to be is neutral equilibrium.

Illustration :

A uniform stick of mass M & length L is suspended from the ceiling through two vertical strings of equal lengths fixed at the ends. A small object of mass m is placed on the stick at a distance of x from the left end. Find the tensions in the two strings.

Sol.



By torque balancing about end's

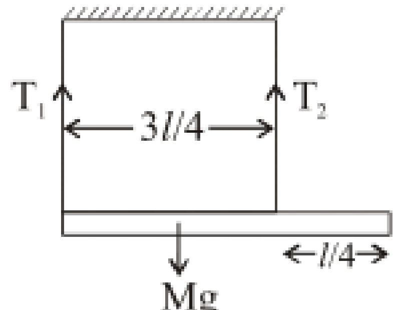
$$T_1 x = mg(L-x) + Mg \times \frac{L}{2}$$

$$T_2 = mgx + Mg\,\frac{L}{2} \qquad \text{....(i)}$$

$$= \left(\frac{M}{L} + \frac{x}{L}m\right)g,\ \left(\frac{M}{2} + \frac{(L-x)}{L}m\right)g$$

Illustration :

A uniform rod of length l and mass m is hung from two strings of equal length from a ceiling as shown, Determine the tension in the string.



Sol. *Balancing Torque about its left end*

$$T_2 \times \frac{3L}{4} = mg \times \frac{L}{2}$$

$T_2 = 2\,mg/3$

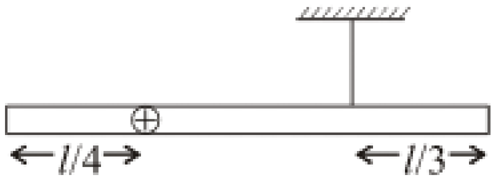
$T_1 + T_2 = 2mg/3$

$T_1 = mg/3$

$$= \left(\frac{mg}{3}, \frac{2mg}{3}\right)$$

Illustration :

Calculate hinge force in the following diagram.

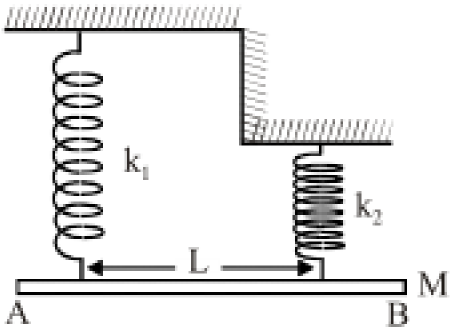


Sol. *By balancing torque about point where string is connected*

$$N \times \frac{5L}{12} = mg \times \frac{L}{6}$$

$$N = \frac{2}{5} mg$$

Q.8 *When a mass M hangs from a spring of length l, it stretches the spring by a distance x . Now the spring is cut in tow parts of lengths l/3 and 2l/3, and the two springs thus formed are connected to a straight rod of mass M which is horizontal in the configuration shown in figure. Find the stretch in each of the spring.*



Sol. *As it is given that the mass M stretches the original spring by a distance x, we have*

$$kx = Mg$$

$$x = \frac{Mg}{k}$$

The new force constants of the two springs can be given by using equation

$$k_1 = 3k \qquad \text{and} \qquad k_2 = \frac{3k}{2}$$

Let we take the stretch in the two springs be x_1 and x_2, we have for the equilibrium of the rod

$$k_1x_1 + k_2x_2 = Mg$$

$$3kx_1 + \frac{3k}{2}x_2 = Mg$$

From equation, we have

$$x_1 + \frac{x_2}{2} = \frac{x}{3}$$

As the rod is horizontal and in static equilibrium, we have net torque acting on the rod about any point on it must be zero. Thus we have torque on it about end A are

$$k_2x_2\, L = Mg\,\frac{L}{2}$$

$$x_2 = \frac{Mg}{2k_2} = \frac{Mg}{3k} = \frac{x}{3}$$

Using this value of x_2 in equation, we have

$$x_1 = \frac{x}{6}$$

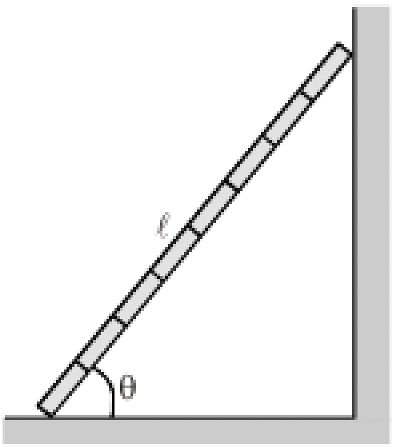
This can also be directly obtained by using torque zero about point B on the rod as

$$k_1x_1\, L = Mg\,\frac{L}{2}$$

$$x_1 = \frac{Mg}{2k_1} = \frac{Mg}{6k} = \frac{x}{6}$$

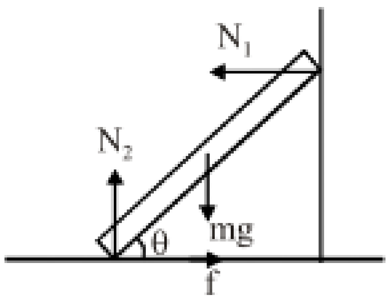
Illustration :

A uniform ladder of length ℓ rests against a smooth, vertical wall (figure). If the mass of the ladder is m and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.375$ *and the minimum angle θ_{min} at which the ladder does not slip.*



A uniform ladder at rest, leaning against a smooth wall. The ground is rough.

Sol.



$$N_2 = mg$$
$$N_1 = f = \mu N_2 = \mu mg$$

By rotational equilibrium

$$mg\ \frac{\ell}{5}\cos\theta = N_1\ \ell \sin\theta$$

$$\mu = \frac{\cot\theta}{2} = \cot\theta = 0.75$$

Illustration :

Two small kids weighting 10 kg and 15 kg are trying to balance a see saw of total length 5.0m with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit?

Sol. *By rotational equilibrium*

$$15x = 10 \times \frac{5}{2}$$

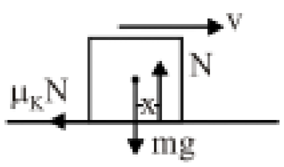
$$x = \frac{5}{3} = 1.7m$$

Illustration :

A block of height h is projected along a rough surface of coefficient of friction μ. Find the point of application of the normal force on the block for $\mu_k = 0.5$.

Sol.

$$\mu_K N \frac{h}{2} = Nx$$

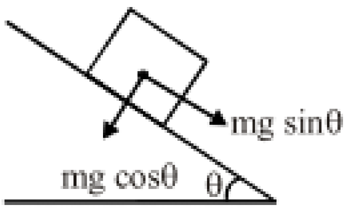


$$x = \frac{\mu_K h}{2}$$

$$x = \frac{h}{4}$$

Illustration :

A cubical block of mass m and edge a slides down a rough inclined plane of inclination with a uniform speed. Find the torque of the normal force acting on the block about its centre.



Sol. $f = mg \sin\theta$

Torque due to friction $= mg \sin\theta \, a/2$

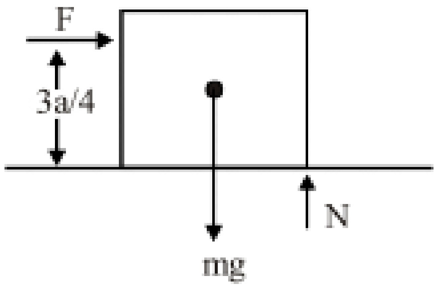
$= 0.5 \, mg \, a \sin \theta$

Toppling

Illustration :

A uniform cube of side 'a' and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height 3a/4 above the base. What is the minimum value of F for which the cube begins to tip about an edge?

Sol.

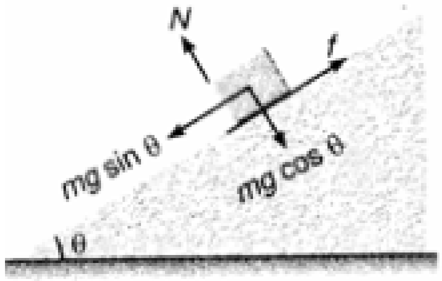


Normal shift upto extreme Right then balancing toqure about that point

$N \times 3a/4 = mg \times a/2$ *then* $N = \dfrac{2mg}{3}$

Illustration :

A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will slide before toppling.

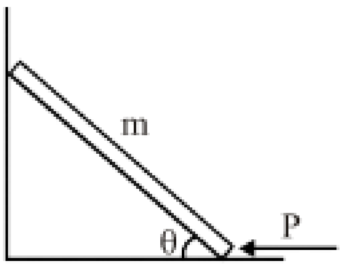


Sol. $f \dfrac{h}{2} < Nr \quad \Rightarrow \quad \mu N \dfrac{h}{2} < Nr$

$$\mu < \frac{2r}{h}$$

Practice Exercise

Q.1 Assuming frictionless contacts, determine the magnitude of external horizontal force P applied at the lower end of equilibrium of the rod. The rod is uniform and its mass 'm'



Q.2 A uniform metre stick of mass 200 g is suspended from the ceiling through two vertical strings of equal lengths fixed at the ends. A small object of mass 20 g is placed on the stick at a distance of 70 cm from the left end. Find the tensions in the two strings.

Answers

Q.1 $P = \frac{W}{2}\cot\theta$ or $P = \frac{mg}{2}\cot\theta$ Q.2 1.04 N in the left string and 1.12 N in the right.

Rotation about fixed axis

Since torque is a rotational analong of force, therefore, Newton's second law for rotational motion is given by

$$\tau_{net} = I\alpha \qquad \text{.... (i)}$$

Note that the above equation (i) is not a vector equation.

It is valid in two situation :

(i) The axis is fixed in position and direction.

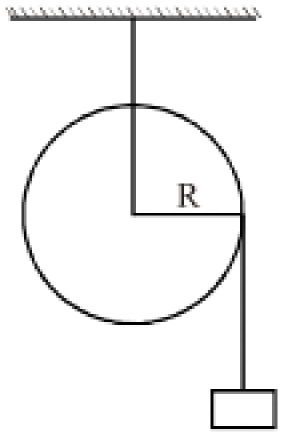
(ii) The axis passes through the center of mass and is fixed in direction only the equation

$$\tau_{cm} = I_{cm}\alpha_{cm} \qquad \text{.... (ii)}$$

is valid even if the center of mass is accelerating.

Illustration :

A disc - shaped pulley has mass $M = 4$ kg and radius $R = 0.5m$. It rotates freely on a horizontal axis, as in figure. A block of mass $m = 2$ kg hangs by a string that is tightly wrapped around the pulley.



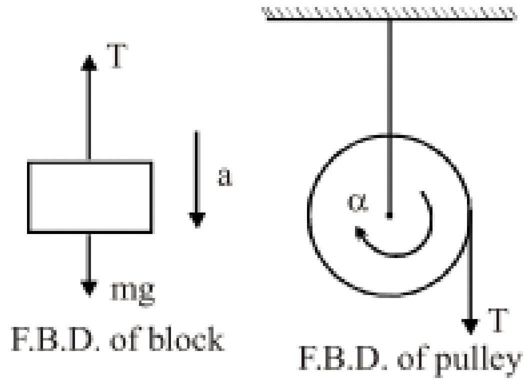
(a) What is the angular velocity of the pully 3 s after the block is released ?

(b) Find the speed of the block after it has fallen 1.6m. Assume the system starts at rest.

Sol. Since the string is tangential to the pulley, the torque on it due to-the tension is $\tau = TR$. The two forms of Newton's second law for the block and the pulley yield

Block ($F = ma$) $\qquad mg - T = ma$

Pulley ($\tau = I\alpha$) $\qquad TR = \left(\frac{1}{2}MR^2\right)\alpha$



Applying Newton's second Law

For the disc, $I = \frac{MR^2}{2}$

$$\therefore \quad TR = \left(\frac{MR^2}{2}\right)\alpha \quad \text{or} \quad T = \frac{MR\alpha}{2} \qquad ...(i)$$

Applying Newton's second Law on the block

$$F_{net} = ma$$

$$\therefore \quad mg - T = ma \qquad ...(ii)$$

Since the block and the rim of the pulley have the same speed (the string does not slip), we have $v = \omega R$. Thus, form equation (i) we find

$$T = \frac{1}{2}Ma \qquad ....(iii)$$

adding (ii) and (iii) leads to

$$a = \frac{mg}{m + \frac{M}{2}} \qquad ....(iv)$$

Putting $m = 2$ kg ; $M = 4$ kg ; $R = 0\ 0.5$ m;

we get $\qquad a = 5\ m/s^2$

(a) To find ω after 3 s, we use equation

$$\omega = \omega_0 + at = 0 + \left(\frac{a}{R}\right)t = 30\ rad/s$$

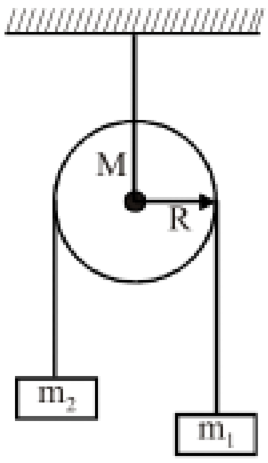
(b) T find the speed of the block we use

$$v_2 = v_0^2 + 2ay = 0 + 2\ (5\ m/s^2)\ (1.6m)$$

Thus $\qquad v = 4\ m/s$

Illustration :

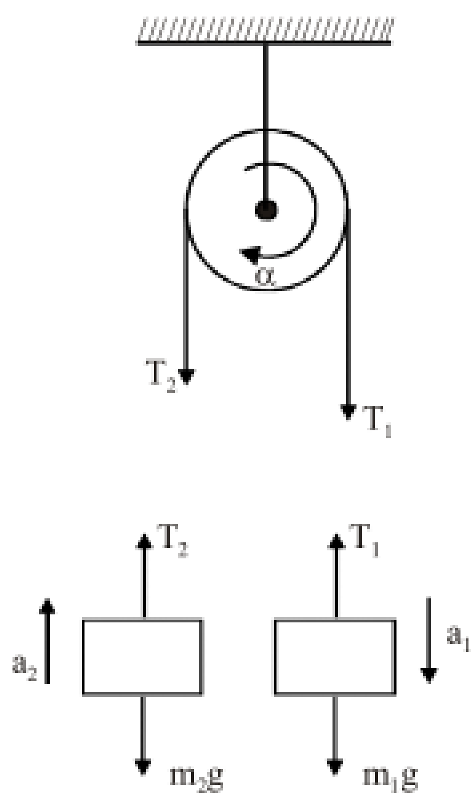
For the arrangement shown in the figure, the string is slightly wrapped over the pulley. Find the acceleration of each when released from rest. The string is not slipping over the pulley.



Sol. *The free body diagrams of the pulley and the blocks shown in the figure.*

Note that tension on two sides of the pulley are different. Why ?

Applying Newton's second law on the pulley, we ge



$$\tau = T_1 R - T_2 R = (T_1 - T_2)\, R$$

Since $$\tau = I\alpha = \left(\frac{MR}{2}\right)\alpha$$

Therefore, $$(T_1 - T_2)\, R = \left(\frac{MR^2}{2}\right)\alpha$$

or $$T_1 - T_2 = \left(\frac{MR}{2}\right)\alpha \quad .....(i)$$

Applying Newton's Law on the blocks, we get

$$T_2\, mg = m_2 a_2 \quad ......(ii)$$

$$m_1 g - T_1 = m_1 a_1 \quad ......(iii)$$

Since the string is tightly wrapped over the pulley, therefore,

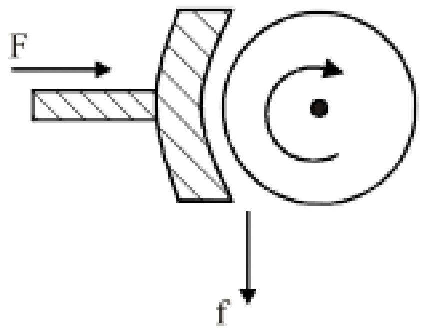
$$a_1 = a_2 = aR = a \quad ......(iv)$$

Solving equation (i), (ii), (iii) and (iv), we obtain

$$a = \left[\frac{m_1 - m_2}{m_1 + m_2 + \dfrac{M}{2}}\right] g$$

Illustration :

A fly wheel of mass M = 2 kg and radius R = 40 cm rotates freely at 600 rpm. Its moment of inertia is $\frac{1}{2}MR^2$. A brake applies a force F = 10 N radially inward at the edge as shown in the figure. If the coefficient of frictions $\mu_k = 0.5$, how many revolutions does the wheel make before coming to rest ?



A wheel is slowed down by the application of force F. With the chosen positive sense, the frictional torque is negative.

Sol. *We choose the initial sense of the angular velocity as positive. The force of friction is $f = \mu_k F$ and its (counterclockwise) torque is $\tau = -fR$.*

Using $\tau = I\alpha$, we have

$$-(\mu_k F)R = \left(\frac{1}{2}MR^2\right)\alpha$$

or $$\alpha = -\frac{2\mu_k F}{MR} = 12.5 \text{ rad/s}^2$$

The angular rotation θ of the wheel is given by

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Here $$\omega_0 = \frac{2\pi N}{60} = \frac{2\pi(600)}{60} = 20\pi \text{ rad/s}$$

$\therefore$ $0 = (20\ \pi\ rad/s)^2 + 2\ (-12.5\ rad/s^2)\theta$

Thus $\theta = 16\pi^2$ *rad.*

The number of revolutions $(16\pi^2$ rad$)$ $(1$ rev$/2\pi$ rad$) = 8\pi$ revolutions.

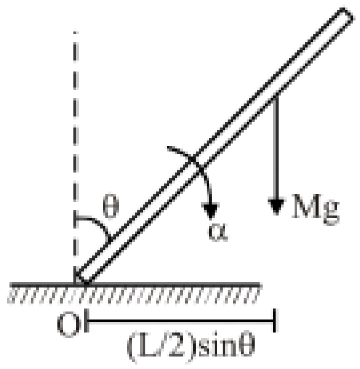
Illustration :

A uniform rod of length L and mass M is pivoted freely at one end as shown in the figure.

(a) what is the angular acceleration of the rod when it is at angle θ to the vertical ?

(b) What is the tangential linear acceleration of the free end when the rod is horizontal ?

The moment of inertia of the rod about one end is $\frac{1}{3}ML^2$.



The angular acceleration of the rod is produced by the torque due to its weight.

Sol. *Figure shown the rod at an angle θ to the vertical.*
Net torque about the point O is

$$\tau_0 = Mg\,\frac{L}{2}\sin\theta$$

Using II law of motion

$$\tau_0 = I_0\alpha$$

$$\frac{MgL}{2}\sin\theta = \frac{ML^2}{3}\alpha$$

Thus, $$\alpha = \frac{3g\sin\theta}{2L}$$

(b) When the rod is horizontal $\theta = \frac{\pi}{2}$ *and* $\alpha = \frac{3g}{2L}$. *From equation the tangential linear accelera-tion is*

$$a_t = \alpha L = \frac{3g}{2}$$

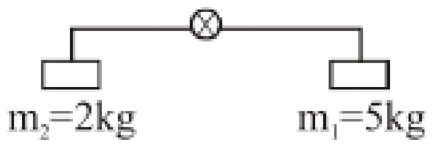
This is greater than the acceleration of an object in free-fall !

Practice Exercise

Q.1 A light rod of length 1 m is pivoted at its centre and two masses of 5 kg and 2 kg are hung from the ends as shown in figure

(a) Find the initial angular acceleration of the rod assuming that it was horizontal in the beginning.

(b) If the rod has a mass of 1 kg distributed uniformly over its length.



(i) Find the initial angular acceleration of the rod

(ii) Find the tension in the supports to the blocks of mass 2 kg and 5 kg.

Q.2 A meter stick is held vertically with one end on a rough horizontal floor. It is gently allowed to fall on the floor . Assuming that the end the floor does not slip, find the angular speed of the rod when it hits the floor.

Answers

Q.1 (a) $\dfrac{2g(m_1 - m_2)}{\ell(m_1 + m_2)} = \dfrac{60}{7} = 8.4\,\text{rad}/\text{s}^2$

(b) (i) $\dfrac{2g(m_1 - m_2)}{\ell(m_1 + m_2 + m_3/3)} = \dfrac{90}{22} = 8.4\,\text{rad}/\text{s}^2$, (ii) $(m_1 g - m_1\,\alpha\,\dfrac{\ell}{2}) = 29$ N; $(m_2 g + m_2\,\alpha\,\dfrac{\ell}{2}) = 27.6$ N

Q.2 $\sqrt{\dfrac{3g}{\ell}} = 5.4\ \text{rad}/\text{s}$

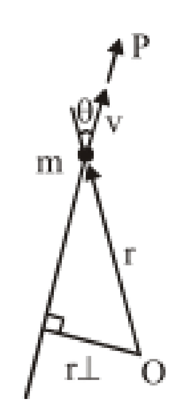
Angular Monentum

The orbital angular momentum : Irrespective of the path or trajectory of the particle, be it a straight line, curved path or a closed orbital path, the orbital angular momentum $\vec{L}$ of the particle at any position w.r.t. a reference point is

$$\vec{L} = \vec{r} \times \vec{P}$$

$$|\vec{L}| = rp\sin\phi$$

$$= r_{\perp} \times mv$$



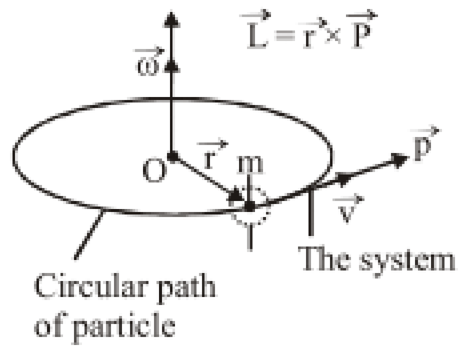
The r sin f is know as the moment arm, or lever arm designated as $r_{\perp}$.

The orbital angular momentum of particle in circular motion is expressed as

$$\vec{L} = mr^2\vec{\omega}$$

Note that direction of angular momentum vector $\vec{L}$ is parallel to

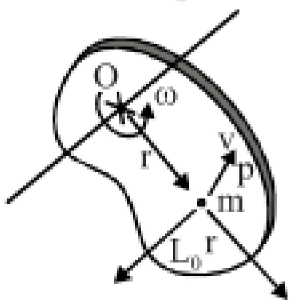
angular velocity $\vec{\omega}$. Figure shows the right hand thumb rule for determining direction of angular momentum. Curl your finger in rotrational sense from $\vec{r}$ vector to $\vec{p}$ vector, then the thumb points in the direction of angular momentum.



Spin angular momentum of a rigid body

We consider two cases:

(i) Axis of rotation passes through centre of mass of the body, referred to as centroidal roation.



(according to right hand thumb rule)

(ii) Axis of rotation is shifted from centre of mass, but passes through the body, referred to as non-centroidal rotation.

For non-centroidal rotation. $\vec{L} = I_0\vec{\omega}$

For non-centroidal rotation. $\vec{L} = I\vec{\omega}$

Where I_0 is moment of inertia about centre of mass and I is moment of inertia about rotational axis, to be calculated with the help of parallel axis theorem.

Simultaneous spin and orbital motion

The total angular momentum is the vector sum of the spin and orbital angular momentum

$$\vec{L}_{total} = \vec{L}_{spin} + \vec{L}_{orbit}$$

$$= I_{CM}\,\vec{\omega}_{spin} + mr_{\perp}^2\,\vec{\omega}_{orbit}$$

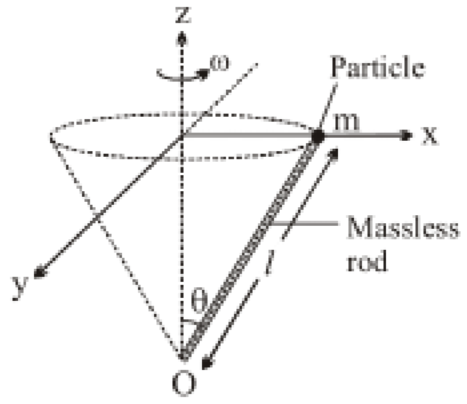
Angular momentum of an inverted conical pendulum

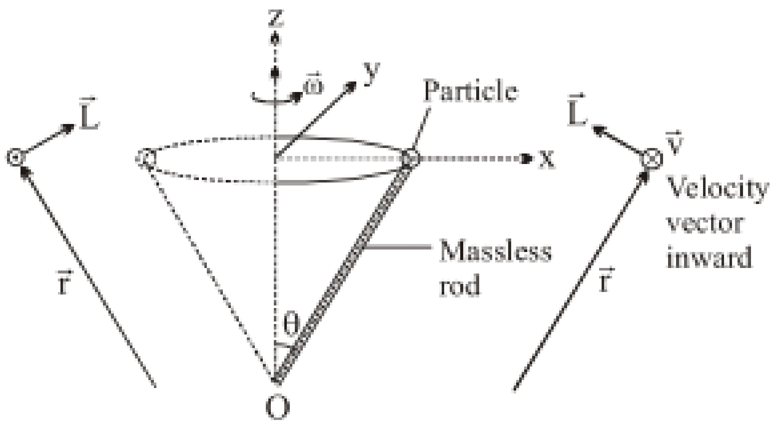
Angular momentum about O,

$\vec{r} = l\sin\theta\hat{i} + l\cos\theta\hat{k}$

$\vec{u} = (l\sin\theta)\omega\hat{j}$

$\vec{L} = \vec{r} \times m\vec{v} = ml^2\sin^2\theta\,\omega\hat{k} - ml\omega^2\sin\theta\cos\theta\,\hat{j}$





Concept : Angular momentum vector $\vec{L}$ is perpendicular to position vector $\vec{r}$ as well as momentum vector $\vec{p}$. The magnitude of $\vec{L}$ is constant but its direction is continuously varying. As the particle swings, $\vec{L}$ vector sweeps out a cone. The z-component of $\vec{L}$ is constant but the horizontal component travels around the circle with the particle.

Torque and Angular Momentum

When a number of force act on a particle, the net torque about origin 0 is sum of the torques due to each force.

$$\tau_{net} = \vec{r}\times\vec{F}_1 + \vec{r}\times\vec{F}_2 + ... = \vec{r}\times\sum_i \vec{F}_1 = \vec{r}\times\vec{F}_{net}$$

From Newton's second Law the net force is equal to rate of change of linear momentum. So we have

$$\tau_{net} = \vec{r}\times\vec{F}_{net} = \vec{r}\times\frac{d\vec{p}}{dt} \qquad ....(i)$$

As rate of change of angular momentum

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} \qquad ....(ii)$$

As $$\frac{d\vec{r}}{dt}\times\vec{p} = \vec{\upsilon}\times m\vec{\upsilon} = 0$$

Thus eqn. (ii) becomes

$$\frac{d\vec{L}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} \qquad .....(iii)$$

On comparing eqns. (i) and (iii), we get

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \qquad ......(iv)$$

Illustration :

What is the angular momentum of a particle of mass m = 2 kg that is located 15 m from the origin in the direction 30° south of west and has a velocity v = 10 m/s in the direction 30° east of north ?

Sol. *In the figure, the x-axis points east. We know r = 15 m; p = mv = 20kg m/s. The angle between r and p is*

$(180^o - 30^o) = 150^o$

Thus, $L = rp\sin\theta = (15)(20)\sin 150^o = 150 kg\ m^2/s$

We could also have used the moment arm $r\perp = 15\sin 30^o = 7.5\ m$

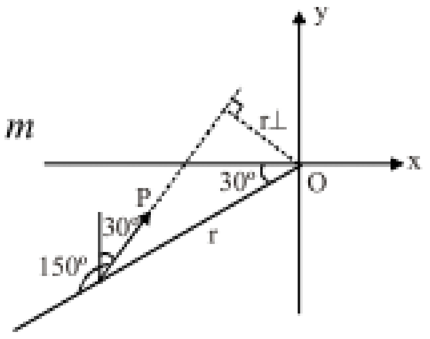
$L = r\perp p = (7.5)\times(20) = 150\ kg\ m/s$

In unit vector notation,

$$\vec{r} = 15\cos 30^o\ i - 15\sin 30^o\ j m$$

$$\vec{p} = 20\sin 30^o\ i + 20\cos 30^o\ j\ \ kg\ m/s$$

Therefore,



The angular momentum of each particle may be found by using unit vector notation or by finding the magnitude from r,p and the direction of the right - hand rule.

$$L = \left(-\frac{15\sqrt{3}}{2}i - \frac{15}{2}j\right)\times(10i + 10\sqrt{3}j) = -150\ kg\ m^2/s$$

Illustration :

A disc of mass M and radius R rotating at an angular velocity ω about an axis perpendicular to its plane at a distance R/2 from the center, as shown in the figure. What is its angular momentum ?

The moment of inertia of a disc about the central axis is $\frac{1}{2}MR^2$.

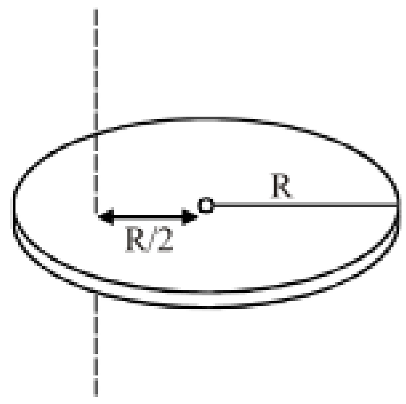
Sol. *The moment of inertia of the disc about the given axis may be found from the parallel axes theorem, equation $I = I_{cm} + Mh^2$, where h is the distance between the given axis and a parallel axis through the center of mass.*

Here $h = \frac{R}{2}$, *therefore,*

$$I = \frac{1}{2}MR^2 + M\left(\frac{R}{2}\right)^2 = \frac{3}{4}MR^2$$

The angular momentum is

$$L = I\omega = \frac{3}{4}MR^2\,\omega$$

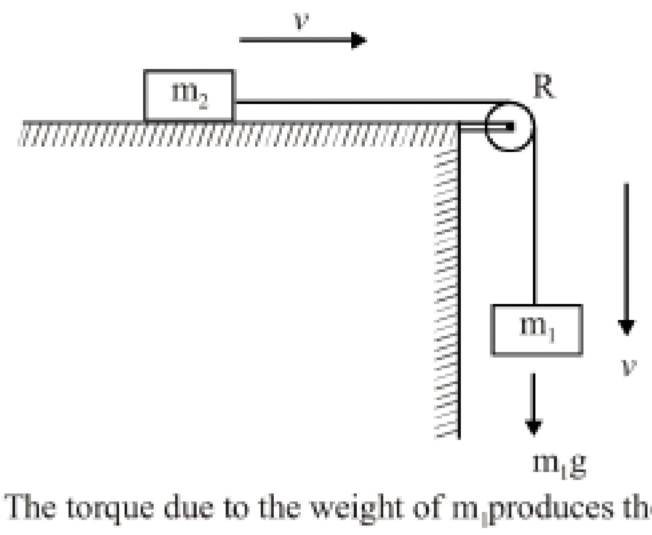


The axis of rotation at a distance R/2 from the center of the disc

Illustration :

Two blocks with masses $m_1 = 3$ kg and $m_2 = 1$ kg are connected by a rope that passes over a pulley of radius R = 0.2 m and mass M = 4 kg. The moment of inertia of the pulley about its center is $I = \frac{1}{2}MR^2$.

Use the concept of angular momentum to find the linear acceleration of the blocks. There is no friction. Assume that the c.m. of the block of mass m_2 is at a distance R above the center of the pulley.



The torque due to the weight of m_1 produces the change in angular momentum of the system

Sol. *If we take the origin at the center of the pulley, the angular momenta of the block are m_1vR and m_2vR and that of the pulley is $I\omega$. Therefore, the angular momentum is*

$$L = m_1vR + m_2vR + I\omega \qquad .....(i)$$

If the rope does not slip, then $v = \omega R$.

$$\therefore \quad L = (m_1 + m_2)vR + \frac{MR}{2}v$$

The net external torque about the center of the pulley is due to the weight of m_1

$$t_{ext} = r \perp F = R\,(m_1g) \qquad ....(ii)$$

Applying equation, $\tau_{ext} = \frac{d\vec{L}}{dt}$

We obtain

$$Rm_1g = (m_1 + m_2)\, Ra + \frac{MR}{2}\, a$$

$$or \qquad a = \frac{m_1 g}{m_1 + m_2 + M/2}$$

putting $m_1 = 3\ kg\ ;\ m_2 = 1\ kg;\ M = 4\ kg\ ;\ R = 0.2\ m$

$$a = \frac{(3)(10)}{3 + 1 + 4/2} = 5\ m/s^2$$

Conservation of Angular Momentum

If the next externa torque on a systme is zero, the total angular momentum is constant in magnitude and direction.

That is , if $\tau_{ext} = 0$ $\qquad \frac{dL}{dt} = 0$

Thus, L = constant

For rigid body rotating about a fixed axis.

$$L_f = L_i$$

or $\qquad L_f\,\omega_f = I_i\,\omega_i$

Angular Impulse

In complete analogy with the linear momentum, angular impulse is defined as

$$\tau = \int \tau_{ext}\, dt$$

Using Newton's second law for rotation motion,

$$t_{ext} = \frac{dL}{dt}$$

$\therefore \qquad \vec{\tau}\, t = \Delta\vec{L}_f - \vec{L}_i$

The net angular impulse acting on a rigid body is equal to the change in angular momentum of the body. This is called the impulse – momentum theorem for rotational dynamics.

Illustration :

A disc of moment of inertia 4 kg m^2 is spinning freely at 3 rad/s. A second disc of moment of inertia 2 kg m^2 slides down the spindle and they together.

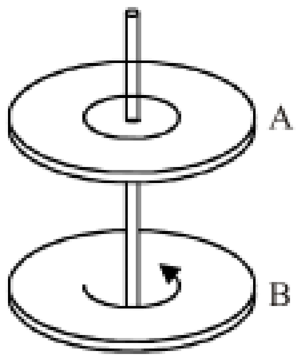
(a) What is the angular velocity of the combination ?

(b) What is the change in kinetic energy of the system ?

Sol. *Since there are no external torques acting, we may apply the conservation of angular momentum.*

$$I_f\omega_f = I_i\omega_i$$
$$6\omega_f = 4\times 3$$
$$\omega_f = 2 \text{ rad/s}$$

(b) The kinetic energies before and after the collision are



Disc A, initially not rotating, slip down a spindle into disc B that initially rotating freely

$$K_i = \frac{1}{2}I_i\,\omega_i^2 = 18\ J;$$

$$K_f = \frac{1}{2}I_f\,\omega_f^2 = 12\ J$$

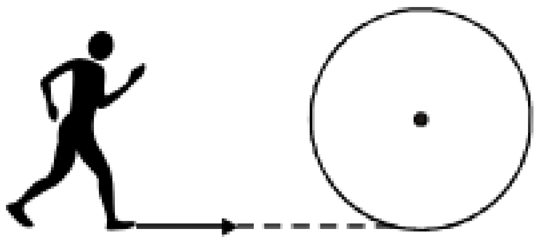
The change is

$$\Delta K = K_f - K_i = -6\ j.$$

In order for the two discs to spin together at the same rate, there had to be friction between them, The lost kinetic energy is converted with thermal energy.

Illustration :

A man of mass m = 80 kg runs at a speed u = 4 m/s along the tangent to disc-shaped horizontal platform of mass M = 160 kg and radius R = 2m. The platform is initially at rest but can rotate freely about an axis through its center.



Take $I = \frac{1}{2}MR^2$.

(a) Find the angular velocity of the platform after the man jumps on it.

(b) He then walk to the center. Find the new angular velocity. Treat the man as a point particle.

Sol. *Can we apply the conservation of linear momentum ?*

No, it con not be applied because the axle exerts an external force on the system man + platform.

Can we apply the conservation of angular momentum ! Yes, since the axle dose not exert any torque, we may use the conservation of angular momentum.

Can we apply kinetic energy for the collision between that man and the platform ? Why ?

(a) We choose the origin at the center of platform as shown in figure. When the man runs in a straight line, his initial angular momentum about this origin is $L = r_\perp p$,

where in this case $r_\perp = R$

so $L_i = muR$

After he jump on, one must take into account his contribution mR^2 to the moment of inertia. The final angular momentum, $L = I\omega$, *is*

$$L_f = \left(\frac{1}{2}MR^2 + mR^2\right)\omega$$

When we use set $L_f = L_i$, we find

$$\omega = \frac{mu}{(M/2+m)R}$$

Putting $m = 80$ kg ; $M = 160$ kg; $u = 4$ m/s ; $R = 2$ m

$$\text{We get } \omega = \frac{(80)(4)}{\left(\frac{160}{2}+80\right)2} = 1 \text{ rad/s}$$

(b) When the man reaches the center, his contribution to the moment of inertia is zero. The final angular momentum of part (a) is the initial value for (b);

$$L_i = \left(\frac{1}{2}MR^2 + mR^2\right)\omega_1 = 640 \text{ kg m}^2\text{/s}$$

$$L_f = \left(\frac{MR^2}{2}\right)\omega_2 = 320\,\omega_2$$

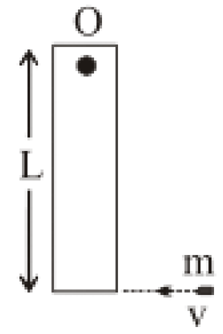
we get $\omega_2 = 2$ rad/s

Table : Analogy between Rotational Dynamics and Linear Dynamics.

	Quantity	Linear	Rotational
1.	Inertia	m	$S\, m_i r_i^2$
		or	$\int r^2 dm$
2.	Newton's Second Law	$F_{ext} = ma$	$\tau_{ext} = I\alpha$
		$\vec{F}_{ext} = \frac{d\vec{p}}{dt}$	$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$
3.	Work	$W_{lin} = \int \vec{F} d.\vec{s}$	$W_{rot} = \int \vec{\tau}\, .d\theta$
4.	Kinetic Energy	$K_{lin} = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$
5.	Work Energy Theorem	$W_{lin} = \Delta K_{line}$	$W_{rot} = \Delta K_{rot}$
6.	Impulse	$I = \int F_{ext}.dt$	$J = \int \tau_{ext}.dt$
7.	Momentum	$p = mv$	$L = I\omega$
8.	Impulse momentum Theorem	$\vec{I} = \Delta\vec{p}$	$\vec{J} = \Delta\vec{L}$
9.	Power	$P = \vec{F}.\vec{v}$	$P = \vec{\tau}.\vec{\omega}$

Practice Exercise

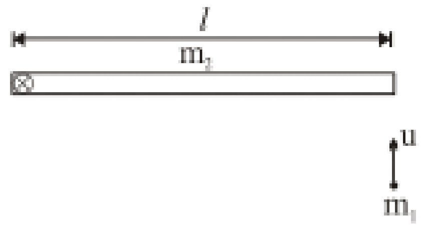
Q.1 A wooden long of mass M and length L is hinged by a frictionless nail at O. A bullet of mass m strikes with velocity v and sticks to it. Find angular velocity of the system, immediately after the collision, about O.



Q.2 A disc of mass M and radius r is rotating about its axis with angular velocity ω. Now if a mass m falls vertically on its rim and sticks to it. Then find final angular velocity of the combined system.

Q.3 A man of mass 100 kg stands at the rim of a turn-table of radius 2m, moment of inertia 4000 kg-m^2 mounted on a vertical frictionless shaft at its centre. The whole system is initially at rest. The man walks along the outer edge of the turn-table with a velocity of 1m/s relative to the earth.
(a) With what angular velocity and in what direction does the turn-table rotate?
(b) Through what angle will it have rotated when the man reaches his initial position on the turn-table?
(c) Through what angle will it have rotated when the man reaches his initial position relative to earth?

Q.4 In horizontal smooth plane. If particle sticks after collision to the rod, find



(a) Final angular velocity
(b) The impulse on particle

Q.5 A particle having mass 2 kg is moving along straight line 3x + 4y = 5 with speed 8 m/s. Find angular momentum of the particle about origin. x and y are in meters.

Q.6 A particle having mass 2 kg is moving with velocity $(2\hat{i}+3\hat{j})$ m/s. Find angular momentum of the particle about origin when it is at (1, 1, 0).

Q.7 A wheel of moment of inertia 0.500 kg-m^2 and radius 20.0 cm is rotating about its axis at an angular speed of 20.0 rad/s. It picks up a stationary particle of mass 200 g at its edge. Find the new angular speed of the wheel.

Answers

Q.1 $\omega = \dfrac{3mv}{L(M+3m)}$ Q.2 $\omega' = \left(\dfrac{M}{M+2m}\right)\omega$

Q.3 (a) $-\dfrac{1}{20}$ rad/s (b) $-\dfrac{2\pi}{11}$ radian (c) $\theta_t = -\dfrac{\pi}{5}$ radian Q.4 (a) $\omega = \dfrac{3m_1u}{(m_2+3m_1)}$ (b) $\dfrac{m_1m_2u}{m_2+3m_1}$

Q.5 16 kg m^2/s Q.6 $2\hat{k}$ kg m^2/s Q.7 19.7 rad/s

Instantaneous Axis of rotation (IAOR or ICR)

It is the axis about which the motion of a rigid body undergoing plane motion is assumed to be pure rotational motion. It is always perpendicular to the plane of motion of rigid body and instantaneously remains at rest. The point of intersection of instantaneous axis of rotation with the plane of motion of the remains at rest. The point of intersection of instantaneous axis of rotation with the plane of motion of the rigid body is called instantaneous centre of rotation (ICR) about which all points of the rigid body are assumed to be going in circles of different radii equal to their respective distances from ICR with the same ω and α as that about CM of the rigid body at that instant.

$\therefore$ Velocity of IAR = 0   i.e. $v_A = 0$

By finding the position of IAR, we can easily find the velocity of any point of rigid body at that instant.

$\vec{v}_P = \vec{\omega} \times \vec{r}_{P,A}$

We can also find the acceleration of any point P of rigid body at that instant provided the acceleration of IAR should be know.

$$\vec{a}_P = \vec{a}_{P,A} + \vec{a}_A$$

(i) Kinetic energy of the rigid body is $K = \frac{1}{2} I_A \omega^2$

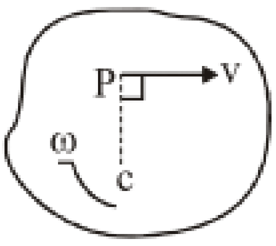
(ii) Angular momentum of rigid body about IAR is $L_A = I_A \omega$

(iii) $\tau_A = I_A \alpha$, where τ_A includes the torque of pseudo force acting on the CM about IAR also.

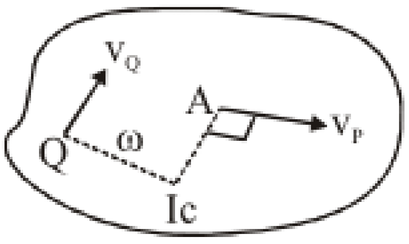
How to find the position of ICR

The position of ICR can be fund out in the following cases.

Case-I :



If the velocity of a point of the body and angular velocity are given.



Draw a line perpendicular to $\vec{v}$, the instantaneous centre must be lying on this line a distance 'r' given by $r = v/\omega$

Case II :

If the lines of action of two non-parallel velocities of two points of the rigid boy are given.

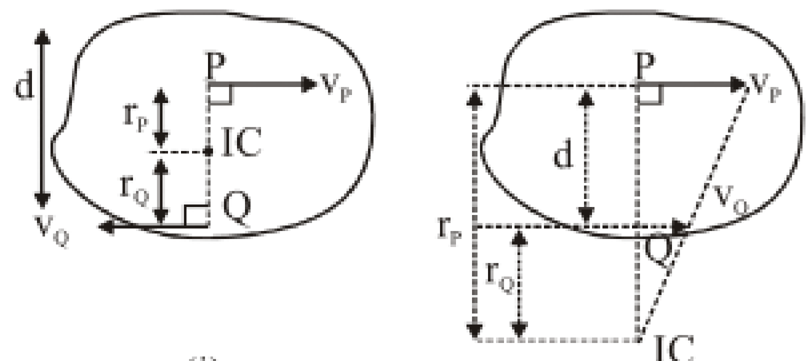
Draw the normals on the two non-parallel velocities $\vec{v}_P$ and $\vec{v}_Q$ at points P and Q, respectively. The point of intersection of these normals is the instantaneous centre at that instant.

Case-III :

If magnitudes and direction of two parallel velocities are given.

In figure (i), if the two velocities $\vec{v}_P$ and $\vec{v}_Q$ are in the opposite direction, then IC must be lying in between P and Q.

$$\frac{v_P}{v_Q} = \frac{r_P}{r_Q} \qquad \text{and } r_P + r_Q = d$$



(i) (ii)

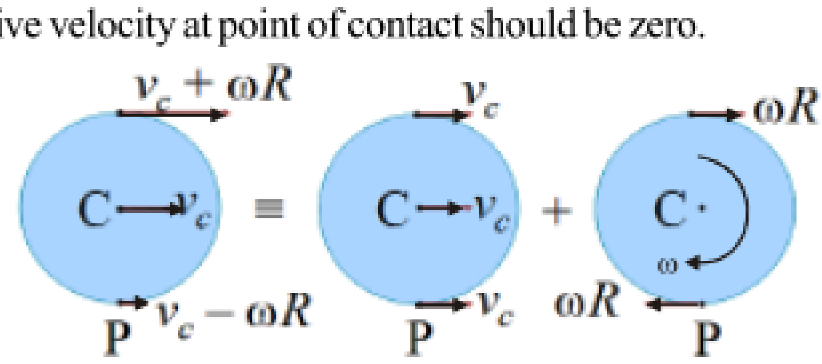
In figure (ii), if the two velocities $\vec{v}_P$ and $\vec{v}_Q$ are in the same direction, the IC must be lying outside PQ (near P if $v_P < v_Q$ and near Q if $v_p > v_Q$)

$$\frac{v_P}{r_Q} = \frac{r_P}{r_Q} \text{ and } r_P - r_Q = d$$

Rolling Motion

Pure rolling means no sliding. Now, the motion of any body can be divided into pure translation & pure rotation. And we can see rotation about any axis. So; for a wheel rolling on a horizontal surface, I m taking its COM as reference point to study its motion. If C is the reference point, then the wheel can be considered rotating about C. And the point C would be translating with velocity vc.

Sliding refers to the condition under which two bodies in contact have relative velocity. And under pure rolling, the relative velocity at point of contact should be zero.



Now for the wheel shown, point P is in contact with the ground. The point P on the ground has zero velocity. Thus, the point P on the wheel should also have zero velocity.

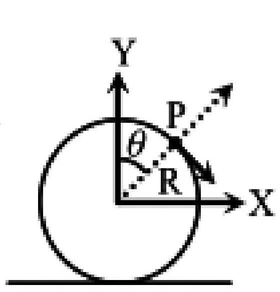
Trajectory of a point on a periphery of the wheel is a cycloid

$$\vec{v} = \frac{d\vec{r}}{dt} = v\left[\left(1+\cos\omega t\right)\hat{i} + \sin\omega t\,(-\hat{j})\right]$$

$$\vec{r} = \int d\vec{r} = v\int\left[\left(1+\cos\omega t\right)\hat{i} + \sin\omega t\,(-\hat{j})\right]dt$$

$$= \omega R\left(t + \frac{1}{\omega}\sin\omega t\right)\hat{i} + \frac{1}{\omega}\cos\omega t(\hat{j})$$

$$= R\left[\left(\omega t + \sin\omega t\right)\hat{i} + \cos\omega t\,(\hat{j})\right]$$



$$\frac{d\vec{v}}{dt}=\frac{dv}{dt}\left[(1+\cos\omega t)\hat{i}+\sin\omega t\left(-\hat{j}\right)\right]+\omega v\left[-\sin\omega t\,(\hat{i})-\cos\omega t\,(\hat{j})\right]$$

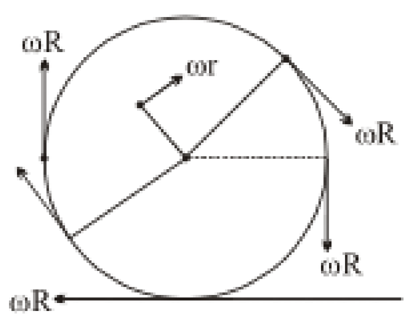
$$=\frac{dv}{dt}(\hat{i})+\frac{dv}{dt}\left[(\cos\omega t)\hat{i}+\sin\omega t\left(-\hat{j}\right)\right]+\frac{v^2}{R}\left[-\hat{r}\right]$$

$$=a_c\left(\hat{i}\right)+\alpha R\,(\hat{\tau})+\frac{v^2}{R}\,(-\hat{r})$$

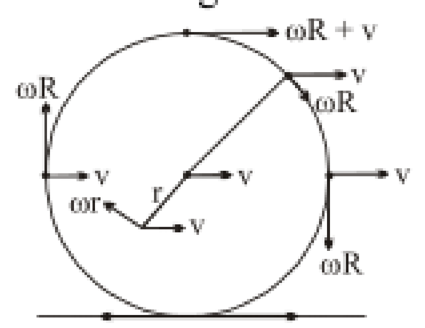
Kinematics of Boby in pure Rolling

1. **Velocity**

wrt centre

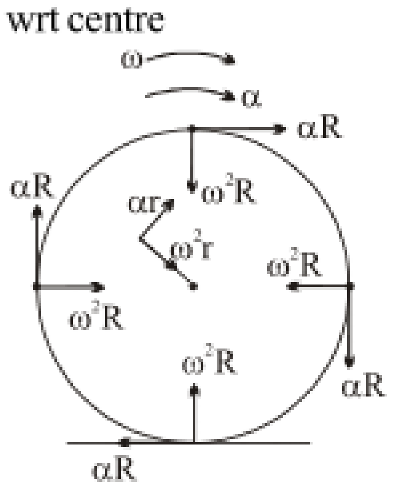


wrt ground



2. **Acceleration**

wrt centre



wrt ground

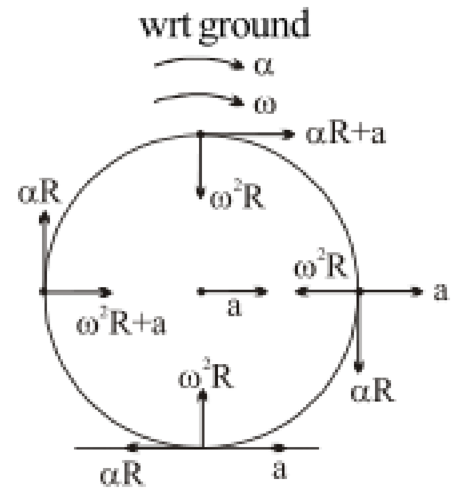
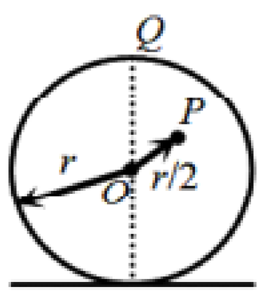


Illustration:

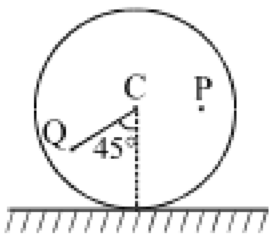
A disc of radius r rolls without slipping on a rough horizontal floor. If velocity of its center of mass is vo, then find the velocity of point P, $(OP = r/2$ *and* $\angle QOP = 60^o)$.



Ans: 7vo/2

Illustration:

A disc is rolling without slipping with angular velocity ω. P and Q are two points equidistant from the centre C. Find the order of magnitude of velocity ?



Ans. $v_P > v_C > v_Q$

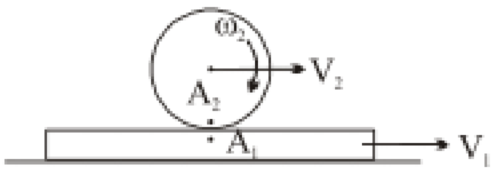
Constraint Equation:

Written at contact points where no slipping takes place.

$$\begin{bmatrix} \text{velocity of a contact} \\ \text{point on } 1^{st} \text{ rigid body} \end{bmatrix} = \begin{bmatrix} \text{velocity of same contact} \\ \text{point on } 2^{nd} \text{ rigid body} \end{bmatrix}$$

Write contraint equation for following exmaples:

Illustration :



Given velocity of C.M. w.r.t. ground is V_2 and V_1 is velocity of platform (w.r.t. ground). Find ω_2 is angular velocity of body about C.M.

Sol. $V_2 - \omega_2 R = V_1$

Friction and rolling

To get the direction of friction in pure rolling, we set two criteria's:

(a) Acceleration : Its direction should be such that vector sum all forces comply with it.

(b) Angular acceleration: Its direction should be such that vector sum all torques comply with it.

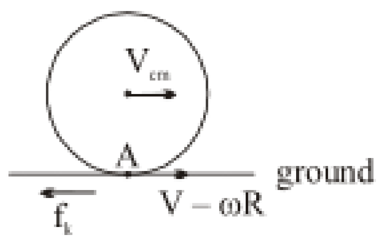
So, from above point we understand that if we show wrong direction of friction in our free body diagram, we will get a negative answer. So, direction of friction force in pure rolling should not be cause of concern.

Here it is also to be noted that in case of sliding, (kinetic friction) we can not take any arbitrary direction of friction and solve that question. In case of rolling static friction is unknown so after sloving, its value comes out negative if taken wrongly, but in case of kinetic friction its value is known beforehand (= μN), so there is no case of value of a known quantity coming negative after solving.

Here are few examples wherein, one might try to guess the direction of friction force.

Rolling with sliping

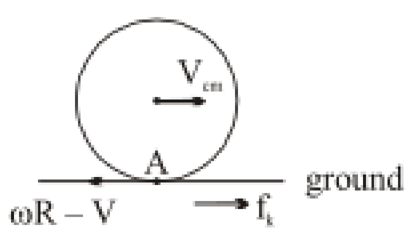
Case-I $V_{cm} > \omega R$



Note : The friction will be kinetic in nature & its magnitude can be determined using $f_k = \mu_k N$. Direction will be opposite to V_{cm} (because pt. of contact A is moving forward w.r.t. ground)

(This is how barkes stop a car)

Case -2 $V_{cm} < \omega R$



(This is how a car accelerate because of friction)

Case -3 $V_{cm} = \omega r$

Known as perfect rolling

Maximum problems we come across this situation

Note : 1. The friction will be be static in nature. Its magnitude be determined, it will very from O to $\mu_s N$.

2. Even its direction can not be perdicted

3. The total work done by static friction is zero. Thus mechanical cnergy of the system will remain conserved.

Application of Newton's Second Law in Rolling Motion

1. Write $F_{net} = M\, a_{cm}$ for the object as if it were a point-mass, that is, ignoring ratation.

2. Write $\tau = I_{cm}\, \alpha$ as if the object were only rotating about the centre of mass, that is ignoring translation.

3. Use of no-slip condition

4. Solve the resulting equations simultaneously for any unknown.

Caution :

• In general, it is not the case that $f = \mu N$

• Be certain that the sign convention of forces and torques are consistent.

Illustration :

Figure shown a shpere of mass M and radius R that rolls without slipping down an incline. Its moment of inertia about a central axis is $\frac{2}{5}MR^2$.

(a) Find the linear acceleration of the center of mass.

(b) What is the minimum coefficient of friction required for the sphere to roll without slipping ?

Sol. *Since the sphere is not driven by a chain or an axle the force of friction must be directed backward, up the slope.*

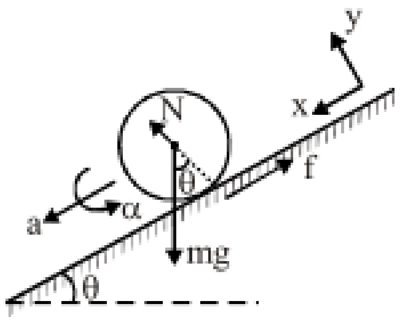
If there is no slipping, the point of contact is simultaneously at rest and so the friction is static.

The linear acceleration a of the c.m. and the angular acceleration α are assumed as shown in the figure.

Applying Newon's Second Law

$$\Sigma F_x = ma \qquad Mg\sin\theta - f = Ma \qquad \text{......(i)}$$

$$\Sigma\tau = I\alpha \qquad fR = I\alpha = \frac{2}{5}MR^2\,\alpha \qquad \text{......(ii)}$$



A shpere rolls down an incline. Since the sphere is not driven, the force of friction is direction up the incline.

Since the sphere rolls without slipping, the speed of the centre is $v = \omega R$.

By differentiating it with respect to time, we get

$$a = \alpha R. \qquad \text{......(iii)}$$

Solving equation (i), (ii) & (iii)

we get $$f = \frac{2}{5}Ma \qquad \text{......(iv)}$$

and $$a = \frac{5}{7}g\sin\theta \qquad \text{......(v)}$$

(b) Substituting (v) into (iv) yields

$$f = \frac{2}{7}Mg\sin\theta \qquad \text{......(vi)}$$

We use equation (vi) to find the minimum coefficient of friction required for the sphere to roll without slipping.

By definition, $f = \mu N$ *where* $N = Mg\cos\theta$. *Combining this with equation (vi) we have*

$$\mu = \frac{2}{7}\tan\theta$$

If the coefficient of static friction is less than this value, the sphere will slip as it rolls down the incline.

Illustration:

A sphere of mass m and radius R is placed at rest on a plank of mass M which is placed on a smooth horizontal surface as shown in the figure. The coefficient of friction between the sphere and the plank is μ. At t = 0, a horizontal velocity v_0 is given to the plank. Find the time after which the sphere starts rolling.

Sol. *Sphere :*

$$a_c = \frac{f}{m} = \mu g$$

$$\alpha = \frac{\tau_C}{I_C} = \frac{f\ R}{\frac{2}{5}mR^2} = \frac{5}{2}\frac{\mu g}{R}$$

After time t

$$v_c = a_c t = \mu g t$$

$$\omega = \alpha t = \frac{5}{2}\frac{\mu g}{R}t$$

The velocity of the point of contact is

$$v = v_c + \omega R = vgt + \frac{5}{2}\mu g t = \frac{7}{2}\mu g t$$

Planck :

Retardation $a = \dfrac{f}{M} = \dfrac{\mu m g}{M}$

Instantaneous velocity $v = v_0 - \dfrac{\mu m g}{M}t$

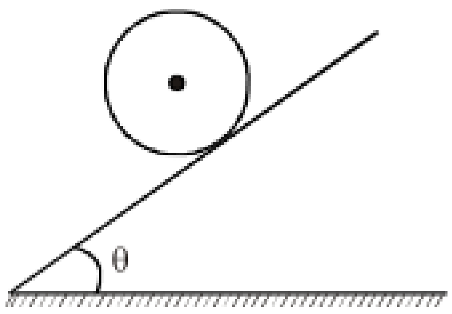
Condition of pure rolling

$$v = \frac{7}{2}\mu g t = v_0 - \frac{\mu m g}{M}t$$

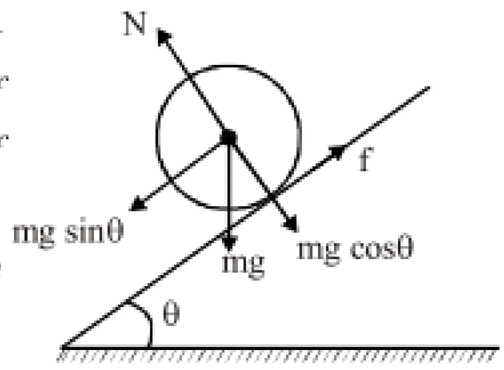
or $$t = \frac{v_0}{\left[\frac{7}{5} + \frac{m}{M}\right]\mu g}$$

Illustration :

A uniform disc of mass m and radius R is rolling without slipping up a rough incline plane which makes an angle 30° with the horizontal. If the coefficient of static and kinetic friction are each equal to μ and the only force acting on the disc are gravitational and frictional, then find direction and magnitude of the frictional force acting on it.



Sol. *Since disc does not slip hence frictional force is static and static friction can have any value between 0 and μN. Component of mg parallel to the plane is mg sinθ which is opposite to the direction of motion of the centre of the disc, and hence speed of the centre of mass decreases. For pure rolling the relation $v_{c.m.} = \omega R$ must be obeyed. Therefore ω must decrease. Only frictional force can provide a torque about the centre.*



Torque due to friction must be opposite to the $\vec{\omega}$. There frictional force will act up the plane Now, for translational motion

$$mg\sin\theta - f = ma_{c.m.} \qquad \text{....(i)}$$

For rotational motion

$$fR = I\alpha, \text{ where } I = \text{M.I. of the disc about centre.}$$

$$= I\frac{a}{R}, \text{ as } a = \alpha R$$

$$\Rightarrow \qquad a_{c.m.} = \frac{fR^2}{I} \qquad \text{....(ii)}$$

For (i) and (ii) we get,

$$f = \frac{mg\sin\theta}{1 + \frac{mR^2}{I}}$$

Putting the value of θ and I we get

$$f = mg/6$$

Kinetic Energy of a Rolling Body

Since the rolling motion is a combination of linear velocity of the center and rotational motion about the center. Therefore, the total kinetic energy of a rolling body is given by

$$K = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2 \quad \text{(i)}$$

Where $\frac{1}{2}mv_c^2$ is the translational kinetic energy and

$\frac{1}{2}I_c\omega^2$ is the rotational kinetic energy about the center of mass

In pure rolling motion, $\nu_c = \omega R$

$$\therefore \quad K = \frac{1}{2}m(\omega R)^2 + \frac{1}{2}I_c\omega^2$$

or $\quad K = (I_c + mR^2)\,\omega^2$

Using parallel axes therem, the term $I_c + mR^2$ given the moment of inertia about the point of contact, therefore,

$$I_0 = I_c + mR^2$$

and $\quad K = \frac{1}{2}I_0\omega^2 \qquad \text{(ii)}$

Note that equation (ii) give the rotational kinetic energy of the wheel aobut the point of contact.

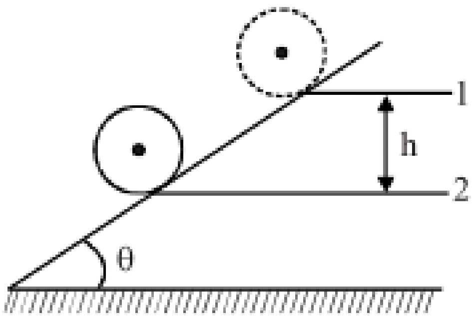
Illustration

A solid cylinder of mass m and radius r starts rolling down an inclined plane f inclination θ. friction is enough to prevent slipping. Find the speed of tis centre of mass when, its centre of mass has fallen a height h.

Sol. *Consider the two shown positions of the cylinder. As it does not slip, total mechanical energy will be conserved.*

Energy at position 1 is $E_1 = mgh$

Energy at position 2 is $E_2 = \frac{1}{2}mv^2_{c.m.} + \frac{1}{2}I_{c.m.}\,\omega^2$



$$\because \quad \frac{V_{c.m.}}{r} = \omega, \text{ and } I_{c.m.} = \frac{mr^2}{2}$$

$$\Rightarrow \quad E_2 = \frac{3}{2}mv^2_{c.m.}$$

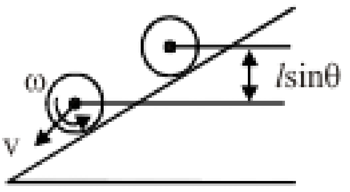
From COE, $E_1 = E_2$

$$\Rightarrow \quad V_{c.m.} = \sqrt{\frac{3}{2}gh}$$

Illustration :

A sphere of radius r starts rolling down an incline of inclination. Find the speed of its CM when it has covered a distance l.

Sol. $mg\,l\sin\theta = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$



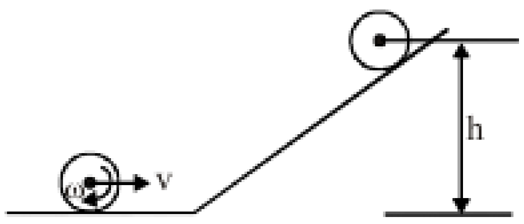
$v = \omega r$

By sloving we get

$v = \sqrt{10 g l \sin\theta / 7}$

Illustration :

A ball of radius R and mass m is rolling without slipping on a horizontal surface with velocity of its centre of mass v. It then rolls without sloping up a hill to a height h before momentarily coming to rets. Find h.



Sol. $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$

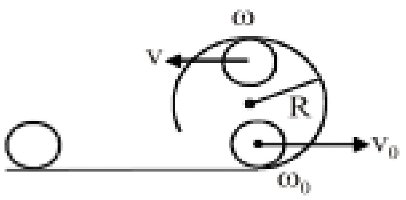
$v = \omega R$, $I = \frac{2}{7}mR^2$

$h = \frac{7v_{CM}^2}{10g}$

Illustration:

Figure shows a rough track, a portion of which is in the form of a cylinder of radius R. With what minimum linear speed should a sphere of radius r be set rolling on the horizontal part so that it completely goes round the circle on the cylindrical part?

Sol.

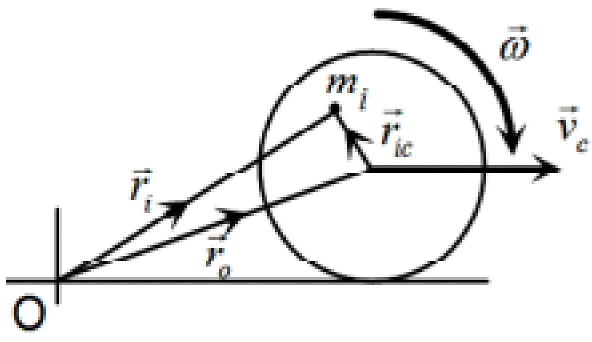


$mg = \frac{mv_C^2}{R-r}$

$\frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2 = mg2(R-r) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

By solving we get $\sqrt{27g(R-r)/7}$

Angular momentum of a rigid body in planar motion about O



$$\vec{l}_i = \sum_{i=1}^{n} m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \sum_{i=1}^{n} m_i \left[(\vec{r}_o + \vec{r}_{ic}) \times (\vec{v}_c + \vec{\omega} \times \vec{r}_{ic})\right]$$

$$= \sum_{i=1}^{n} m_i (\vec{r}_o \times \vec{v}_c) + \vec{r}_o \times \sum_{i=1}^{n} m_i (\vec{\omega} \times \vec{r}_{ic}) + \sum_{i=1}^{n} m_i (\vec{r}_{ic} \times \vec{v}_c) + \sum_{i=1}^{n} m_i \left[\vec{r}_{ic} \times (\vec{\omega} \times \vec{r}_{ic})\right]$$

$$= \left[\sum_{i=1}^{n} m_i\right] (\vec{r}_o \times \vec{v}_c) + \vec{r}_o \times \sum_{i=1}^{n} m_i \vec{v}_{ic} + \left[\sum_{i=1}^{n} m_i \vec{r}_{ic}\right] \times \vec{v}_c + \sum_{i=1}^{n} m_i \left[\vec{r}_{ic}^{\,2} \vec{\omega} - (\vec{r}_{ic} \cdot \vec{\omega}) \vec{r}_{ic}\right]$$

$$= M (\vec{r}_o \times \vec{v}_c) + \vec{\omega} \sum_{i=1}^{n} m_i \vec{r}_{ic}^{\,2} = M (\vec{r}_o \times \vec{v}_c) + I_{COM} \vec{\omega}$$

Angular momentum of a rigid body in planar motion about O: $\vec{l} = M (\vec{r}_o \times \vec{v}_c) + I_{COM} \vec{\omega}$

Spin and Orbital Angular Momentum

For a rigid body undergoing linear and rotational motion, the total angular momentum may be split into two pats – the orbital angular momentum and the spin angular momentum. The orbital angular momentum L_0 is the angular momentum relative to the center of mass.

The orbital term treats the system as a point particle at the center of mass, where as the spin term is the sum of the angular momenta of the particles relative to the center of mass. The total angular momentum relative to the origin O in an inertial frame is the sum:

$$L = L_0 + L_{cm}$$

Illustration :

A solid sphere of mass M and radius R rolls without slipping on a horizontal surface as shown in the figure. Find the total angular momentum of the sphere with respect to the origin O fixed on the ground.

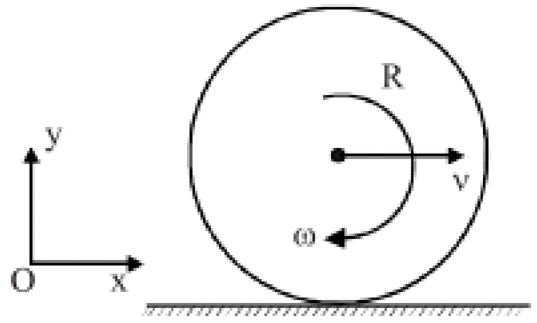
Sol. *Let us assume the clockwise sense of rotation positive.*

Orbital angular momentum about O is

$$L_0 = MvR$$

Spin angular momentum about c.m. is

$$L_{c.m.} = I\omega = \frac{2}{5} MR^2 \omega$$



The total angular momentum is

$$L = L_0 + L_{cm} = M v R + \frac{2}{5} M R^2 \omega$$

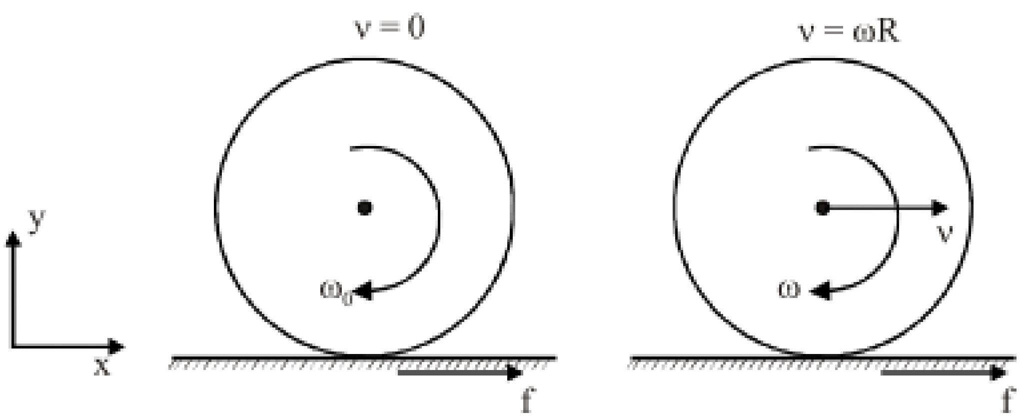
For pure rolling, $v = \omega R$, *therefore,* $L = \frac{7}{5} M v R$

Illustration :

A wheel is held by a handle on its axle and given initial angular velocity ω_0. *The wheel is then placed in contact with the ground. At first the wheel remains stationary, spraining in place. After a short time it begins to move forward and eventually reaches the point where it rolls without slipping. Find the final velocity of the wheel in terms of the initial angular velocity* w_0.

Take $I = \frac{MR^2}{2}$.

Sol. *We assume that wheel is initially rotating clockwise. Let the wheel starts rolling after a time t. Then, Using Impulse – Momentum Theorem For translation*



(a) A spinning wheel is placed on a horizontal surface with zero initial velocity. The friction force acts forward.
(b) After a time t the wheel starts rolling.

$$\text{Impulse} = \Delta p = p_f - p_i$$

or $\quad f\,t = M v - 0 \qquad (i)$

For rotation

Angular Impulse $= \Delta L = L_f - L_i$

or $\quad -f R t = I\omega - I\omega_0$

Note that clockwise angular momentum is considered as positive.

Since $I = \frac{MR^2}{2}$, *therefore,* $-fRt = \frac{MR^2}{2}\,(\omega - \omega_0) \qquad (ii)$

condition of pure rolling

$$v = \omega R \qquad (iii)$$

Using equations (i), (ii) and (iii), we get

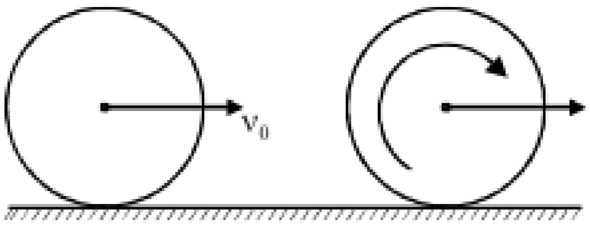
$$-MwR^2 = \frac{MR^2}{2}\,(\omega - \omega_0)$$

or $\quad \omega = \frac{\omega_0}{3}$

The linear momentum of the wheel is $v = \omega R = \frac{\omega_0 R}{3}$

Illustration :

A unifrom disc of mass m and radius r is projected horizontally with velocity v_0 on a routh horizotnal floor so that it starts off with a purely sliding motion at t = 0. At t = t_0 seconds it acquries a purely rolling motion.



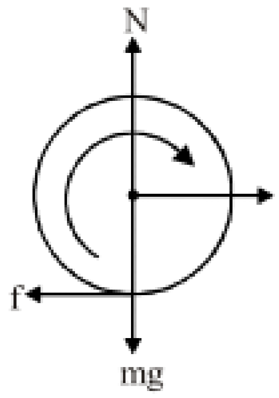
(a) Calculate the velocity of the centre of mass of the disc at $t = t_0$.

(b) Assuming coefficient of friction to be μ calculate t_0.

(c) The work done by frictional force as a function of time

(d) Total work done by the frictional over a time t much longer than t_0.

Sol. *F.B.D. of the disc.*



When the disc is projected it starts sliding and hence there is relative motion between the points of contact. Therefore frictional force acts on the disc in the direction opposite to the motion.

(a) *Now for translational motion*

$$a_{c.m.} = \frac{f}{m}$$

$$f = \mu N \text{ (as it slides)}$$

$$= \mu mg$$

$\Rightarrow$ $a_{c.m.} = -\mu g$, *negative sign indicates that $a_{c.m.}$ is opposite $v_{c.m.}$*

$$\Rightarrow \quad v_{c.m.(t)} = v_0 - \mu g t_0$$

$$\Rightarrow \quad t_0 = \frac{(v_0 - v)}{\mu g}, \text{ where } v_{c.m.(t_0)} = v \qquad (i)$$

For rotational motion about centre

$$\tau_f + \tau_{mg} = I_{c.m.}\,\alpha \qquad \Rightarrow \qquad \mu mgr = \frac{mr^2}{2}\alpha$$

$$\Rightarrow \quad a = \frac{2mg}{r} \qquad (ii)$$

Therefore $\omega_{(t_0)} = 0 + \frac{2mg}{r} t_0$, using $\omega_t = \omega_0 + at$

$$\Rightarrow \quad \omega = \frac{2(v_0 - v)}{r} \qquad \text{(iii) using (i)}$$

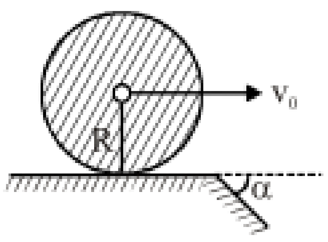
$$v_{c.m.} = \omega r$$

$$\Rightarrow \quad v = 2\,(v_0 - v) \text{using (iii)}$$

$$\Rightarrow \quad v = \frac{2}{3} v_0$$

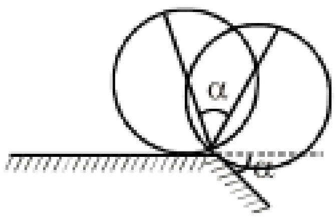
Illustration :

A uniform solid cylinder of radius R =15 cm rolls over a horizontal plane passing into an inclined plane forming an angle $\alpha = 30^o$ with the horizontal. Find the maximum value of the velocity v_0 which still permits the cylinder to roll onto the inclined plane section without a jump. The sliding is assumed to be absent.



Sol. Initial energy $E_i = \frac{1}{2} m v_0^2 + \frac{1}{2} I_{c.m.} \omega^2 + mgR$

For rolling $\frac{v_0}{R} = \omega$



$$\Rightarrow \quad E_1 = \frac{1}{2} m v_0^2 + \frac{1}{2} \cdot \frac{1}{2} mR^2 \frac{v_0^2}{R^2} + mgR$$

$$= \frac{3}{4} m v_0^2 + \frac{1}{2} I_{c.m.} w^2 + mgR \cos\alpha$$

From COE (conservation of energy)

$$\Rightarrow \quad mv^2 = mv_0^2 + \frac{4}{3} mgR\,(1 - \cos\alpha)$$

F.B.D. of the cylinder when it is at the edge.

Centre of mass of the cylinder describes circular motion about P.

Hence $mg\cos\alpha - N = mv^2/R$

$$\Rightarrow \quad N = mg\cos a - mv^2/R$$

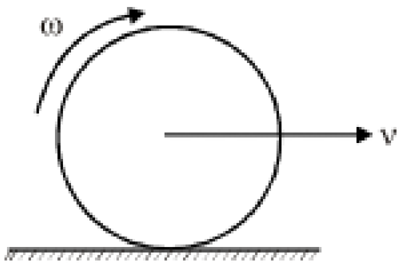
$$= mg\cos\alpha - \frac{mv_0^2}{R} - \frac{4}{3} mg + \frac{4}{3} mg\cos\alpha$$

For no jumping, $N \geq 0$

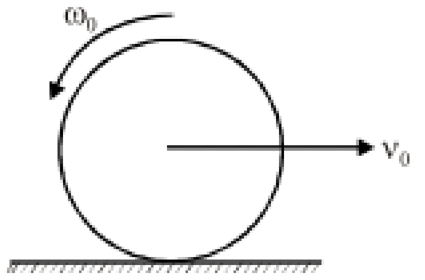
$$\Rightarrow \quad \frac{7}{3} mg\cos a - \frac{4}{3} mg - \frac{mv_0^2}{R} \geq 0 \qquad \Rightarrow \quad v_0 \leq \sqrt{\frac{7gR}{3}\cos\alpha - \frac{4}{3}g}$$

Practice Exercise

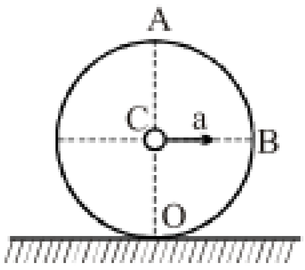
Q.1 A disc of mass and radius R is rolling with slipping on a rough horizontal surface which has coefficient of friction μ. At some instant the velocity of its center of mass is v and angular speed about center of mass is ω. The torque of the frictional force about the point which is instantaneously at rest at this moment is ?



Q.2 A uniform circular disc of radius r placed on a rough horizontal surface has initially a velocity v_0 and angular velocity ω_0 as shown in the figure. The disc comes to rest (neither translates nor rotates) after moving some distance. Then $\frac{v_0}{r\omega_0}$ is ?

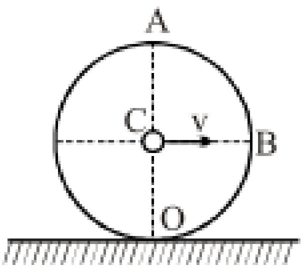


Q.3 A ball of radius R = 10.0 cm rolls without slipping down an inclined plane so that its centre moves with constat acceleration a = 2.50 cm/s^2 ; t = 2.00 s after the beginning of motion its position corresponds to that shown in figure. Find :

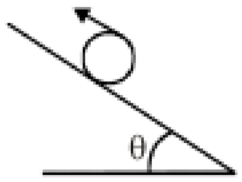


(A) the velocities of the points A, B and O; (B) the acceleration of these points

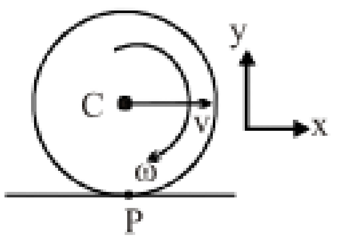
Q.4 A cylinder rolls without slipping over a horizotal plane. The radius of the cylinder is equal to r. Find the curvature radii of trajectories traced out by the points A and B in figure.



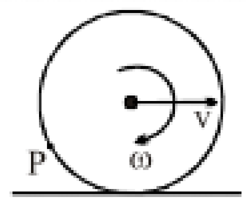
Q.5 A sphere kept on a rough inclined plane is in equilibrium by a string wrapped over it. If the angle of inclination is θ, the tension in the string will be equal to



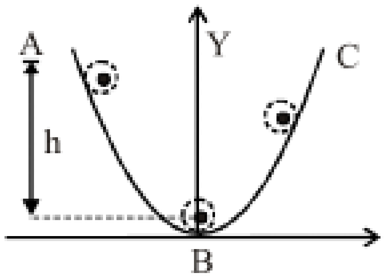
Q.6 A wheel of radius R is rolling without slipping on a stationary horizontal surface. Find the acceleration of point of contact P at the instant shown.



Q.7 A uniform circular disc of radius R rolls without slipping with a constant velocity on a stationary horizontal surface. Find the distance covered by a point P on its circumference in one full rotation.

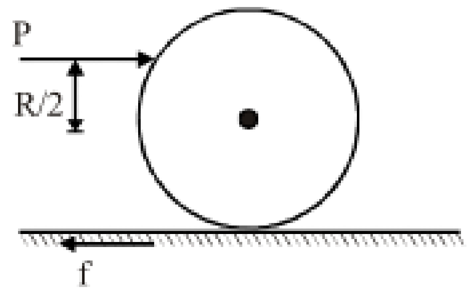


Q.8 A uniform cylinder rolls from rest from A down the side of a trough whose vertical dimension y si given by the equation $y = kx^2$. The cylinder does not slip form A to B but the surface of trough is frictionless from B to C. Then height of ascent of cylinder towards C is



Q.9 A sphere of mass M and radius r slips on a rough horiozntal plane. At some instant it has translational velocity v_0 and rotational velocity about the center $v_0/2r$. The rotational velocity when the sphere starts pure rolling is

Q.10 A solid cylinder of mass M and radius R lying on a rough horizontal surface for which coefficient of friction is μ is pushed by applying a horizontal force P at a distance R/2 from the center as shown in figure. The frictional force acting at the contact is



Answers

Q.1 $\mu mg\left(R - \frac{v}{\omega}\right)$ Q.2 $\frac{1}{2}$

Q.3 (A) $v_A = 10$ cm/s, $v_B = 7.1$cm/s $v_0 = 0$ (B) $a_A = 5.6$ cm/s^2, $a_B = 2.5$ cm/s^2, $a_0 = 2.5$ cm/s^2

Q.4 $R_A = 4r$, $R_B = 2\sqrt{2}r$ Q.5 $\frac{mg\sin\theta}{2}$ Q.6 $\vec{a}_p = \omega^2 R\,\hat{j}$ Q.7 8R

Q.8 $\frac{2h}{3}$ Q.9 $\frac{6}{7}v_0$ Q.10 zero

ELASTICITY

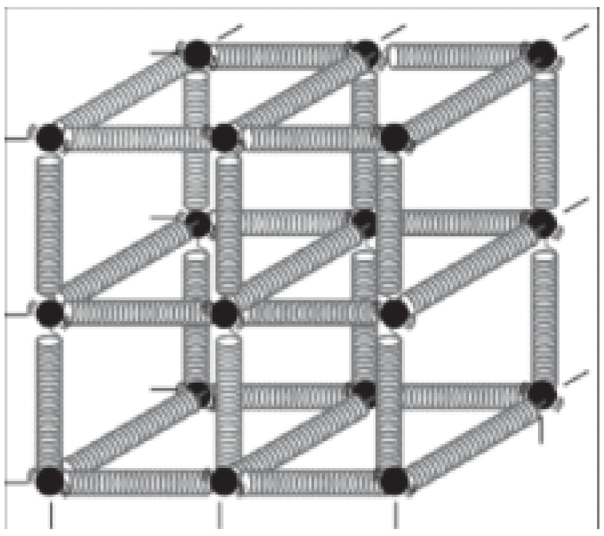
Elasticity

The property of material body by virtue of which its regain its original configuration, when external force is removed is called elasticity.

The property of the material body by virtue of which it does not regain its original configuration when the external force is removed is called plasticity.

Cause of Elasticity

In a solid atoms and molecules are arranged in such a way that each molecule is acted upon by the forces due to the neighbouring molecules. When no deforming force is applied on the body, each molecule of the solid is in its equilibrium position and the inter molecular forces of the solid are maximum. On applying deforming force, the molecules are displaced from their equilibrium position. Inter molecular force gets changed and restoring forces are developed. It is explained by using spring- ball model. Deforming force is removed, these restoring force bring the molecule to its equilibrium positions. Thus the body regains its original shape and size.



Spring-ball model for the illustration of elastic behaviour of solids.

The restoring mechanism can be visualised by taking a model of spring-ball system shown above. Here the balls represent atoms and springs represent interatomic forces.

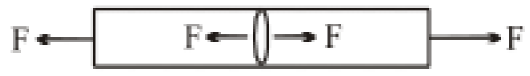
If you try to displace any ball from its equilibrium position, the spring system tries to restore the ball back to its original position. Thus elastic behaviour of solids can be explained in terms of microscopic nature of the solid. When a body is subjected to a deforming force, a restoring force is developed in the body. This restoring force is equal in magnitude but opposite in direction to the applied force.

Stress(σ)

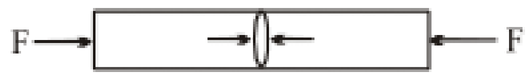
When deforming force is applied on the body then the equal restoring force in opposite direction is developed inside the body. The restoring force per unit area is called stress.

$$\text{Stress}(\sigma) = \frac{\text{restoring force}}{\text{Area of cross section of the body}}$$

Stress can be tensile or compressive as given below–

Tensile 

$\sigma = \dfrac{F}{A}$ as $F \propto A$

Compressive 

Strain

Suppose we stretch a wire by applying tensile forces of magnitude F to each end. The length of the wire increases from L to $L + \Delta L$. The fractional length change is called the strain. It is a dimensionless quantity.

$$\text{strain} = \frac{\Delta L}{L}$$

Hooke's law for tensile and compressive forces

Suppose we had wires of the same composition and length but different thicknesses. It would require larger tensile forces to stretch the thicker wire the same amount as the thinner one. We conclude that the tensile force required is proportional to the cross-sectional area of the wire ($F \propto A$). Thus, the same applied force per unit area produces the same deformation on wires of the same length and composition.

Hooke's Law

$$\text{stress} \propto \text{strain}$$

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

equation still says that the length change (ΔL) is proportional to the magnitude of the deforming forces (F). Stress and strain account for the effects of length and cross-sectional area ; the proportionality constant Y depends only on the inherent stiffness of the material from which the object is composed ; it is independent of the length and cross-sectional area.

Comparing equation $F = k\Delta L$ and $\dfrac{F}{A} = Y\dfrac{\Delta L}{L}$, $F = Y\dfrac{\Delta L}{L}A$. Y is called the elastic modulus or Young's modulus, Y has the same units as those of stress (Pa or N/m^2) since strain is dimensionless.

Young's modulus can be thought of as the inherent stiffness of a material ; it measures the resistance of the material to elongation or compression. Material that is flexible and stretches easily (for example, rubber) has a low Young's modulus. A stiff material (such as steel) has a high Young's modulus. It takes a larger stress to produce the same strain.

Hooke's law holds up to a maximum stress called the proportional limit. For many materials, Young's modulus has the same value for tension and compression. Some composite materials, such as bone and concrete, have significantly different Young's moduli for tension and compression. The different properties of these two substances lead to different values of Young's modulus for tensile and compressive stress.

Illustration :

A light wire of length 4m is suspended to the ceiling by one of its ends. If its crossectional area is19.6 mm^2, what is its extension under a load of 10kg. Young's modulus of steel = 2×10^{11} Pa.

Sol. *Given quantities – original length L = 4m ; force F = 10 × 9.8 = 98 N ; and Y = 2×10^{11} Nm^{-2}*

l = ?

Using the relation,

$$\text{Young's modulus (Y)} = \frac{longitudinal\ stress}{longitudinal\ strain}$$

We have

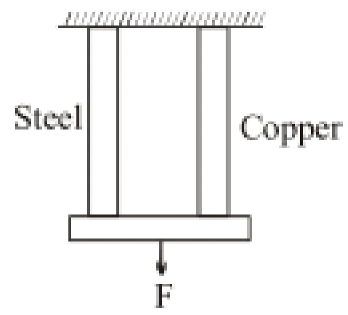
$$Y = \frac{F/A}{l/L} \qquad \Rightarrow l = \frac{FL}{YA}$$

$$\therefore \qquad l = \frac{98 \times 4}{2 \times 10^{11} \times 19.6 \times 10^{-6}} = 1 \times 10^{-4}\ m$$

$$= 0.1\ mm$$

Illustration :

Two vertical rods of equal lengths, one of steel and the other of copper, are suspended from the ceiling, at a distance l apart and are connected rigidly to a rigid horizontal light bar at their lower ends.



If A_S and A_C be their respective cross sectional areas, and Y_S and Y_C, their respective Young's modulii of elasticities, find where should a vertical force F be applied to the horizontal bar, in order that the bar remains horizontal. (Fig.)

Sol. *Let the force F be applied at a distance x from the steel bar, measured along the horizontal bar.*

Let F_S and F_C be the loads on steel and copper rods respectively, so

$$F_S + F_C = F \qquad \text{... (i)}$$

Since the rigid horizontal bar remains horizontal so, the extensions produced in the two rods and hence strains remains same.

i.e.,

$$\frac{F_S}{A_S Y_S} = \frac{F_C}{A_C Y_C} \qquad \text{... (ii)}$$

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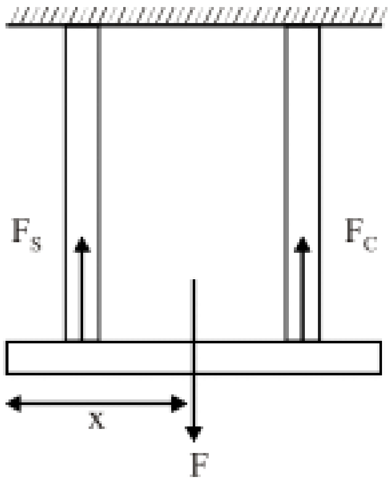


Solving (i) and (ii) $F_S = \dfrac{FA_SY_S}{A_SY_S + A_CY_C}$

and $$F_C = \dfrac{FA_CY_C}{A_SY_S + A_CY_C}$$

Now, taking moments about the steel bar.

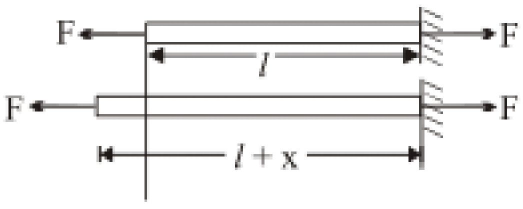
$$F_C l = Fx \qquad \Rightarrow x = \frac{F_C}{F}\, l \;\Rightarrow\; \frac{A_CY_Cl}{A_SY_S + A_CY_C}$$

or $$x = l \Big/ \left[1 + \left(\frac{A_S}{A_C}\right)\left(\frac{Y_S}{Y_C}\right)\right]$$

Elastic potential energy

It is the potential energy stored inside the body due to change their configuration. If F force is applied on a body as shown below.

For differential change in length dx the work done by restoring force F is dw



$$dw = -Fdx \qquad \therefore \left(F = \frac{AY}{L}x\right)$$

$$dw = -\frac{AY}{L}\,x\,dx$$

$$W_{elastic} = -\frac{AY}{L}\int_0^l x\,dx$$

$$\Delta U = -W = \frac{AYl^2}{2L} \quad = \frac{1}{2}\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)(AL) \qquad \therefore U_i = 0, U_f = U$$

Elastic potential energy (U) = $\frac{1}{2}$ (stress) (strain) (volume)

Elastic potential energy per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

The above formula holds good for any type of strain. Change in equilibrium, restoring force = external force F

Then $$U = \frac{1}{2}\left(\frac{YA}{L}\right)l^2$$

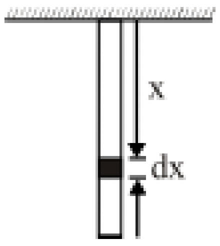
$$= \frac{1}{2}\left(\frac{Y}{L}l\right)Al = \frac{1}{2}Fl$$

Illustration :

A uniform heavy rod of weight W, cross- sectional area A and length L is hanging from a fixed support. Young's modulus of the material of the rod is Y. Neglect the lateral contraction. Find the elongation of the rod.

Sol. *Consider a small length dx of tne rod at a distance x from the fixed end. The part below this small element has* length L – x. The tension T of the rod at the element equals the weight of the rod below it.

$$T = (L - x)\,\frac{W}{L}$$



Elongation in the element is given by

elongation = original length × stress / Y

$$= \frac{Tdx}{AY} = \frac{(L-x)Wdx}{LAY}$$

$$\text{The total elongation} = \int_0^L \frac{(L-x)Wdx}{LAY}$$

$$= \frac{W}{LAY}\left(Lx - \frac{x^2}{2}\right)_0^L = \frac{WL}{2AY}$$

Illustration:

A wire having a length $l = 2m$, and cross sectional area $A = 5mm^2$ is suspended at one of its ends from a ceiling. What will be its strain energy due to its own weight, if the density and Young's modulus of the material of the wire be $d = 9g/cm^3$ and $Y = 1.5 \times 10^{11}\ Nm^{-2}$?

Sol. *Consider an elemental length of the wire of length dx, at a distance x from the lower end. Clearly, this length is acted upon by the external force equal to the weight of the portion of wire below it = xAdg. In equilibrium, the restoring force f = xAdg.*

$$\therefore \quad stress = \frac{f}{A} = xdg.$$

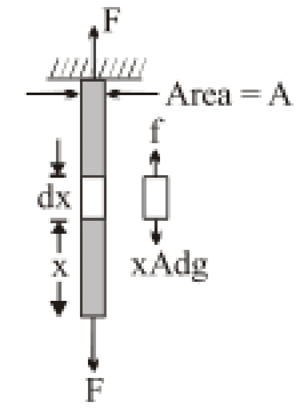
Now, elastic potential energy stored in the elemental length will be

$$dU = \frac{1}{2} \times stress \times strain \times volume$$

$$= \frac{1}{2} \times \frac{(stress)^2}{Y} \times volume$$

$$= \frac{1}{2} \times \frac{(xdg)^2}{Y} \times Adx = \frac{1}{2}\,\frac{d^2g^2A}{Y}\,x^2\,dx$$

$\therefore$ *Total elastic potential energy* $U = \int dU$



$$= \int_0^l \frac{1}{2}\frac{d^2g^2A}{Y}\,x^2\,dx$$

$$= \frac{1}{6}\,d^2g^2\,\frac{Al^3}{Y}$$

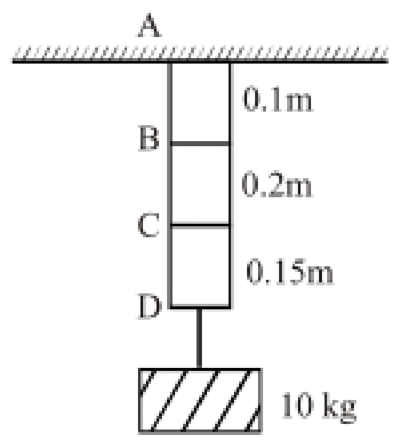
Substituting the values,

$$U = \frac{1}{6} \times \frac{(9\times10^3)\,(9\cdot8)^2 \times 5 \times 10^{-6} \times 2^3}{1.5 \times 10^{11}}$$

$$= 3.46 \times 10^{-7}\ J$$

Illustration :

Find out the shift in point B, C and D.



$Y_{AB} = 2.5 \times 10^{10}\ N/m^2$

$Y_{BC} = 4 \times 10^{10}\ N/m^2$

$Y_{CD} = 1 \times 10^{10}\ N/m^2$

$A = 10^{-7}\ m^2$

Sol. $$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\,\Delta\,L}$$

$$\Rightarrow \quad \Delta L = \frac{FL}{AY} = \frac{MgL}{AY}$$

Shift of point B $(\Delta L_B) = \Delta L_{AB} = \dfrac{10\times10\times0.1}{10^{-7}\times2.5\times10^{10}} = 4 \times 10^{-3}\ m = 4\ mm$

Shift of point C $(\Delta L_C) = \Delta L_B + \Delta L_{BC} = 4 \times 10^{-3} + \dfrac{100\times0.2}{10^{-7}\times4\times10^{10}}$

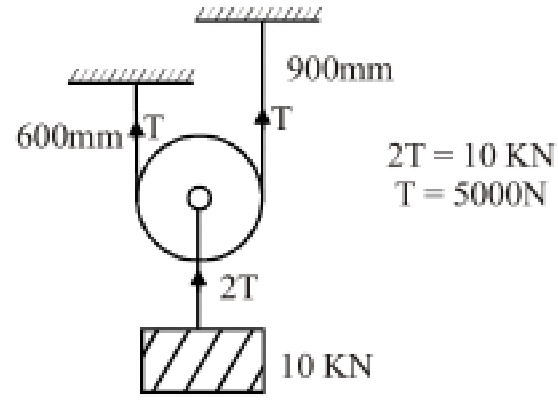
$= 9 \times 10^{-3}\ m = 9\ mm$

Shift of point D $(\Delta L_D) = \Delta L_C + \Delta L_{CD} = 9 \times 10^{-3} + \dfrac{100\times0.5}{10^{-7}\times1\times10^{10}}$

$= 9 \times 10^{-3} + 15 \times 10^{-3} = 24\ mm$

Illustration :

A load of 10 KN is supported from a pulley which in turn is supported by a rope of cross-sectional area 1×10^3 mm² and modulus of elasticity 10^3 N/mm², as shown in figure. Neglecting the friction at the pulley determine the deflection of load.



Sol. *longitudinal stress in the rope is*

$$\sigma = \frac{T}{A} = \frac{5\times 10^3}{10^3\, mm^2} = 5\ N/mm^2$$

$$\text{Extension in the rope} = \frac{stress}{Y} \times L$$

$$= \frac{5 N/mm^2}{10^3\ N/mm^2} \times 1500$$

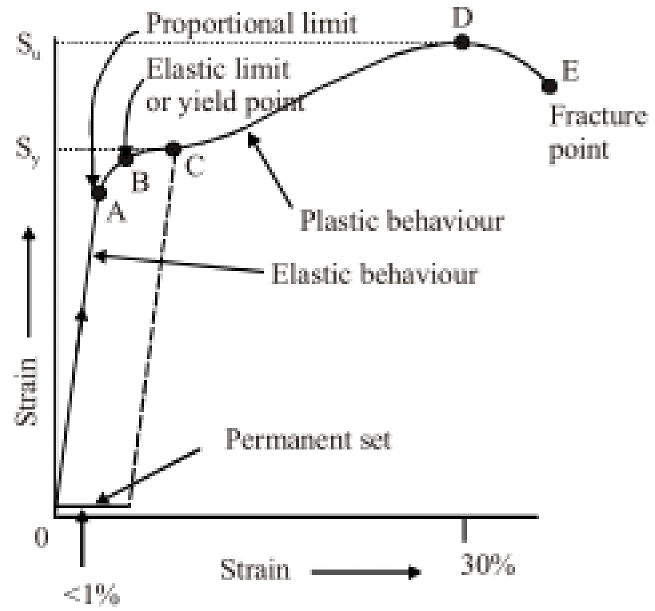
$$= 7.5\ mm$$

$$\text{Deflection in the load} = \frac{7.5}{2}$$

$$= 3.75\ mm$$

Stress-strain curve

The relation between the stress and the strain for a given material under tensile stress can be found experimentally. The applied force is gradually increased in steps and the change in length is noted. A graph is plotted between the stress and the strain produced. The stress-strain curves vary from material to material. These curves help us to understand how a given material deforms with increasing loads. From the graph, we can see that in the region between O to A, the curve is linear. In this region, Hooke's law is obeyed. The body regains its original dimensions when the applied force is removed. In this region, the solid behaves as an elastic body.



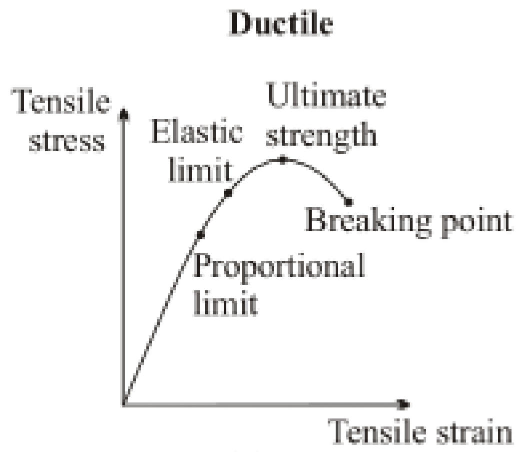
Stress-strain curve for steel.

Beyond hooke's law

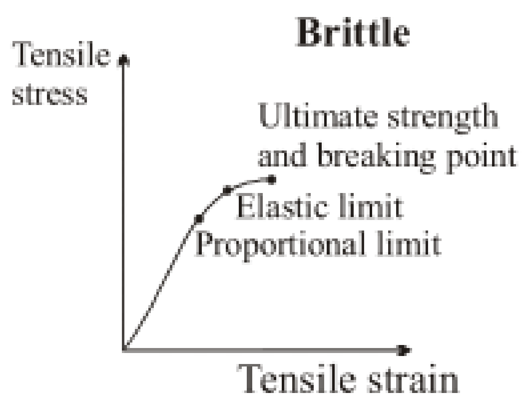
In the region from A to B, stress and strain are not proportional. Nevertheless, the body still returns to its original dimension when the load is removed. The point B in the curve is known as **yield point** (also known as **elastic limit**) and the corresponding stress is known as **yield strength** (S_y) of the material. If the tensile or compressive stress exceeds the proportional limit, the strain is no longer proportional to the stress. The solid still returns to its original length when the stress is removed as long as the stress does not exceed the elastic limit.

If the stress exceeds the elastic limit, the material is permanently deformed. For still larger stresses, the solid factures when the stress reaches the breaking point. The maximum stress that can be withstood without breaking is called the ultimate strength. The ultimate strength can be different for compression and tension ; then we refer to the compressive strength or the tensile strength of the material. A ductile material continues to stretch beyond its ultimate tensile strength without breaking ; the stress then decreases from the ultimate strength (fig. (a)). Examples of ductile solids are relatively soft metals, such as gold, silver, copper, and lead. These metals can be pulled like taffy, becoming thinner and thinner until finally reaching the breaking point.

While as Brittle material can not stand beyond ultimate strength



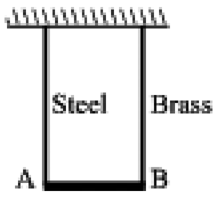
(a)



(b)

Practice Exercise

Q.1 A light rigid bar is suspended horizontally from two vertical wires, one of steel and one of brass, as shown in figure. Each wire is 2.00 m long. The diameter of the steel wire is 0.60 mm and the length of the bar AB is 0.20 m. When a mass of 10 kg is suspended from the centre of AB bar remains horizontal.



(i) What is the tension in each wire?

(ii) Calculate the extension of the steel wire and the energy stored in it.

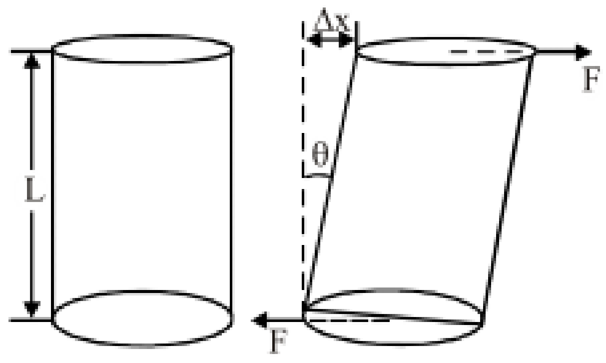
(iii) Calculate the diameter of the brass wire.

(iv) If the brass wire were replaced by another brass wire of diameter 1 mm, where should the mass be suspended so that AB would remain horizontal? The Young modulus for steel = 2.0×10^{11} Pa, the Young modulus for brass = 1.0×10^{11} Pa.

Answers

Q.1 (i) 50 N, (ii) 0.045 J, (iii) 8.4×10^{-4} m, (iv) x = 0.12 m

Shearing Stress



A cylinder subjected to shearing (tangential) stress deforms by an angle θ.

However, if two equal and opposite deforming forces are applied parallel to the cross-sectional area of the cylinder, as shown in fig, there is relative displacement between the opposite faces of the cylinder. The restoring force per unit area developed due to the applied tangential force is known as **tangential** or **shearing stress**.

As a result of applied tangential force, there is a relative displacement Δx between opposite faces of the cylinder as shown in the fig. The strain so produced is known as **shearing strain** and it is defined as the ratio of relative displacement of the faces Δx to the length of the cylinder *L*.

$$\text{Shearing strain} = \frac{\Delta x}{L} = \tan\theta$$

where θ is the angular displacement of the cylinder from the vertical (θ is very samll $\tan\theta \simeq \theta$).

Volume Deformation

Since the fluid presses inward on all sides of the object (figure), the solid is compressed-its volume is reduced. The fluid pressure P is the force per unit surface area ; it can be through of as the volume stress on the solid object. Pressure has the same units as the other kinds of stress: N/m^2 or Pa.

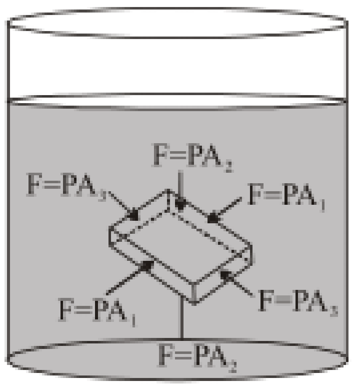


Fig. Forces on an object when submerged in a fluid

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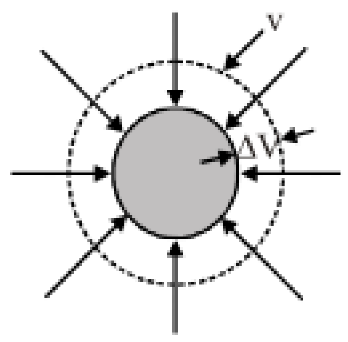
$$\text{volume stress} = \text{pressure} = \frac{F}{A} = P$$

The resulting deformation of the object is characterized by the volume strain, which is the fractional change in volume :

$$\text{volume strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\Delta V}{V}$$

Bulk Modulus (B)



In fig., a solid sphere placed in the fluid under high pressure is compressed uniformly on all sides. The force applied by the fluid acts in perpendicular direction at each point of the surface and the body is said to be under hydraulic compression. This leads to decrease in its volume without any change of its geometrical shape. The body develops internal restoring forces that are equal and opposite to the forces applied by the fluid (the body restores its original shape and size when taken out from the fluid). The internal restoring force per unit area in this case is equal to the hydraulic pressure (applied force per unit area). The strain produced by a hydraulic pressure is called **volume strain** and is defined as the ratio of change in volume (ΔV) to the original volume (V).

$$\text{Volume strain} = \frac{\Delta v}{v}$$

We have seen that when a body is submerged in a fluid, it undergoes a hydraulic stress (equal in magnitude to the hydraulic pressure). This leads to the decrease in the volume of the body thus producing a strain called volume strain.

$$\Delta P = -B\frac{\Delta V}{V} \qquad \text{(Hooke's law for volume deformation)}$$

where V is the volume at atmospheric pressure. The negative sign. equation $\Delta P = -B\frac{\Delta V}{V}$ allows the bulk modulus to be positive. The bulk moduli of liquids are generally not much less than those of solids, since the atoms in liquids are nearly as close together as those in solids.

Gases are much easier to compress than solids or liquids, so their bulk moduli are much smaller. The bulk moduli of a few common materials are given in Table

Material	B (10^9 Nm^{-2} or GPa)
Solids	
Aluminium	72
Brass	61
Copper	140
Glass	37
Iron	100
Nickel	260
Steel	160
Liquids	
Water	2.2
Ethanol	0.9
Carbon disulphide	1.56
Glycerine	4.76
Mercury	25
Gases	
Air (at STP)	1.0×10^{-4}

Table : Bulk moduli *(B)* of some common Materials

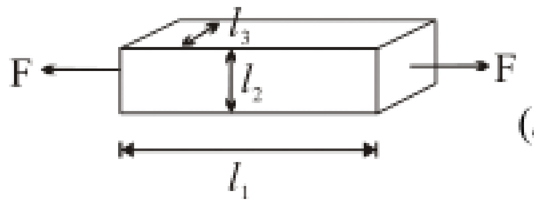
Compressibility (k)

The reciprocal of the bulk modulus is called compressibility and is denoted by k. It is defined as the fractional change in volume per unit increase in pressure.

$$k = \frac{1}{B} = -\frac{1}{V}\left(\frac{\Delta V}{\Delta P}\right)$$

Poisson's ratio

When an elongation is produced by longitudinal stresses, a change is produced in the lateral dimensions of the strained substance. Thus, when a wire is stretched, its diameter diminishes ; and when the longitudinal strain is small, the lateral strain is proportional to it. The ratio of the lateral strain to the longitudinal strain is called Poisson's ratio.



(l_1, l_2, and l_3 are the dimentional when no strain. Δl_1, Δl_2, and Δl_3 are the change in length of l_1, l_2, and l_3 respectively)

$$Y = \frac{\frac{F}{A}}{\frac{\Delta l_1}{l_1}}$$

$$\frac{\Delta l_2}{l_2} = \frac{\Delta l_3}{l_3} = -\sigma\,\frac{\Delta l_1}{l_1}$$

Illustration:

. *A uniform bar of length L and cross sectional area A is subjected to a tensile load F. If Y be the Young's modulus of the material of the bar and σ be its poisson's ratio, then determine the volumertic strain.*

Sol. *Longitudinal stress* $= \dfrac{F}{A}$.

Longitudinal strain $= \dfrac{F}{AY} = \varepsilon_1$ *(say)* ... (i)

Now, by definition of Poisson's ratio,

$$\sigma = \frac{-\text{lateral strain}}{\text{longitudinal strain}} = \frac{-\delta r / r}{\delta L / L}$$

or $\delta r / r = -\sigma \delta L / L \quad \Rightarrow -\dfrac{\sigma F}{AY}$ *[From eqn. (i)]*

Since Volumetric strain = Strain in length + Twice strain in radius.

$\therefore$ *Volumetric strain* $= \dfrac{\delta L}{L} + \dfrac{2\delta\delta}{r}$

$$= \frac{F}{AY} + 2\left(-\frac{\sigma F}{AY}\right) \qquad = \frac{F}{AY}(1 - 2\sigma).$$

Calorimetry

Units of heat & Mechanical equivalent of heat (J)

It was early 19th century when "James Prescott Joule" accidentally did an experiment which made two very important contribution in the scientific world. And it was Herman Von Helmholtz (a German) who later proved that indeed Joule was right.

Joules contribution bridged two major gaps in the scientific world.

i) Energy conservation principle was well grounded.

ii) The missing link between heat and energy was rectified.

Yes, heat was not thought to be a form of energy, rather it was known to be a fluid substance that flows. And that fluid was named calorie. They would say that when an iron rod is heated at one end, the other end also becomes hot as some calorie has flown to the rod. It was a very detailed mathematical theory.

Now, lets see the problem of energy conservation. We have seen many examples where energy in the form of K + U = constant; but not always. We know many places where in energy doesn't seem to be conserved. One of the examples is a box sliding on a rough surface. The box eventually stops because of friction. Thus, the KE of the box is lost. Where did it go? Today we can say that it got converted into heat energy, but earlier heat was not known as energy, but heat. Thus for them it was lost. And so energy conservation principle doesn't hold true.

A system is said to be isolated if no exchange or transfer of heat occurs between the system and its surroundings. When different parts of an isolated system are at different temperature, a quantity of heat transfers from the part at higher temperature to the part at lower temperature. The heat lost by the part at higher temperature is equal to the heat gained by the part at lower temperature. Calorimetry means measurement of heat. When a body at higher temperature is brought in contact with another body at lower temperature, the heat lost by the hot body is equal to the heat gained by the colder body, provided no heat is allowed to escape to the surroundings.

As heat is just energy in transit, its unit in SI is joule. However, another unit of heat "calorie" is in wide use. This unit was formulated much before it was recognised that heat is a form of energy. The old day definition of calorie is as follows :

The amount of heat needed to increas the temperature of 1g of water from 14.5°C to 15.5°C at a pressure of 1 atm is called 1 calorie.

The calorie is now defined in terms of joule as 1 cal = 4.186 joule.

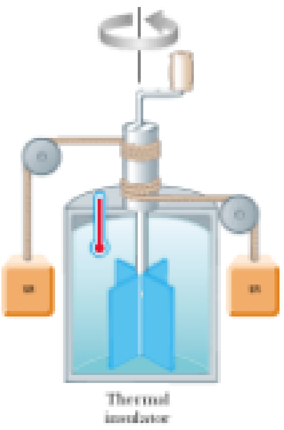
Illustation :

What is the kinetic energy of a 10 kg mass moving at a speed of 36 km/h in calorie ?

Sol. KE = $\frac{1}{2} \times 10 \times 10^2 = 500\text{J} \simeq 120\ cal$

Principle of Calorimetry

A device in which heat measurement can be made is called a **calorimeter**. It consists a metallic vessel and stirrer of the same material like copper or aluminium. The vessel is kept inside a wooden jacket which contains heat insulating materials like glass wool etc. The outer jacket acts as a heat shield and reduces the heat loss from the inner vessel. There is an opening in the outer jacket through which a mercury thermometer can be inserted into the calorimeter.



From this experiment he came up with new physical quantities.

Heat capacity (C'): $Q = \int C' dT$

specific heat capacity: $Q = m \int_{T_i}^{T_f} s \, dT = m \, s_{avg} \Delta T$

Molar heat Capacity: $C = \frac{C'}{n}$ (n - no. of moles)

The branch of thermodynamics which deals with the measurement of Heat is called colorimetry. When two bodies at different temperature are mixed, heat will be transferred from body at higher temperature to a body at lower temperature till both acquire same temperature. Principle of colorimetry represents the law of conservation of Heat Energy.

Heat lost = Heat gained

Specific Heat capacity

The amount of heat needed to raise the temperature of unit mass of a material by unit degree of measurement is known as the specific heat capacity of that material. If Q amount of heat raises the temperature of mass m of a material by ΔT, then its specific heat capacity is given as :

$$s = \frac{Q}{m\Delta T} \quad \Rightarrow \quad Q = ms\Delta T$$

Also the amount of heat supplied per unit increase in temperature for any body is known as

its heat capacity, $c = \frac{Q}{\Delta T} = ms$.

Latent Heat

Heat required for the change of phase or state. No chage in temperature is involved when substance changes its state or phase. ($Q = mL$, L = Latent Heat)

Latent Heat of Fusion : The Heat supplied to a substance which changes it from solid to liquid state at its melting point and 1 atm. pressure is called latent Heat of fusion. ($Q = mL_f$)
Latent heat of fusion of Ice (L_f) = 80 cal/gm.

Latent Heat of Vapourization : The Heat supplied to a substance which changes it from liquid to vapour state at its boiling point and 1 atm pressure is called latent heat of vapourization. ($Q = mL_V$)
Latent heat of vapouriztion of water (L_V)= 540 cal/g.

Illustration :

The temperature of equal masses of three different liquids A, B, and C are 12°C, 19°C and 28°C respectively. The temperature when A and B are mixed is 16°C and when B and C are mixed it is 23°C. What should be the temperature when A and C are mixed ?

Sol. *Let m be the mass of each liquid and S_A, S_B, S_C be specific heats of liquids A, B and C respectively. When A and B are mixed. The final temperature is 16°C.*

$\therefore$ *Heat gained by A = heat lost by B*

i.e. $mS_A(16 - 12) = mS_B(19 - 16)$

i.e. $S_B = \frac{4}{3}S_A$ .....(i)

When B and C are mixed. Heat gained by B = heat lost by C

i.e. $mS_B(23 - 19) = mS_C(28 - 23)$

i.e. $S_C = \frac{4}{5}S_B$ .....(ii)

From eq. (i) and (ii) $S_C = \frac{4}{5} \times \frac{4}{3}S_A = \frac{16}{15}S_A$

When A and C are mixed, let the final temperature be θ

$\therefore$ $mS_A(\theta - 12) = mS_C(28 - \theta)$

i.e. $\theta - 12 = \frac{16}{15}(28 - \theta)$

By solving, we get $\theta = \frac{628}{31} = 20.26^\circ C$

Illustation:

An object A of mass 2 kg is moving on a frictionless horizontal track has perfectly inelastic collision with another object B of mass 3 kg made of the same material and moving in front of A in same direction. Their common speed after the collision is 4 m/s. Due to the collision the temperature of the two objects, which was initially the same, is increased, though only by 0.006°C. The specific heat capacities of the two objects are the same : 0.5 kJ/kg°C. What was the initial speed (in m/s) of the colliding object A before the collision?

Sol.

$$2(V_1) + 3(V_2) = (2 + 3)V$$

$$\Rightarrow \quad 2V_1 + 3V_2 = 20 \qquad \text{........(i)}$$

loss in KE = heat energy

$$KE_i - KE_f = M_{total}\, S\Delta T$$

$$\frac{1}{2}2(V_1)^2 + \frac{1}{2}3(V_2)^2 - \frac{1}{2}(2 + 3)V^2 = (2 + 3)S\Delta T$$

$$V_1^2 + \frac{3}{2}V_2^2 - 40 = 15 \text{ or } V_1^2 + \frac{3V_2^2}{2} = 55 \qquad \text{........(ii)}$$

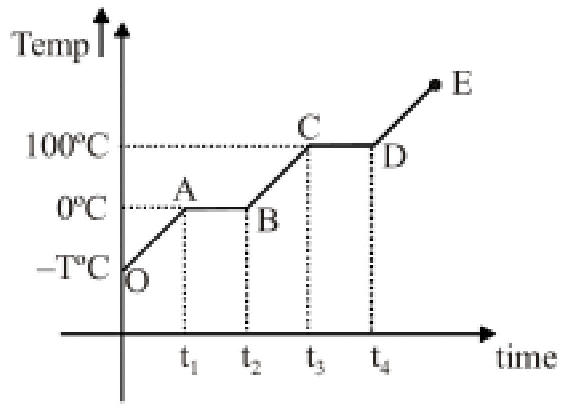
Solving equation (i) and (ii), we get,

$V_1 = 1$m/s or 7 m/s and $V_2 = 6$m/s or 2 m/s for collision $V_1 > V_2$. So $V_1 = 7$ m/s and $V_2 = 2$ m/sec

The following example provides a method by which the specific heat capacity of a given solid can be determinated by using the principle, heat gained is equal to the heat lost.

Heating Curve

If to a given mass (m) of a solid (Ice), Heat is supplied at constant rate P and a graph is poltted between temperature and time



(1) In the region OA

Temperature of solid is changing with time

$$Q = m\, s\, \Delta T$$

$$P(\Delta t) = m\, s\, \Delta T \qquad (\Delta t = t_1 - 0,\ \Delta T = 0 - (-T))$$

$$\frac{P}{ms} = \frac{\Delta T}{\Delta t} \qquad \frac{\Delta T}{\Delta t} = \text{slope of line OA}$$

(2) In the region AB

Temperature is constant, here substance changes its phase solid to liquid, between A and B.

$$Q = mL_f$$
$$P\,\Delta t = m\,L_f$$
$$L_f = \frac{P(t_2 - t_1)}{m}$$
L_f = length of line AB

Latent Heat of fusion is proportional to length of line .

(3) In the Region BC

Temp. of liquid is increasing with time

$$Q = m\,s\,\Delta T \qquad (\Delta t = t_3 - t_2,\ \Delta T = 100 - 0)$$
$$P\,\Delta t = m\,s\,\Delta T$$

(4) In the regin CD, temperature is constant, so it represents change of state.

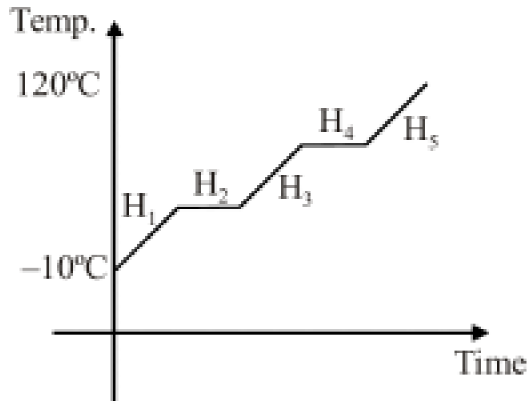
$$Q = mL_V$$
$$\frac{P(t_4 - t_3)}{m} = L_V$$
L_V = length of line CD

(5) The line DE represents gaseous state of substance with its temperature increasing linearly with time. The reciprocal of slope of line will be proportional to specific heat of substance in vapour state.

Illustration:

How many colories are required to change exactly 1 gm of ice at – 10 °C to steam of 120°C at atmospheric pressure.

Sol.



$$H_1 = ms_{ice}\Delta T = 1 \times \frac{1}{2} \times 10 = 5cal$$
$$H_2 = mL_f = 1 \times 80 = 80\ cal$$
$$H_3 = ms_w\Delta T = 1 \times 1 \times 100\ cal$$
$$H_4 = mL_V = 1 \times 540 = 540\ cal$$
$$H_5 = ms_{steam}\Delta T = 1 \times 0.48 \times 20 = 9.6\ cal$$

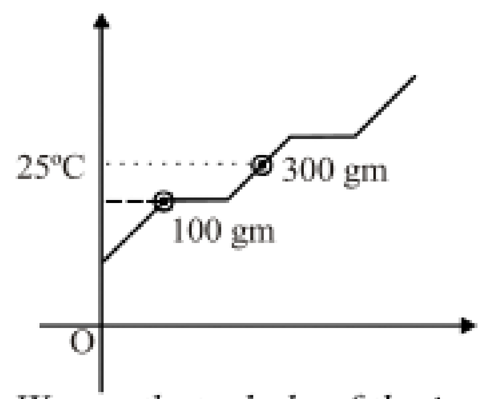
Total colories reqd. $\Rightarrow H_1 + H_2 + H_3 + H_4 + H_5$

$$H_{Total} \Rightarrow 734.6\ cal$$

Illustration :

300 gram of water at 25°C is added to 100 gm of ice at 0°C. The final temp. of the mixture is _____?

Sol.



Heat released by water

$Q_1 = ms\Delta T$; $Q_1 = 300 \times 1 \times 25$

$Q_1 = 7500\ cal$

Heat required by Ice for completely melt $Q_2 = mL_f$

$Q_2 = 8000\ cal$

$Q_2 > Q_1$

We see that whole of the ice cannot be melted as the required amount of heat is not provided by the water. Also, the heat is enough to bring the ice to 0°C. Thus the final temperature of the mixture is 0°C with some of the ice melted.

Water equivalent

It is a equivalent mass of water (w) that has same heat capacity as that of the given body (b). In other words,

$$C = m_w s_w = m_b s_b$$

It is a convinent way to represent the heat capacity of the calorimeter

Illustation:

A sphere of aluminium of 0.047 kg placed for sufficient time in a vessel containing boiling water, so that the sphere is at 100 °C. It is then immediately transferred to 0.14 kg copper calorimeter containing 0.25 kg of water at 20 °C. The temperature of water rises and attains a steady state at 23 °C. Calculate the specific heat capacity of aluminium.

Sol. *In solving this example we shall use the fact that at a steady state, heat given by an aluminium sphere ill be equal to the heat absorbed by the water and calorimeter. Mass of aluminium sphere* $(m_1) = 0.047$ kg

Initial temp. of aluminium sphere = 100 °C

Final temp. = 23 °C

Change in temp $(\Delta T) = (100\ °C - 23\ °C) = 77\ °C$

Let specific heat capacity of aluminium be s_{Al}

The amount of heat lost by the aluminium

sphere $= m_1 s_{Al} \Delta T = 0.047 \text{ kg} \times s_{Al} \times 77°C$

Mass of water (m_2) 0.25 kg

Mass of calorimeter $(m_3) = 0.14$ kg

Initial temp. of water and calorimeter = 20°C

Find temp. of the mixture = 23°C

Change in temp. $(\Delta T_2) = 23°C - 20°C = 3°C$

Specific heat capacity of water $(s_w) = 4.18 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$

Specific heat capacity of copper calorimeter $= 0.386 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$

The amount of heat gained by water and calorimeter $= m_2 s_w \Delta T_2 + m_3 s_{cu} \Delta T_2$

$$= (m_2 s_w + m_3 s_{cu})(\Delta T_2)$$

$$= (0.25 \text{ kg} \times 4.18 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} + 0.14 \text{ kg} \times 0.386 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1})(23°C - 20°C)$$

In the steady state heat lost by the aluminium sphere = heat gained by water + heat gained by calorimeter.

So, $0.047 \text{ kg} \times s_N \times 77°C$

$$= (0.25 \text{ kg} \times 4.18 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} + 0.14 \text{ kg} \times 0.386 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1})(3°C)$$

$$s_{Al} = 0.911 \text{ kJ kg}^{-1} \text{ K}^{-1}$$

Practice Exercise

Q.1 1 kg of ice at – 10ºC is mixed with 1 kg water at 100ºC, then find the equilibrium temperature and mixture content.

Q.2 5 gm of ice at 0°C is mixed with 10 gm of steam at 100°C. Find the final temperature and composition of the mixture if the mixing is done in a calorimeter of water equivalent 13 gm, initially at 0°C.

Q.3 A lumb of ice of 0.1 kg at –10ºC is put in 0.15 kg of water at 20ºC. How much water and ice will be found in the mixtre when it has reached thermal equilibrium.

Answers

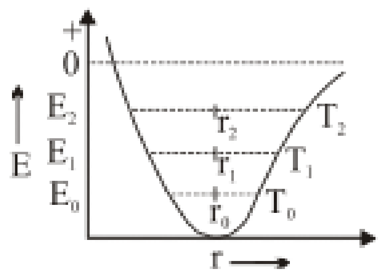
Q.1 T = 7.5ºC Q.2 245/27 gm of water, 160/27 gm of steam

Q.3 Amount of water amd Ice are 181.25 gm and 68.75 gm respetively.

Thermal expansion

Thermal Expansion

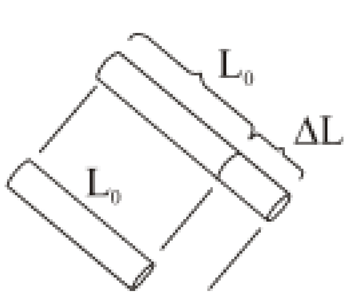
When matter is heated without change in state, it usually expands. According to atomic theory of matter, asymmetry in potential energy curve is responsible for thermal expansion as with rise in temperature say from T_1 to T_2 the amplitude of vibration and hence energy of atoms increases from E_1 to E_2 and hence the average distance between atoms increases from r_1 to r_2.



Due to this increase in distance between atoms, the matter as a whole expands. Had the potential energy curve been symmetrical, no thermal expansion would have taken place in spite of heating.

Linear Expansion of solids

To varying extents, most materials expand when heated and contract when cooled. The increase in any one dimension of a solid is called linear expansion, linear in the sense that the expansion occurs along a line. A rod whose length is L_0 when the temperature is T_0 when the temperature increases to $T_0 + \Delta T$, the length becomes $L_0 + \Delta L$, where ΔT and ΔL are the magnitudes of the changes in temperature and length, respectively.
Conversely, when the temperature decreases to $T_0 - \Delta T$, the length decreases to $L_0 - \Delta L$.
For small temperature changes, experiments show that the change in length is directly proportional to the change in temperature ($\Delta L \propto \Delta T$). In addition, the change in length is proportional to the initial length of the rod,



Equation $\Delta L = \alpha L_0 \Delta T$ expresses the fact that ΔL is proportional to both L_0 and $\Delta T (\Delta L \propto L_0 \Delta T)$ by using a proportionality constant α, which is called the coefficient of linear expansion. Common unit for the coefficient of linear expansion $(C°)^{-1}$.

Illustration :

A circular hole of radius 2 cm is made in a iron plate at 0ºC. What will be its radius at 100ºC ? α for iron = 11 ×10⁻⁶/ ºC.

Sol. $R_{100} = R_0 (1 + \alpha\Delta T) = (2)\ [1 + (11 \times 10^{-6} / ^\circ C)\ (100^\circ C)]$
$= (2)\ (1+11\times10^{-4}) = 2.0022cm$

Illustration

A brass scale correc*tly calibrated at 15°C is employed to measure a length at a temperature of 35°C. If the scale gives a reading of 75cm, find the true length. (Linear expansively of brass = 2.0 × 10⁻⁵C°)⁻¹)*

Sol. *Let the distance between two fixed divisions on the scale at 15°C be* L_1 *and that at 35°C be* L_2.

Clearly, $(L_2 - L_1) = \alpha L_1 (35 - 15)$

or $L_2 = L_1 (1 + 20 \times 2.0 \times 10^{-5})$

$= L_1 (1.0004)$

i.e., at 35°C, an actual length of L_2 *will be read as* L_1, *due to the increased separation of the divisions of the scale. In other words, the observed length will be less than the actual length.*

Given : $L_1 = 75$ cm

$\therefore$ $L_2 = 75\ (1.0004)$ cm

$= 75.03$ cm

Illustration

Estimate the time lost or gained by a pendulum clock at the end of a week when the atmospheric temperature rises to 40°C. The clock is known to give correct time at 15°C and the pendulum is of steel. (Linear expansively of steel is 12 × 10⁻⁶ / °C).

Sol. *Time period of pendulum clock* $T_0 = 2\pi\sqrt{\dfrac{\ell_0}{g}}$

ℓ_0 *length of pendulum wire at temperature 0ºC, temperature increased to tºC chage in temperature* $\Delta\theta = (t - 0)^\circ C$

Time period at tºC,

$$T_t = 2\pi\sqrt{\frac{\ell_t}{g}} = 2\pi\sqrt{\frac{\ell_0(1+\alpha\Delta\theta)}{g}}$$

α for wire of pendulum

$$T_t = 2\pi\sqrt{\frac{\ell_0}{g}}\sqrt{(1+\alpha\Delta\theta)}$$

$$\frac{T_t}{T_0} = (1+\alpha\Delta\theta)^{1/2} \approx (1+\frac{\alpha(\Delta\theta)}{2}) \qquad \text{(by using binomial approximation)}$$

$$\frac{T_t - T_0}{T_0} = \frac{\alpha(\Delta\theta)}{2} \qquad \Rightarrow \qquad \frac{\Delta T}{T_0} = \frac{\alpha(\Delta\theta)}{2}$$

Thus, time lost per second $= \dfrac{\alpha(\Delta\theta)}{2}$

rise in temperature $\Delta\theta = 40 - 15 = 25^\circ C$

$$\text{Time lost per second} = \frac{1}{2}\,\alpha\Delta\theta$$
$$= \frac{1}{2} \times (12 \times 10^{-6}\,/\,°C) \times (25°C)$$
$$= 150 \times 10^{-6}\ s/s$$

Therefore, time lost per week (i.e., 7×86400 s)

$$= 150 \times 10^{-6}\ s/s \times 7 \times 86400\ s$$
$$= 90.72\ s$$

Illustration :

A glass rod when measured with a zinc scale, both being at 30°C, appears to be of length 100cm. If the scale shows correct reading at 0°C, determine the true length of the glass rod at (a) 30°C and (b) 0°C. *('α' for glass = 8×10^{-6} /°C and for zinc 26×10^{-6} / °C)*

Sol. *At 30°C, although the reading shown by the zinc scale corresponding to the length of the glass rod is 100cm, but the actual length would be more than 100cm, the reason being the increased separation between the markings, owing to a rise in temperature (from 0°C to 30°C).*

Now, an actual (original at 0°C) length of 100cm on the zinc scale (or more precisely, two markings or divisions on the scale, separated by a distance of 100cm) would, at a temperature of 30°C, correspond to a length given by

$$l = 100\,(1 + 26 \times 10^{-6} \times 30)\ cm$$
$$= 100.078\ cm$$

∴ The true length of the glass rod at 30°C is 100.078 cm.

Now, at 0°C, the length of glass rod would be lesser than that at 30°C,

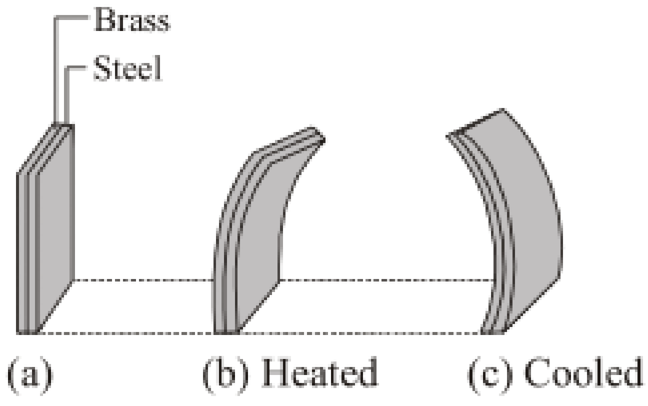
∴ *Using* $l_t = l_0\,(1 + \alpha t)$, $l_0 = \dfrac{l_t}{1+\alpha t}$

∴ *The length of the rod at 0°C, will be*

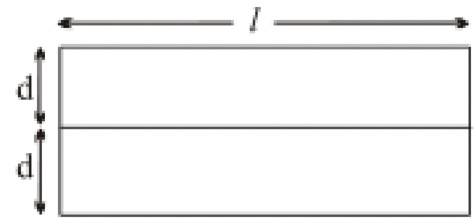
$$l_0 = \frac{100.078cm}{(1+8\times10^{-6}\times30)} = 100.054\ cm$$

Thermal expansion of bimetallic strip

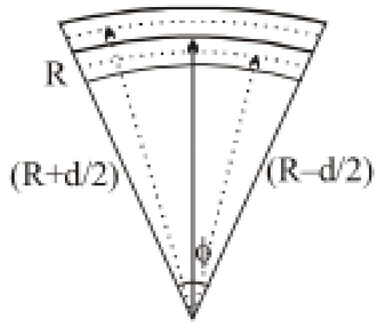
A bimetallic strip is made from two thin strips of metal that have different coefficients of linear expansion, as fig. (a) A bimetallic strip and how it behaves when (b) heated and (c) cooled



Often brass [$\alpha = 19 \times 10^{-6}\,(C°)^{-1}$] and steel [$\alpha = 12 \times 10^{-6}\,(C°)^{-1}$] are selected. The two pieces are welded or riveted together. When the bimetallic strip is heated, the brass, having the larger value of α, expands more than the steel. Since the two metals are bonded together, the bimetallic strip bends into an arc as in fig. (b), with the longer brass piece having a larger radius than the steel piece. When the strip is cooled, the bimetallic strip bends in the opposite direction, as in fig. (c).



on heating the bimetallic strip bends into an are as shown below



Mathematical analysis

$$\left(R + \frac{d}{2}\right)\phi = L_0\,(1 + \alpha_1\Delta\theta) \qquad (\Delta\theta \text{ increase in temp.})$$

$$\left(R - \frac{d}{2}\right)\phi = L_0\,(1 + \alpha_2\Delta\theta)$$

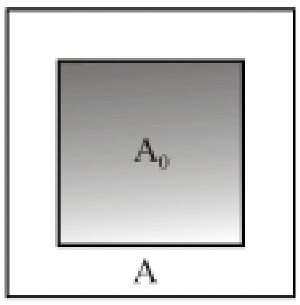
On dividing above equations we get

$$\frac{R + \frac{d}{2}}{R - \frac{d}{2}} = \frac{1 + \alpha_1\,\Delta\theta}{1 + \alpha_2\,\Delta\theta}$$

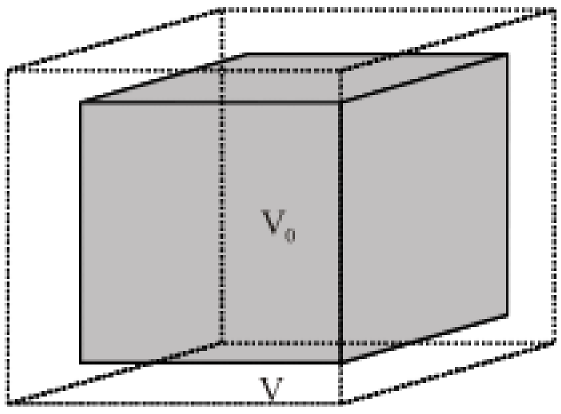
by above euqation we can find mean radius R of bimetallic strip.

$$R = \frac{d}{(\alpha_1 - \alpha_2)\Delta\theta}$$

Area and Volume Expansion



Area Expansion



Volume Expansion

If the temperature of a two-dimensional object (lamina) is changed, its area changes. If the coefficient of linear expansion of the material of lamina is small and constant, then its final area is given by $A = A_0\,(1 + \beta\Delta T)$, where A_0 is the initial area. ΔT is the change in temperature and β is the area coefficient of thermal expansion. For isotropic bodies it can be shown the $\beta = 2\alpha$.

The volume V_0 of an object change by an amount ΔV when its temperature changes by an amount ΔT. $\Delta V = \gamma V_0 \Delta T$ where γ is the coefficient of volume expansion. Common Unit for the coefficient of volume Expansion : $(C°)^{-1}$. The unit for γ, like that for α, is $(C°)^{-1}$. Values for γ depend on the nature of the material. The values of γ for liquids are substantially larger than those for solids, because liquids typically expand more than solids, given the same initial volumes and temperature expansion is three times greater than the coefficient of linear expansion : $\gamma = 3\alpha$.

If a cavity exists within a solid object, the volume of the cavity increases when the object expands, just as if the cavity were filled with the surrounding material. The expansion of the cavity is analogous to the expansion of a hole in a sheet of material. Accordingly, the change in volume of a cavity can be found using the relation $\Delta V = \gamma V_0 \Delta T$, where γ is the coefficient of volume expansion of the material that surrounds the cavity.

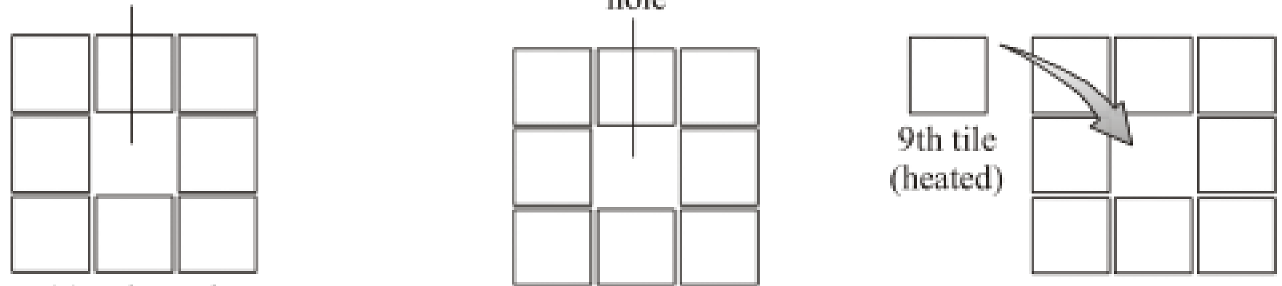
Similar (Here $\gamma \approx 3\alpha$) is known as the coefficient of volume expansion

$\alpha : \beta : \gamma :: 1 : 2 : 3$

***Illustration :* (The expansion of holes)**

Do holes expand or contract when the temperature increases?

Figure (a) shows eight square tiles that are arranged to form a square pattern with a hole in the centre. If the tiles are heated, what happens to the size of the hole?



(a) unheated (b) Heated (c)

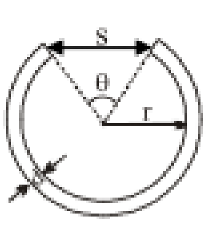
Sol. We can analyze this problem by disassembling the pattern into separate tiles, heating, it is evident from figure (b) that the heated pattern expands and so does the hole in the centre. In fact, if we had a ninth tile that was identical to and also heated like the others, it would fit exactly into the centre hole, as figure (c) indicates. Thus, not only does the hole in the pattern expand, but it expands exactly as much as one of the tiles. Since the ninth tile is made of the same material as the others, we see that the hole expands just as if it were made of the material of the surrounding tiles.

The thermal expansion of the hole and the surrounding material is analogous to a photographic enlargement ; in both situations everything is enlarged, including holes.

Thus, it follows that a hole in a piece of solid material expands when heated and contracts when cooled, just as if it were filled with the material that surrounds it. If the hole is circular, the equation $\Delta L = \alpha L_0 \Delta T$ can be used to find the change in any linear dimension of the hole, such as its radius or diameter. Example illustrates this type of linear expansion.

Illusration :

A thin cylindrical metal rod is bent into a ring with a small gap as shown in figure. On heating the system



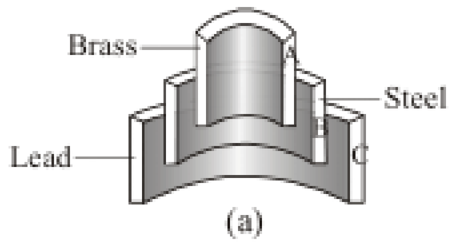
(A) θ and s decreases, r and d increases (B) θ and r increases, d and s decreases

(C) θ, r, s and d all increases (D) θ is constant, d, s and r increases

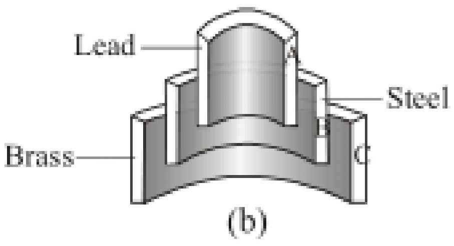
Sol. θ remains constant d, s and r increases.

Illustration :

Figure shows a cross-sectional view of three cylinders, A, B and C. Each is made from a different material ; one is lead, one is brass, and one is steel. All three have the same temperature, and barely fit inside each other. As the cylinders are heated to the same, but higher, temperature, cylinder C falls off, While cylinder A becomes tightly wedged to cylinder B. Given, lead has the greatest coefficient of linear expansion, followed by brass, and then by steel. Which cylinder is made from which material?



(a)



(b)

Sol. We need to consider how the outer and inner diameters of each cylinder change as the temperature is raised. With respect to the inner diameter, we will be guided by the fact that a hole expands as if it were filled with the surrounding material. These data indicate that the outer and inner diameters of the lead cylinder change the most, while those of the steel cylinder change the least.

Since the steel cylinder expands the least, it cannot be the outer one, for if it were, the greater expansion of the middle cylinder would prevent the steel cylinder from falling off. The steel cylinder also cannot be the inner one, because then the greater expansion of the middle cylinder would allow the steel cylinder to fall out, contrary to what is observed. The only place left for the steel cylinder is in the middle, which leads to the two possibilities in figure.

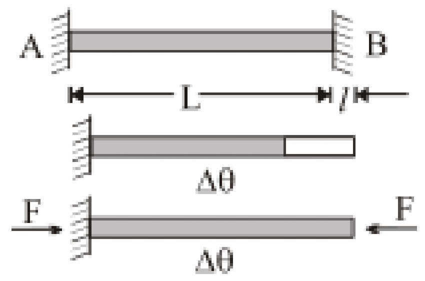
In part (a), lead is on the outside and will fall off as the temperature is raised, since lead expands more than steel. On the other hand, the inner brass cylinder expands more than the steel that surrounds it and becomes tightly wedged, as observed. Thus, one possibility is A = brass, B = steel, and C = lead.

In part (b), of the drawing, brass is on the outside. As the temperature is raised, brass expands more than steel, so the outer cylinder will again fall off. The inner lead cylinder has the greatest expansion and will be wedged against the middle steel cylinder. A second possible answer, then is A = lead, B = steel, and C = brass.

Thermal Stress

A change in shape/size i.e., dimensions need not necessarily imply a strain. For example, if a body is heated to expand, its volume change, as it acquires a new size, due to expansion. However, the strain remains zero. Unless and until, internal elastic forces operate, to bring the body to the original state, no strain exists. When a body is heated, the total energy of molecule increase, owing to an increase in the kinetic energy of the molecules. This results in a shift (increment) of the "equilibrium distance" of molecules and the body acquires a new shape and size, in the expanded form, whereby the molecules are in "zero force" state. Hence, there is no strain. However, if the body is resistricted to expand, during the process of heating, then the molecules become "strained", and even if there is no apparent change in dimensions of the body, there is strain. In such cases, strain is measured as the ratio. In dimension that would have occured, and the change in dimension that would have occured, had the body been free to expand or contract, to the original dimension.

When a metal rod is heated or cooled it tends to expand or contract. If it is left free to expand or contract, no temperature stresses will be induced. However, if the rod be restricted to change its length, then temperature stresses are generated within it. Stress induced due to temperature change can be understood as follows:



Consider a uniform rod AB fixed rigidly between two supports. (fig.) If L be its length, α the coefficient of linear expansion, then a change in temperature of Δθ, would tend to bring a change in its length by $l = L\alpha\Delta\theta$. Had the rod been free (say one of its ends) its length would have changed by l. Now, let a force be gradually applied so as to restore the natural length. Since the rod, tends to remain in the new state, due to a change in temperature, so when a force F is applied, thermal stress is induced. In equilibrium.

$$\frac{F}{A} = \frac{l}{(L \pm l)} Y \qquad [\because \text{stress} = \text{strain} \times Y]$$

Neglecting l in comparison to L,

$$F = \frac{lA}{L} Y = AY\ \alpha\,\Delta\theta$$

Now, if the two ends remain fixed, then this external force is provided from the support.

$$\text{Clearly strain} = \frac{\ell}{L} = \alpha\Delta\theta$$

Illustration :

A brass rod of length 1m is fixed to a vertical wall, at one end, with the other end free to expand. When the temperature of the rod is increased by 120°C, the length increases by 2cm. What is the strain?

Sol. *After the rod expands, to the new length there are no elastic forces developed internally in it. So, strain = 0.*

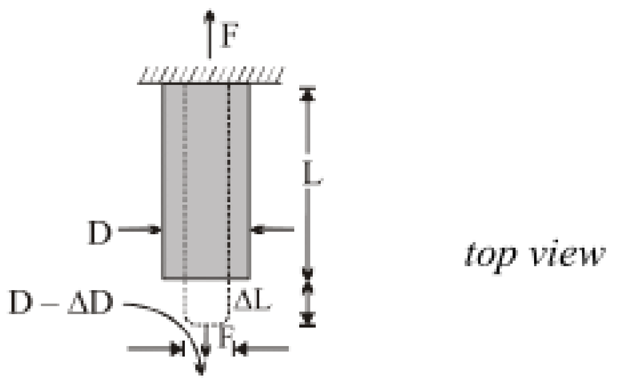


Illustration :

A rod of length 2m is at a temperature of 20°C. Find the free expansion of the rod, if the temperature is increased to 50°C. Find the temperature stresses produced when the rod is (i) fully prevented to expand, (ii) permitted to expand by 0.4 mm. $Y = 2 \times 10^{11}\ Nm^{-2}$; $\alpha = 15 \times 10^{-6}$ per °C.

Sol. Free expansion of the rod = $\alpha \Delta \theta$

$= (15 \times 10^{-6} /°C) \times (2m) \times (50° - 20°)C$

$= 9 \times 10^{-4}\ m = 0.9\ mm$

(i) If the expansion is fully prevented

then strain $= \dfrac{9 \times 10^{-4}}{2} = 4.5 \times 10^{-4}$

$\therefore$ temperature stress = strain $\times$ Y

$= 4.5 \times 10^{-4} \times 2 \times 10^{11} = 9 \times 10^{7}\ Nm^{-2}$

(ii) If 0.4 mm expansion is allowed, then length restricted to expand = 0.9 – 0.4 = 0.5 mm

$\therefore$ Strain $= \dfrac{5 \times 10^{-4}}{2} = 2.5 \times 10^{-4}$

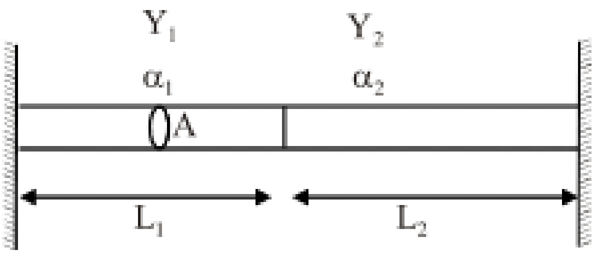
$\therefore$ Temperature stress = strain $\times$ Y

$= 2.5 \times 10^{-4} \times 2 \times 10^{11} = 5 \times 10^{7}\ Nm^{-2}$

Illustration :

Two rods made of different materials are placed between massive walls. The cross section of both the rods are same A, their lengths L_1 and L_2 coefficients of linear expansion α_1 and α_2 and the modulii of elasticity of their materials Y_1 and Y_2 respectively. If the rods are heated by t°C, find the force F with which the rods act on each other.

Sol.



Let the first rod expand slightly (say by length δl) and the second rod get compressed by the same amount (since net elongation / compression of the rods is zero.)

$\therefore$ Natural increase in length of the first rod (after being heated) when free to expand would have been $\alpha_1 L_1 t$. The expansion allowed is just δl (where $\delta l < \alpha_1 L_1 t$).

$\therefore$ Amount of elongation restricted $= \alpha_1 L_1 t - \delta l$

$\therefore$ Strain $= \dfrac{\text{elongation restricted}}{\text{orignal length}} = \dfrac{\alpha_1 L_1 t - \delta l}{L_1(1 + \alpha_1 t)}$

Since $\alpha_1 t << 1$

$\therefore$ $1 + \alpha_1 t \simeq 1$

$\therefore$ Strain $= \dfrac{\alpha_1 L_1 t - \delta l}{L_1}$

$\therefore$ Stress = strain $\times$ Y $= \left(\dfrac{\alpha_1 L_1 t - \delta l}{L_1}\right) Y_1$.

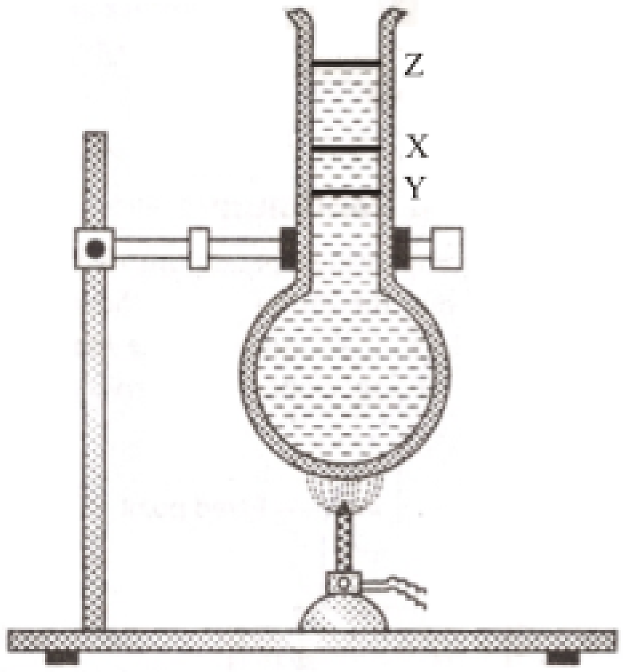
$$or \qquad F = stress \times A = \frac{(\alpha_1 L_1 t - \delta l)}{L_1} Y_1 A \qquad \ldots (i)$$

$$Similarly,\ F = \frac{(\alpha_1 L_1 t - \delta l)}{L_2} y_2 t_2 \qquad or \qquad \delta l = \alpha_1 L_1 t - \frac{FL_1}{Y_1 A} = \frac{FL_2}{Y_1 L_2} - \alpha_2 L_2 t$$

$$or \qquad F = \frac{(\alpha_1 L_1 + \alpha_2 L_2)\, t}{\left(\frac{L_1}{Y_1 A} + \frac{L_2}{Y_1 A}\right)}$$

Expansion of liquids

Like solids, liquids also, in general, expand on heating ; however, their expansion is much large compared to solids for the same temperature rise. A noteworthy point to be taken into account during the expansion of liquid is that they are always contained in a vessel or a container and hence the expansion of the vessel also comes into picture. Further, linear or superficial expansion in case of a liquid does not carry any sense. Consider a liquid contained in a round bottomed flask fitted with a long narrow stem as shown in fig. Let the initial level of the liquid be X. When it is heated the level falls initially to Y.



However, after sometime, the liquid level eventually rises to Z. The entire phenomenon can be understood as follows: Upon being heated, the container gets heated first and hence expands. As a result, the capacity of the flask increases and hence the liquid level falls.

After sometime, the heat gets conducted from the vessel to the liquid and hence liquid also expands thereby rising its level eventually to Z. Since, the volume expansivity of liquids, in general, are far more than that of solids, so the level Z will be above the level X.

Effect of temperature on density

When a solid or liquid is heated, it expands, with mass remaining constant. Density being the ratio of mass to volume, it decreases. Thus, if V_0 and V_t be the respective volumes of a substance at 0°C and t°C and if the corresponding values of densities be ρ_0 and ρ_t, then the mass m of the substance is given by

$$m = V_0\rho_0 = V_t\rho_t$$

But $\quad V_t = V_0 (1 + \gamma t)$, so $\rho_t = \rho_0 (1 + \gamma t)^{-1}$

Illustration :

The volume of a glass vessel filled with mercury is 500 cc, at 25°C. What volume of mercury will overflow at 45°C?

the coefficients of volume expansion of mercury and glass are 1.8×10^{-4} / °C and 9.0×10^{-6} / °C respectively.

[Sol. The volume of mercury overflowing will be the expansion of mercury relative to the glass vessel (i.e., apparent expansion).

Now, since $\quad (\Delta V)_a = (\Delta V)_l - (\Delta V)_c$

Apparent expansion $(\Delta V)_a$ will be

$$(\Delta V)_a = V_l\gamma_l\Delta T - V_C\gamma_C\Delta T$$

$$= 500\ cc\ (180 - 9) \times \frac{10^{-6}}{°C}\ (45 - 25)°C$$

$$= 1.71\ cc$$

Thus, 1.71 cc of mercury overflows.

Illustration :

A sphere of diameter 4 cm and mass 150g floats in a bath of liquid. As temperature is raised, the sphere begins to sink at temperature 50°C. If the density of the liquid is 6.5 g/cm³ at 0°C, find the coefficient of cubical expansion of the liquid, neglecting the expansion of the sphere.

Sol. (i) When sphere is floating i.e. at temperature < 50° C

weight of body = Thrust

$$mg = V_{in}\,\sigma g \qquad (\sigma \text{ density of liquid at temp. } 0°C) \qquad .....(i)$$

(ii) When body just sinks i.e. at temperature 50°

$$mg = V'\sigma g$$

$$\therefore \quad \sigma' = \frac{m}{V} \qquad (\sigma' \text{ density of liquid at temp. } 50°C)$$

$$= \frac{150}{\frac{4}{3}\pi(2)^2} = 4.48\ g/cm^2$$

Now, $\quad \sigma' = \dfrac{\sigma}{1+\gamma\Delta T}$

$$\therefore \quad 1+\gamma\Delta T = \frac{\sigma}{\sigma'} = \frac{6.5}{4.48}$$

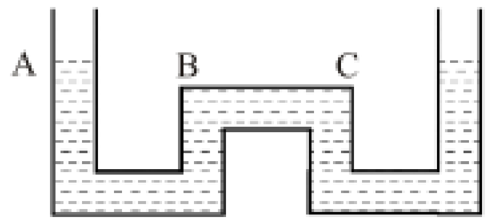
$$1 + \gamma\,[50 - 0] = \frac{6.5}{4.48}$$

$$\therefore \quad \gamma = \frac{2.02}{50 \times 4.48}$$

or $\gamma = 9.02 \times 10^{-3}/°C$

Practice Exercise

Q.1 The apparatus shown in fiugre consists of four glass colums connected by horizontal sections. The heights of two central columns B and C are 49 cm each. The two outer column A and D are open to atmosphere. A and C are maintained at a temperature of 95ºC, while the column B and D are maintained at 5ºC. The heights of liquid in A and D measured from base line are 52.8 cm and 51cm respectively. Determine the coefficient of thermal expansion of the liquid.



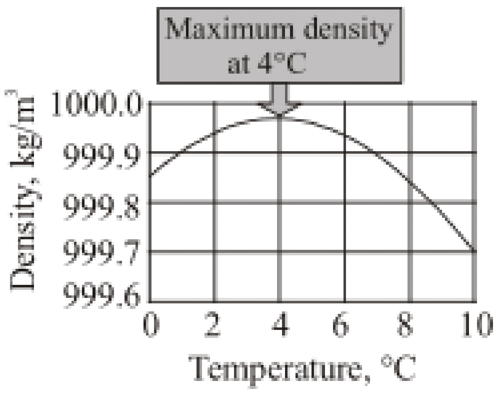
Answers

Q.1 $\gamma = 1.96 \times 10^{-4}$ / ºC

Anomolous expansion of water

While most substances expand when heated, a few do not. For instant, if water at 0°C is heated, its volume decreases until the temperature reaches 4°C. Above 4 °C water behaves normally, and its volume increases as the temperature increases.

Because a given mass of water has a minimum volume at 4°C, the density (mass per unit volume) of water is greatest at 4 °C, as figure shows.

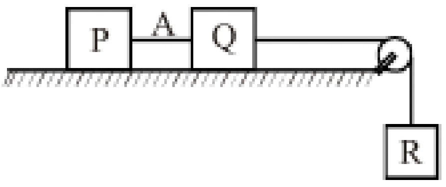


The density of water in the temperature range from 0 to 10°C. Water has a maximum density of 999.973kg/m^3 at 4°C. (This value for the density is equivalent to the often quoted density of 1.000 grams per milliliter)

When the air temperature drops, the surface layer of water is chilled. As the temperature of the surface layer drops toward 4°C, this layer becomes more dense than the warmer water below. The denser water sinks and pushes up the deeper and warmer water, which in turn is chilled at the surface. This process continues until the temperature of the entire lake reaches 4°C. Further cooling of the surface water below 4°C makes it less dense than the deeper layers ; consequently, the surface layer does not sink but stays on top. Continued cooling of the top layer to 0°C leads to the formation of ice that floats on the water, because ice has a smaller density than water at any temperature. Below the ice, however, the water temperature remains above 0°C. The sheet of ice acts as an insulator that reduces the loss of heat from the lake, especially if the ice is covered with a blanket of snow, which is also an insulator. As a result, lakes usually do not freeze solid, even during prolonged cold spells, so fish and other aquatic life can survive.

Solved Examples

Q.1 Each of the three blocks P,Q and R shown in figure has a mass of 3 kg. Each of th wires A and B has cross-sectional area 0.005 cm^2 and Young's modulus 2×10^{11} N/m^2. Neglect friction. Find the longitudinal strain developed in each of the wires. Take $g = 10\ m/s^2$.



Sol. The block R will descend vertically and the blocks P and Q will move on the frictionless horizontal table. Let the common magnitude of the acceleration be a. Let the tensions in the wires A and B be T_A and T_n respectively.

Writing the equation of motion of the blocks P, Q and R, we get,

$$T_A = (3)a \quad .....(i)$$

$$T_B - T_A = (3)a \quad ....(ii)$$

and $(3)g - T_B = (3)a$ .....(iii)

By (i) and (ii)

$$T_B = 2T_A.$$

By (i) and (iii)

$$T_A + T_B = (3)g = 30\ N$$

or $3T_A = 30\ N$

or $T_A = 10\ N$ and $T_B = 20\ N$.

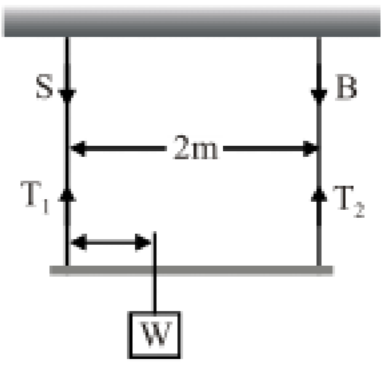
$$\text{Longitudinal strain} = \frac{\text{Longitudinal stress}}{\text{Young's modules}}$$

$$\text{Strain in wire A} = \frac{10\ N / 0.005\ cm^2}{2\times10^{11}\ N/m^2} = 10^{-4}$$

$$\text{Strain in wire B} = \frac{20 N / 0.005 cm^2}{2\times10^{11} N/m^2} = 2\times10^{-4}$$

Q.2 A light rod of length 200 cm is suspended from the ceiling horizontally by means of two vertical wires of equal length, tied to its ends. One of the wires is made of steel and is of cross-section 0.1 cm^2 and the other of brass of cross-section 0.2 cm^2. Along the rod at which distance may a weight be hung to produce (a) equal stresses in both the wires (b) equal strains in both the wires ? Y for brass and steel are 10×10^{11} and 20×10^{11} dyne/cm^2 respectively.

Sol. (a) According to the problem stresses are equal, so we have



$$\frac{T_1}{A_1} = \frac{T_2}{A_2},$$

i.e. $$\frac{T_1}{T_2} = \frac{A_1}{A_2} = \frac{0.1}{0.2}$$

or $$T_2 = 2T_1 \quad ....(i)$$

As the rod is in equilibrium,

$$\Sigma F_y = T_1 + T_2 - W = 0$$

or $$T_1 + T_2 = W \quad ....(ii)$$

From equation (i) and (2), we get

$$T_1 = (W/3) \text{ and } T_2 = (2W/3) \quad ....(iii)$$

Let x be the distance of weight W from steel wire, Torque balance for rotational equilibrium of rod.

$$\Sigma\tau = T_1 x - T_2 (2 - x) = 0$$

or $$(W/3)\, x = (2W/3)\,(2 - x),$$

i.e. $$x = (4/3) \text{ m}$$

(b) According to the problem, strains are equal.

$$\frac{T_1}{A_1 Y_1} = \frac{T_2}{A_2 Y_2} \quad \left[\text{as strain} = \frac{\text{stress}}{Y}\right]$$

So, $$\frac{T_1}{T_2} = \frac{A_1 Y_1}{A_2 Y_2} = \frac{0.1\times20\times10^{11}}{0.2\times10\times10^{11}} = 1$$

$$T_1 = T_2 \quad ......(iv)$$

i.e. from equation (ii) and (iv), we get

$$T_1 = T_2 = (W/2)$$

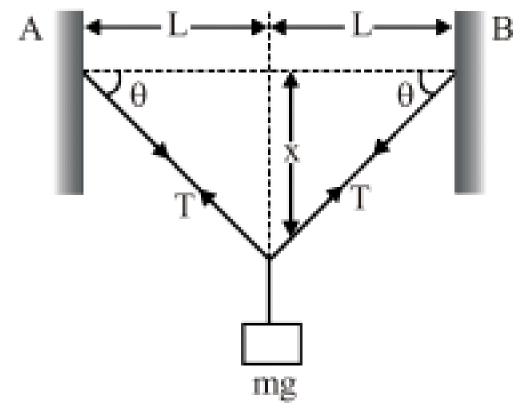
and for rotational equilibrium of rod

$$\Sigma\tau = T_1\, x - T_2\,(2 - x) = 0$$

or $$(W/2)\, x = (W/2)(2 - x), \qquad \text{i.e. } x = 1 \text{ m}$$

Q.3 A steel wire of diameter 0.8 mm and length 1 m is clamped firmly at two points A and B which are 1 m apart and in the same plane. A body is hung from the middle point of the wire such that the middle point sages 1 cm lower from the original position. Calculate the mass of the body. Given that Young's modulus of the material of wire is 2×10^{11} N/m^2.

Sol. As shown in figure, mass M is in equilibrium



$$Mg = 2T\sin\theta \qquad .....(i)$$

But from the geometry of figure, for small angle θ, we have

$$\sin\theta \approx \tan\theta = (x/L) \qquad ....(ii)$$

and by definition of Young's modulus, we have

$$T = \frac{YA}{L}\Delta L = \frac{YA}{L}\left[\left(L^2 + x^2\right)^{1/2} - L\right] \approx \frac{YAx^2}{2L^2} \qquad ....(iii)$$

So substituting the values of sin θ and T from equation and (iii) in (i), we get

$$Mg = 2 \times \frac{YAx^2}{2L^2} \times \frac{x}{L}$$

$$M = \frac{YAx^3}{gL^3} \qquad ....(iv)$$

Now as here 2L = 1 m, x = 1 cm = 10^{-2} m

and $A = \pi r^2 = p\,[(0.8/2)\times10^{-3}]^2$

$= \pi \times (4 \times 10^{-4})^2\ m^2$

$$\text{so } M = \frac{2\times10^{11}\times\pi(4\times10^{-4})^2\times(10^{-2})^3}{9.8\times(1/2)^3}\,kg = 82g$$

from equation (iv), we have

$$x = L\left(\frac{Mg}{YA}\right)^{1/3}$$

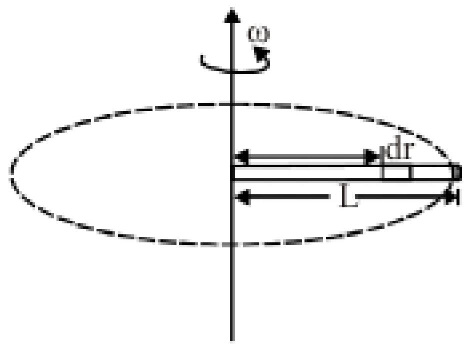
So the angle θ can be determined from

$$\theta = \frac{x}{L} = \left(\frac{Mg}{YA}\right)^{1/3}$$

Q.4 A thin unifrom metallic rod of length 0.5 m and radius 0.1 m rotates with an angular veloicty 400 rad/s in a horizontal plane about a vertical axis passing through one of its ends. Calculate (a) the tension in the rod and (b) the elongation of the rod. The density of the meterial of the rod is 10^4 kg/m^3 and the Young's modulus is 2×10^{11} N/m^2.

Sol. (a) We take a differential element of the length dr at a distance r form the axis of rotation as shown in figure. The centripetal force acting on this element is

$$dT = dmr\omega^2 = (\rho A dr) r\omega^2$$



The tension in the rod at a distance r from the axis of rotation will be due to the centripetal force due to all elements between x = r to x = L,

i.e., $$T = \int_r^L \rho A\omega^2 r dr = \frac{1}{2}\rho A\omega^2[L^2 - r^2] \qquad .....(i)$$

Thus, tension as function of r,

$$T = \frac{1}{2} \times 10^4 \times \pi \times 10^{-2} \times (400)^2 \left[\left(\frac{1}{2}\right)^2 - r^2\right]$$

$$= 8\pi \times 10^6 \left[\frac{1}{4} - r^2\right] N$$

Note that tension in the rod is minimum at r = L and maximum at r = 0

(b) Let dy be the eleongation in the element of length dr at position r due to tension, T. From definition of Young's modulus,

$$\text{Strain} = \frac{\text{stress}}{Y}$$

so $$\frac{dy}{dr} = \frac{T}{AY}$$

From equation (i), we have $dy = \frac{1}{2}\frac{\rho\omega^2}{Y}[L^2 - r^2]\, dr$

So the elongation of the entire rod,

$$\Delta L = \frac{\rho\omega^2}{2Y}\int_0^L [L^2 - r^2]\, dr$$

$$= \frac{1}{3}\frac{\rho\omega^2 L^3}{Y}$$

Here $$\Delta L = \frac{1}{3} \times \frac{10^4 \times (400)^2 (0.5)^3}{2 \times 10^{11}}$$

$$= \frac{1}{3} \times 10^{-3} m$$

Q.5 Estimate the pressure deep inside the sea at a depth h below the surface. Assume that the density of water is ρ_0 at sea level and its bulk modulus is B.

Sol. In a static fluid the pressure variation is given by

$$\frac{dP}{dh} = -\rho g \qquad ...(i)$$

The bulk modulus is defined as

$$B = -\frac{dP}{dV/V} \qquad ...(ii)$$

Where dV/V is fractional change in volume of a element subjected to isotropic pressure increase dP. We consider a sample of the fluid having mass M, its volume $V = M/\rho$, so that

$$dv = \frac{-M}{\rho^2} d\rho$$

Hence $$\frac{dV}{V} = -\frac{d\rho}{\rho} \qquad ....(iii)$$

combining equations (ii) and (iii), we get

$$\frac{B d\rho}{\rho} = \rho g \ dh$$

or $$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = \int_0^h \frac{g dh}{B}$$

or $$\frac{1}{\rho^0} - \frac{1}{\rho} = \frac{gh}{B} \qquad ....(iv)$$

as $$dP = -\frac{B dV}{V} = B\frac{d\rho}{\rho}$$

Hence $$\int_{P_0}^{P} dP = \int_{\rho_0}^{\rho} B\frac{d\rho}{\rho}$$

or $$P - P_0 = B \, \ell n \frac{\rho}{\rho_0} \qquad ....(v)$$

On multiplying equation (iv) by ρ_0, we get

$$1 - \frac{\rho_0}{\rho} = \frac{\rho_0 gh}{B}$$

so that $$\text{In} \frac{\rho}{\rho_0} = -\ell n\left(1 - \frac{\rho_0 gh}{B}\right)$$

Substituting this in equation (v), we get

$$P = P_0 - B \, \ell n\left(1 - \frac{\rho_0 gh}{B}\right)$$

This is the required expression for P.

Q.6 A 5g piece of ice at – 20°C is put into 10g of water at 30°C. Assuming that heat is exchanged only between the ice and the water, find the final the final temperature of the mixture. Specific heat capacity of ice = 2100 J/kg-°C specific heat capacity of water = 4200 J/kg-°C and latent heat of fusion of ice = 3.36×10^5 J/kg.

Sol. The heat given by the water when it cools down from 30°C to 0°C is

$$(0.01 \text{ kg})(4200 \text{ J/k-°C})(30°\text{C}) = 1260 \text{ J}.$$

The heat required to bring the ice to 0°C is $(0.005 \text{ kg})(2100 \text{ J/kg-°C})(20°\text{C}) = 210$ J.

The heat required to melt 5g of ice is $(0.005 \text{ kg})(3.36 \times 10^5 \text{ J/kg}) = 1680$.

We see that whole of the ice cannot be melted as the required amount of heat is not proved by the water. Also, the heat is enough to bring the ice to 0°C. Thus the final temperature of the mixture is 0°C with some of the ice melted.

Q.7 Steam at 100ºC is passed into 1.1 kg of water contained in a calorimeter of water equivalent 0.02 kg at 15ºC till the temperature of the calorimeter and its contents riseds to 80ºC. The mass of the steam condensed in kilogram is

(A) 0.130 (B) 0.065 (C) 0.260 (D) 0.135

Sol. Heat required to bring water and calorimeter from 15ºC to 80ºC

$$Q = mC\Delta T$$
$$Q = (1.1+0.02) \times 1 \times (80–15)$$
$$Q = 72.8 \text{ kcal}$$

Amount of steam condensed to provide 72.8 kcal heat = $m_1 L$

$$\Rightarrow \quad 72.8 \times 1000 = m_1 [540]$$
$$m_1 = 134.8 \text{ gm}$$
$$m_1 = 0.1348 \text{ kg}$$

Q.8 A bullet of mass of 10 gm moving with a speed of 20 m/s hits an ice block of mass 990 gm kept on a frictionless floor and gets stuck in it. How much ice will melt if 50% of the lost kinetic energy goes to ice? (Temperature of ice block = 0ºC)

Sol. Velocity of bullet + ice block, $V = \dfrac{(10\text{gm}) \times (20\text{m/s})}{1000\text{gm}} = 0.2\text{m/s}$

Loss of K.E. $= \dfrac{1}{2}mv^2 - \dfrac{1}{2}(m+M)V^2$

$$= \frac{1}{2}\left[0.01 \times (20)^2 - 1 \times (0.2)^2\right] = \frac{1}{2}[4 - 0.04] = 1.98\text{J}$$

$\therefore$ Heat generated $= \dfrac{1.98}{4.2}$ Cal

$\therefore$ Heat recived by ice block $= \dfrac{1.98}{4.2 \times 2}$ Cal $= 0.24$ cal

$\therefore$ Mass of ice malted $= \dfrac{(0.24\text{Cal.})}{(80\text{Cal/gm})} = 0.003\text{gm}$

Q.9 A calorimeter of water equivalent 1 kg contains 10 kg of ice & 10 kg of water in thermal equilibrium. The atmospheric temperature is 15° below freezing point due to which the calorimeter loses heat. As a result ice is formed inside the calorimeter at a rate of 10.8 gm per second. To try to compensate for this heat loss, steam at 100°C is supplied to the calorimeter at a rate of r. (L_V = 540 cal/gm, L_f = 80 cal/gm, sp heat of water 1 cal/gm °C.) Column-I gives the value of r and column-II gives the situation just after the introduction of steam.

	Column-I		**Column-II**
(A)	r = 1.6 gm/sec	(P)	Amount of ice in calorimeter increases.
(B)	r = 1.35 gm/sec	(Q)	Amount of water in calorimeter increases.
(C)	r = 1.2 gm/sec	(R)	Amount of ice remains constant at 10 kg
(D)	r = 1 gm/sec	(S)	Amount of water remains constant at 10 kg
		(T)	Amount of ice in calorimeter decreases.

[Ans. (A) Q, T ; (B) Q, R ; (C) P, S ; (D) P]

Sol. Rate of heat loss = 80 × 10.8 = 54 × 16 cal/sec.

(A) r = 1.6

$\Rightarrow$ rate of heat supplies by forming steam to water at 0° = 1.6 × 640 > 54 × 16

$\therefore$ additional ice will melt. Correct options are Q and T

(B) Rate of heat loss = 54 × 16 = 64 × 13.5 cal/sec.

r = 1.35

= rate of heat supplied for converting steam to water at 0°C = 1.35 × 640 = 13.5 × 64.

no additional ice will melt or water will fuse. Correct options are Q and R

(C) Rate of heat loss = 54 × 16 = 72 × 12 cal/sec.

Rate of heat supplied by converting steam to ice at 0°C = 1.20 × 720 = 12 × 72 cal/sec

no additional ice will melt or water will fuse. Correct options are P and S

(D) Additional water will fuse to ice. Correct option is P.

Q.10 An ice cube of mass 0.1kg at 0°C is placed in an isolated container which is at 227°C. The specific Heat S of the container varies with temperature T according to relation S=A+BT, where A = 100 cal/kg-k and B= 2×10^{-2} cal/kg-k^2. If the final temperature of the container is 27°C, Determine the mass of the container.

Sol. Specific heat of container is temperature dependent so we have to calculate heat lost for a small temperature change dT and then integrate it from initial temperature to final temperature.

$$\text{Heat lost by container} = -\int_{500}^{300} m_C(A+B+T)dT$$

$$= -m_C\left[AT + \frac{BT^2}{2}\right]_{500}^{300} \qquad = 21600m_C$$

Heat gained by Ice = mL + mCΔT

$= 0.1\times1000\times80+0.1\times1000\times27$

$= 10700$ Cal

from principle of colorimetry

Heat lost by container = Heat gained by Ice

$21600\, m_C = 10700$

$m_C = 0.495$ kg

Q.11 A certain clock with an iron pendulum is made so as to keep correct time at 10°C. Given $\alpha_{iron}=12\times10^{-6}$ per °C. How will the rate alter if the temperature rises to 25° C?

Sol. When the pendulum keeps correct time, its period of vibration is 2 sec and so it makes

$$\frac{20\times60\times60}{2} = 43200 \text{ Vibrations/day}$$

If length of pendulum at 10°C is ℓ_{10} and at 25°C is ℓ_{25}

$\therefore \quad \ell_{25} = \ell_{10}[1+\alpha(25-10)] = \ell_{10}[1+15\alpha]$

as $\quad T = 2\pi\sqrt{\frac{\ell}{g}}$

i.e. $\quad T \propto \sqrt{\ell}$

i.e. $\quad n \propto \frac{1}{\sqrt{\ell}}$, n is no. of vibrations per sec.

$\therefore \quad \frac{n_{25}}{n_{10}} = \sqrt{\frac{\ell_{10}}{\ell_{25}}} = [1+15\alpha]^{-1/2} \approx 1-\frac{15}{2}\alpha$

$\therefore \quad n_{25} = n_{10}\left(1-\frac{15}{2}\times12\times10^{-6}\right)$

$= 43200\,[1-0.00009] = 43196.12$

That is the clock makes (43200 – 43196.12) = 3.88 vibration loss per day. That is clock losses 3.88 ×2 = 7.76 sec per day

Q.12 A sphere of silver is floating in a mercury bath. If temperature is increased will the sphere sink deeper or rise ? It is given $\gamma_{silver} > \gamma_{mercury}$

Sol. Rise

Q.13 A glass vessel of volume V_0 is completely filled with a liquid and its temperature is raised by ΔT. What volume of the liquid will overflow ? Coefficient of linear expansion of glass = α_g and coefficient of volume expansion of the liquid = γ.

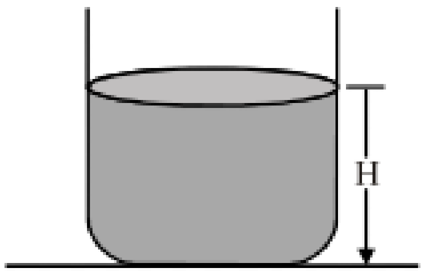
Sol. Volume of the liquid over flown

= increase in the volume of the liquid – increase in the volume of the container

$= [V_0(1+\gamma_\ell\Delta T) - V_0] - [V_0(1+\gamma_g\Delta T) - V_0]$

$= V_0\Delta T(\gamma_\ell - \gamma_g) = V_0\Delta T(\gamma_\ell - 3\alpha_g) \qquad (\because \gamma \approx 3\alpha)$

Q.14 A liquid having coefficient of volume expansion γ_0 is filled in a glass vessel. The coefficient of linear expansion of glass is α. When the arrangement is heated to raise the temperature of the liquid and the glass container by ΔT, exapnsion takes place in both. The expansion may be different or equal. Depending on the values of γ_0 and α you may find that level of the liquid rises with respect to ground or it may fall with repect to ground.



(i) What is the relation between γ_0 and α so that the fraction of volume of container occupied by the liquid does not change with rise in temperature ?

(ii) What is the relation between γ_0 and α so the liquid does not change with respect to ground ?

(iii) What is the relation between γ_0 and α so that the level of the liquid does not change with respect to the container itself ?

Sol. (i) Let V_C be the volume of the container and V_ℓ be the volume of the liquid. According to the condition

$$\frac{V_\ell}{V_C} = \text{constant (i.e. independent of temperture)}$$

$$\Rightarrow \quad \frac{V'_\ell}{V'_C} = \frac{V_\ell}{V_C} \quad \text{or} \quad \frac{V'_\ell}{V_\ell} = \frac{V'_C}{V_C} \quad \text{(Here } V'_\ell \text{ and } V'_C \text{ are volume on heating)}$$

$$\Rightarrow \quad \frac{V'_\ell - V_\ell}{V_\ell} = \frac{V'_C - V_C}{V_C}$$

$$\Rightarrow \quad \frac{\Delta V_\ell}{V_\ell} = \frac{\Delta V_C}{V_C}$$

$$\gamma_0 \Delta T = (3\alpha)\Delta T \qquad \text{[For container, coefficient of volume expansion will be } 3\alpha]$$

(ii) If on heating, H does not change, than the increase in volume of the liquid is accomodated by the increase in base area of the vessel. Let the area be A.

$$\text{Initial volume of liquid} = V_\ell = A \times H$$

$$\text{Final volume of liquid} = V'_\ell = A' \times H$$

$$\Rightarrow \quad \frac{V'_\ell}{V_\ell} = \frac{A'}{A} \Rightarrow \frac{\Delta V'_\ell}{V_\ell} = \frac{\Delta A'}{A}$$

$$\Rightarrow \quad \gamma_0\, \Delta T = \beta \Delta T \text{ or } \gamma_0 = 2\alpha \qquad (\because \beta = 2\alpha)$$

(iii) This is exactly same as part (i)

$$\gamma = 3\alpha.$$

Heat Transfer

Heat may be transported from one point to another by any of three possible mechanisms : conduction, convection, and radiation. We study the rate of energy transfer between bodies due to temperature difference between them.

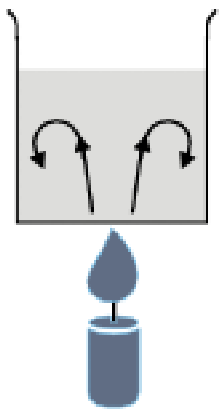
Convection

Convection is the process in which heat is carried from place to place by the bulk movement of a fluid. In liquid and gases, the atoms or molecules can move from point to point. The transfer of heat that accompanies mass transport is called convection.

In forced convection, a fan or pump sets up fluid currents. For examples, a fan blows air, or a pump circulates water in a hot-water heating system in a house.

In free convection, it occurs because the density of a fluid varies with its temperature.

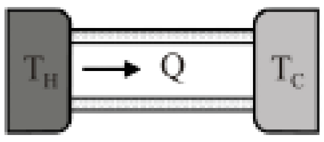
An example of convection currents in a pan of water being heated on a gas burner. The currents distribute the heat from the burning gas to all parts of the water. The direction of convection current is opposite to acceleration due to gravity as shown in figure.



In convection, heat transfer accompanies the movement of a fluid

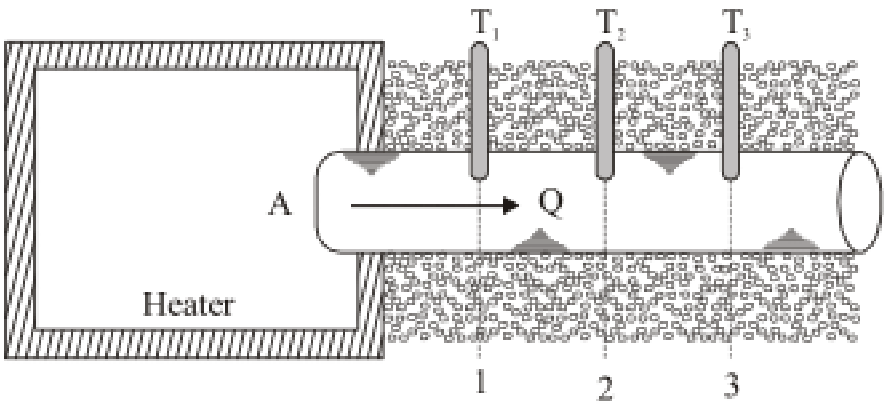
Conduction

A rod whose ends are in thermal contact with a hot reservoir at temperature T_H and a cold reservoir at temperature T_C. The sides of the rod are covered with insulation meterial, so the transport of heat is along the rod, not through the sides. The molecules at the hot reservoir have greater vibrational energy. This energy is trransferred by collisions to the atoms at the end face of the rod. These atoms in turn transfer enery to their neighbors futher along the rod. Such transfer of heat through a substance is called conduction as shown in figure.

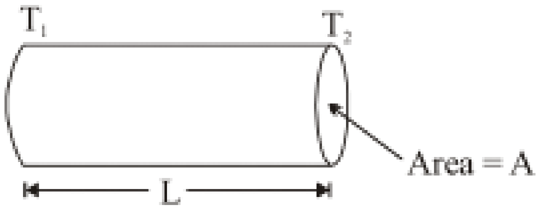


Heat is conducted through an insulated bar whose ends are in thermal contact with two reservoirs

Steady and Transient State :



Consider a metal rod AB, with one end A inserted into a chamber containing a heater with other end B left free and exposed to the surrounding as shown in figure. The rod is thermally insulated sideways with some bad conductor of heat say cotton. Three thermometers are installed in the rod at three distinct sections numbered (1), (2) and (3). Initially, the enitre system is at the room temperature and the three thermometers show the same room temperature. The heater is then switched on. The end A first gets heated up and simultaneously heat is conducted to the adjacent sections towards end B. Due to heat absorption at each sections. The corresponding temperatures start rising with $T_1 > T_2 > T_3$. Such a state, encountered initially is known as a transient state. In this state, the heat coming through end A, is continuously absorbed at each sections with a temperature rise as time elapses. After some time when the temperature of end B becomes equal to that of surrounding and thus becomes constant. Similarly, the temperature of each of the sections of the rod (for example 1, 2, 3) becomes constant or steady. But these steady values at different sections are different.



Consider a portion of the rod of cross sectional area A as shown in figure. Let the temperatures of the two sections separated by a length L be T_1 and T_2 respectively (with $T_1 > T_2$).

Temperature gradient (fall in temperature per unit length) along the length of the rod will be $\frac{T_1 - T_2}{L}$.

Experiments show that the conduction rate (energy transferred per unit time) is given by: Fourier's Law of Heat Conduction

$$H = \frac{\partial Q}{\partial t} = KA\frac{d(-T)}{dx}$$

(Where K : Thermal conductivity of material

H : Thermal current

$\frac{dT}{dx}$: Temperature gradient

A : cross-sectional area of heat path)

The reciprocal of thermal conductivity (K) is called thermal resistivity or thermal specific resistance. Substances having high values of K are good conductors of heat.

Temperature distribution along a conductor :

In order to study conduction in more detail consider figure (i), which shows a metal bar AB whose ends have been soldered into the walls of two metal tanks H and C. Tanks H contains boiling water and C contains ice-water. Heat flows along the bar from A to B and when conditions are steady the temperature θ of the bar is measured at points along its length.

The curve in the upper part of the figure shows how the temperature falls along the bar, less and less steeply from the hot end to the cold. So the temperature gradient decreases from the hot end to the cold.

The figure (ii) shows how the temperature varies along the bar, if the bar is well lagged with a bad conductor, such as asbestos or wool. It now falls uniformly from the hot to the cold end, so the temperature gradient along the bar is constant.

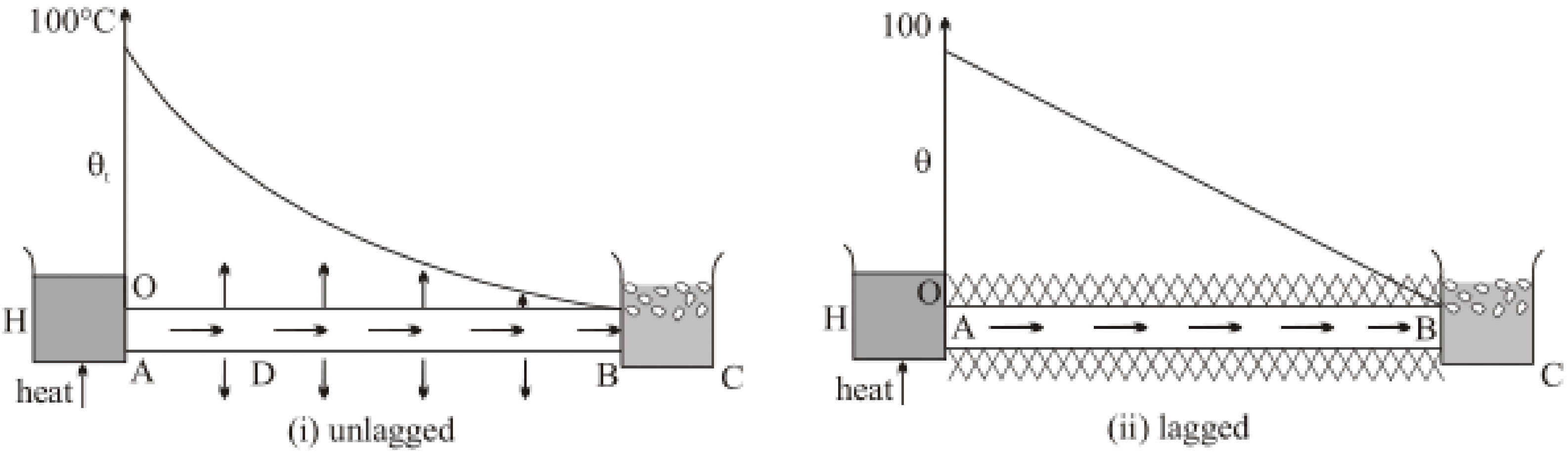
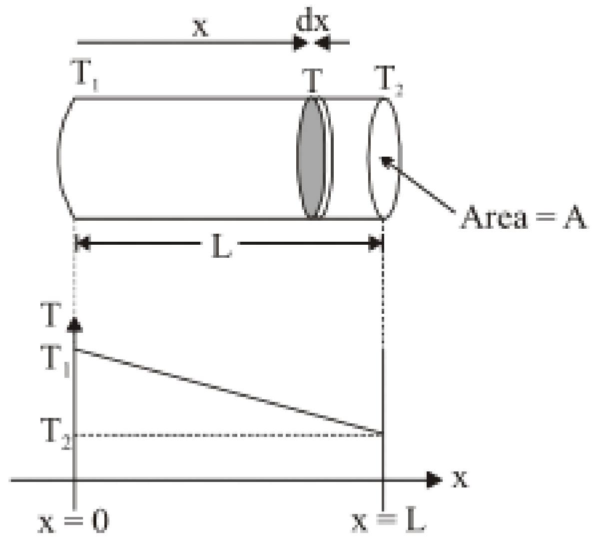


Figure : Temperature fall along lagged and unlagged bars

The difference between the temperature distributions is due to the fact that, when the bar is unlagged, heat escapes from its sides, by convection in the surrounding air, figure (i). The arrows in the figure represent the heat escaping per second from the surface of the bar, and the heat flowing per second along its length. The heat flowing per second along the length decreases from the hot end to the cold. But when the bar is lagged, the heat escaping from its sides is negligible, and the flow per second is now constant along the length of the bar, figure (ii).

Steady State Heat Conduction :



Temperature variation
along length of rod

At steady state, energy transferred through one cross-section of the rod during a certain time interval is equal to the energy transferred by at the other cross-section of the rod during the same time interval.

$$H = \frac{\Delta Q}{\Delta t} = KA\left(\frac{\Delta T}{\Delta x}\right) = KA\left(\frac{T_1 - T_2}{L}\right)$$

Temperature distribution across the rod :

Let at distance x we take element of length dx having a cross-sectional area A and temperature T (As shown in figure). In steady state, rate of heat flow H remains constant

$$H = -KA\frac{dT}{dx}$$

$$\int_{T_1}^{T} dT = -\int_0^x \frac{H}{KA}dx$$

$$T - T_1 = -\frac{Hx}{KA} \qquad \left(\because \frac{H}{KA} = \frac{T_1 - T_2}{L}\right)$$

$$T = T_1 - \frac{x}{L}(T_1 - T_2)$$

The variation has been plotted above.

Thermal Resistance :

The heat transfer by conduction due to temperature difference has an analogy with flow of electric current through a wire when a potential difference is applied. In that case, electrical resistance is defined as

$$R = \frac{V}{i}$$

Similarly, thermal resistance is defined as

$$R = \frac{(T_1 - T_2)}{H}$$

For a rod having length L , area of cross-section A and thermal conductivity K,

$$R = \frac{(T_1 - T_2)}{H}$$

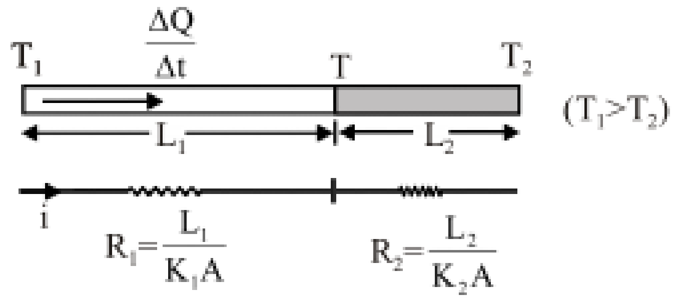
$$= \frac{(T_1 - T_2)}{KA(T_1 - T_2)/L}$$

$$R = \frac{L}{KA}$$

Having calculated the thermal resistance, we can now apply the results of series combination and parallel combination of resistors. It has been explained below.

Composite Rods :

Series Connection : If same heat current are flowing both the rods in steady state, they are said to be in series.



$$(\because R_{eq} = R_1 + R_2 = \Sigma R)$$

Where A - cross-section area of rods

T - Temperature at the juction or Interface temperature

K_1 & K_2 - Themal conductivities of rods having lenghts L_1 and L_2 respectively.

In steady state, heat current is constant throughout the rods.

$$i = \frac{\Delta Q}{\Delta t} = \frac{T_1 - T}{R_1} = \frac{T - T_2}{R_2}$$

$$\therefore\ T_1 - T = iR_1 \qquad \text{...(i)}$$

$$T - T_2 = iR_2 \qquad \text{...(ii)}$$

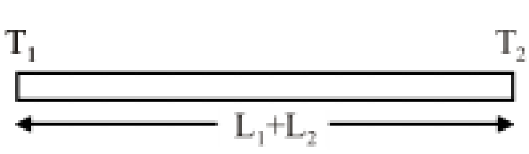
From (i) & (ii)

$$\frac{T_1 - T_2}{R_1 + R_2} = i \quad \text{and} \quad T = \frac{(T_1 R_2 + T_2 R_1)}{R_1 + R_2}$$

$$i = \frac{\Delta T}{R_{eq}}, \text{ in series } R_{eq} = R_1 + R_2$$

Equivalent conductivity of composite Rods (K_{eq}) :

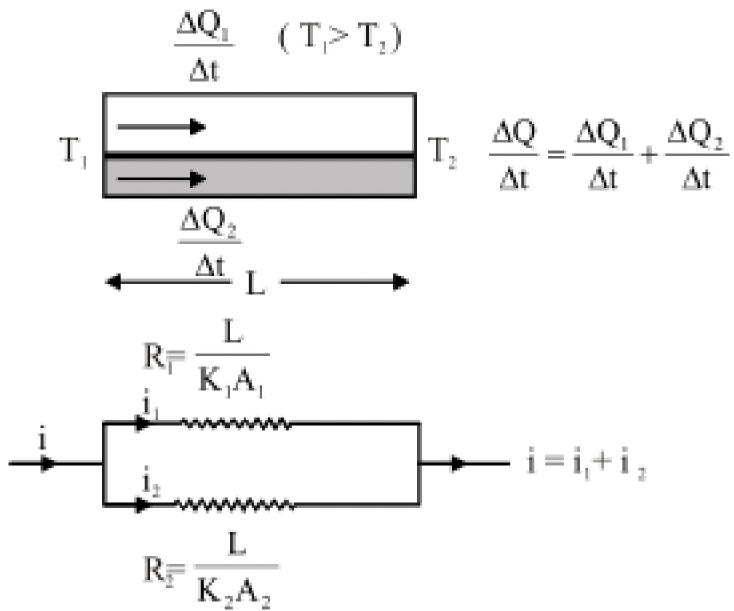
If this rod is replaced by a single rod, then $i = (T_1 - T_2)/R_{eq}$



$$\therefore \qquad R_{eq} = R_1 + R_2 = \frac{L_1}{K_1 A} + \frac{L_2}{K_2 A} = \frac{L_1 + L_2}{K_{eq} A}$$

$$K_{eq} = \frac{L_1 + L_2}{\dfrac{L_1}{K_1} + \dfrac{L_2}{K_2}}$$

Parallel Connection : If the tow rods have the same temperature difference across it, they are said to be in parallel.



$$\left(\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \Sigma \frac{1}{R}\right)$$

$$i_1 = \frac{T_1 - T_2}{R_1}, \; i_2 = \frac{T_1 - T_2}{R_2}$$

$$\therefore \; i = i_1 + i_2 = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

In parallel, $\dfrac{1}{R_{eq}} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

If the two rods are repleaced by a single rod, then K_{eq} will be

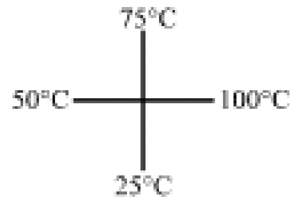
$$K_{eq} = \frac{L}{R_{eq}(A_1 + A_2)} \text{ and } i = \frac{T_1 - T_2}{R_{eq}}$$

Thus, the heat current in thermal resistances in terms of total thermal current is given by :

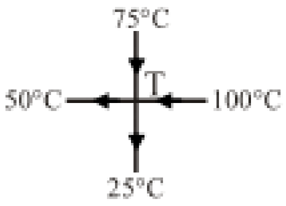
$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right) \times i \qquad \text{and} \qquad i_2 = \left(\frac{R_1}{R_1 + R_2}\right) \times i$$

Illustration :

Two identical rods are joined at their middle points. The ends are maintained at constant temperatures as indicated. The temperature of the junction is _______?



Sol. *Let junction temperature be T*



Accourding to kirchouff's junction law,

Net input thermal current is equal to net output thermal current on a junction. i.e.

$$\sum\left(\frac{\Delta Q}{\Delta t}\right)_{in} = \sum\left(\frac{\Delta Q}{\Delta t}\right)_{out}$$

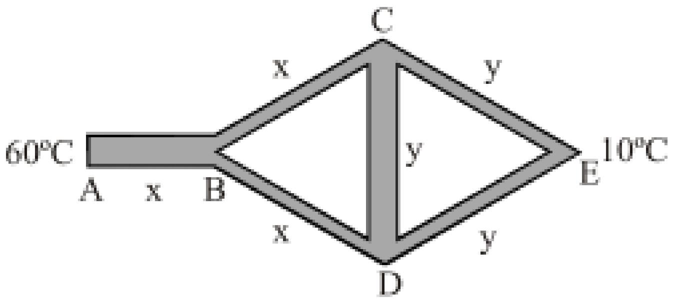
$$\frac{(100-T)}{(R/2)} + \frac{(75-T)}{(R/2)} = \frac{(T-50)}{(R/2)} + \frac{(T-25)}{(R/2)}$$

$175 + 75 = 4T$

$T = 62.5°C$

Illustration :

Three rods of material x and three of material y are connected as shown in figure. All the rods are identical in length and cross-sectional area. If the end A is maintained at 60ºC and the junction E at 10ºC, calculate the temperature of the junction B. The thermal conductivity of x is 800 W/n – ºC and that of y is 400W/m –ºC.



Sol. *It is clear from the symmetry of the figure that the points C and D are equivalent in all respect and hence, they are at the same temperature, say T. No heat will flow through the rod CD. We can, therefore neglect this rod in further analysis. (Treated as balance wheat stone bridge)*

Let L and A be the length and the area of cross-section of each rod. The thermal resistnces of AB, BC and BD are equal. Each has a value

$$R_1 = \frac{1}{K_x}\frac{L}{A} \qquad ...(i)$$

Similarly, thermal resistances of CE and DE are equal, each having a value

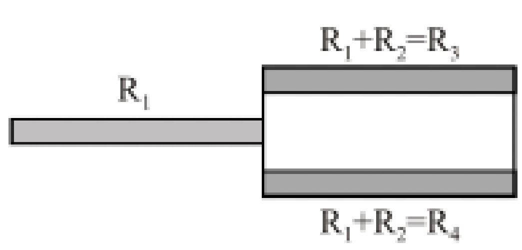
$$R_2 = \frac{1}{K_y}\frac{L}{A} \qquad ...(ii)$$

As the rod CD has no effect, we can say that the rods BC and CE are joined in series. Their equivalent thermal resistance is

$$R_3 = R_{BC} + R_{CE} = R_1 + R_2.$$

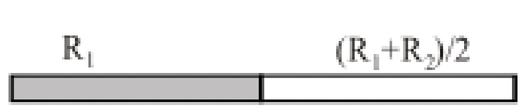
Also, the rods BD ad DE together have an equivalent thermal resitance

$$R_4 = R_{BD} + R_{DE} = R_1 + R_2.$$



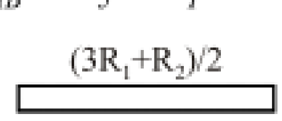
The resistances R_3 and R_4 are joined in parallel and hence their equivalent thermal resistance is given by

$$\frac{1}{R_5} = \frac{1}{R_3} + \frac{1}{R_4} = \frac{2}{R_3} \qquad or \qquad R_5 = \frac{R_3}{2} = \frac{R_1 + R_2}{2}$$



This resistance R_5 is connected in series with AB. Thus, the total arrangement is equivalent to a thermal resistance.

$$R = R_{AB} + R_5 = R_1 + \frac{R_1 + R_2}{2} = \frac{3R_1 + R_2}{2}$$



The heat current through A is

$$i = \frac{T_A - T_E}{R} = \frac{2(T_A - T_E)}{3R_1 + R_2} \qquad ...(i)$$

This current passes through the rod AB. We have

$$i = \frac{T_A - T_B}{R_{AB}} \qquad ...(ii)$$

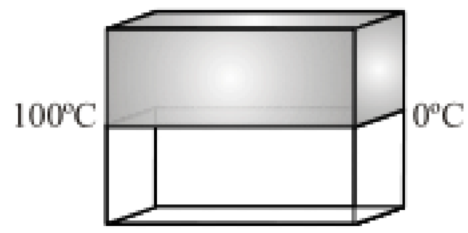
by using (i) and (ii) we get

$$T_A - T_B = \frac{2K_y(T_A - T_E)}{K_x + 3K_y} = \frac{2 \times 400}{800 + 3 \times 400} = 20^\circ C$$

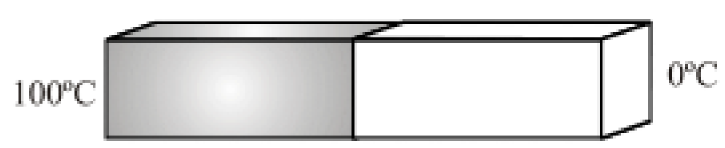
$$T_B = T_A - 20^\circ C = 40^\circ C$$

Illustration :

Two identical rectangular rods of metal are welded as shown in figure (1) and 20 J of heat flows through the rods in 1 min. How long would it take for 20 J heat to flow through the rods if they are welded as shown in figure (2).



(Figure - 1)



(Figure - 2)

Sol. *Let R be the thermal resistance of each rod.*

$\therefore$ *In first case* $\frac{1}{R_1} = \frac{1}{R} + \frac{1}{R}$ *or* $R_1 = \frac{R}{2}$

So the rate of flow of heat in this situation will be

$$\frac{\Delta Q}{\Delta t} = \frac{\Delta T}{R_1} = \frac{100 - 0}{R/2} = \frac{20}{60}$$

$$R = 600\ ^\circ C/W$$

Now for case (2)

$$R_2 = R + R = 600 + 600 = 1200\ ^\circ C/W$$

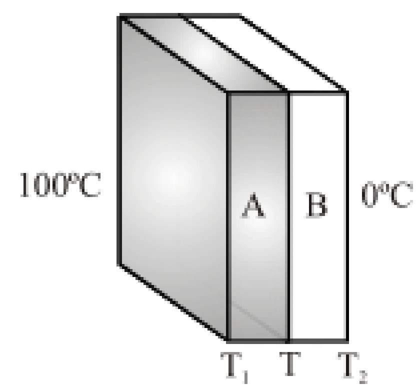
$$\therefore \quad \frac{\Delta Q}{\Delta t} = \frac{\Delta T}{R_2}$$

$$\frac{20}{t} = \frac{100}{1200}$$

$$t = 240 \text{ sec.}$$

Illustration :

Two parallel plates A and B are joined together to form a compound plate (in figure). The thicknesses of the plates are 4.0 cm and 2.5 cm respectively and the area of cross-section is 100 cm^2 for each plate. The thermal conductivities are K_A = 200 W/m-° for the plate A and K_B = 400 W/m-°C for the plate B. The outer surface of the plate A is maintained at 100°C and the outer surface of the plate B is maintained at 0°C. Find (a) the rate of heat flow through any cross-section, (b) the temperature at the interface and (c) the equivalent thermal conductivity of the compound plate.



Sol. (a) *Let the temperature of the interface be T.*

The area of cross-section of each pate is A = 100 cm^2 = 0.01m^2. The thicknesses are x_A = 0.04 m and x_B = 0.025 m

The thermal resistance of the plate A is

$$R_A = \frac{x_A}{K_A A}$$

and that of the plate B is

$$R_B = \frac{x_B}{K_B A}$$

The equivalent thermal resistance is

$$R_{eq} = R_A + R_B = \frac{1}{A}\left(\frac{x_A}{K_A} + \frac{x_B}{K_B}\right) \qquad \text{... (i)}$$

Thus, $$\frac{\Delta Q}{\Delta t} = \frac{T_1 - T_2}{R_{eq}} = \frac{A(T_1 - T_2)}{x_A/K_A + x_B/K_B}$$

$$= \frac{(0.01\,m^2)\,(100\,°C)}{(0.04\,m)/(200\,W/m-°C) + (0.025m)/(400\,W/m-°C)} = 3810\ W.$$

(b) *We have* $$\frac{\Delta Q}{\Delta t} = \frac{A(T - T_2)}{x_B/K_B}$$

or, $$3810\ W = \frac{(0.01\,m^2)\,(T - 0°C)}{(0.025\,m)\,/\,(400\,W/m-°C)}$$

or, $T = 25°C$

(c) *If K is the equivalent thermal conductivity of the compound plate, its thermal resistance is*

$$R_{eq} = \frac{1}{A}\,\frac{x_A + x_B}{K_{eq}}$$

Comparing with (i),

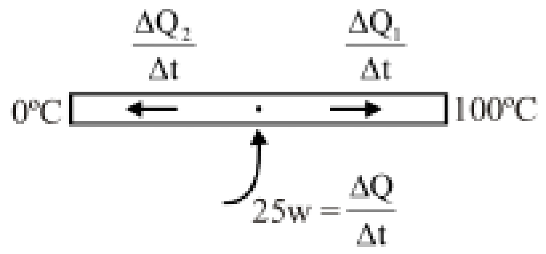
$$\frac{x_A + x_B}{K_{eq}} = \frac{x_A}{K_A} + \frac{x_B}{K_B}$$

or, $$K_{eq} = \frac{x_A + x_B}{x_A / K_A + x_B / K_B} = 248\ W/m\text{–}°C$$

Illustration :

The ends of copper rod of length 1 m and area of cross section 1 cm^2 are maintained at 0°C and 100°C. At the centre of the rod there is a source of heat of power 25 W. Thermal conductivity of copper is 400 W/m-K. In steady state, the temperature at the section on rod at which source is supplying heat, will be__________?

Sol.



Net thermal current supplied by source $\left(\frac{\Delta Q}{\Delta t}\right)$ *then*

$$\frac{\Delta Q}{\Delta t} = \frac{\Delta Q_1}{\Delta t} + \frac{\Delta Q_2}{\Delta t}$$

$$25 = \frac{kA}{0.5}(T - 100) + \frac{kA}{0.5}(T - 0)$$

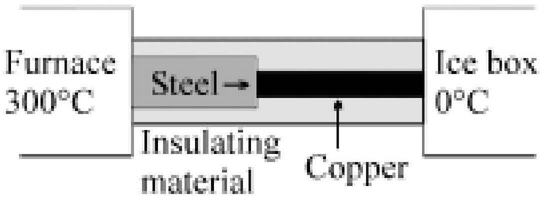
$$\frac{25 \times 0.5}{400 \times 1 \times 10^{-4}} = 2T - 100$$

$$\therefore\ 2T = \frac{1250}{4} + 100$$

$$T = 206.25\ °C$$

Illustration :

What is the temperature of the steel-copper junction in the steady state of the system shown in the figure. Length of the steel rod = 25 cm, length of the copper rod = 50 cm, temperature of the furnace = 300 °C, temperature of the other end = 0°C. The area of cross section of the steel rod is twice that of the copper rod. (Thermal conductivity of steel = 50 J s^{-1} m^{-1} K^{-1} and of copper = 400 J s^{-1} m^{-1} K^{-1})



Sol. *Let temperature be T, in steady state series connection heat transfered* $\left(\frac{\Delta Q}{\Delta t}\right)$ *through each rod is same.*

$$\frac{\Delta Q}{\Delta t} = \frac{k_1A_1(T_1 - T)}{L_1} = \frac{k_2A_2(T - T_2)}{L_2}$$

$$300 - T = \left(\frac{L_1}{L_2}\right)\left(\frac{k_2}{k_1}\right)\left(\frac{A_2}{A_1}\right)(T - 0)$$

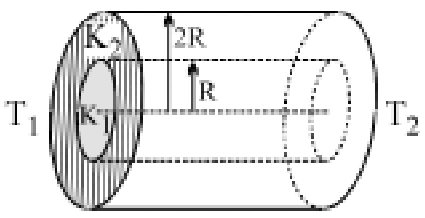
$$300 - T = 2T$$
$$T = 100°C$$

Practice Exercise

Q.1 Three identical metal rods A, B and C are placed end to end and a temperature difference is maintained between the free ends of A and C. If the thermal conductivity of B (K_B) is twice that of C (K_C) and half that of A (K_A), (K_A = 49 w/mK)calculate the effective thermal conductivity of the system ?

Q.2 Two identical rectangular rods of metal are welded end to end in series between temperatures of 0° C and 100°C and 10J of heat is conducted (in a steady state process) through the rods in 2.0min. How long would it take for 10J to be conducted through the rods if they are welded together in parallel across the same temperatures?

Q.3 A composite cylinder is made of two materials having thermal conductivities K_1 and K_2 as shown. Temperature of the two flat faces of cylinder are maintained at T_1 and T_2. For what ratio K_1/K_2 the heat current through the two materials will be same. Assume steady state and the rod is lagged (insulated from the curved surface).



Answers

Q.1 21 w/mK   Q.2 30 secs.   Q.3 $\frac{K_1}{K_2} = 3$

Radiation

Radiation is the process in which energy is transferred by means of electromagnetic waves.
All bodies continuously radiate energy in the form of electromagnetic waves. It does not require a material medium. Electromagnetic waves from the sun, for example, travel through the void of space during their journey to earth. Even an ice cube radiates energy, although so little of it is in the form of visible light that an ice cube cannot be seen in the dark. The surface of an object plays a significant role in determining how much radiant energy the object will absorb or emit.

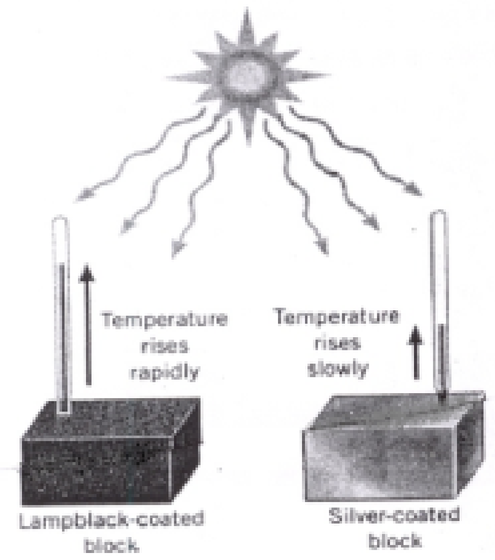


Figure : The temperature of the block coated with lampblack rises faster than the temperature of the block coated absorbs radiant energy from the sun at the greater rate

The two blocks in sunlight in figure, for example, are identical, except that one has a rough surface coated with lamblack (a fine black soot), while the other has a highly polished silver surface. As the thermometers indicate, the temperature of the black block rises at a much faster rate than that of the silver block. This is because lampblack absorbs about 97% of the incident radiant energy, while the silvery surface absorbs only about 10%. The remaining part of the incident energy is reflected in each case. We observe the lampblack as black in color because it reflects so little of the light falling on it, while the silvery surface looks like a mirror because it reflects so little of the light falling on it, while the silvery surface looks like a mirror because it reflects so much light. Since the color black is associated with nearly complete absorption of visible light, the term perfect blackbody or, simply, blackbody is used when referring to an object that absorbs all the electromagnetic waves falling on it.

Black body:

The experiments described before lead us to the idea of a perfectly black body, one which absorbs all the radiation that falls upon it, and reflects and transmits none. The experiments also lead us to suppose that such a body would be the best possible radiator.

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Radiation

Radiation is the process in which energy is transferred by means of electromagnetic waves.

All bodies continuously radiate energy in the form of electromagnetic waves. It does not require a material medium. Electromagnetic waves from the sun, for example, travel through the void of space during their journey to earth. Even an ice cube radiates energy, although so little of it is in the form of visible light that an ice cube cannot be seen in the dark. The surface of an object plays a significant role in determining how much radiant energy the object will absorb or emit.

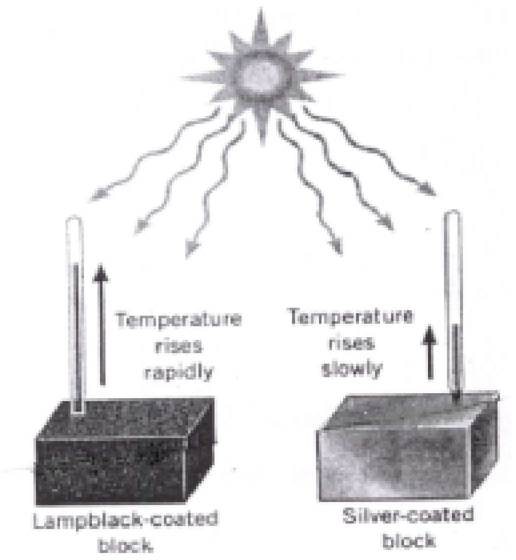


Figure : The temperature of the block coated with lampblack rises faster than the temperature of the block coated absorbs radiant energy from the sun at the greater rate

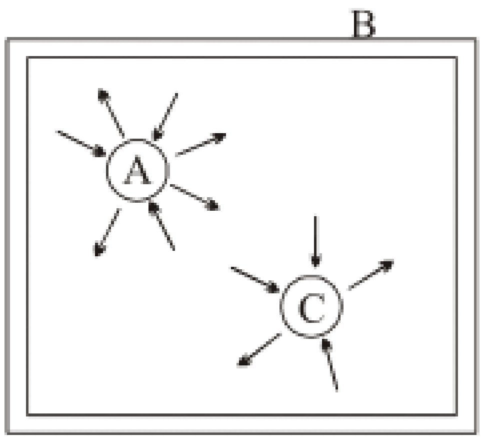
The two blocks in sunlight in figure, for example, are identical, except that one has a rough surface coated with lamblack (a fine black soot), while the other has a highly polished silver surface. As the thermometers indicate, the temperature of the black block rises at a much faster rate than that of the silver block. This is because lampblack absorbs about 97% of the incident radiant energy, while the silvery surface absorbs only about 10%. The remaining part of the incident energy is reflected in each case. We observe the lampblack as black in color because it reflects so little of the light falling on it, while the silvery surface looks like a mirror because it reflects so little of the light falling on it, while the silvery surface looks like a mirror because it reflects so much light. Since the color black is associated with nearly complete absorption of visible light, the term perfect blackbody or, simply, blackbody is used when referring to an object that absorbs all the electromagnetic waves falling on it.

Black body:

The experiments described before lead us to the idea of a perfectly black body, one which absorbs all the radiation that falls upon it, and reflects and transmits none. The experiments also lead us to suppose that such a body would be the best possible radiator.

Prevost's theory of exchange:

Any body having temperature greater then zero kelvin, must emit or aborb radiation.



A is placed in an evacuated enclosure B, at lower temperature than A, then A cools until it reaches the temperature of B. If a body C, cooler than B, is put in B, then C warms up to the temperature of B. We conclude that radiation from B falls on C, and therefore also on A, even through A is at a higher temperature. Thus A and C each come to equilibrium at the temperature of B when each is absorbing and emitting radiation at equal rates.

If Q is the total incident energy on a body, Q_1 is the part absorbed, Q_2 is the part reflected and Q_3 is the part transmitted then

$Q = Q_1 + Q_2 + Q_3$

Absorption coefficient or absorptive power $a = Q_1/Q$

Reflection coefficient $r = Q_2/Q$

Transmission coefficient $t = Q_3/Q$

Thus $a + r + t = 1$

If, for a body, $r = t = 0$ and $a = 1$, i.e. it absorbs all the energy falling on it, such bodies are known as black bodies.

Emissive Power:

Emissive power of a surface is the quantity of heat energy emitted per second, per unit area of surface through unit solid angle. It depends on the nature and the temperature of the surface.

Emissivity:

Emissivity of a surface is the ratio of the emissive power of that surface to the emissive power of a black body at the same temperature.

Kirchhoff's Law:

At a given temperature, the ratio of emissive power to absorptive power of any body is equal to the emissive power of a black body at that temperature. Thus,

$$\frac{E_1}{a_1} = \frac{E_2}{a_2} = E_{\text{Black body}}$$

From Kirchhoff's law, it can be deduced that good absorbers are also good emitters

Stefan's radiation law

An idealized body that absorbs all the radiation incident upon it is called a blackbody. A blackbody absorbs not only all visible light, but infrared, ultraviolet, and all other wavelengths of electromagnetic radiation. It turns out that a good absorber is also a good emitter of radiation. A blackbody emits more radiant ower per unit surface area than any real object at the same temperature. The rate at which a blackbody emits rdiation per unit surface area is proportional to the fourth power of the absolute temperature.

$$P = \frac{dQ}{dt} = \sigma A T^4 \qquad \text{(for a black body)}$$

In equation, A is the surface area and T is the surface temperature of the blackbody in kelvins. Since Stefan's law involves the absolute temperature and not a temperature difference, °C cannot be substituted. The universal constant σ (Greek letter sigma) is called Stefan's constant :

$$\sigma = 5.670 \times 10^{-8}\ \mathrm{W/(m^2.K^4)}$$

The fourth-power temperature dependence implies that the power emitted is extremely sensitive to temperature changes. If the absolute temperature of a body doubles, the energy emitted increases by a factor of $2^4 = 16$.

Since real bodies are not perfect absorbers and therefore emit less than a blackbody, we define the emissivity (e) as the ratio of the emitted power of the body to that of a blackbody at the same temperature. Then Stefan's law becomes.

$$P = e\sigma A T^4 \qquad \text{(for a non-black body)}$$

The emissivity ranges from 0 to 1.

e = 1 for a perfect radiator and absorber (a blackbody).

e = 0 for a perfect reflector.

Hot object placed in isothermal enclosure:

Consider a body at a temperature of T_0 and T_e is the temperature of the room or enclosure containing the body. If A is the surface area of the body and emissivity (e).

Since the body is in temperature equilibrium, the energy per second it radiates must equal the energy per second it absorbs. then, from Stefan's law,

energy per second emitted $(P_{emit}) = e\sigma A T_0^4$

energy per second absorbed $(P_{absorbed}) = e\sigma A T_e^4$

$P_{emit} = P_{aborbed} \Rightarrow T_e = T_o$

Now suppose the body X is heated electrically by a heater of power W watts and finally reaches a constant temperature T. In this case, from Prevost's theory,

energy per second from heater, W = net energy per second radiated by X

The net energy per second radiated by X = $e\sigma A T^4 - e\sigma A T_0^4$. So

$$W = e\sigma A T^4 - e\sigma A T_0^4 = e\sigma A (T^4 - T_0^4)$$

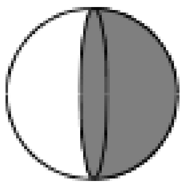
Illustration:

A copper sphere is suspended in an evacuated chamber maintained at 300 K. The sphere is mantained at a constant temperature of 500 K by heating it electrically. A total of 300 W of electric power is needed to do it. When half of the surface of the copper sphere is completely blackened, 600W is needed to maintain the same temperature of the sphere. Calculate the emissivity of copper.

Sol. *Applying Stefan's Law*

Initially $T = 300K$ and $T_s = 500K$

$$300 = \sigma e A\,[500^4 - 300^4] \qquad ....(1)$$

afterwards  *half of the surface of sphere is completely blackened*

$$600 = \frac{\sigma e A}{2}[500^4 - 300^4] + \frac{\sigma A}{2}[500^4 - 300^4] \qquad ....(2)$$

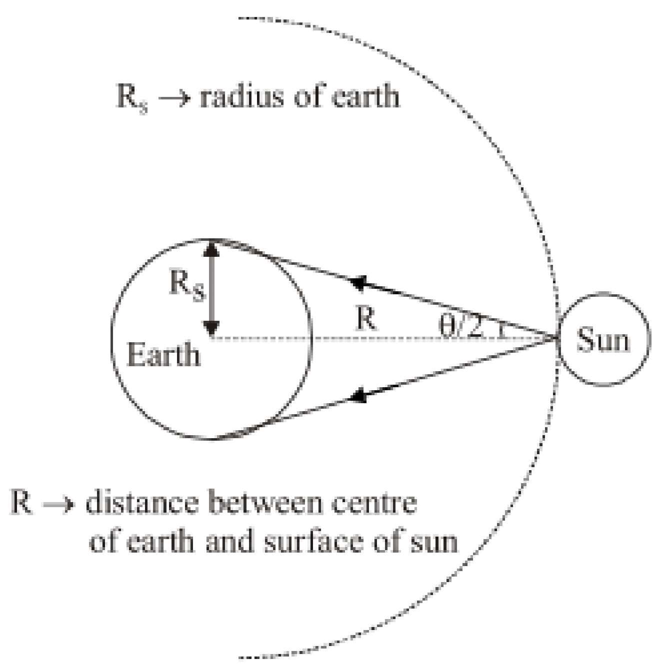
dividing (2) by (1)

$$2 = \frac{\left(\frac{e}{2} + \frac{1}{2}\right)}{e} \quad \Rightarrow \quad 2e = \frac{e}{2} + \frac{1}{2} \quad \Rightarrow \quad e = \frac{1}{3}$$

Illustration:

The solar constant for a planet is S. The surface temperature of the sun is T K. The sun subtends an angle θ at the planet. Find S.

Sol. *Solar constant (S) is defined the rate at which radiations are received from sun per unit surface area.*



$$\tan(\theta/2) = \frac{Rs}{R}$$

$$\tan(\theta/2) \simeq \theta/2 = \frac{Rs}{R}$$

$$\left[\frac{Rs}{R} = \theta/2\right]$$

$$\text{Solar constant } (S) = \frac{\text{Power received}}{\text{Surface Area}}$$

$$S = \frac{\sigma(4\pi Rs^2)T^4}{4\pi R^2} = \sigma\left(\frac{Rs}{R}\right)^2 T^4$$

Using small angle approximation we get

$$S = \sigma\left(\frac{\theta^2}{4}\right)T^4$$

$$S = \frac{\sigma T^4 \theta^2}{4}$$

Illustration:

A highly conducting solid sphere of radius R, density ρ and specific heat s is kept in an evacuated chamber. A parallel beam of electromagnetic radiation having uniform intensity I is incident on its surface. Assuming surface of the sphere to be perfectly black and its temperature at t = 0 to be equal to T_0, Calculate maximum attainable temperature of the sphere. (Stefan's constant = σ)

Sol. *At maximum temperature,*

heat received by solid sphere from electromagnetic radiation = heat radiated by solid sphere

$$I \times \pi R^2 = \sigma(4\pi R^2)\ (T_{max})^4.$$

(∵ Power received per second (P_{abs}) = Intensity (I) × Projection area of sphere)

$$T_{max} = \left(\frac{I}{4\sigma}\right)^{1/4}$$

Illustration:

The distance of the Earth from the Sun is 4 times that of the planet Mercury from the Sun.The temperature of the Earth in radiative equilibrium with the Sun is 290 K. Find the radiative equilibrium temperature of the Mercury. Assume all three bodies to be black body.

Sol. $$P_{received} = \left(\pi R_p^2\right)\left(\frac{P_{sun}}{4\pi r_s^2}\right)$$

$$P_{emitted} = \sigma(e)4\pi R_p^2 T_p^4$$

In Thermal Equilibrium

$$P_{received} = P_{emitted}$$

$$\Rightarrow \quad (T_P)^2 \propto \frac{1}{r_s}$$

$$\frac{T_{earth}}{T_{mercury}} = \sqrt{\frac{r_{mercury}}{r_{earth}}}$$

$$T_{mercury} = (290K)\left(\frac{4}{1}\right)^{1/2}$$

$$= 580\ K$$

Illustration :

The tungsten filament of an electric lamp has a length of 0.5m and a diameter 6×10^{-5} m. The power rating of the lamp is 60W. Assuming the radiation from the filament is equivalent to 80% that of a perfect black body radiator at the same temperature, estimate the steady temperature of the filament. (Stefan constant = $5.7 \times 10^{-8}\ W\,m^{-2}\,K^{-4}$)

Sol. *When the temperature is steady,*

power radiated from filament = power received = 60W

$\therefore \quad 0.8 \times 5.7 \times 10^{-8} \times 2\pi \times 3 \times 10^{-5} \times 0.5 \times T^4 = 60$

since surface area of cylindrical wire is $2\pi rh$ with the usual notation.

$$\therefore \quad T = \left(\frac{60}{0.4 \times 5.7 \times 10^{-8} \times 2\pi \times 3 \times 10^{-5}} \right)^{1/4} = 1933\ K$$

Newton's law of cooling

For small temperature differences, the rate of cooling, due to conduction, convection, and radiation combined, is proportional to the difference in temperature. It is a valid approximation in the transfer of heat from a radiator to a room, the loss of heat through the wall of a room, or the cooling of a cup of tea on the table.

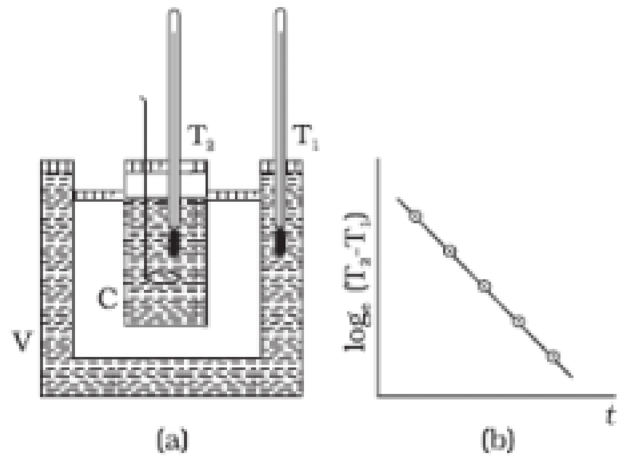


Figure : Verification of Newton's Law of cooling

Suppose, a body of surface area A at an absolute temperature T is kept in a surrounding having a lower temperature T_0. The net rate of loss of thermal energy from the body due to radiation is

$$\Delta u_1 = e\sigma A(T^4 - T_0^4)$$

If the temperature difference is small, we can write

$$T = T_0 + \Delta T$$

or, $$T^4 - T_0^4 = (T_0 + \Delta T)^4 - T_0^4$$

$$= T_0^4 \left(1 + \frac{\Delta T}{T_0}\right)^4 - T_0^4$$

$$= T_0^4 \left[1 + 4\frac{\Delta T}{T_0} + \text{higher powers of } \frac{\Delta T}{T_0}\right] - T_0^4$$

$$\approx 4T_0^3\,\Delta T = 4T_0^3\,(T - T_0)$$

Thus, $$\Delta u_1 = 4e\sigma A T_0^3\,(T - T_0)$$

$$= b_1 A\,(T - T_0)$$

The body may also lose thermal energy due to convection in the surrounding air. For small temperature difference, the rate of loss of heat due to convection is also proportional to the temperature difference and the area of the surface. This rate may, therefore, be written as

$$\Delta u_2 = b_2 A\,(T - T_0)$$

The net rate of loss of thermal energy due to convection and radiation is

$$\Delta u = \Delta u_1 + \Delta u_2 = (b_1 + b_2)\,A\,(T - T_0).$$

If s be the specific heat capacity of the body and m its mass, the rate of fall of temperature is

$$\frac{-dT}{dt} = \frac{\Delta u}{ms} = \frac{b_1 + b_2}{ms}\,A\,(T - T_0)$$

$$= bA\,(T - T_0)$$

Thus, for small temperature difference between a body and its surrounding, the rate of cooling of the body is directly proportional to the temperature difference and the surface area exposed. We can write

$$\frac{dT}{dt} = -\,bA\,(T - T_0)$$

Cooling curve:

The law holds good only for small difference of temperature. Also, the loss of heat by radiation depends upon the nature of the surface of the body and the area of the exposed surface. We can write

$$-\frac{dT}{dt} = k\,(T - T_s)$$

where *k* is a positive constant depending upon the area and nature of the surface of the body. Suppose a body of mass *m* and specific heat capacity *s* is at temperature T. Let T_s and T_o be the temperature of the surroundings and body respectively. If the temperature falls by a small amount dT in time dt, then the amount of heat lost is

$$dQ = msdT$$

∴ Rate of loss of heat is given by

$$\frac{dQ}{dt} = ms\,\frac{dT}{dt}$$

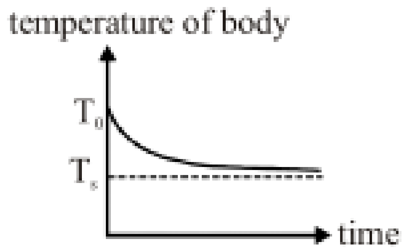
from equation

$$-\frac{dQ}{dt} = k\,(T - T_s) \text{ and } \frac{dQ}{dt} = ms\,\frac{dT}{dt}$$

we have $$-ms\frac{dT}{dt} = k(T - T_s)$$

$$\frac{dT}{T - T_s} = -\frac{k}{ms}\,dt = -\,Kdt \qquad (\text{where } K = k/ms)$$

On integrating,



$$ln\left(\frac{T - T_s}{T_0 - T_s}\right) = -\,kt$$

$$T = T_s + (T_0 - T_s)e^{-kt}$$

enables you to calculate the time of colling of a body through a particular range of temperature.

Illustration :

The temperature of a body falls from 52°C to 36°C in 10 minutes when placed in a surrounding of constant temperature 20°C. What will be the temperature of the body after another 10 min. (Use Newton's law of cooling)

Sol. *Applying newton's law of cooling*

$$\frac{\Delta T}{\Delta t} = -\frac{b}{mc}\left(T_{avg} - T_S\right)$$

$$\frac{52-36}{10\,\text{min}} = \frac{-b}{mc}\left[\frac{52+36}{2} - 20\right]$$

Let the tempature of the body after another 10 min be T

$$\frac{36-T}{10\,\text{min}} = \frac{-b}{mc}\left[\frac{36+T}{2} - 20\right]$$

solving we get T = 28°C

Illustration :

A metal block is placed in a room which is at 10°C. It is heated by an electric heater of power 500 W till its temperature becomes 50°C. Its initial rate of rise of temperature is 2.5°C/sec. The heater is switched off and now a heater of 100W is required to maintain the temperature of the block at 50°C.(Assume Newtons Law of cooling to be valid)

(i) *What is the heat capacity of the block?*

(ii) *What is the rate of cooling of block at 50°C if the 100W heater is also switched off?*

(iii) *What is the heat radiated per second when the block was 30°C?*

Sol.(i) $P_{heater} = P_{given\ to\ block} + P_{Loss\ to\ surroundings}$

Initial $P_{Loss} = 0$

$$\therefore \quad 500 = ms\frac{dT}{dt} + 0$$

$$500 = C\ 2.5$$

$$\therefore \quad C = 200\ J/°C$$

(ii) *At 50°, power loss to surroundings = 100W*

$$0 = ms\frac{dT}{dt} + 100$$

$$\frac{dT}{dt} = -\frac{100}{200} = -0.5\ s/sec$$

(iii) *Given at 50°C : Newton's Law of cooling*

$$\text{Power Loss} = 100 = k(50-10) \quad \Rightarrow \quad k = \frac{10}{4} = \frac{5}{2}$$

$$\therefore \text{At } 30°C \quad P_{Loss} = k(30-10) = \frac{5}{2} \times 20 = 50\ W$$

Practice Exercise

Q.1 A metal sphere with a black surface and radius 30 mm, is cooled to – 73 C (200 K) and placed inside an enclosure at a temperature of 27°C (300K). Calculate the initial rate of temperature rise of the sphere, assuming the sphere is a black body. (Assume density of metal = 8000 kg m^{-3} specific heat capacity of metal = 400 J kg^{-1} K^{-1}, and Stefan constant = 5.7×10^{-8} W m^{-2} K^{-4}).

Q.2 A pan filled with hot food cools from 94 °C to 86 °C in 2 minutes when the room temperature is at 20 °C. How long will it take to cool from 71 °C to 69 °C?

Q.3 A body cools from 50° C to 40°C in 5 minutes. The surrounding temperature is 20°C. What will be its temperature 5 minutes after reading 40°C? Use approximate method.

Answers

Q.1 0.012 K s^{-1} (approx). Q.2 42 sec Q.3 $T = \frac{100}{3}$ °C

Wien's Displacement Law

The wavelength corresponding to highest intensity m is inversely proportional to the absolute temperature. Thus

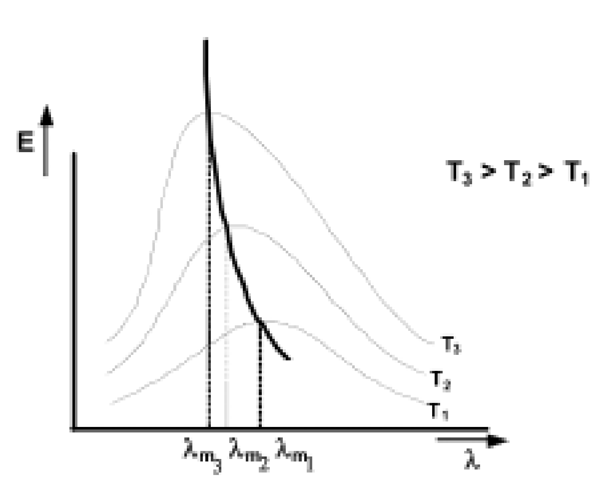
$$\lambda_m = \frac{b}{T}$$

where b (= 2.89×10^{-3} meter Kelvin) is known as the Wien's constant.

When the temperature of a black body is increased, the contribution of low wavelength radiation increases. This explains why a body on heating first appears red, then orange, then white and finally blue. This law also helps us in determining the temperatures of the stars.

Energy Distribution in Black Body Radiation:

The radiation emitted by a black body at any temperature is a mixture of all wavelengths. The graph shows qualitative variation in intensity wavelength, at different temperatures.



Spectral emmissive power:

To speak of the intensity of a single wavelength is meaningless. The slit of the spectrometer always gathers a band of wavelengths the narrower the slit the narrower the band –and we always speak of the intensity of a given band. We express it as follows :

energy radiated $m^{-2}\ s^{-1}$, in band λ to $\lambda + \Delta\lambda = E_\lambda\ \Delta\lambda$

The quantity E_λ is called emissive power of a black body for the wavelength λ and at the given temperature ; its definition follows from equation λ to $\lambda + \Delta\lambda = E_\lambda\ \Delta\lambda$:

$$E_\lambda = \frac{\text{energy radiated } m^{-2}\ s^{-1}, \text{ in band } \lambda \text{ to } \lambda + \Delta\lambda}{\text{band width}, \Delta\lambda}$$

$$E_\lambda = \frac{\text{power radiated } m^{-2} \text{ in band } \lambda \text{ to } \lambda + \Delta\lambda}{\Delta\lambda}$$

In the figure, E_λ is expressed in watts per m^2 per nanometre (10^{-9} m).

The quantity $E_\lambda\ \Delta\lambda$ in equation λ to $\lambda + \Delta\lambda = E_\lambda\ \Delta\lambda$ is the area beneath the radiation curve between the wavelength λ and $\Delta\lambda$ (figure). Thus the energy radiated per meter² per second between those wavelengths in proportional to that area.

Similarly, the total radiation emitted per metre² per second over all wavelengths is proportional to the area under the whole curve.

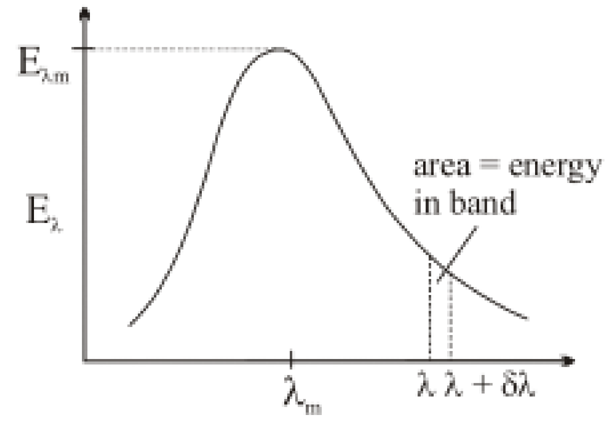


Figure : Definition of $E \cdot \lambda_m$ and $E_{\lambda m}$

Laws of black body radiation:

The curves of figure can be explained only Planck's quantum theory of radiation, which is outside our scope. Both theory and experiment lead to three generalisations, which together describe well the properties of black body radiation.

(i) If λ_m is the wavelength of the peak of the curve for T (in K), then

$$\lambda_m T = \text{constant} \qquad \text{... (2)}$$

The value of the constant is 2.9×10^{-3} m K. In figure the dotted line is the locus of the peaks of the curves for different temperatures.

The relationship in (2) is sometimes called Wien's displacement law.

(ii) If $E_{\lambda m}$ is the height of the peak of the curve for the temperature T, then

$$E_{\lambda m} \propto T^5 \quad \text{... (3)}$$

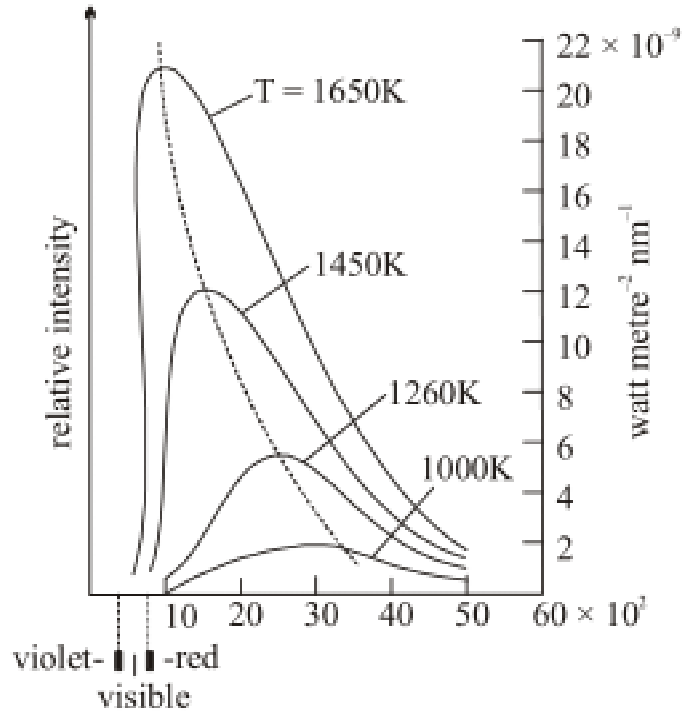


Figure : Distribution of intensity in black body radiation

(iii) If E is the total energy radiated per metre² per second at a temperature T, which is represented by the total area under the particular $E_\lambda - \lambda$ curve, then

$$E = \sigma T^4$$

So in figure, which shows four $E_\lambda - \lambda$ graphs at different temperatures T, the total area below the graphs should be proportional to the corresponding value of T^4.

Illustration:

Estimate the surface temperature of sun. Given for solar radiations, $\lambda_m = 4753$ Å.
($b = 2.89 \times 10^{-3}$ meter Kelvin)

Sol. *From Wien's displacement law*

$\lambda_m T = b$

$T = 6097$ K.

Illustration :

The energy radiated by a black body at 2300 K is found to have the maximum at a wavelength 1260 nm, its emissive power being 8000 Wm^{-2}. When the body is cooled to a temperature T K, the emissive power is found to decrease to 500 Wm^{-2}. Find:

(i) the temperature T

(ii) the wave length at which intensity of emission in maximum at the temperature T.

Sol. *(i)* $\lambda m_1 = 1260$ nm

$\lambda m_1 = 1260 \times 10^{-9}$m, $T_1 = 2300$ K, $T_2 = T$ K

$$\frac{E_1}{E_2} = \left(\frac{T_1}{T_2}\right)^4$$

$$\left(\frac{E_1}{E_2}\right)^{1/4} = \frac{T_1}{T_2},\ T_2 = T_1 \times \left(\frac{E_2}{E_1}\right)^{1/4}$$

$$T_2 = T_1 \times \left(\frac{E_2}{E_1}\right)^{1/4}$$

$$T_2 = 2300 \times \left[\frac{500}{8000}\right]^{1/4} = 2300 \times \frac{1}{2} = 1150\ k$$

$T_2 = 1150\ k$ *Ans.*

(ii) *by using Wein's law*

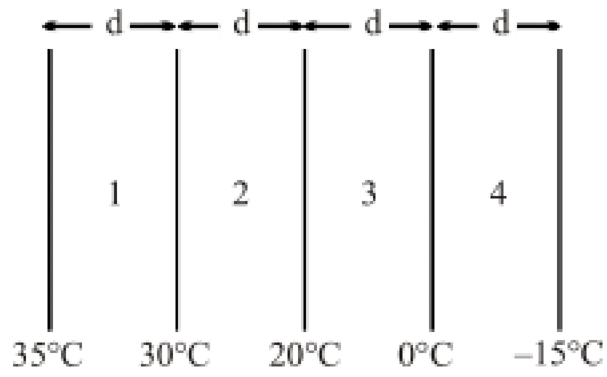
$$T_1\lambda_{m_1} = T_2\lambda_{m_2}\ ;\ \lambda_{m_2} = \left(\frac{T_1\lambda_{m_1}}{T_2}\right)$$

$$\lambda_{m_2} = \frac{2300 \times 1260 \times 10^{-9}}{1150} = 2520 \times 10^{-9}\ M$$

$$\lambda_{m_2} = 2520\ nm$$

Solved Example

Q.1 The diagram shows four slabs of different materials with equal thickness, placed side by side. Heat flows from left to right and the steady-state temperatures of the interfaces are given. Rank the materials according to their thermal conductivities, smallest to largest.



Sol In steady state heat transfered through each slab is same.

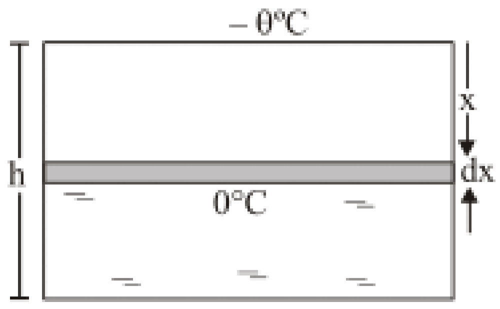
Than $$\frac{\Delta Q}{\Delta t} = \frac{k_1 A \Delta T_1}{d} = \frac{k_2 A \Delta T_2}{d} = \frac{k_3 A \Delta T_3}{d} = \frac{k_4 A \Delta T_4}{d}$$

$$k \propto \frac{1}{\Delta T} \text{ thus } 3, 4, 2, 1$$

Q.2 **(Growth of ice on Pond)**

On a cold winter day, the atmospheric temperature is $-\theta$ (on Celsius scale) which is below 0°C. A cylindrical drum of height h made of a bad conductor is completely filled with water at 0°C and is kept outside without any lid. Calculate the time taken for the whole mass of water to freeze. Thermal conductivity of ice is K and its latent heat of fusion is L. Neglect expansion of water on freezing.

Sol.



Suppose, the ice starts forming at time t = 0 and a thickness x is formed at time t. The amount of heat flown from the water to the surrounding in the time interval t to t = dt is

$$\Delta Q = \frac{KA\theta}{x}\ dt$$

The mass of the ice formed due to the loss of this amount of heat is

$$dm = \frac{\Delta Q}{L} = \frac{KA\theta}{x}\ dt.$$

The thickness dx of ice formed in time dt is

$$dx = \frac{dm}{A\rho} = \frac{K\theta}{\rho x L}\,dt$$

or, $$dt = \frac{\rho L}{K\theta}\,x\,dx$$

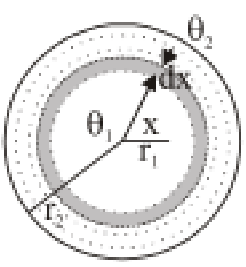
Thus, the time T taken for the whole mass of water to freeze is given by

$$\int_0^T dt = \frac{\rho L}{K\theta}\int_0^h x\,dx$$

$$T = \frac{\rho L h^2}{2K\theta}$$

Q.3 Two thin metallic spherical shells of radii r_1 and r_2 ($r_1 < r_2$) are placed with their centres coinciding. A material of thermal conductivity K is filled in the space between the shells. The inner shell is maintained at temperature θ_1 and the outer shell at temperature θ_2 ($\theta_1 < \theta_2$). Calculate the rate at which heat flows radially through the material.

Sol. Let us draw two spherical shells of radii x and x + dx concentric with the given system. Let the temperatures at these shells be θ and $\theta + d\theta$ respectively. The amount of heat flowing radially inward through the material between x and x + dx is



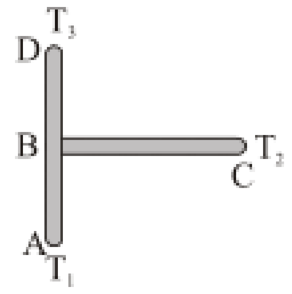
$$\frac{\Delta Q}{\Delta t} = \frac{K4\pi x^2\,d\theta}{dx}$$

Thus, $$K4\pi\int_{\theta_1}^{\theta_2} d\theta = \frac{\Delta Q}{\Delta t}\int_{r_1}^{r_2}\frac{dx}{x^2}$$

$$K4\pi(\theta_2 - \theta_1) = \frac{\Delta Q}{\Delta t}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\frac{\Delta Q}{\Delta t} = \frac{4\pi K r_1 r_2(\theta_2 - \theta_1)}{r_2 - r_1}$$

Q.4 Three rods AB, BC and BD having thermal conductivities in the ratio 1 : 2 : 3 and lengths in the ratio 2 : 1 : 1 are joined as shown is figure. The ends A, C and D are at temperatures T_1, T_2 and T_3 respectively. Find the temperature of the junction B. Assume steady state.



Sol. Let the thermal conductivities of the rods AB, BC and BD be K, 2K and 3K respectively. Also, let their lengths be 2L, L and L.

If T be the required temperature of the junction B and assuming $T_1 > T > T_2 > T_3$, we have

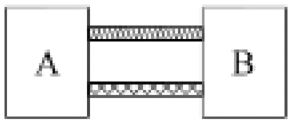
$$\left.\frac{\Delta Q}{\Delta t}\right]_{AB} = \left.\frac{\Delta Q}{\Delta t}\right]_{BC} + \left.\frac{\Delta Q}{\Delta t}\right]_{BD}$$

i.e. $$\frac{KA(T_1 - T)}{2L} = \frac{2KA(T - T_2)}{L} + \frac{3KA(T - T_3)}{L}$$

or $$\frac{T_1 - T}{2} = 2(T - T_2) + 3(T - T_3)$$

or $$T = \frac{1}{11}(T_1 + 4T_2 + 6T_3)$$

Q.5 The container A is constantly maintained at 100°C and insulated container B of the figure initially contains ice at 0°C. Different rods are used to connect them. For a rod made of copper, it takes 30 minutes for the ice to melt and for a rod of steel of same cross-section taken in different experiment it takes 60 minutes for ice to melt. When these rods are simultaneously connected in parallel. Find the time interval in which ice melts ?



Sol. $Q = it$ where i = heat flow rate $= \frac{\Delta T}{R} = \frac{100}{R}$

For copper rod :

$$Q = \left(\frac{100}{R_1}\right)(30) \quad \Rightarrow \quad R_1 = \left(\frac{100}{Q}\right) \times 30$$

Also for steel rod :

$$Q = \left(\frac{100}{R_2}\right) \times 60 \quad \Rightarrow \quad R_2 = \left(\frac{100}{Q}\right) \times 60$$

Now, $$Q = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) t_{reqd.}$$

$$\therefore \quad t_{reqd} = \frac{Q}{\left(\frac{Q}{100}\right) \times \frac{1}{30} + \left(\frac{Q}{100}\right) \times \frac{1}{60}} = 20 \text{ min}$$

Q.6 **(Temperature of Sun)**

Estimate the temperature T_e of the earth, assuming it is in radiative equilibrium with the sun. (Assume radius of sun, $r_s = 7 \times 10^8$ m, temperature of solar surface = 6000 K, distance of earth from sun, $R = 1.5 \times 10^{11}$ m)

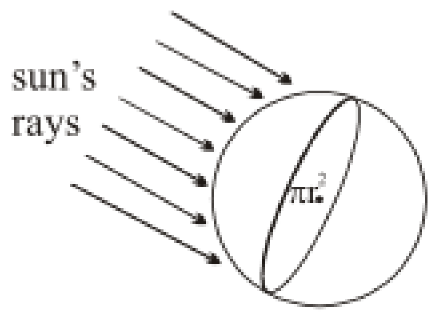
Sol. Power radiated from sun = $\sigma \times$ surface area $\times T^4$

$$= \sigma \times 4\pi r_s^2 \times T_s^4$$

Power received by earth = $\dfrac{\pi r_e^2}{4\pi R^2} \times$ power radiated by sun

since πr_e^2 is the effective area of the earth on which the sun's radiation is incident normally. figure and $4\pi R^2$ is the total area on which the sun's radiation falls at a distance R from the sun where the earth is situated.

Now power radiated by earth = $\sigma \cdot 4\pi r_e^2 \cdot T_e^4$



Assuming radiative equilibrium

power radiated by earth = power received by earth

$$\therefore \quad \sigma \cdot 4\pi r_e^2 \cdot T_e^4 = \sigma \cdot 4\pi r_s^2 \cdot T_s^4 \times \frac{\pi r_e^2}{4\pi R^2}$$

Cancelling r_e^2 and simplifying, then

$$T_e^4 = T_s^4 \times \left(\frac{r_s^2}{4R^2}\right)$$

$$\therefore \quad T_e = T_s \times \left(\frac{7 \times 10^8}{2 \times 1.5 \times 10^{11}}\right)^{1/2} = 209 \text{ K}$$

Note that the calculation is approximate, for example, the earth and the sun are not perfect black body radiators and the earth receives heat from its interior.

Q.7 A wood-burning stove stands unused in a room where the temperature is 18°C (291 K). A fire is started inside the stove. Eventually, the temperature of the stove surface reaches a constant 198°C (471 K), and the room warms to a constant 29°C (302 K). The stove has an emissivity of 0.900 and a surface area of 3.50 m^2. Determine the net radiant power generated by the stove when the stove (a) is unheated and has a temperature equal to room temperature and (b) has a temperature of 198°C.

Sol. **Reasoning :** The stove emits more radiant power heated than when unheated. In both cases, however, the Stefan-Bolfzmann law can be used to determine the amount of power emitted. Power is energy per unit time or Q/t. But in this problem we need to find the net power produced by the stove. The net power is the power the stove emits minus the power the stove absorbs. Then power the stove absorbs comes from the wall, ceiling, and floor of the room, all of which emit radiation.

(a) Remembering that temperature must be expressed in kelvins when using the Stefan-Boltzmann law, we find that

$$\text{Power emitted by unheated} = \frac{Q}{t} = e\sigma T^4 A$$

$$= (0.900)\,[5.67 \times 10^{-8}\ \text{J/(s}\cdot\text{m}^2\cdot\text{K}^4]\,(291\ \text{K})^4\,(3.50\ \text{m}^2) = 1280\ \text{W}$$

The fact that the unheated stove emits 1280 W of power and yet maintains a constant temperature means that the stove also absorbs 1280 W of radiant power from its surroundings. Thus, the net power generated by the unheated stove is zero.

$$\text{Net power generated by stove at } 18°\text{C} = \underbrace{1280\ \text{W}}_{\substack{\text{Power emitted}\\ \text{by stove at}\\ 18°\text{C}}} - \underbrace{1280\ \text{W}}_{\substack{\text{Power emitted by}\\ \text{room at } 18°\text{C and}\\ \text{absorbed by stove}}} = 0\ \text{W}$$

(b) The hot stove (198 °C) or 471 K) emits more radiant power than it absorbs from the cooler room. The radiant power the stove emits is

$$\text{Power emitted by stove at } 198°\text{C} = \frac{Q}{t} = e\sigma T^4 A$$

$$= (0.900)\,[5.67 \times 10^{-8}\ \text{J/(s}\cdot\text{m}^2\cdot\text{K}^4)]\,(471\ \text{K})^4\,(3.50\ \text{m}^2) = 8790\ \text{W}$$

The radiant power the stove absorbs from the room is identical to the power that the stove would emit at the constant room temperature of 29°C (302 K). The reasoning here is exactly like that in part (a).

$$\begin{array}{l}\text{Power emitted by}\\ \text{room at } 29°\text{C and}\\ \text{absorbed by stove}\end{array} = \frac{Q}{t} = e\sigma T^4 A$$

$$= (0.900)\,[5.67 \times 10^{-8}\ \text{J/(s}\cdot\text{m}^2\cdot\text{K}^4)]\,(302\ \text{K})^4\,(3.50\ \text{m}^2) = 1490\ \text{W}$$

The net radiant power the stove produces from the fuel it burn is

$$\begin{array}{l}\text{Net power}\\ \text{generated by}\\ \text{stove at } 198°\text{C}\end{array} = \underbrace{8790\ \text{W}}_{\substack{\text{Power emitted}\\ \text{by stove at}\\ 198°\text{C}}} - \underbrace{1490\ \text{W}}_{\substack{\text{Power emitted by}\\ \text{room at } 29°\text{C and}\\ \text{absorbed by stove}}} = 7300\ \text{W}$$

Q.8 The room heater can provide only 16ºC in the room when the temperature outside is -20ºC. It is not warm and comfortable, that is why the electric stove with power of 1 kW is also plugged in. Together these two devices maintain the room temperature of 22ºC. Determine the thermal power of the heater.

Sol. Rate of heat loss with only room heater

$$P_h = \frac{\Delta Q}{\Delta t} = C\,(16 + 20) \qquad \text{[where C = constant]}$$

while both heater and stove it is

$$P_h + P_s = \left(\frac{\Delta Q}{\Delta t}\right)' = C\,(22 + 20)$$

$$\therefore \frac{P_h}{P_h + P_s} = \frac{36}{42} \Rightarrow 7P_h = 6\,P_h + 6P_s$$

$$\Rightarrow P_h = 6P_s = 6 \text{ kW}$$

Q.9 A hot body palced in air is cooled down according to Newton's law of cooling, the rate of decrease of temperature being k times the temperature difference from the surrounding. Starting from $t = 0$, find the time in which the body will loose half the maximum heat it can lose.

Sol. We have,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

where θ_0 is the temperature of the surrounding and θ is the temperature of the body at time t. Suppose $\theta = \theta_1$ at $t = 0$

Then,

$$\int_{\theta_1}^{\theta} \frac{d\theta}{\theta - \theta_0} = -k\int_0^t dt$$

or $$\ln \frac{\theta - \theta_0}{\theta_1 - \theta_0} = -kt$$

or, $$\theta - \theta_0 = (\theta_1 - \theta_0)\, e^{-kt} \qquad \text{...(i)}$$

The body continues to lose heat till its temperature becomes equal to that of the surrounding. The loss of heat in this entire period is

$$\Delta Q_m = ms\,(\theta_1 - \theta_0).$$

This is the maximum heat the body can looses. If the body loses half this heat, the decreases in its temperature will be,

$$\frac{\Delta Q_m}{2ms} = \frac{\theta_1 - \theta_0}{2}$$

If the body loses this heat in time t_1, the temperature at t_1 will be

$$\theta_1 - \frac{\theta_1 - \theta_0}{2} = \frac{\theta_1 + \theta_0}{2}$$

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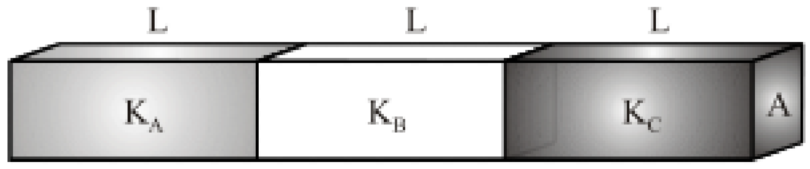
Putting these values of time and temperature in (i),

$$\frac{\theta_1 + \theta_0}{2} - \theta_0 = (\theta_1 - \theta_0)e^{-kt}$$

or, $$e^{-kt} = \frac{1}{2}$$

or, $$t_1 = \frac{\ln 2}{k}$$

Q.10 Three identical metal rods A, B and C are placed end to end and a temperature difference is maintained between the free ends of A and C. If the thermal conductivity of B (K_B) is twice that of C (K_C) and half that of A (K_A), ($K_A = 49$ w/mK)calculate the effective thermal conductivity of the system ?

Sol. 

$$K_B = K_A/2$$
$$K_C = K_A/4$$

$$\frac{1}{R_{eq}} = \frac{L}{A}\left(\frac{1}{K_A} + \frac{1}{K_B} + \frac{1}{K_C}\right) = \frac{3L}{A}\left(\frac{1}{K_{eff}}\right)$$

$$K_{eff} = \frac{3K_A}{7} = 21 \text{ w/mK}$$

Kinetic Theory Of Gases

An ideal gas is collection of small, hard, randomly moving atoms that occasionally collide and bounce off each other but otherwise do not interact. Never the less, gases if two condition are met:

1. The density is low (i.e., the atoms occupy a volume much smaller that of the container), and
2. The temperature is well above the condensation point.

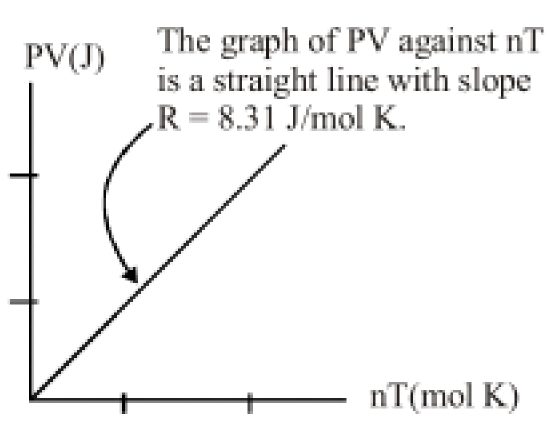
If the density gets two high, or the temperature too low, then the attractive forces between the atoms came interaction. These are the forces that are responsible under the right conditions, for the gas condensing into a liquid.

The Ideal-Gas Law

Experiments shown that there is a specific relationship between the state variables. (P, V, T)

There is a very clear proportionality between the quantity PV and the quantity nT. If we designate the slope of the line in this graph as R, then we can write relationship as

$$PV = R \times (nT)$$



A graph of PV versus nT for an ideal gas.

It is customary to write the relationship in a slightly different form, namely

$$PV = nRT \qquad \text{(ideal-gas law)} \qquad \text{...(i)}$$

equation (i) is ideal gas law.

The constant R, which determined experimentally as the slope of the graph in figure is called the universal gas constant. Its value in SI units, is

$$R = 8.31 \text{ J-mol K}$$

If the gas is initially in state i, characterizes buy the state variables P_i, V_i and T_i and at some later time in a final in a final state f, the state variables for these two states are related by

$$\frac{P_f V_f}{T_f} = \frac{P_i V_i}{T_i} \qquad \text{(ideal gas in a scaled container)}$$

The mathematical relation between the state variables of a system is called the equation of state. Ideal gas will always follow ideal gas equation.

Symbols and constants :

P = Pressure of gas ; V = Volume of gas ; T = Temperature of gas
n = no. of moles ; M = mol. wt. of gas ; m_0 = mass of each atom or molecule
N = total no. of molecule ; N_0 = Avogadro no.
K = Boltzmann gas constant = $1.38 \times 10^{-23}\ JK^{-1}$
R = Universal gas constant
μ = Specific gas constant = $\frac{R}{M}$; m = mass of gas

Different forms of Ideal gas of equation

(a) $PV = nRT$

(b) $P = \frac{d\ RT}{M}$ (d is density of gas)

(c) $PV = NKT$ (per molecule)

(d) $PV = m\mu T$

* $n = \frac{m}{M} = \frac{N}{N_0}$

* $M = m_0 N_0$

* $K = \frac{R}{N_0}$

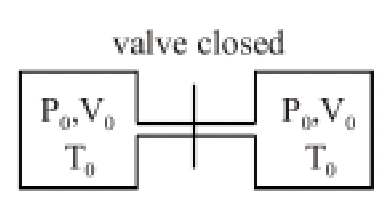
If mass of the gas is not constant then we can use

$$\frac{P_1V_1}{n_1T_1} = \frac{P_2V_2}{n_2T_2} = R \text{ (Universal gas constant)}$$

Illustration :

Two closed vessels of equal volume contain air at P_0 pressure, T_0 temperature and are connected through a narrow tube. If one of the vessels is now maintained at T_0 and other at T, what will be the pressure in the vessels?

Sol. *Lets say volume of each vessel be V_0.*
lets say initially n moles where present in each vessel



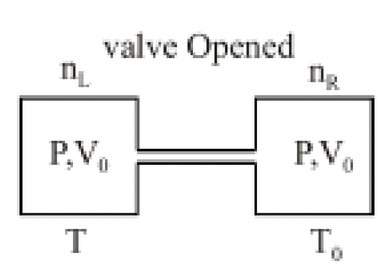
$$n = \frac{P_0V_0}{RT_0}$$

thus total no. of moles in the system $= \frac{2P_0V_0}{RT_0}$

Finally lets say n_L moles where present in left vessel and n_R moles where present in right vessel.

$$n_L = \frac{PV_0}{RT} \quad \& \quad n_R = \frac{PV_0}{RT_0}$$

here P is the pressure in vessel



$$n_L + n_R = 2n \quad \text{(no. of moles is constant)}$$

$$\frac{PV_0}{RT} + \frac{PV_0}{RT_0} = \frac{2P_0V_0}{RT_0}$$

$$P = \frac{2P_0}{T_0\left(\frac{1}{T} + \frac{1}{T_0}\right)}$$

Symbols and constants :

P = Pressure of gas ; V = Volume of gas ; T = Temperature of gas
n = no. of moles ; M = mol. wt. of gas ; m_0 = mass of each atom or molecule
N = total no. of molecule ; N_0 = Avogadro no.
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Different forms of Ideal gas of equation

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* $M = m_0 N_0$

* $K = \dfrac{R}{N_0}$

If mass of the gas is not constant then we can use

$$\frac{P_1V_1}{n_1T_1} = \frac{P_2V_2}{n_2T_2} = R \text{ (Universal gas constant)}$$

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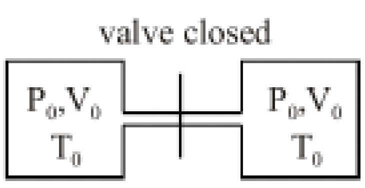


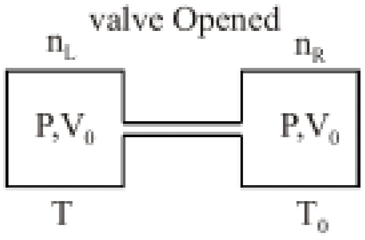
$$n = \frac{P_0V_0}{RT_0}$$

thus total no. of moles in the system $= \dfrac{2P_0V_0}{RT_0}$

Finally lets say n_L moles where present in left vessel and n_R moles where present in right vessel.

$$n_L = \frac{PV_0}{RT} \quad \& \quad n_R = \frac{PV_0}{RT_0}$$

here P is the pressure in vessel



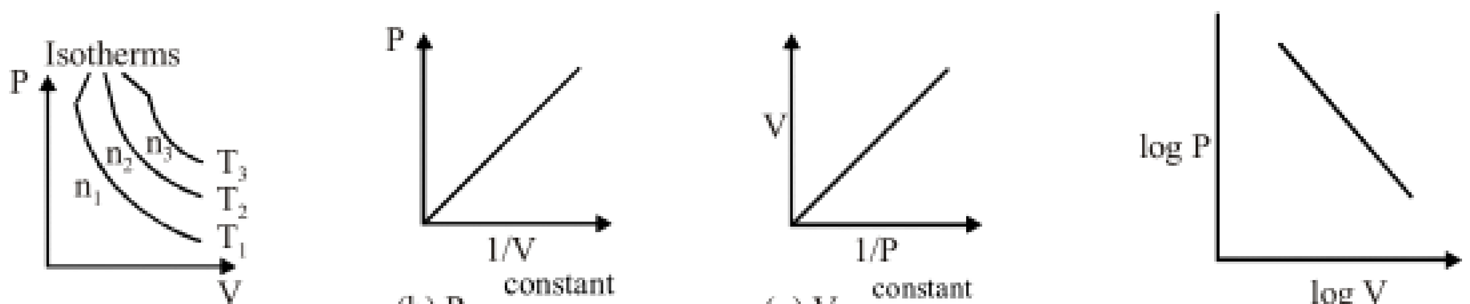
$$n_L + n_R = 2n \quad \text{(no. of moles is constant)}$$

$$\frac{PV_0}{RT} + \frac{PV_0}{RT_0} = \frac{2P_0V_0}{RT_0}$$

$$P = \frac{2P_0}{T_0\left(\dfrac{1}{T} + \dfrac{1}{T_0}\right)}$$

Boyle's Law : Relation Between Pressure and volume of a Gas

At constant temperature, volume of a fixed mass of a gas is inversely proportional to its pressure.



(a) PV = constant (b) $P = \frac{\text{constant}}{V}$ (c) $V = \frac{\text{constant}}{P}$ (d) log P = constant – logV

$$V \propto \frac{1}{P}$$

$$PV = \text{constant}$$

$$P_1 V_1 = P_2 V_2 \qquad \text{(T, n are constant)}$$

Charles' Law: Relation Between Temperature and volume of a Gas

When a gas is heated at constant pressure, its volume is a linear function of the temperature and can be expressed by the equation for a straight line

$$V = mt + C$$

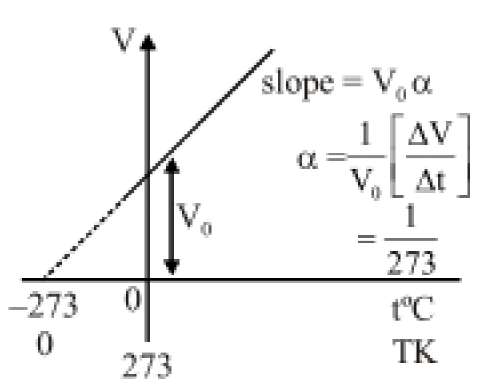
Where t is the temperature in °C and m and C are constants. The intercept on the vertical axis, C, is V_0 which is the volume at t = 0°C. The slope of the line is $m = \frac{\Delta V}{\Delta t}$

Thus

$$V_t = V_0 + \left(\frac{\Delta V}{\Delta t}\right)t \qquad \text{(n, p are constant)}$$

$$\frac{\Delta V}{\Delta t} = \text{increase in volume per degree}$$

$$\alpha = \frac{1}{V_0}\frac{\Delta V}{\Delta t} = \text{relative increase in volume per degree}$$

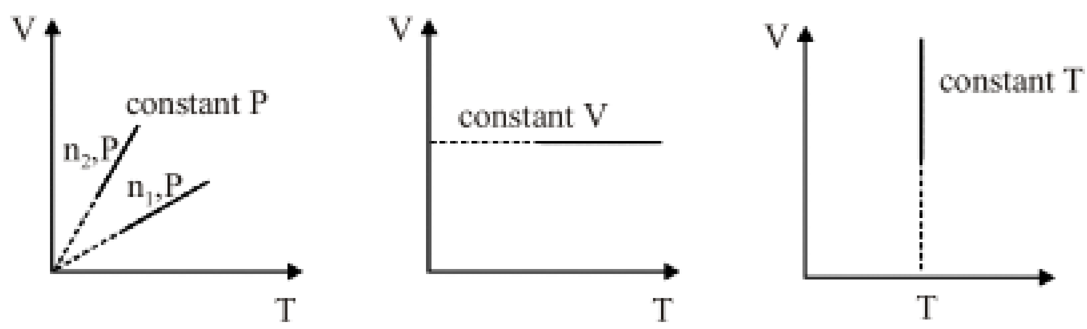


Thus $V_t = V_0 + V_0\,\alpha t = V_0\,(1+\alpha t)$

α is called coefficient of expansion. It is approximately $\frac{1}{273}$ for all the gases.

$$V = V_0\left(1 + \frac{t}{273}\right)$$

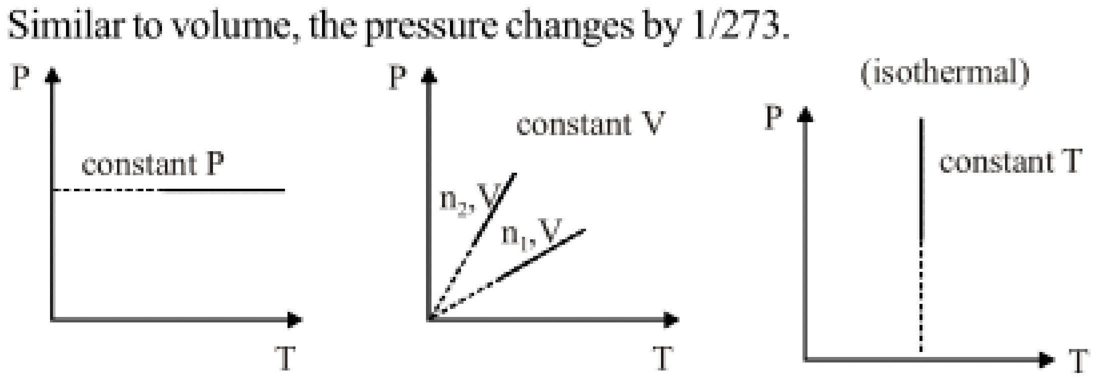
Thus, an increases the temperature of a fixed volume of a gas at constant pressure increases the volume by $\frac{1}{273}$ of the volume at 0ºC.



$$\frac{V}{V_0} = \frac{T}{T_0} \qquad \text{(n, P are constants)}$$

Gay-Lussac's Law

When the temperature of a gas is changed keeping the volume constant, the pressure of the gas changes. Similar to volume, the pressure changes by 1/273.



$$\beta = \frac{1}{P_0}\left[\frac{\Delta F}{\Delta t}\right] = \frac{1}{273}$$

$$P_t = P_0\left[1 + \frac{t}{273}\right] = P_0\left(\frac{273+t}{273}\right)$$

or $$\frac{P_t}{P_0} = \frac{T}{T_0} \qquad \text{(n, V are constants)}$$

Dalton's Law of partial Pressures

This law states that the total pressure exerted by a mixture of non-reacting gases is equal to the sum of the pressure which each component would exert if placed independently in the container.

$$P_{total} = P_1 + P_2 + P_3 + ... \qquad \text{(T, V are constants)}$$

$$P_{total} = \sum_i P_i$$

Where the symbol $\sum_i$ stands for the summation over all the components present in the mixture.

The partial pressure P_i of component i is defined as the pressure that the gas would exert if it were present along in the same volume and the same temperature.

$$p_1 = \frac{n_1RT}{V}$$

$$p_2 = \frac{n_2RT}{V}$$

The total pressure of the system can be written as

$$P_{total} = \sum_i p_i$$

$$= \frac{RT}{V}\sum_i n_i = n_{total}\frac{RT}{V}$$

and partial pressure of i^{th} component can be written as

$$P_i = P_{total}\frac{n_i}{n_{total}}$$

When $\frac{n_i}{n_{total}}$ is the mole fraction of the respective component

Equation of State :

The equation of state for an ideal gas :

$$PV = nRT$$

In this expression, known as the ideal gas law, R is a universal constant that is the same for all gases and T is the absolute temperature in kelvins. Experiments on numerous gases show that as the pressure approaches zero, the quantity PV/nT approaches the same value R for all gases. For the reason, R is called the universal gas constant. In SI units, in which pressure is expressed in pascals ($1\ Pa = 1\ N/m^2$) and volume in cubic meters, the product PV has units of newton meters, or joules, and R has the value

$$R = 8.315\ J/mol\ K$$

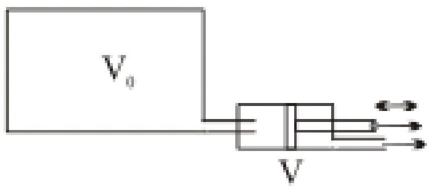
If the pressure is expressed in atmospheres and the volume in liters ($1L = 10^3 = 10^{-3}\ m^3$), then R has the value

$$R = 0.08214\ L\ atm/mol\ K$$

Using this value of R we find that the volume occupied by 1 mol of any gas at atmospheric pressure and at 0ºC (273 K) is 22.4 L.

Illustration :

A big container having ideal gas at pressure P and of volume V_0 is being evacuated by using vacuum pump of cylinder volume V. Find pressure in the big container after n strokes. Assume the whole process to be isothermal.



Sol. *For first stroke, pressure becomes P_1.*

$$PV_0 = P_1(V_0 + V)$$

$$P_1 = \frac{PV_0}{(V_0 + V)}$$

for 2^{nd} stroke, pressure becomes P_2

$$P_1V_0 = P_2(V_0 + V)$$

$$P_2 = \frac{P_1V_0}{(V_0 + V)}$$

$$P_2 = \frac{PV_0^2}{(V_0 + V)^2}$$

for n^{th} stroke, pressure becomes P_n

$$P_n = \frac{PV_0^n}{(V_0 + V)^n}$$

Illustration :

Consider the lung capacity to be 500 cm^3 and the pressure thereing to be equivalent of 761 mm of Hg; estimate the number of molecules per breath.

Sol. *From ideal gas equation,*

$$PV = NkT$$

Normal body temperature is 98.6°F = 37 °C = 310 K.

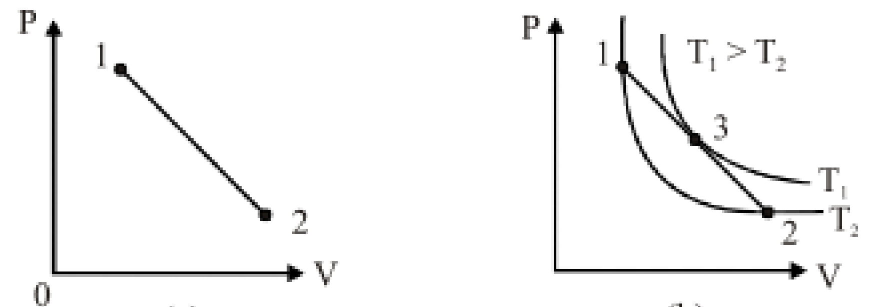
$$N = \frac{PV}{kT}$$

$$= \frac{(101.46)(500 \times 10^{-6})}{(1.3807 \times 10^{-23} \times 310)}$$

$$= 1.19 \times 10^{19} \text{ molecules}$$

Illustration :

A gas is enclosed in a cylinder with a freely movable piston. The load on the piston is gradually decreased. The temperature of the gas can be changed by placing the cylinder on hot and cold heat reservoirs. The figure shown P-V graph of such a cylinder. What inference can be drawn the nature of change in the temperature of the gas ?

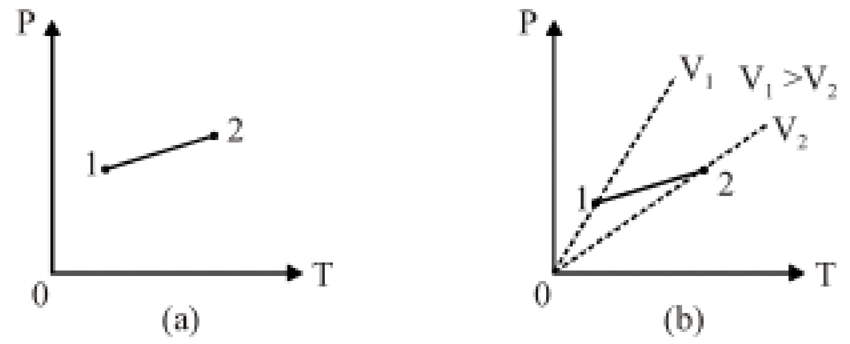


(a) (b)

Sol. *We draw two isotherms representing constant temperatures T_1 and T_2, such that T_2 passes through initial and final points 1 and 2, and T_1 passes through certain intermediate point 3. The curve closer to origin represents lower temperature. Hence the gas is heated in the section 1 – 3 and cooled in the section1–2.*

Illustration :

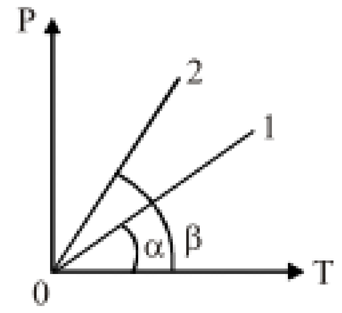
Figure (a) shows P-T curve of a ideal gas during a process. Does compression or expansion takes place when the gas in heated ?



Sol. *We draw constant volume lines (isochores) through the initial and final points 1 and 2 (see example 71). The volume V_2 is gmeater than V_1. Hence during heating of the gas expansion took place.*

Illustration :

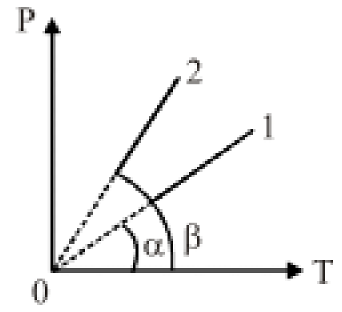
A certain mass of a gas was heated in a constant volume vessel; its P-T curve 3 is 1; similarly another mass of the gas was heated in the same vessel ; its P-T curve is 2. If tan $\beta = 2\tan a$, *what is the ratio of masses of gas in the two experiments ?*



Sol. *From ideal gas equation,*

$$P = \left(\frac{nR}{V}\right)T = \left(\frac{mR}{MV}\right)T$$

The constant volume curves are straight lines with slope mR/MV. The slope of lines is proportional to the mass of the gas.



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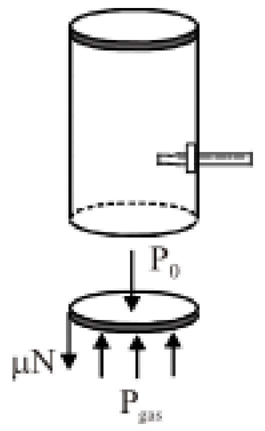
The constant volume curves are straight line with slope mR/MV. The slope of line is proportional ot the mass of the gas.

$$\frac{(slope)_1}{(slope)_2} = \frac{m_1}{m_2} = \frac{\tan\alpha}{\tan\beta} = \frac{1}{2}$$

Hence the mass m_2 is twice of m_1

Illustration :

Figure shown a cylindrical tube of radius r and length l, fitted with a cork. The friction coefficient between the cork and the tube is μ. The tube contains an ideal gas at temperature T, and atmospheric pressure P_0. The tube is slowly heated ; the cork pipe out when temperature is doubled. What is normal force per unit length exerted by the cork on the periphery of tube ? Assume uniform temperature throughout gas any instant.



Sol. *Since volume of the gas is constant,*

$$\frac{P_i}{T_i} = \frac{P_f}{T_f}$$

$$P_f = P\left(\frac{T_f}{T_i}\right) = 2P_i = 2P_0$$

The forces acting on the cork are shown in the figure in equilibrium.

$$P_0 \times A + \mu N = 2P_0 A$$

$$N = \frac{P_0 \mathrm{A}}{\mu}$$

N is the total normal force exerted by the tube on the cork; hence contact force unit length is

$$\frac{dN}{dl} = \frac{N}{2\pi r} = \frac{P_0 A}{2\pi \mu r}$$

Assuption of kinetic theory of gases

1. All gases are made of molecule moving randomly in all directions.

2. The size of a molecule is much smaller than the average separation between the molecules.

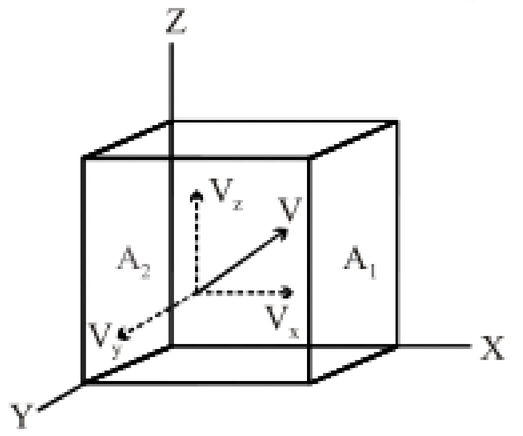
3. The molecules exert no force on each other or on the walls of the container except during collision.

4. All collisions between two molecules or between a molecule and a wall are perfectly elastic. Also, the time spent during a collision is negligibly small.

5. The molecules obey Newton's laws of motion.

6. When a gas is left for sufficient time, it comes to a steady state. The density and the distribution of position, direction and time. This assumption may be justified if the number of molecules is very large.

Calculation of the pressure of an ideal gas



Consider an ideal gas enclosed in a cubical vessel of edge L. Take a corner of the vessel as the origin O and the X-, Y-, Z- axes along the edges. Let A_1 and A_2 be the parallel faces perpendicular to the X-axis. Consider a molecule moving with velocity $\vec{\upsilon}$. The components of the velocity along the axes are υ_x, υ_y and υ_z. When the molecule collides with the force A_1, the x-component of the velocity is reversed whereas the y- and the z-components remain unchanged. This follows from our assumption that the collisions of the molecules with the wall are perfectly elastic. The change in momentum of the molecule is

$$\Delta p = (-m\upsilon_x) - (m\upsilon_x) = -2m\upsilon_x.$$

As the momentum remains conserved in a collision, the change in momentum of the wall is

$$\Delta p' = 2m\upsilon_x \qquad \text{...(i)}$$

After rebound, this molecule travels towards A_2 with the x-component of velocity equal to $-\upsilon_x$. Any collision of the molecule with any other face (except for A_2) does not change the value of υ_x. So, it travels between A_1 and A_2 with a constant x-component of velocity which is equal to υ_x. Note that we can neglect any collision with the other molecules in view of the last assumption discussed in the previous section.

The distance travelled parallel to the x-direction between A_1 and A_2 = L. Thus, the time taken by the molecule to go from A_1 to $A_2 = L/\upsilon_x$. The molecule rebounds from A_2, travels towards a_1 and collides with it after another time interval L/υ_x. Thus, the time between two consecutive collisions of this molecule with A_1 is $\Delta t = 2\,L/\upsilon_x$. The number of collisions of this molecule with A_1 in unit time is

$$n = \frac{1}{\Delta t} = \frac{\upsilon_x}{2L} \qquad \text{...(ii)}$$

The momentum imparted per unit time to the wall by this molecule is, from (i) and (ii),

The momentum imparted per unit time to the wall by this molecule is from (i) and (ii),

$$\Delta F = n\Delta p'$$

$$= \frac{\upsilon_x}{2L} \times 2m\upsilon_x = \frac{m}{L}\upsilon_x^2$$

This is also the force exerted on the wall A_1 due to this molecule. The total force on the wall A_1 due to all the molecules is

$$F = \sum \frac{m}{L}\upsilon_x^2$$

$$= \frac{m}{L}\Sigma\upsilon_x^2 \qquad \text{...(iii)}$$

As all directions are equivalent, we have

$$\Sigma\upsilon_x^2 = \Sigma\upsilon_y^2 = \Sigma\upsilon_z^2$$

$$= \frac{1}{3}\Sigma(\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2)$$

$$= \frac{1}{3}\Sigma\upsilon^2$$

Thus, from (iii), $F = \frac{1}{3}\frac{m}{L}\Sigma\upsilon^2$.

If N is the total number of molecules in the sample, we can write

$$F = \frac{1}{3}\frac{mN}{L}\frac{\Sigma\upsilon^2}{N}$$

The pressure is force per unit area so that

$$p = \frac{F}{L^2}$$

$$= \frac{1}{3}\frac{mN}{L^3}\frac{\Sigma\upsilon^2}{N}$$

$$= \frac{1}{3}\frac{M}{L^3}\frac{\Sigma\upsilon^2}{N} = \frac{1}{3}\rho\frac{\Sigma\upsilon^2}{N}$$

where M is the total mass of the gas taken and ρ is its density. Also $\Sigma\upsilon^2/N$ is the average of the speeds squared. It is written as u^2 and is called mean square speed. Thus, the pressure is

$$p = \frac{1}{3}\rho\upsilon^2 \qquad ...(1)$$

or $$pV = \frac{1}{3}M\upsilon^2 \qquad ...(2)$$

or, $$pV = \frac{1}{3}Nm\upsilon^2 \qquad ...(3)$$

RMS Speed

The square root of mean square speed is called root-square speed or rms speed. It is denoted by the symbol υ_{rms} Thus,

$$\upsilon_{rms} = \sqrt{\Sigma\upsilon^2 / N}$$

or, $\upsilon^2 = (\upsilon_{rms})^2$.

Equation (1) may be written as

$$p = \frac{1}{3}\rho\upsilon_{rms}^2$$

so that $$\upsilon_{rms} = \sqrt{\frac{3p}{\rho}} = \sqrt{\frac{3pV}{M}} = \sqrt{\frac{3RT}{M}} \qquad (M \rightarrow \text{molecular weight})$$

Avg. speed $\upsilon_{avg} = \sqrt{\frac{8KT}{\pi m_0}} = \sqrt{\frac{8RT}{\pi M}}$

Most probable speed $\upsilon_p = \sqrt{\frac{2KT}{m_0}} = \sqrt{\frac{2KT}{M}}$

Translational kinetic energy of gas :

Kinetic energy of any molecule is k_1

$$k_1 = \frac{1}{2} m_0 V_1^2$$

Total K.E. of all molecular is k

$$k = \Sigma \frac{1}{2} m_0 V^2$$

$$= \frac{1}{2} m_0 \Sigma V^2 \qquad = \frac{1}{2} m_0 N \left(\frac{\Sigma V^2}{N}\right)$$

$$= \frac{m}{2} V_{rms}^2 \qquad = \frac{m}{2} \times \frac{3RT}{M}$$

$$k = \frac{3nRT}{2} \qquad \left(\frac{m}{M} = n\right)$$

$$k = \frac{3\ PV}{2}$$

$$\text{K.E. / volume} = \frac{3}{2} P$$

Illustration :

Calculate the rms speed of nitrogen at STP (pressure = 1 atm and temperature = 0°C). The density of nitrogen in these conditions is 1.25 kg/m³.

Sol. *At STP, the pressure is 1.0×10^5 N/m². The rms speed is*

$$\upsilon_{rms} = \sqrt{\frac{3p}{\rho}}$$

$$= \sqrt{\frac{3 \times 10^5\ N/m^2}{1.25\ kg/m^3}}$$

$$= 490\ m/s$$

Translational Kinetic Energy of a Gas

The total translational kinetic energy of all the molecules of the gas is

$$K = \Sigma \frac{1}{2} m\upsilon^2 = \frac{1}{2} mN \frac{\Sigma \upsilon^2}{N} = \frac{1}{2} M\upsilon_{rms}^2 \qquad ...(4)$$

The average kinetic energy of a molecule is

$$K/N = \frac{1}{2}\frac{M}{N}\upsilon_{rms}^2 = \frac{1}{2} m\upsilon_{rms}^2$$

From equation (2)

$$pV = \frac{2}{3}\cdot\frac{1}{2} M\upsilon_{rms}^2$$

$$pV = \frac{2}{3} K$$

$$K = \frac{3}{2} pV$$

Illustration :

Assume that the temperature remains essentially constant in the upper part of the atmosphere. Obtain an expression for the variation pressure in the upper atmosphere with height. The mean molecular weight of air is M.

Sol. *Suppose the pressure at height h is p and the at h + dh is p + dp. Then*

$$dp = -\rho g\, dh. \qquad ...(i)$$

Now considering any small volume ΔV of air mass, Δm,

$$p\Delta V = nRT = \frac{\Delta m}{M} RT$$

or, $$p = \frac{\Delta m}{\Delta V}\frac{RT}{M} = \frac{\rho RT}{M}$$

or, $$\rho = \frac{M}{RT}$$

Putting in (i)

$$dp = -\frac{M}{RT} pg dh$$

or, $$\int_{p_0}^{p} \frac{dp}{p} = \int_0^h -\frac{M}{RT} g dh$$

or, $$In \frac{p}{p_0} = -\frac{Mgh}{RT}$$

Where p_0 is the pressure at h = 0

Thus, $p = p_0\, e^{-\frac{Mgh}{RT}}$.

Illustration:

A barometer tube contains a mixture of air and saturated water vapour in the space above the mercury column. It reads 70 cm when the actual atmospheric pressure is 76 cm of mercury. The saturation vapour pressure at room temperature is 1.0 cm of mercury. The tube is now lowered in the reservoir till space above the mercury column is reduced to half its original volume. Find the reading of the barometer. Assume that the temperature remains constant.

Sol. *The pressure due to the air+vapour is 76cm – 70 cm = 6 m of mercury. The vapour is saturated and the pressure due to it is 1 cm of mercury. The pressure due to the air is therefore, 5 cm of mercury.*

As the tube is lowered and the volume above the mercury is decreased, some of the vapour will condense. The remaining vapour will again exert a pressure of the volume is halved. Thus, $p_{air} = 2\times5$ *cm = 10 cm of mercury. The pressure due to the air + vapour reading is 76 cm – 11cm = 65 cm.*

Internal Energy

Degree of freedom :

No of ways in which molecule can passes energy is known as degree of freedom.

S. N.	*Gas*	*Translatory*	*Rotation*	*Vibration*	*Total*
1.	Monoatomic	3	0	0	3
2.	Diatomic	3	2	0	5

Law of equipartition of energy :

Statement : For an ideal gas average energy associated with its any molecule for each degree of freedom is $\frac{KT}{2}$. (Where temperature T in kelvin)

Let f be the degree of freedom for a gas. Average energy associated with its any molecule = $\frac{f\,KT}{2}$

Total kinetic energy of a gas = $\frac{N\,f\,KT}{2}$

Kinetic energy of one mole of gas = $\frac{f\,RT}{2}$

Value of internal energy of a gas

Internal energy of gas should be sum of K.E. and P.E. of its constitute molecules.
But for ideal gas we has assumed P.E. = 0 (Since force of interaction between molecules is zero)
Thus, I.E. of a gas = K.E. of molecules
If f is the degree of freedom of a gas molecules than total K.E. [Trans + Rotation of each molecule]

for one mole $\quad U = \frac{f\,RT}{2}$

for n mole of gas and having f degree of freedom are at temperature T_1 kelvin and heated to temperature T_2 then

at T_1 kelvin $\quad U_1 = \dfrac{n f R T_1}{2}$

at T_2 kelvin $\quad U_2 = \dfrac{n f R T_2}{2}$

change in I.E. $\quad \Delta U = \dfrac{n f R}{2} (T_2 - T_1)$

Change in internal energy of ideal gas is a function of temperature and temperature only. It does not depend on how we carry out this change in temperature.

$\dfrac{fR}{2}$ is represented as C_v

$$C_v = \frac{fR}{2}$$

$$U = nC_vT$$

(i) For monoatomic gas (f = 3)

$$U = \frac{n\,3RT}{2} \quad \& \quad \Delta U = \frac{n3R\Delta T}{2}$$

or $\quad C_v = \dfrac{3R}{2}$

(ii) For rigid diatomic molecule (f = 5)

$$U = \frac{n\,5RT}{2} \quad \& \quad \Delta U = \frac{n\,5R\,\Delta T}{2}$$

$$C_v = \frac{5R}{2}$$

Illustration

The temperature of an ideal gas consisting of rigid diatomic molecules is T = 300K. Calculate the angular root mean square velocity of a rotating molecules if its moment of inertial is equal to

$I = 2.1 \times 10^{-39}$ *g.cm*^{2}

Sol. *K.E. associated with rotation* $= \dfrac{1}{2} I\omega^2$

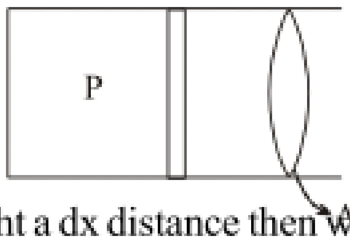
Degree of freedom associated
with rotation = 2

$$\frac{1}{2} I\omega^2 = 2 \times \frac{KT}{2}$$

$$\omega = \sqrt{\frac{2KT}{I}}$$

$$= 6.3 \times 10^{12} \text{ rad } S^{-1}$$

Calculation of work done by gas :



If the piston moves towards right a dx distance then work done by this force is dW.

$$dW = F \,.\, dx$$

$$= P\,(A \,.\, dx)$$

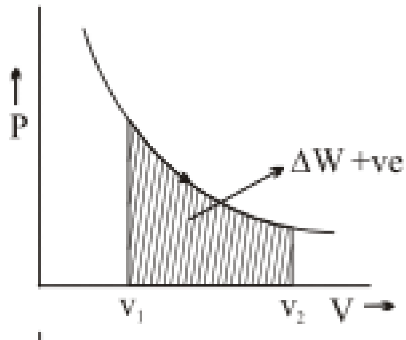
$$dW = PdV$$

$$dW = \int_{V_1}^{V_2} PdV$$

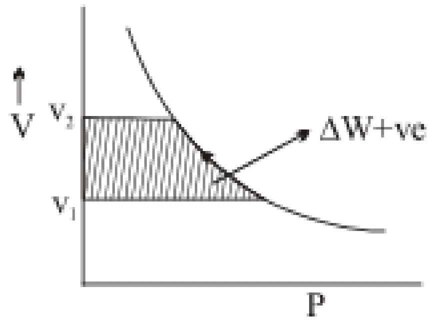
Graphical interpretation :

From above integral it can be understood that area enclosed by PV-curve and V-axis represents the work done by gas.

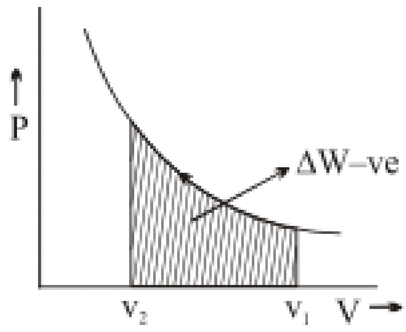
(i)



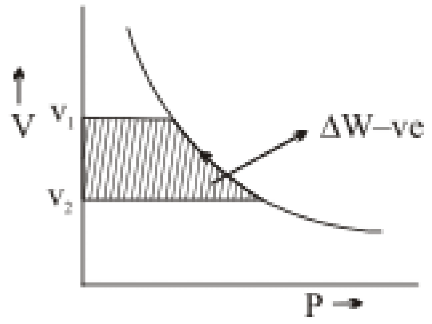
(ii)



(iii)



(iv)



When volume decrease ΔW is –ve

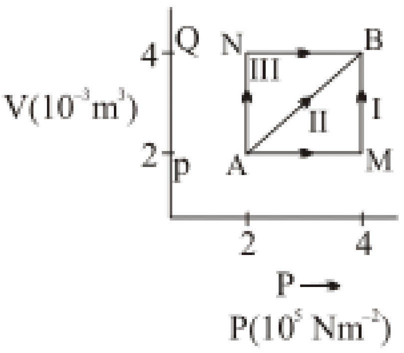
Illustration :

From the given curves find out the work done by the ideal for going from state A to state B. For all the three processes.

Sol. $W_1 = PAMBNQP = 800\ J$

$W_2 = PABNQP = 600\ J$

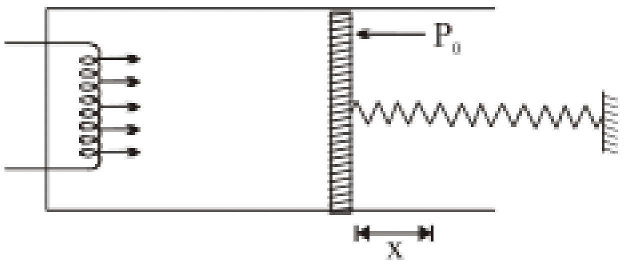
$W_3 = PANQP = 400\ J$



Indirect technique(for quasistatic process) :

Work done by gas + work done by all external agents = 0

⇒ Work done by gas = – (work done by all the external agents)



Work done by gas + WD by spring + W.D. by atm press = 0

W.D. by gas = – (W. D. by atm press + W.D. by spring)

$$= -[(-P_0Ax) + (-1/2\ kx^2 - 0)]$$

$$= P_0Ax + \frac{1}{2}kx^2$$

Heat & Thermodynamics

Definition :

Heat is defined as the amount of energy transfer from one body to another body or from one part of a body to other part of the body by virtue of temperature difference.

Important points :

(a) When two bodies at different temperature are brought in contact the hotter body cools down and the cooler body warms up due to the transfer of heat from hotter body to cooler body.

(b) "Heat is energy in transit due to temperature different between the body" when heat is transferred from one body to another body then after the transfer this heat becomes part of internal energy or is used for doing work.

Unit & Dimension :

SI unit ⟶ Joule

Dimension ⟶ ML^2T^{-2}

Specific heat of gas :

Types of specific heat of gas :

(a) Molar specific heat : Unit mass is taken as one mole

Units ⟶ J / mole – k

(b) Gram specific heat : Unit mass is taken as 1kg or 1 gm.

Units ⟶ J / kg-k or J / gm-k

$$C_{gram} = \frac{C_{molar}}{M}$$

Definition :

Amount of heat required to raise temperature of one unit amount (mass or mole) by one unit temperature through a given process is known as specific heat of that process.

$$C_{process} = \left(\frac{dQ}{n\,\Delta T}\right)_{process}$$

Note : Thus any gas can have infinite specific heats depending on the infinite different processes.

$\Delta Q = \int_{T_1}^{T_2} n\,C\,dT$ where C is the sp. heat for the process through which temperature from T_1 to T_2.

Note : C remains inside the integral because it may be a variable i.e. C may be a function of temperature (directly or indirectly).

Specific heat at constant volume :

Represented as C_v $\qquad C_v = \left(\frac{dQ}{n\,dT}\right)_{const.\,volume}$

Its experimental value is found to be very close to $\frac{f\,R}{2}$

thus $\quad C_v = \frac{f\,R}{2}$

Specific heat at constant pressure :

Represented as C_p

$$C_p = \left(\frac{dQ}{n\,dT}\right)_{const.\ pressure}$$

its experimental value is found to be almost equal to $\frac{(f+2)\,R}{2}$

$$C_p = \frac{(f+2)\,R}{2}$$

Adiabatic exponent (γ) :

$$\gamma = \frac{C_p}{C_v}$$

$\Rightarrow \quad \gamma = 1 + \frac{2}{f} \qquad \Rightarrow \quad C_v = \frac{R}{\gamma - 1} \qquad \Rightarrow \quad C_p = \frac{\gamma R}{\gamma - 1}$

First law of Thermodynamics

Statement : $dQ = dU + dW$ (For zero heat loss)

dQ = heat supplied to the system

dU = change in internal energy

dW = work done by the system

* This law is a form of energy conservation.

The first Law of Thermodynamics (derivation)

Let us introduce certain amount of heat energy dQ into a gas confined inside a cylinder fit with a piston. The gas can either:

(i) store energy as random KE of its molecules (dU), or

(ii) use the energy to do work (dW) in the environment (such as raising a weight on the piston).

Thus, the first law of thermodynamics can be written : $dQ = dU + dW$

Note : In solving thermodynamics problems, always take gas as system.

Sign convention :

(i) Whenever heat is added to the system sign is +ve

(ii) Whenever system rejects the heat sign is -ve

Work (dW) :

(i) ($V\uparrow$) sign is +ve (ii) sign is -ve ($V\downarrow$)

Internal energy (dU) :

(i) When temperature increases sign is +ve

(ii) When temperature decreases sign is -ve

Application of first law of thermodynamics :

Isochoric process :

A process that takes place at constant volume is called isochoric process or isovolumetric process. (Gas will follow Gaylusac's Law)

(i) Conditions $\longrightarrow$ V is constant

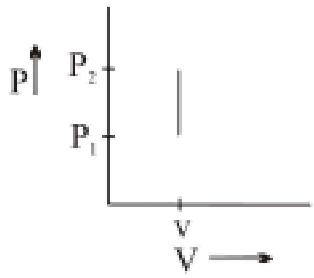
(ii) Process equation $\longrightarrow \frac{P}{T} = \text{constant}$

(iii) If gas is taken from state A having pressure P_1 temperaure T_1 and volume V to state B having Pressure P_2 temperaure T_2 and volume V then

$$\underset{(P_1, V, T_1)}{A} \xrightarrow{\qquad} \underset{(P_2, V, T_2)}{B}$$

$$\frac{P_1}{T_1} = \frac{P_2}{T_2}$$ [While PV = nRT will relate variables in same state]

(iv) P-V curve



(v) $dW = 0$ since $dV = 0$

(vi) $\Delta U = \dfrac{n f R}{2}(T_2 - T_1)$

(vii) $\Delta Q = nC_v\,(T_2 - T_1)$

(viii) Applying first law of thermodynamics (F.L.T.)

$$O + \Delta U = \Delta Q$$

$$\Rightarrow \quad n\left(\frac{fR}{2}\right)(T_2 - T_1) = nC_v\,(T_2 - T_1)$$

$$\Rightarrow \quad C_v = \frac{fR}{2}$$

(ix) Bulk modulus $= -\dfrac{dP}{(dV/V)} = -\infty$

(x) sp. heat $= C_v$

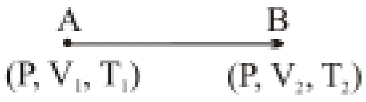
Isobaric process :

A process that takes place at constant pressure is called isobaric process. (Gas will follow Charle's Law)

(i) Condition : P is constant

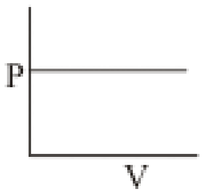
(ii) Process equation $\longrightarrow \dfrac{V}{T}$ is constant

(iii) If gas is taken from state A having pressure P temperaure T_1 and volume V_1 to state B having Pressure P temperaure T_2 and volume V_2 then



$$\frac{V_1}{T_1} = \frac{V_2}{T_2} \qquad\qquad \begin{aligned} PV_1 &= nRT_1 \\ PV_2 &= nRT_2 \end{aligned}$$

(iv) P – V curve



(v) $\Delta W = \int_{v_1}^{v_2} P\, dv$

$= P(V_2 - V_1)$
$= nR(T_2 - T_1)$

(vi) $dU = nC_v\, dT$
$\Delta U = nC_v(T_2 - T_1)$

(vii) $dQ = nC_p dT$
$\Delta Q = nC_p(T_2 - T_1)$

(viii) Applying FLT

$$nC_p(T_2 - T_1) = nC_v(T_2 - T_1) + nR(T_2 - T_1)$$

$$\Rightarrow \quad C_p - C_v = R$$

(known as meyer's relation)

Characteristic gas constant :

Gas constant expressed for particular gas in terms of mass is known as characteristic gas constant of that particular gas.

$$R_{\text{characteristic}} = \frac{R\,(\text{universal gas const.})}{M\,(\text{mol. wt. of gas})}$$

Units J / gm-k or J / kg-k

(ix) Bulk modulus

$$B = \frac{-dp}{(dV/V)} = 0$$

(x) Sp. heat capacity C_p

Isothermal process :

A process that takes place at constant temperature is called isothermal process. (Gas will follow Boyle's Law)

(i) Condition $\longrightarrow$ Temp. is constant

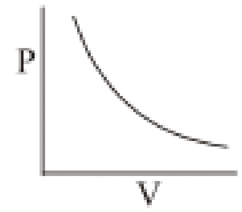
(ii) Process equation $\longrightarrow$ PV = const.

(iii) If gas is taken from state A having pressure P_1 volume V_1 and temperaure T to state B having Pressure P_2 volume V_2 and temperaure T then

$$\underset{(P_1, V_1, T)}{\overset{A}{\bullet}} \longrightarrow \underset{(P_2, V_2, T)}{\overset{B}{}}$$

$$P_1V_1 = P_2V_2 \qquad\qquad P_1V_1 = nRT$$
$$P_2V_2 = nRT$$

(iv) P-V curve



$$\text{slope} = \frac{dP}{dV} = \frac{-P}{V}$$

(v) $\Delta W = \int_{V_1}^{V_2} PdV = k\int_{V_1}^{V_2} \frac{dV}{V}$

$$= PV\, ln\, \frac{V_2}{V_1}$$

$$= P_1V_1\, ln\, \frac{V_2}{V_1} = P_1V_2\, ln\, \frac{V_2}{V_1}$$

$$= nRT\ ln\, \frac{V_2}{V_1} = nRT\, ln\, \frac{P_1}{P_2}$$

(vi) $dU = 0$

(vii) Using FLT

$dQ = dW$

$\Delta Q = nRT\, ln\, \frac{V_2}{V_1}$

(viii) $B = -\frac{dP}{(dV/V)} = P$

(ix) Specific heat $= C = \left(\frac{dQ}{n\,dT}\right) = \infty$

By adiabatic process :

Adiabatic Process (Derivation): For an adiabatic process ($\Delta Q = 0$)

$$0 = C_v + \frac{P}{n}\frac{dV}{dT} = \frac{R}{\gamma-1} + \frac{RT}{V}\ \frac{dV}{dT} \qquad \Rightarrow \qquad (\gamma-1)\int\frac{dV}{V} + \int\frac{dT}{T} = 0$$

$$(\gamma-1)\, ln\, V + ln\, T = ln\, c' \ \therefore\ TV^{\gamma-1} = c'\ ; \text{Also, } T = \frac{PV}{nR}\ \therefore\ PV' = c'$$

$$P\left(\frac{T}{P}\right)^{\gamma} = c'\ \therefore\ \frac{T^{\gamma}}{P^{\gamma-1}} = c' \qquad \therefore\ P = T^{(\gamma/\gamma-1)}$$

In this process net heat supplied to the gas is zero

(i) Condition $dQ = 0$

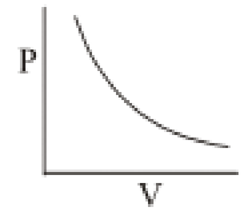
(ii) Process equation $PV^{\gamma} = \text{const.}$

(iii) If gas is taken from state A having pressure P_1 volume V_1 and temperaure T_1 to state B having Pressure P_2 volume V_2 and temperaure T_2 then

$$\underset{(P_1, V_1, T_1)}{A} \longrightarrow \underset{(P_2, V_2, T_2)}{B}$$

$$P_1V_1^{\gamma} = P_2V_2^{\gamma} \qquad P_1V_1 = nRT_1 \qquad P_2V_2 = nRT_2$$

(iv) P-V curve



$$\text{slope} = \frac{dP}{dV} = \frac{-\gamma P}{V}$$

(v) $\Delta W = \int_{V_1}^{V_2} PdV$ (since $Pv^{\gamma} = k$)

$$= k\int_{v_1}^{v_2} V^{-\gamma}\, dV \qquad = \left.\frac{kv^{-\gamma+1}}{(-\gamma+1)}\right|_{v_1}^{v_2} = \frac{P_2V_2 - P_1V_1}{1-\gamma}$$

(vi) $\Delta V = nC_v(T_2 - T_1)$

$$= \frac{nR(T_2 - T_1)}{\gamma - 1} = \frac{P_2V_2 - P_1V_1}{\gamma - 1}$$

(vii) Applying FLT we can see that

$$0 = \Delta W + \Delta U \Rightarrow \Delta U = -\Delta W$$

(viii) $B = -\dfrac{dP}{(dV/V)} = \gamma P$

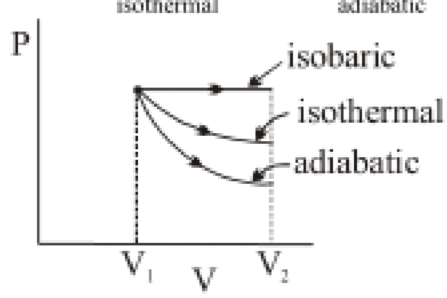
(ix) Sp. heat

$$C = \frac{dQ}{n\,dT} = 0$$

Results :

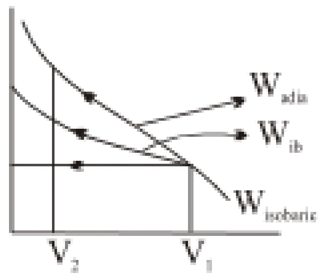
(i) For expansion from same initial vol. & pressure to same final volume $\rightarrow$

$$W_{\text{isobaric}} > W_{\text{isothermal}} > W_{\text{adiabatic}}$$

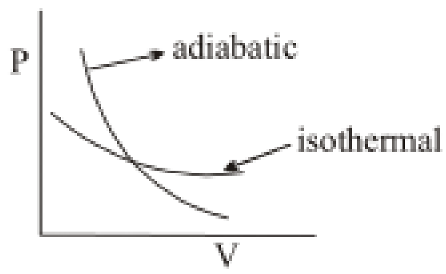


As the area under the P-V graph is largest for isobaric hence its work done is greater than isothermal and in similar manner we can say that work done in isothermal process will be greater than adiabatic process.

(ii) For compression from same initial state to same final volume



(iii) At pt. of interaction (slope of isothermal) x γ = slope of adiabatic.



Polytropic process :

It is a process in which molar heat capacity C = constant

$$C = C_v + \frac{P}{n}\frac{dV}{dT'}; \qquad PdV + VdP = n\,RdT$$

$$\Rightarrow \quad C = C_v + \frac{PRdV}{PdV + VdP}$$

$$\Rightarrow \quad \frac{R}{C - C_v} = 1 + \frac{VdP}{PdV}$$

$$\Rightarrow \quad \frac{R - (C - C_v)}{C - C_v} = \frac{VdP}{PdV} = \frac{C - C_p}{C - C_v} = K$$

$$\Rightarrow \quad \int \frac{dP}{P} + K\int \frac{dV}{V} = 0$$

$\Rightarrow$ $ln\,P + K\,ln\,V = \text{const}$

$\therefore$ $PV^K = \text{const.}$ K : Polytropic constant

Isochoric :	$K = \infty$	$[P^{1/K}V = \text{const.}]$	$C = C_v$
Isocbaric :	$K = 0$		$C = C_p$
Isothermal :	$K = 1$	$K = \frac{C - C_p}{C - C_v} = \frac{1 - C_p/C}{1 - C_v/C}$	$C = \infty$
Adiabatic :	$K = \gamma$	[Put C = o in the equation of K]	$C = 0$

(i) Characteristic equation process

$$PV^x = \text{const.}$$

(ii) $$\Delta W = \int_{V_1}^{V_2} PdV$$

$$= \int_{V_1}^{V_2} \frac{k}{V^x} dV \qquad = k\left(\frac{V_2^{-x+1} - V_1^{-x+1}}{-x+1}\right)$$

$$= \frac{\frac{k}{V^x}V_2 - \frac{k}{V_1^x}V_1}{1-x} 1 \qquad = \frac{P_2V_2 - P_1V_1}{1-x}$$

$$= \frac{nR(T_2 - T_1)}{1-x}$$

(iii) $$\Delta U = \frac{nR(T_2 - T_1)}{(\gamma - 1)}$$

(iv) From FLT

$$dQ = dW + dU$$

$$\Delta Q = nR(T_2 - T_1)\left[\frac{1}{1-x} + \frac{1}{\gamma - 1}\right]$$

(v) Sp. heat

$$\Delta Q = nc(T_2 - T_1)$$

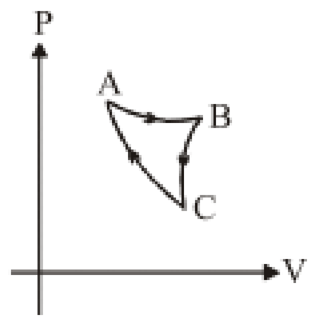
$$C = \frac{R}{1-x} + \frac{R}{\gamma - 1}$$

Illustration

An ideal gas expands isothermally along AB and does 700 J of work.

(a) How much heat does the gas exchange along AB.

(b) The gas then expands adiabatically along BC and does 400 J of work. When the gas returns to A along CA, it exhausts 100 J of heat to its surroudings . How much work is done on the gas along this path.



Sol. *(a) AB is an isothermal process. Hence,*

$$\Delta U_{AB} = 0$$

and $Q_{AB} = W_{AB} = 700\ J$

(b) BC is an adiabatic process. Hence,

$$Q_{BC} = 0$$

$$W_{BC} = 400\ J \qquad \textit{(given)}$$

$\therefore \quad \Delta U_{BC} = -W_{BC} = -400\ J$

ABC is a cyclic process and internal energy is a state function. Therefore,

$$(\Delta U)_{\text{whole cycle}} = 0 = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA}$$

and from first law of thermodynamics,

$$Q_{AB} + Q_{BC} + Q_{CA} = W_{AB} + W_{BC} + \Delta W_{CA}$$

Substituting the values,

$$700 + 0 - 100 = 700 + 400 + \Delta W_{CA}$$

$$\therefore \quad \Delta W_{CA} = -500\,J$$

Negative sign implies that work is done on the gas.

Illustration

An ideal monoatomic gas at 300 K expands adiabatically to twice its volume. What is the final temperature.

Sol. *For an ideal monoatomic gas,*

$$\gamma = \frac{5}{3}$$

In an adiabatic process,

$$TV^{\gamma-1} = \text{constant}$$

$$\therefore \quad T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

$$\text{or} \quad T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1}$$

$$= (300)\left(\frac{1}{2}\right)^{\frac{5}{3}-1}$$

$$= 189\,K$$

Illustration :

1 mole of an ideal monoatomic gas is expanded till the temperature of the gas is doubled under the process V^2T = constant. The initial temperature of the gas is 400 K. In terms of R, find total work done in the process.

Sol. *Given $T_i = 400$ K and $T_f = 2T_i = 800$ K*

$$\Delta T = T_f - T_i = 400\,K$$

$$\Delta U = nC_V\,\Delta T$$

$$= (1)\left(\frac{3}{2}R\right)(400) = 600\,R$$

The given process is V^2T = constant

Putting $T = \dfrac{PV}{R}$ *we get*

$$PV^3 = \text{constant}$$

Comparing this equation with PV^x = constant we have $x = 3$ and molar heat capacity is

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-x} = \frac{R}{\frac{5}{3} - 1} + \frac{R}{1-3}$$

$$\frac{3}{2}R - \frac{R}{2}$$

$\therefore \quad \Delta Q = nC\Delta T = (1)(R)(400) = 400\,R$

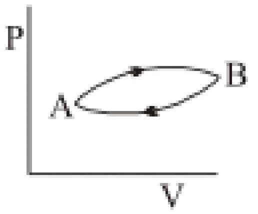
Now, $\Delta W = \Delta Q - \Delta U = -200\,R$

Cyclic process :

It is combination of two or more than two processes in which initial and final state of the system is same

(i) $dU = 0$

Total change in I.E. is zero because initial & final temp. is same



(ii) Applying FLT

$dQ = dW$

(iii) Net work done in a cyclic process is area bounded by process curve on PV diagram.

(iv) Efficiency of cyclic process :

$$\eta = \frac{\text{Total work done (+ve and} - \text{ve both)}}{\text{Total heat supplied (only + ve heat) to the system}} \times 100$$

Carnot Heat Engine :

(A) The Engine :

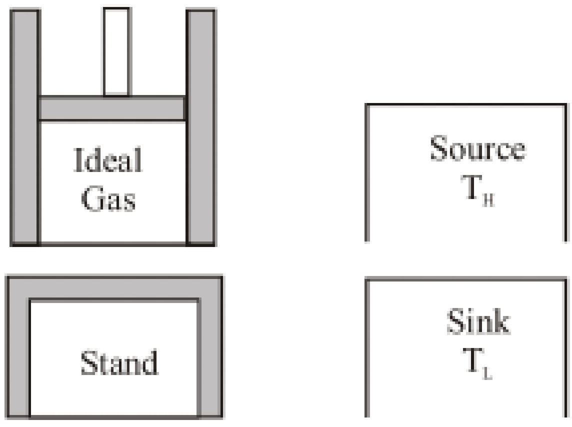
As according to the second law of thermodynamics whole of heat can never be converted into work, the question then arises under what conditions the conversion of heat to work is these questions Carnot developed an ideal heat engine which is supposed to consist of the following four components :

(1) A cylinder with perfectly non-conducting walls and a perfectly conducting base containing a perfect gas as working substance and fitted with a non-conducting frictionaless piston'

(2) A source of infinite thermal capacity maintained at constant higher temperature T_H;

(3) A sink of infinite thermal capacity maintained at constant lower temperature T_L; and

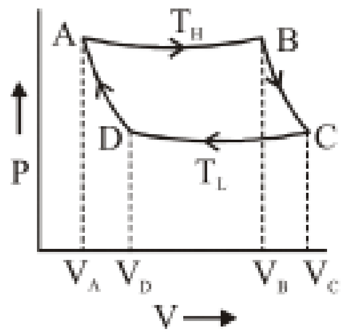
(4) A perfectly non-conducting stand for the cylinder.



Here it is worth mentioning that as all the above mentioned components cannot exist in reality, Carnot engine is an ideal (hypothetical) engine which can never be actually constructed.

(B) Carnot Cycle [or Working of the Engine] :

The working substance in a Carnot engine is taken through a reversible cycle consisting of the folloing four steps :



(i) The cylinder containing ideal gas is placed on the source and the gas is allowed to expand slowly at constant temperature T_H absorbing heat Q_H. This isothermal change is represented by the curve AB in the indicator diagram.

(ii) The cylinder is then placed on the non-conducting stand and the gas is alloweed to expand adiabatically till the temperature falls from T_H to T_L. This adiabatic expansion is represented by the curve BC.

(iii) The cylinder is next placed on the sink and the gas is compressed at constant temperature T_L. This adiabatic expansion is represented by the curve BC.

(iv) Finally the cylinder is again placed on the non-conducting stand and the compression is continued so that the gas returns to its initial stage along DA.

The closed pathABCDA represents the so called **Carnot cycle** and the four stages taken together represent a cyclic process

(C) Efficiency of the Engine :

The efficiency of a engine is defined as the ratio of work done to the heat supplied, i.e.,

$$\eta = \frac{\text{Work done}}{\text{heat input}} = \frac{W}{W_H} \qquad \text{...(i)}$$

But as for cyclic process $\Delta W =$ i.e., $W = Q_H - Q_L$

So eqn. (i) reduces to

$$\eta = \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H} \qquad \text{...(ii)}$$

Now as in an isothermal process internal energy remains constant, in accordance with first law

$$\Delta Q = \Delta W = nRT \log_e (V_F/V_I) \qquad [\text{as } \Delta U = 0]$$

So $\quad Q_H = nRT_H \log_e\left(\frac{V_B}{V_A}\right)$

and $\quad |Q_L| = nRT_L \log\left(\frac{V_C}{R_D}\right) \qquad \text{...(iii)}$

But as for adiabatics BC and DA respectively,

$$T_H V_B^{\gamma-1} = T_L V_C^{\gamma-1} \text{ and } T_H V_A^{\gamma-1} = T_L V_D^{\gamma-1}$$

Dividing these two results,

$$\left(\frac{V_B}{V_A}\right)^{\gamma-1} = \left(\frac{V_C}{V_D}\right)^{\gamma-1}, \text{ i.e., } \frac{V_B}{V_A} = \frac{V_C}{V_D} \qquad \text{...(iv)}$$

Substituting the value of Q_H and Q_L from, equation (iii) in (ii) in the light of (iv), we get

$$\eta = 1 - \frac{T_L}{T_H} \qquad \text{...(v)}$$

This is the required result and from this it is clear that :

(1) Efficiency of a heat engine depends only on temperatures of source and sink and is independent of all other factors.

(2) All reversible heat engines working between same temperatures are equally efficient and no heat engine can be more efficient than Carnot engine (as it is ideal).

(3) As on Kelvin scale temperature can never be negative (as 0 K is defined as lowest possible temperature) and T_H and T_L are finite, efficiency of a heat engine is always lesser than unity, i.e., whole of heat can never be converted into work which is in accordance with second law.

The efficiency of actual engines is much lesser than that of ideal engine. Actually the practical efficiency of a steam engine is about (8-15)% while that of a petrol engine 40%. The efficiency of a diessel engine is maximum and is about (50-55)%.

Illustration :

An inventor claims to have developed an engine that during a certain time interval takes in 110 MJ of heat at 415 K, rejects 50 MJ of heat at 212 while manages to do 16.7kW of work. Do you agree with the inventor's claim ?

Sol. *The claimed efficiency*

$$\eta = \frac{W}{Q_H} = \frac{16.7\ kWh}{10\ MJ}$$

But as $1kWh = 10^3 \times (J/s) \times (60 \times 60s) = 3.6\ MJ$

So $$\eta = \frac{16.7 \times 3.6}{110} = 0.55 = 55\% \qquad \text{....(i)}$$

While maximum possible theoretical efficiency,

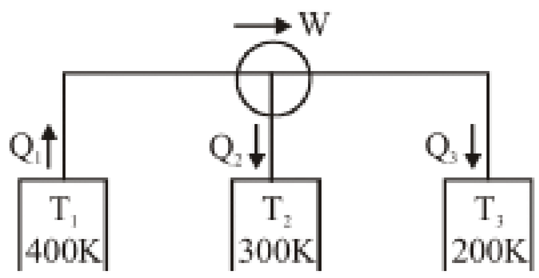
$$\eta_{max} = -\frac{T_L}{T_H} = \frac{T_H - T_L}{T_H} = \frac{415 - 212}{415}$$

$$= .49 = 49\%$$

From eqns. (i) and (ii) it is clear that claimed efficiency is greater than maximum possible theoretical efficiency; so inventor's c claim does not appear to be correct.

Illustration :

During an integral number of complete cycles, a reversible engine (shown by a circle) absorbs 1200 joule from reservoir at 400 k and performs 200 joule of mechanical work,



(a) Find the quantities of heat exchanged with the other two reservoirs. State whether the reservoirs absorb or lose heat.

(b) Find the change of entropy of each reservoir.

(c) What is the change in entropy of the universe ?

Sol. *(a) By conservation of energy*

$$Q_1 = W + Q_2 + Q_3$$

i.e., $Q_2 + Q_3 = Q_1 - W = 1200 - 200 = 1000$ ...(1)

And as change in entropy in a reversible-process is zero

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} + \frac{Q_3}{T_3} = 0 \qquad \text{i.e.,} -\frac{200}{400} + \frac{Q_2}{300} + \frac{Q_3}{200} = 0$$

i.e., $2Q_2 + 3Q_3 = 1800$

Solving equation (i) and (ii) for Q_2 and Q_3, we get

$$Q_2 = 1200\ J \text{ and } Q_3 = -200\ J$$

i.e., the reservoir at temperature T_2 absorbs 1200 J of heat while the reservoir at temperature T_3 lose 200 J of heat

(b) Now as change in entropy at constant temperature is given by $\Delta S = (\Delta Q/T)$

So change in entropy of reservoir at temperatures T_1, T_2 and T_3 will be respectively

$$\frac{-1200}{400} = -3\frac{J}{K}, \frac{1200}{300} = 4\frac{J}{K} \text{ and } \frac{-200}{200} = -1\frac{J}{K}$$

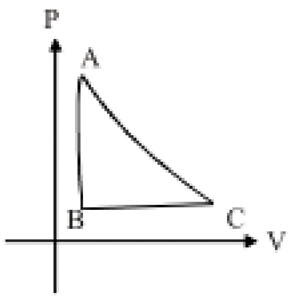
Practice Exercise

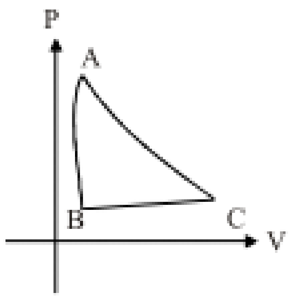
Q.1 The average energy and the rms speed of molecules in a sample of oxygen gas at 300 K are 6.21×10^{-21} J and 484 ms^{-1} respectively. The corresponding values of 600 K are nearly

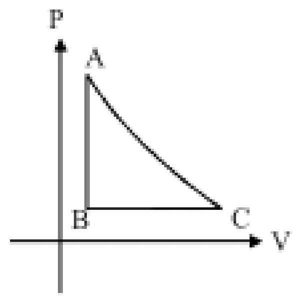
(A) 12.42×10^{-21} J, 968 ms^{-1} (B) 8.78×10^{-21} J, 684 ms^{-1}

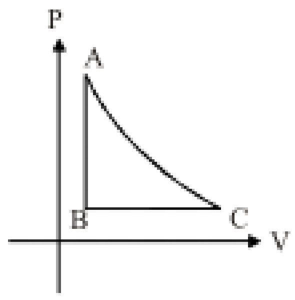
(C) 6.21×10^{-21} J, 968 ms^{-1} (D) 12.42×10^{-21} J, 684 ms^{-1}

Q.2 The P - T diagram for a ideal gas is shown in Figure where AC is an adiabatic process. The corresponding PV diagram is .

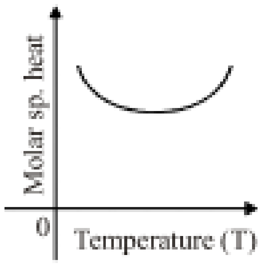
(A)


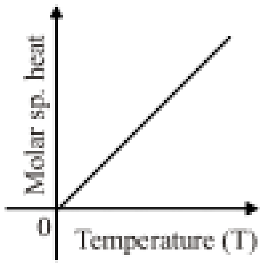
(B)


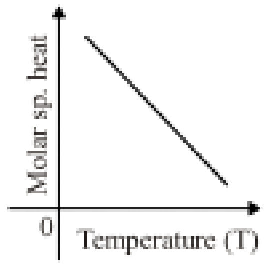
(C)


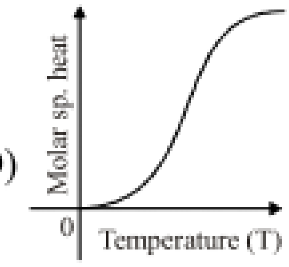
(D)


Q.3 Variation of molar specific heat of a metal with temperature is best depicted by

(A) 

(B) 

(C) 

(D) 

Answers

Q.1 D Q.2 B Q.3 D

Solved Example

Q.1 70 calorie of heat is required to raise the temperture of a diatomic gas at contant pressure from 30 to 35°C. the amount of heat requried (in calorie) to raise the temperature of the same gas through the same range (30 to 35°C) at constant volume is

(A) 30 (B) 60 (C) 50 (D) 70

Sol. $\frac{C_P}{C_V} = \gamma = 1.4$

$$\frac{(\Delta Q)_P}{(\Delta Q)_V} = \frac{nC_P\Delta T}{nC_V\Delta T} = \frac{C_P}{C_V} = 1.4$$

$$\therefore \quad (\Delta Q)_V = \frac{(\Delta Q)_P}{1.4} = 50 \text{ cal}$$

Q.2 A vessel contains 1 mole of O_2 and 1 mole of He. The value of γ of the mixture is

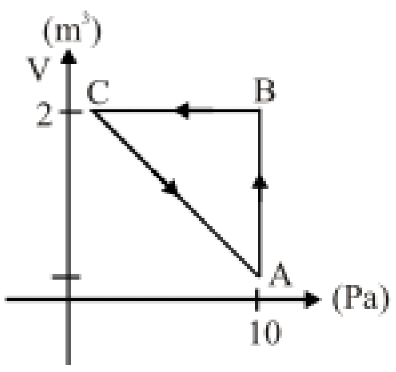
(A) 1.4 (B) 1.50 (C) 1.53 (D) none of these

Sol. $C_{vmix} = \dfrac{\frac{3}{4}R + \frac{5}{2}R}{2} = 2R$

$C_{pmix} = 2R + R = 3R$

$y_{mix} = \frac{C_P}{C_V} = \frac{3}{2}$

Q.3 An ideal is taken through a cycle $A \to B \to C \to A$ as shown in Figure if the net heat supplied in the cycle is 5 J, then work done by the gas in the process $C \to A$ is



(A) – 5 J (B) – 10J (C) – 15 J (D) – 20 J

Sol. Work done = area under the curve 10 J; $5 = W_{CA} + 10$ or $W_{CA} = -5$ J

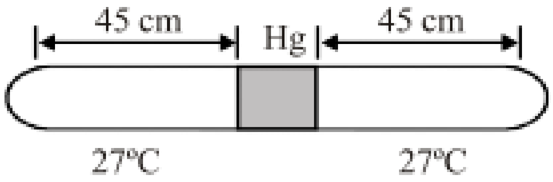
Q.4 A glass tube scaled at both ends is 1 m long. It lies horizontally with the middle 10 cm containing Hg. The two ends of the tube, eqaul in length, contain air at 27° C and pressure 76 cm of Hg. The temperature at one end is kept 0°C and at the other end it 127° C. Neglect the change in length of Hg column. Then the change in length on two sides is

(A) 12.3 cm (B) 10.311 cm (B) 9.9 cm (D) 8.49 cm

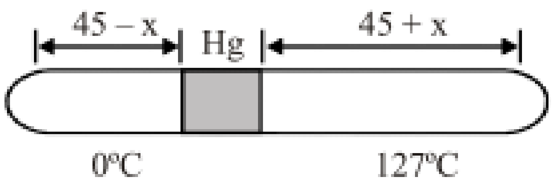
Sol. Initially $l = 45$ cm $(2l+10=100$ cm)

$$P_1 = P_2 = P \text{ (say)} \quad \text{...(i)}$$

(a)



(b)



Applying gas law at end A,

$$\frac{45AP}{300} = \frac{(45-x)AP_1}{273} \quad \text{...(ii)}$$

At end B $\frac{45AP}{300} = \frac{(45+x)AP_2}{400}$ ...(iii)

From (i), (ii) and (iii)

$$\frac{(45-x)}{273} = \frac{45+x}{400} = 8.49\text{cm}$$

Q.5 Find the amont of work done to increase the temperature of one mole of an ideal gas by 30ºC if it is expanding under the condition $V \propto T^{2/3}$.

(A) 166.2J (B) 136.2 J (C) 126.2 J (D) None of these

Sol. $PV = RT$ for 1 mole

$$W = \int PdV = \int \frac{RT}{V} dV$$

$$V = CT^{2/3}$$

$$\therefore \quad dV = \frac{2}{3} CT^{2/3} dT$$

or $$\frac{dV}{V} = \frac{2}{3}\frac{dT}{T}$$

$$\therefore \quad W = \int_{T_1}^{T_2} RT\left(\frac{2}{3}\right)\frac{dT}{T}$$

$$= \frac{2}{3}R(T_2 - T_1) = 166.2\text{J}$$

Q.6 A mercury thermometer read 80ºC when the mercury is at 5.2 cm mark and 60ºC when the mercury is at 3.9 cm mark. Find the temperature when the mercury level is at 2.6 cm mark.

Sol. $\frac{l_1 - l_2}{l_1 - l_3} = \frac{\alpha l_0 (T_1 - T_2)}{\alpha l_0 (T_1 - T_3)} = \frac{T_1 - T_2}{T_1 - T_3}$

or $\frac{5.2 - 3.9}{5.2 - 2.6} = \frac{80 - 60}{80 - T_3}$

$1.3\ (80 - T_3) = 2.6\ (20)$ or $T_3 = 40°C$

Q.7 A vertical cylinder pistion system has cross-section S. It contains 1 mole of an ideal monoatomic gas under a pistion of mass M. At a certain instant a heater is switched on which transmits a heat q per unit time ot the cylinder. Find the velocity v of the pistion under the condition that pressure under the pistionis constant and the system is thermally insulated.

Sol. Gas pressure $= P_0 + \frac{Mg}{S}$, Where M is mass of the pistion.

As $C_V = \frac{3}{2}R$

$\therefore\ \Delta U = \frac{3}{2} R\Delta T = \frac{3}{2} P\Delta V$

$Q = P\Delta V + \Delta U = P\Delta V + \frac{3}{2} P\Delta V = \frac{5}{2} P\Delta V$

$\Delta V = Sdx$

or $Q = q.\ dt = \frac{5}{2} PSdx$

or $\frac{dx}{dt} = \frac{2q}{5PS} = \frac{2q}{5\left(P_0 + \frac{Mg}{S}\right)S}$

Q.8 A typer pumped to a pressure 3.3375 atm at 27ºC suddenly bursts. What is the final temperature (γ = 1.5)?

(A) 27ºC (B) –27ºC (C) 0ºC (D*) –73º

Sol. $T_1^{\gamma} P_1^{1-\gamma} = T_2^{\gamma} P_2^{1-\gamma}$

or $\left(\frac{T_1}{T_2}\right)^{\gamma} = \left(\frac{P_1}{P_2}\right)^{\gamma-1} = \left(\frac{300}{T_2}\right)^{3/2} = \left(\frac{3.375}{1}\right)^{3/2-1}$

or $T_2 = \frac{300}{(3.375)^{1/3}} = 200\ K = -\ 73°C$

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Q.9 3 Moles of a gas mixture having volume V and temperature T is compressed to 1/5th of the initial volume. Find the change in its adiabatic compressibility if the gas obeys $PV^{19/13}$ = constant [R = 8.3 J/mol – K]

Sol. Bulk modulus $B = \gamma P$

$$\text{Compressibility } C = \left(\frac{1}{B}\right) = \frac{1}{\gamma P}$$

and $\Delta C = C - C$

or $DC = \frac{1}{\gamma}\left[\frac{1}{P'} - \frac{1}{P}\right]$

$$PV^{\gamma} = P'\left(\frac{V}{5}\right)^{\gamma}$$

With $\gamma = \frac{19}{13}$ and $P' = 5^{\gamma} P$, 11

$$\Delta C = \frac{1}{\gamma P}\left[\frac{1}{5^{\gamma}} - \frac{1}{1}\right] = \frac{13 \times 0.905}{19P}$$

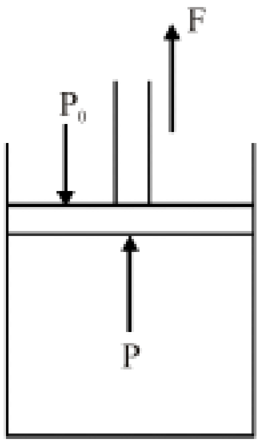
But $PV = nRT$ or $P = \frac{nRT}{V}$

$$\Delta C = \frac{13.(0905)V}{19 \times 3 \times 8.317T} = \frac{-0.0248V}{T}$$

Q.10 One mole of an ideal gas is contained under a weightless pistion of a vertical cylinder at a temperature T. The space over the pistion opens into the atmosphere. What work has to be performed in order to increaseisothermally the gas volume under the pistion η times

Sol. Let A be the area of cross-section

$$F + PA = P_0 A$$
$$F = (P_0 - P)A$$



Work done by the agent

$$W = \int_V^{\eta V} F dx = \int_V^{\eta V} (P_0 - P) A dx$$

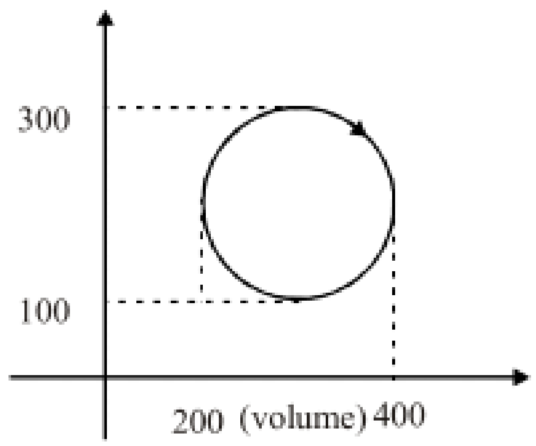
$$= \int_V^{\eta V} (P_0 - P) dV$$

$$= P_0 (\eta - 1)V - \int_V^{\eta V} nRT \frac{dV}{V}$$

$$= RT(\eta - 1) - n\log_e \eta$$

Q.11 Calculate the heat absorbed by the system in going through the process shown in figure

(A) 31.4 J (B) 3.14 J (C) 3.14×10^4 J (D) none



Sol. Heat absorbed = πr^2

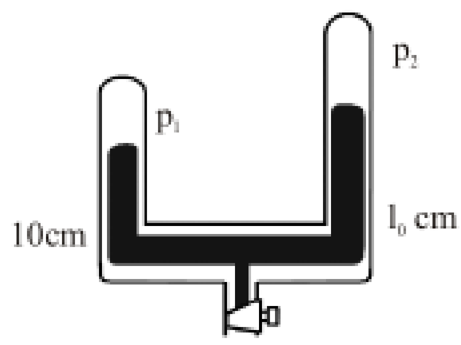
$= \pi (P_r)(V_r)$

$= 3.14\,(100\times10^3)\,(100\times10^{-6})$

$= 31.4$ J

Q.12 A mercury manometer consists of two unequal arms of equal cross-section 1 cm^2 and lengths 100cm and 50cm. The two open ends are sealed with air in the tube at a pressure of 80 cm of mercury. Some amount of mercury is now introduced in the monometer through the stopcock connected to it. If mercury resis in the shorter tube to a length 10 cm in steady state, find the length of the mercury column risen in the longer tube.

Sol. Let p_1 and p_2 and be the pressures in centimetre of mercury in the two arms after introducing mercury in the tube. Suppose the mercury column rises in the second arm to l_0 cm.



Using pV = constant for the shorter arm,

$(80\text{cm})\,(50\text{cm}) = p_1\,(50\text{cm} - 10\text{cm})$

or $p_1 = 100$ cm. ...(i)

Using pV = constant for the longer arm,

$(80\text{ cm})\,(100\text{cm}) = p_2\,(100 - l_0)$ cm ...(ii)

From the figure,

$p_1 = p_2 + (l_0 - 10)$ cm.

Thus by (i),

$$100\text{ cm} = p_2 + (l_0 - 10)\text{ cm}.$$

or $\quad p_2 = 110\text{ cm} - l_0\text{ cm}$

Putting in (ii),

$$(110 - l_0)(100 - l_0) = 8000$$

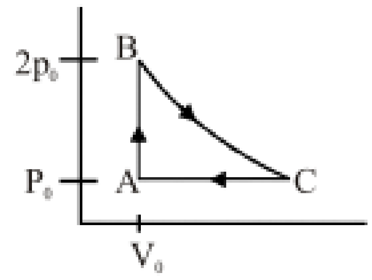
or, $\quad l_0^2 - 210\, l_0 + 3000 = 0$

or $\quad l_0 = 15.5$

The required length is 15.5 cm.

Q.13 An ideal gas has pressure p_0, volume V_0 and temperature T_0. It is taken through an isochoric process till its pressure is doubled. It is now isothermally expanded to get the original pressure. Finally, the gas is isobarically compressed to its original volume V_0. (a) Show the process on a p–V diagram. (b)What is the temperature in the isothermal part of the process ? (c) What is the volume at the end of the isothermal part of the process ?

Sol. (a) The process is shown in a p-V diagram in figure. The process starts from A and goes through ABCA.



(b) Applying $\quad pV = nRT$ at A and B,

$$p_0V_0 = nRT_0$$

and $\quad (2p_0)V_0 = nRT_B$

Thus, $\quad T_B = 2T_0$

This is the temperature in the isothermal part BC.

(c) As the process BC is isothermal, $T_C = T_B = 2T_0$.

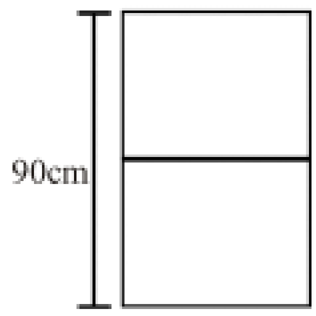
Applying pV = nRT at A and C,

$$p_0V_0 = nRT_0$$

and $\quad p_0V_C\ nR(2T_0)$

$$V_C\ 2V_C.$$

Q.14 Figure shown a vertical cylindrical vessel separated in two parts by a frictionless pistion free to move along the length of the vessel. The length of the cylinder is 90 cm and the pistion divides the cylinder in the ratio of 5 : 4. Each of the two parts of the vessel contains 0.1 mole of an ideal gas. The temperature of the gas is 300 K in each part. Calculate the mass of the piton.



Sol. Let l_1 and l_2 be the lengths of the upper part and the lower part of the cylinder respectively. clearly, l_1 = 50 cm and l_2 = 40 cm. Let the perssures in the upper and lower parts be p_1 and p_2 respectively. Let the area of coss-section of the cylinder be A. The temperature in both parts is T = 300 K.

Consider the equilibrium of the piston. The forces acting on the pistion are

(a) its weight mg

(b) p_1 A downward, by the upper part of the gas and (c) p_2 A upward, by the lower part of the gas.

Thus, $\quad p_2 A = p_1 A + mg \quad$ ...(i)

Using pV = nRT for the upper and the lower parts

$$p_1 l_1 A = nRT \quad \text{...(ii)}$$

and

$$p_2 l_2 A = nRT. \quad \text{...(iii)}$$

Putting p_1 A and p_2 A from (ii) and (iii) into (i),

$$\frac{nRT}{l_2} = \frac{nRT}{l_1} + mg$$

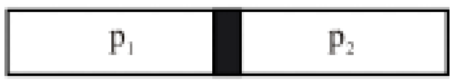
Thus,

$$m = \frac{nRT}{g}\left[\frac{1}{l_1} - \frac{1}{l_2}\right]$$

$$= \frac{(0.1\,\text{mol})(8.3\,\text{J/mol}-\text{K})(300\,\text{K})}{9.8\,\text{m/s}^2}\left[\frac{1}{0.4\,\text{m}} - \frac{1}{0.5\,\text{m}}\right]$$

$$= 12.7\ \text{kg}.$$

Q.15 Figure shows a cylindrical tube of volume V_0 divided in two parts by a frictionless separator. The walls of the tube are adiabatic but the separator is conducting. Ideal gasses are filled in the two parts. When the separator is kept in the middle, the pressures are p_1 and p_2 in the left part and the right part respectively. The separator is slowly slide and is released at a position where it can stay in equilibrium. Find the volume of the two parts.



Sol. As the separator is conducting, the temperatures in the two parts will be the same. Suppose the common temperature is T when the separator is in the middle. Let n_1 and n_2 be the number of moles of the gas in the left part and the right part respectively. Using ideal gas equation,

$$p_1 \frac{V_0}{2} = n_1 RT$$

and

$$p_2 \frac{V_0}{2} = n_2 RT$$

Thus,

$$\frac{n_1}{n_2} = \frac{p_1}{p_2} \quad \text{....(i)}$$

The separator will stay in equilibrium at a position where the pressures on the two sides are equal. Suppose the volume of the left part is V_1 and of the right part is V_2 in this situation. Let the common pressure be p'. Also, let the common temperature in this situation be 'T'.

Using ideal gas equation

$$p'V_1 = n_1 RT'$$

and $$p'V_2 = n_2 RT'$$

or, $$\frac{V_1}{V_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2} \qquad \text{[using (i)]}$$

Also, $$V_1 + V_2 = V_0$$

Thus, $$V_1 = \frac{p_1 V_0}{p_1 + p_2} \text{ and } V_2 = \frac{p_2 V_0}{p_1 + p_2}$$

Q.16 A gas is heated isobarically and the heat used for external work is W. Find the total amount of heat supplied.

Sol. $$\Delta Q = \Delta W + \Delta U$$

$$= nRT + nC_V \,\Delta T = nRT + \frac{nR\Delta T}{\gamma - 1} = (nR\Delta T)\left(\frac{\gamma}{\gamma - 1}\right)$$

$$= \frac{W\gamma}{\gamma - 1} \qquad [\therefore W = nR\Delta T]$$

Q.17 A vessel contains a mixture consisting of $m_1 = 7$g of nitrogen ($M_1 = 28$) and $m_2 = 11$ g of carbon dioxide ($M_2 = 44$) at temperature T = 300 K and pressure $p_0 = 1$ atm. Find the density of the mixture.

Sol. Let V be the volume of the vessel. Then $\rho_{mis} = \dfrac{m_1 + m_2}{V}$

Let p_1 and p_2 be the partial pressure

Then $p_1 V = \dfrac{m_1}{M_1} RT$ and $p_2 V = \dfrac{m_2}{M_2} RT$, $p_0 = p_1 + p_2$

$$\therefore \quad p_0 = \left(\frac{m_1}{M_1} + \frac{m_2}{M_2}\right)\frac{RT}{V}$$

$$\therefore \quad \rho_{mix} = \frac{(m_1 + m_2)M_1 M_2}{m_1 M_2 + m_2 M_1} \times \frac{p_0}{RT}$$

$$= \frac{(7+11)\times 28 \times 44 \times 10^{-3}}{7 \times 44 + 11 \times 28} \times \frac{10^5}{8.3 \times 300}$$

$$= 1.446 \text{kg m}^{-3} = 1.446 \text{ per litre}$$

Fluid Mechanics

This lesson is devoted to fluids (liquids and gases). Find is a Latin word meanins 'to flow'. Liquids and gases can flow to take shape of the vessel that holds them. Another important property of fluids is that a tangential force causes continuous deformation of fluids whereas if we apply a tangential force on a solid there will be a particular deformation in the solid. First section of the chapter is study of fluid at rest called as fluid statics and Second section is study of fluid in motion called as fluid dynamics. In these two sections we shall see the concepts and their application for liquids only but these concepts and their application for liquids only but these concepts are equally good for gases also but for low-pressure variation.

What is a Fluid ?

A **fluid,** in contrast to a solid, is a substance that can flow. Fluids conform to the boundaries of any container in which we put them. They do so because a fluid cannot sustain a force that is tangential to its surface. a fluid is a substance that flows because it cannot withstand a shearing stress. It can, however, exert a force in the direction perpendicular to its surface.

FLUID MECHANICS

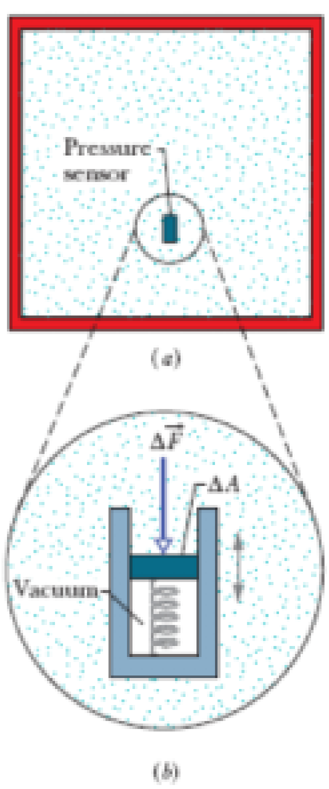
- FLUID STATICS (Fluid remains stationary wrt container)
- FLUID DYNAMICS
- VISCOUS FLUID

Fluid includes property → (A) Density (B) Viscosity (C) Bulk modulus of elasticity (D) Pressure (E) Specific gravity.

Assumptions used in fluid mechanics

1. Fluid is incompressible means density remains constant and volume also remains constant.
2. Fluid is non viscous. There is no tangential force between two layers.

Pressure



Let a small pressure-sensing device be suspended inside a fluid-filled vessel, as in Fig. 14-1a. The sensor consists of a piston of surface area ΔA riding in a close-fitting cylinder and resting against a spring. A readout arrangement allows us to record the amount by which the (calibrated) spring is compressed by the surrounding fluid, thus indicating the magnitude ΔF of the force that acts normal to the piston. We define the **pressure** on the piston from the fluid as

$$p = \frac{\Delta F}{\Delta A}$$

In theory, the pressure at any point in the fluid is the limit of this ratio as the surface area ΔA of the piston, centered on that point, is made smaller and smaller. However, if the force is uniform over a flat area A, we can write Eq. as

$$p = \frac{F}{A} \text{ (pressure of uniform force on flat area)}$$

where F is the magnitude of the normal force on area A. (When we say a force is uniform over an area, we mean that the force is evenly distributed over every point of the area.) We find by experiment that at a given point in a fluid at rest, the pressure p defined by Eq. has the same value no matter how the pressure sensor is oriented. Pressure is a scalar, having no directional properties. It is true that the force acting on the piston of our pressure sensor is a vector quantity, but Eq. involves only the magnitude of that force, a scalar quantity.

Let us look first at the increase in pressure with depth below the water's surface. We set up a vertical y axis in the tank, with its origin at the air–water interface and the positive direction upward. We next consider a water sample contained in an imaginary right circular cylinder of horizontal base (or face) area A, such that y_1 and y_2 (both of which are negative numbers) are the depths below the surface of the upper and lower cylinder faces, respectively.

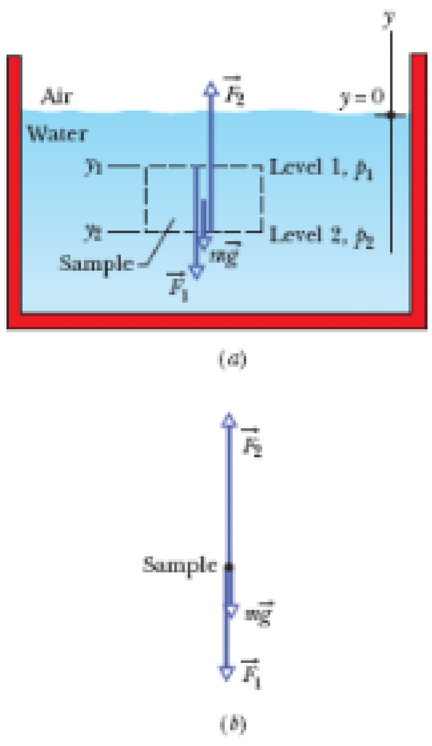


Figure shows a free-body diagram for the water in the cylinder. The water is in static equilibrium; that is, it is stationary and the forces on it balance. Three forces act on it vertically: Force $\vec{F}_1$ acts at the top surface of the cylinder and is due to the water above the cylinder. Similarly, force $\vec{F}_2$ acts at the bottom surface of the cylinder and is due to the water below the cylinder. The gravitational force on the water in the cylinder is represented by $m\vec{g}$, where m is the mass of the water in the cylinder. The balance of these forces is written as

$$F_2 = F_1 + mg$$

We want to transform Eq. into an equation involving pressures. From Eq., we know that

$$F_1 = p_1 A \text{ and } F_2 = p_2 A$$

The mass m of the water in the cylinder is, from Eq., $m = \rho V$, where the cylinder's volume V is the product of its face area A and its height $y_1 - y_2$. Thus, m is equal to $\rho A(y_1 - y_2)$. Substituting this and Eq. into Eq., we find

$$p_2 A = p_1 A + \rho A g(y_1 - y_2)$$

or

$$p_2 = p_1 + \rho g(y_1 - y_2)$$

This equation can be used to find pressure both in a liquid (as a function of depth) and in the atmosphere (as a function of altitude or height). For the former, suppose we seek the pressure p at a depth h below the liquid surface. Then we choose level 1 to be the surface, level 2 to be a distance h below it and p_0 to represent the atmospheric pressure on the surface. We then substitute

$$y_1 = 0,\ p_1 = p_0 \text{ and } y_2 = -h,\ p_2 = p$$

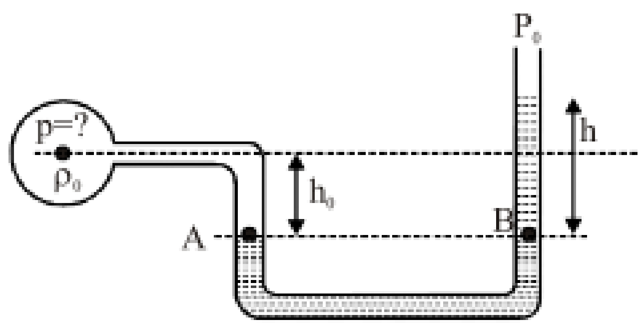
$$p = p_0 + \rho g h \text{ (Pressure at depth h)}$$

Note that the pressure at a given depth in the liquid depends on that depth but not on any horizontal dimension.

The pressure at a point in a fluid in static equilibrium depends on the depth of that point but not on any horizontal dimension of the fluid or its container.

Pressure Measuring Devices

A manmenter is a tube open at both the ends and bent into the shape of a "U" and partially filled with mercury. When one end of the tube is subjected to an unkonwon pressure p, the mercury level drops on that side of the tube and rises on the other so that the difference in murcury level is h as shown in the figure. When we move down in a fluid pressure increases with depth and when we move up the pressure decreases with depth. When we move horizontally in a fluid pressure remains constant.



Therefore,

$$p + \rho_0 g h_0 - \rho_m g h = p_0$$

Where p_0 the atmosheric pressure and

r_0 is the density of the fluid inside the vessel

The mercury barometer

It is a straight glass tube (closed at one end) completely filled with mercury and inserted into a dish which is also filled with mercury as shown in the figure. Atmosheric pressure supports the column of mercury in the tube to a height h. The pressure between the closed end of the tube and the column of mercury is zero. p = 0.

Therefore, pressure at points A and B are equal and thus $p_0 = 0 + \rho_m gh$

At the sea level, p_0 can support a column of mercury about 76 cm in height

Hence $p_0 = (13.6 \times 10^3)(9.81)(0.76) = 1.01\times^5 \text{ Nm}^2$ (or Pa)

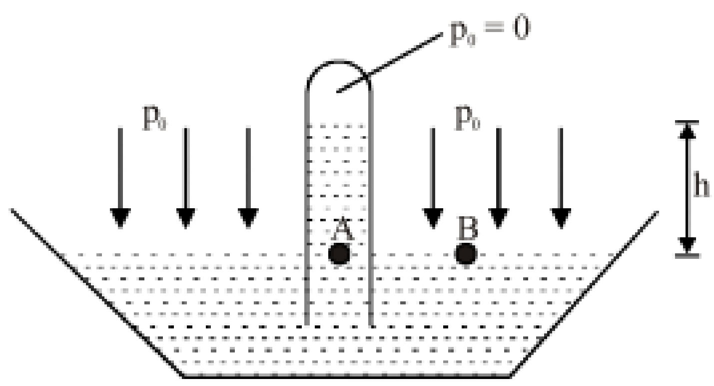
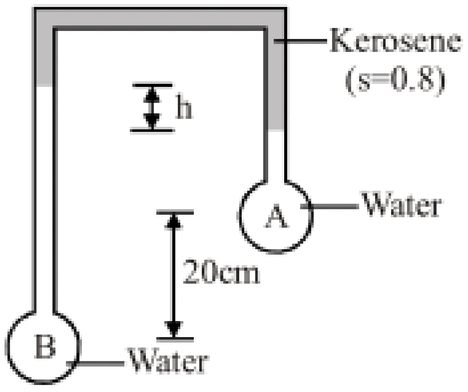


Illustration :

For the arrangement shown in the figure, determine h if the pressure difference between the vessels A and B is 3 kN/m²



Sol. *Let pressure in the horizontal tube is P So in left vertical tube*

$$P + \rho_k g h_1 + \rho_w g h_o = P_B$$

$$P + \rho_k g h_2 + \rho_w g(h_0 - h - 0.2) = P_A$$

here, $P_B - P_A = 3 \times 10^3\ N/m^2$,

$$\rho_W = 10^3\ kg/m^3$$

$$\rho_k = 800\ kg/m^3$$

Thus, $h = 0.5\ m = 50\ cm$

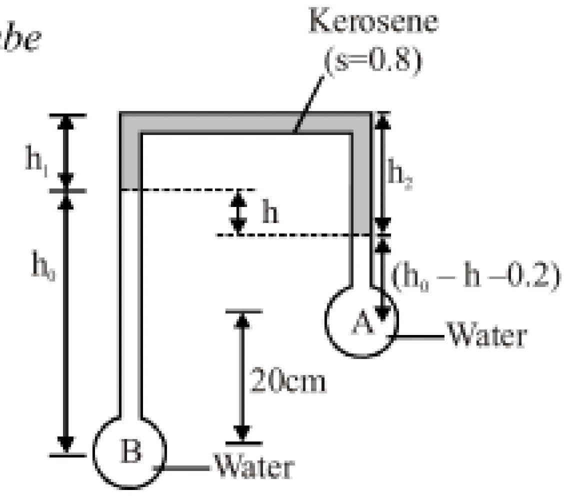


Illustration :

What must be the length of a barometer tube used to measure atmospheric pressure if we are to use water instead of mercury.

Sol. *We know that*

$$p_0 = \rho_m g h_m = \rho_w g h_w$$

where ρ_w *and* h_w *are the density and height of the water column supporing the atmospheric pressure* p_0.

$$\therefore \quad h_w = \frac{\rho_m}{\rho_w} h_m$$

Since $\frac{\rho_m}{\rho_w} = 13.6$ *and* $h_m = 0.76\ m$

$$\therefore \quad h_w = (13.6)(0.76) = 10.33\ m$$

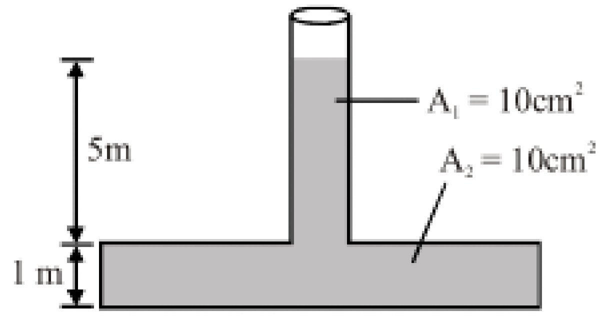
Illustration :

In the figure shown, find

(a) the total force on the bottom of the tank due to the water pressure.

(b) the total weight of water.

Why is there a difference between the two ?



Sol.

(a) Pressure at the base due to water is

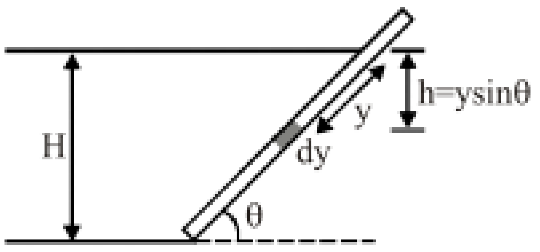
$$p = \rho_w g\,[5 + 1] = (10^3)\,(10)\,(5 + 1) = 6 \times 10^4\ N/m^2$$

$\therefore$ Force $= pA_2 = (6 \times 10^4)\,(100 \times 10^{-4}) = 600\ N$

(b) Weight of water $= \rho_w\, g\,[5A_1 + A_2] = 10^4\,[5 \times 10 \times 10^{-4} + 100 \times 10^{-4}] = 150\ N$

Illustration :

Find the force acting per unit width on a plane wall inclined at an angle θ with the horizontal as shown in the figure.



Sol. *Consider a small element of thickness dy at a distance y measured along the wall from the free surface. The pressure at the position of the element is*

$$p = \rho g h = \rho g y \sin\theta$$

The force is given by

$$dF = p\,(b\,dy) = \rho g b\,(y\,dy)\sin\theta$$

The total force per unit width b is given by

$$\frac{F}{b} = \rho g \sin\theta \int_0^{H/\sin\theta} y\,dy = \rho g \sin\theta \left[\frac{y^2}{2}\right]_0^{H/\sin\theta}$$

or $$\frac{F}{b} = \frac{1}{2}\rho g \frac{H^2}{\sin\theta}$$

Note that the above formula reduces to $\frac{1}{2}\rho g H^2$ *for a vertical wall* $(\theta = 90^o)$

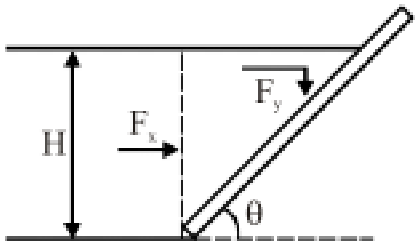
Alternatively, the force on the inclined wall may be obtained in two parts viz. Horizontal and vertical.

The horizontal force F_x acts on the vertical projection of the incline wall,

i.e. $F_x = \frac{1}{2}\rho gbH^2$

The vertical force F_y acts to weight of the liquid supported by the wall, i.e.

$$F_y = \frac{1}{2}\rho gb(H)(H\cot\theta) = \frac{1}{2}\rho gbH^2\cot\theta$$



The magnitude of the resultant force is given by

$$F = \sqrt{F_x^2 + F_y^2}\ \frac{1}{2}\rho gbH^2 \operatorname{cosec}\theta$$

or $F = \frac{1}{2}\rho g\frac{bH^2}{\sin\theta}$

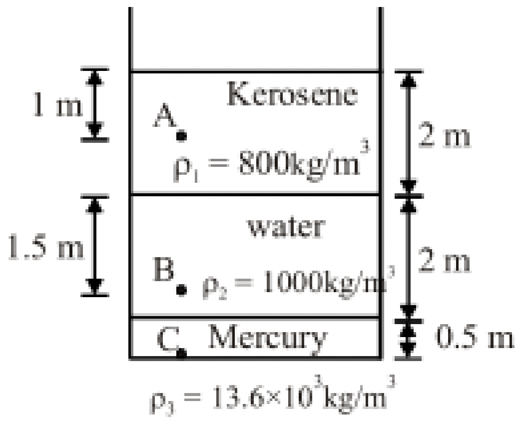
Illustration :

Atmosphericpressure is about 1.01×10^5 Pa. How large a force does the atmosphere exert on a $2cm^2$ area on the top of your head?

Sol. *Because p = F / A. where F is perpendicular to A, we have F = pA. Assuming that $2\ cm^2$ of your head is flat (nearly correct) and the force due to the atmosphere is perpendicular to the surface (as it is), we have*

$$F = pA = (1.01\times10^5\ N/m^2)\ (2\times10^{-4}m^2) \approx 20\ N$$

(1)6. *Fin the absolute pressure and gauge pressure at pint A, B and C as hown in the figure (1 atm = 10^5 Pa)*

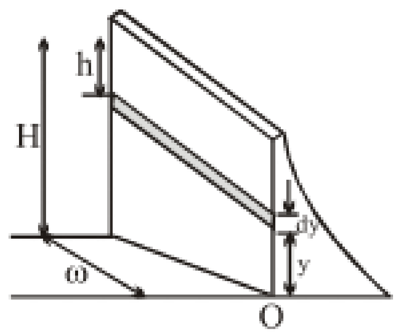


Sol. $p_{atm} = 10^5$ Pa

Points	Gauge Pressure	Abolute Pressure
A	$pA = \rho_1 gh_A = (800)(10)\ 1 = 8\ k\ Pa$	$p'A = pA + p_{atm} = 108\ kPa$
B	$pB = \rho_1 g(2) + \rho_2 g(1.5) = (800)(10)(2) + (10^3)(10)(1.5) = 31\ kPa$	$p'B = pB + p_{atm} = 131\ kPa$
C	$pC = \rho_1 g(2) + \rho_2 g(2) + \rho_3 g\ (0.5) = (800)(10)(2) + (10^3)(10)(2) + (13.6\times10^3)(10)(0.5) = 104\ kPa$	$p'C = PC + p_{atm} = 204\ kPa$

Illustration :

Water is filled to a height H behind a dam of width w (fig.). Determine the resultant force exerted by the water on the dam.



Water exerts a force on a dam

Sol. *Let's consider a vertical y axis, starting from the bottom of the dam. Lets consider a thin horizontal strip at a height y above the bottom, such as shown in Fig. We need to consider force due to the pressure of the water only as atmospheric pressure acts on both sides of the dam.*

The pressure due to the water at the depth h : $P = \rho gh = \rho g(H - y)$

The force exerted on the shaded strip of area $dA = wdy$:

$$dF = P\,dA = \rho g(H - y)\,\omega\,dy$$

Integrate to find the total force on the dam :

$$F = \int P\,dA = \int_0^H \rho g(H-y)\omega\,dy = 1/2\ \rho g\omega H^2$$

Illustration :

In the previuous example find the total torque exerted by the water on dam about a horizontal axis through O. Also find the effective line of action of the total force exerted by the water is at a distance 1/3 H above O.

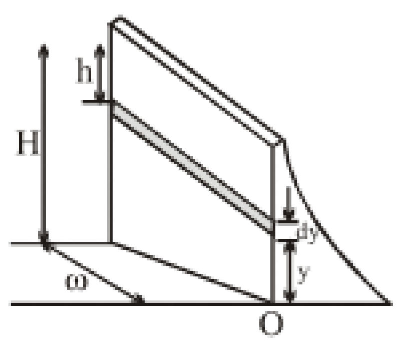
Sol. *The torque is* $\tau = \int d\tau = \int r dF$

From the figure $$\tau = \int_0^H y[\rho g(H-y)wdy] = \frac{1}{6}\ \rho gwH^3$$

The total force is given as $\frac{1}{2}\rho gwH^2$

If this were applied at a height y_{eff} *such that the torque remains unchanged, we have*

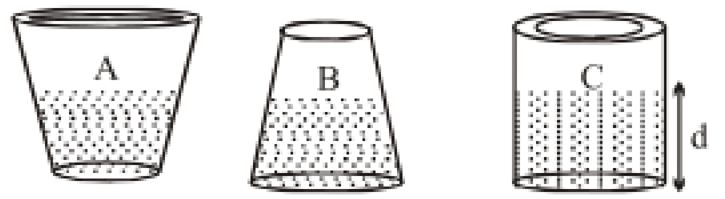
$$\frac{1}{6}\ \rho gwH^3 = y_{eff}\left[\frac{1}{2}\rho gwH^2\right] \quad \text{and} \quad y_{eff} = \frac{1}{3}\ H$$



Water exerts a force on a dam

Illustration :

Three vessels having different shapes are as shown in the figure below, they have same base area and the same weight when empty (Fig.). The vessels are filled with mercury to the same level. Neglect the effect of the atmosphere. (a) Which have the largest and which have the smallest pressures at the bottom of the vessel or are they same? (b)Which show the highest weight when weighed on a weighing scale or are they same?



Three differently shaped vessels filled with water to same level.

Sol. *(a) The mercury at the bottom of each vessel is at the same depth d below the surface. Neglecting the pressure at the surface, the pressures at the bottom must be equal hence:*

$$P = \rho g d$$

(b) The weight of each filled vessel is equal to the weight of the vessel itself plus the weight of the mercury inside. The vessels themselves are of equal weights, but vessel A holds more mercury than C, while vessel B holds less mercury than C. Vessel A weighs the most and vessel B weighs the least.

Illustration :

As the mercury exerts the same downward force on the bottom of each vessel, then why does the vessels weigh differently ?

Sol. *In vessel C ,forces due to fluid pressure on the sides of the container are horizontal. Forces on any two diametrically opposite points on the walls of the container are equal and opposite; thus, the net force on the container walls is zero. The force on the bottom is*

$$F = PA = (\rho g d)\ (\pi r^2)$$

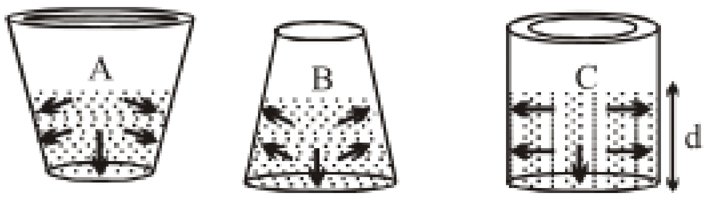
The volume of water in the cylinder is $V = \pi r^2 d$, so

$$F = \rho g V = (\rho V)\ g = mg$$

The force on the bottom of vessel C is equal to the weight of the water, as expected. The forces due to fluid pressure on the sides of the containers A and B have vertical components also. Hence the force between the fluid and the base of container will not be equal to the weight of the fluid. These containers support the fluid by exerting an upward force equal in magnitude to the weight of the fluid but some force is being applied by the sidewalls and the remaining by the bottom. Fig. ( ) shows the forces acting on each container due to the water.

The force on the bottom of vessel A is less than the weight of the mercury in the container, while

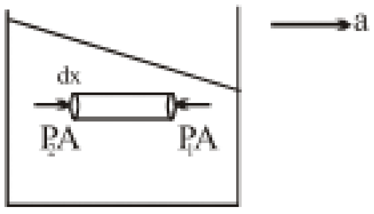
the force on the bottom of vessel B is greater than the weight of the mercury. In vessel A, the forces on the container walls have downward components as well as horizontal components. The sum of the downward components of the forces on the walls and the downward force on the bottom of the container is equal to the weight of the water. Similarly, the forces on the walls of vessel B have upward components. In each case, the total force on the bottom and sides of the container due to the water is equal to the weight of the water.



Forces exerted on the containers by the water.

Linear Accelerated Motion :

We consider an open container of a liquid that is moving along a straight line with a constant acceleration a as shown in Fig.



Lets consider a small horizontal cylinder of length dx and crossectional area A located y below the free surface of the fluid. This cylinder is accelerating in ground frame with accleration a hence the net horizontal force acting on it should be equal to the product of mass(dm) and acceleration.

$dm = Adx\rho$

$$P_2A - P_1A = (Adx\rho)a \qquad P_2 - P_1 = Padx$$

If we say that the right face of the cylinder is y below the free surface of the fluid then the left surface is y + dy below the surface of liquid. Thus

$$P_2 - P_1 = \rho g dy$$

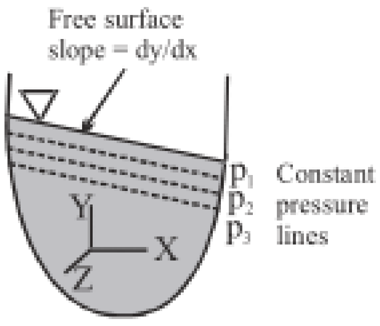
$$\therefore \qquad \frac{dy}{dx} = \frac{a}{g}$$

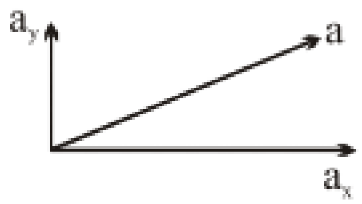
Since the slope of the free surface is coming out to be constant we can say that it must be straight line.

$$\tan\theta = \frac{a}{g}$$

If the container have acceleration along y also than the slope of this line is given by the relationship.

$$\frac{dy}{dx} = -\frac{a_x}{g + a_y}$$





Along a free surface the pressure is constant, so that for the accelerating mass shown in Fig. ( ) the free surface will be inclined if $a_x \neq 0$. In addition, all lines parallel to the free surface will have same presure. For the special circumstance in which $a_x = 0, a_y \neq 0$. which corresponds to the mass of fluid accelerating in the vertical direction, Eq. ( ) indicates that the fluid surface will be horizontal. However, from Eq. ( ) we see that the pressure variation is not ρgdy, but is given by the equation.

$$dP = \rho(g + a_y)dy$$

Thus, the pressure on the bottom of a liquid-filled tank which is resting on the floor of an elevator that is accelerating upward will be more than, if the, tank would have been at rest (or moving with a constant velocity). It is to be noted that for a freely falling fluid mass ($a_y = -g$), the pressure variation in all three coordinate directions are zero, which means that the pressure throughout will be same. The pressure throughout a "blob" of a liquid floating in an orbiting space shuttle (a form of free fall) is zero. The only force holding the liquid together is surface tension.

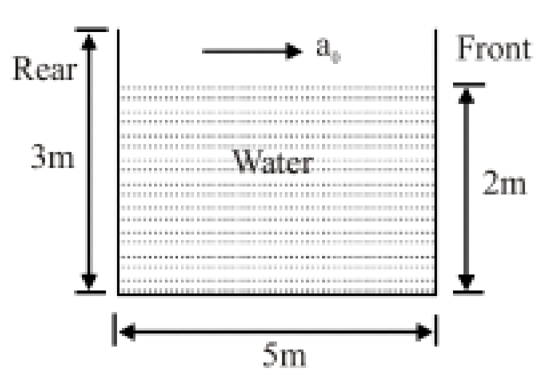
Illustration :

An open rectangular tank 5 m × 4 m × 3m high containing water upto a height of 2 m is accelerated horizontally along the langer side.

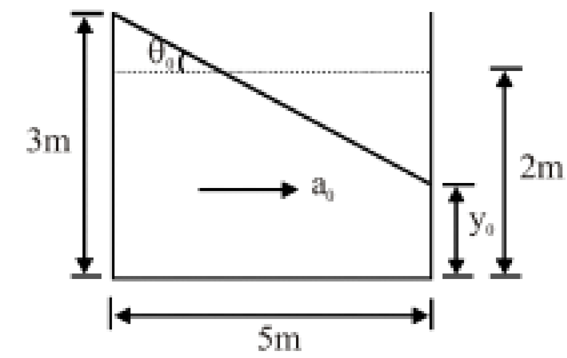
(a) Determine the maximum acceleration that can be given without spilling the water.

(b) Calculate the percentage of water spilt over, if this acceleration is increased by 20%

(c) If initially, the tank is closed at the top and is acceleration horizontally by 9 m/s², find the gauge pressure at the bottom of the front and rear walls of the tank. (Take g = 10 m/s²)



Sol. *(a) Volume of water inside that tank remains constant*



$$\left(\frac{3+y_0}{2}\right) 5 \times 4 = 5 \times 2 \times 4$$

or $y_0 = 1m$

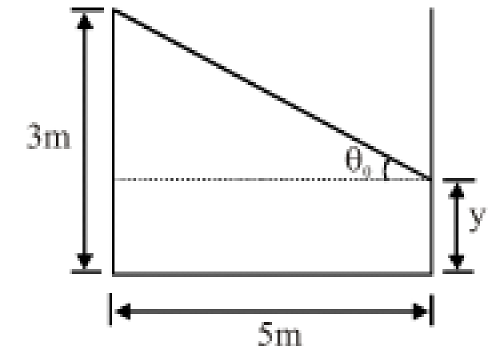
$\therefore \quad \tan\theta_0 = \dfrac{3-1}{5} = 0.4$

Since, $\tan\theta_0 = \dfrac{a_0}{g}$, *therefore* $a_0 = 0.4g = 4\ m/s^2$

(b) When acceleration is increased by 20%

$a = 1.2\ a_0 = 0.48\ g$

$\therefore \quad \tan\theta = \dfrac{a}{g} = 0.48$



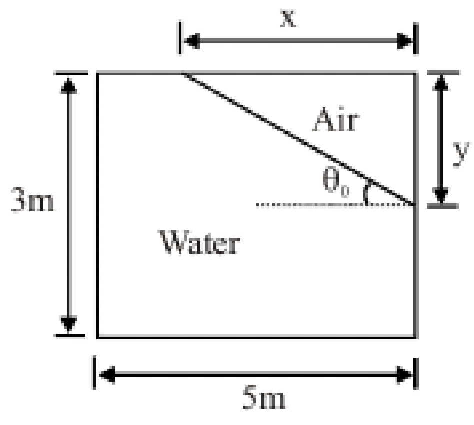
Now, $y = 3–5 \tan\theta = 3 - 5\ (0.48) = 0.6\ m$

Fraction of water spilt over

$$= \frac{4\times2\times5\frac{(3+0.6)}{2}\times5\times4}{2\times5\times4} = 0.1$$

Percentage of water spilt over = 10%

(c) $a' = 0.9\,g$



$$tan\theta' = \frac{a'}{g} = 0.9$$

volume of air remins constant

$$4\times\frac{1}{2}yx = (5)(1)\times4$$

Since $y = x\ tna\ \theta' = 5$

or $x = 3.33\ m\ ;\ y = 3.0\ m$

Gauge presure at the bottom of the

(i) Front wall p_f = *zero*

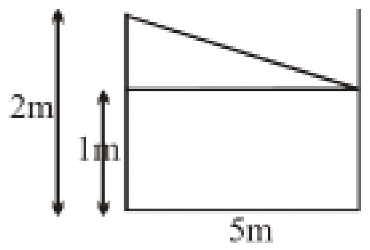
(ii) Rear wall $p_r = (5tan\theta')\,\rho_w g = 5\,(0.9)(10^3)(10) = 4.5\times10^4\ Pa$

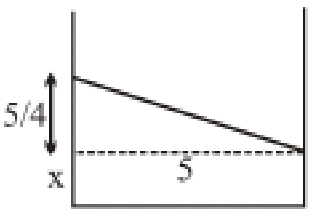
Illustration :

The cross section of a tank kept on a vehicle is shown in Fig.. The rectangular tank is open to the atmosphere. During motion of the vehicle, the tank is subjected to a constant linear acceleration, $a = 2.5\ m/s^2$. *How much fluid will be left inside the tank if initially the tank is half filled. The vessel is 5m wide and 2m high.*

Sol. *If the height of the liquid on the left wall is greater than 2m the fluid will be spilled out. we can find the angle that the fluid will make with the horizontal.*

$$tan\ \theta = \frac{2.5}{10} = \frac{1}{4}$$





Lets assume that the dimension of tank in the plane perpendicular to the page is d.
From the geometry its easy to see that free surface on RHS will go down and will rise on LHS. Thus if we assume that fluid on RHS has not touched the floor, we will have fluid taking ashape as described in the diagram. The cuboid part will have volume $x\times5\times d$, *where x is the height above the bottom.*

The wedge part will have the volume $\frac{1}{2}\times h\times5\times d$ *where h can be found as following*

$(h/5) = tan\ \theta = (1/4)$

Thus total volume will be $\frac{1}{2} \times (5/4) \times 5 \times d + x \times 5 \times d$ *and if we assume there is no spilling than it must beequal to the final volume.*

$$\frac{1}{2} \times (5/4) \times 5 \times d + x \times 5 \times d = 1 \times 5 \times d$$

solving we get $x = \frac{3}{8}$

$\therefore$ *Total length* $\frac{5}{4} + \frac{3}{8} = \frac{10+3}{8} = \frac{13}{8} < 2$

Thus, height is less than 2.

Hence water will not spill.

Illustration :

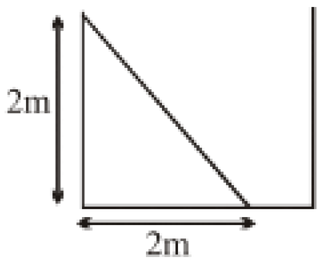
In previous question how much fluid will be left inside the tank if the vehicle accelerates at accleration, $a = 10\ m/s^2$ *?*

Sol. *(a) If the height of the liquid on the left wall is greater than 2m the fluid will be spilled out.*

If dimenion of tank in the plane perpendicular to page is d

$$\tan\theta = \frac{10}{10} = 1, \text{ thus } \theta = \frac{\pi}{4}$$

In this case fluid can not remain inside. Fluid having an inclined free surface at 45^0 *angle, and covering the bottom of length 5m, will also be 5 m high. This will require the wall to be of 5 m height, which is just 2m for the given vessel. Instead if we think it other way round to keep in contact with the LHS wall, bottom will have to be covered only 2m with the fluid as as shown in the diagram,*



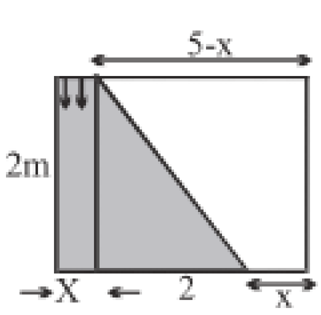
Fluid Inside $= (1/2) \times 2 \times 2 \times d\ m^3$

Remain inside $= 2d\ m^3$

$\therefore$ *Thus volume of fluid gone Outside* $= 3d\ m^3$

(b) *If the vessel is closed from the top and now accelerated at* $10\ m/s^2$*, then what length of the of the floor will be uncovered?*

Sol. *Looking at the figure it is clear*



$X + 2\ m + X = 5\ m$

$X = 1.5\ m$ *Ans*

Rotating Vessel

Consider a cylinderical vessel, rotating at constant angular velocity about its axis. If it contains fluid then after an initial irregular shape, it will rotate with the tank as a rigid body. The acceleration of fluid particles located at a distance r from the axis of rotation will be equal to $\omega^2 r$, and the direction of the acceleration is toward the axis of rotation as shown in the figure. The fluid particles will be undergoing circular motion.

Lets consider a small horizontal cylinder of length dr and crossectional area A located y below the free surface of the fluid and r from the axis. This cylinder is accelerating in ground frame with accleration $\omega^2 r$ towards the axis hence the net horizontal force acting on it should be equal to the product of mass (dm) and acceleration.

$dm = Adr\rho$

$$P_2A - P_1A = (Adr\rho)\omega^2 r$$

If we say that the left face of the cylinder is y below the free surface of the fluid then the right surface is y + dy below the surface of liquid. Thus

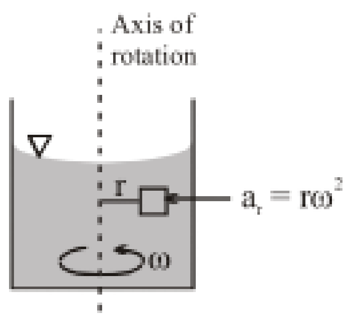
$$P_2 - P_1 = \rho g dy$$

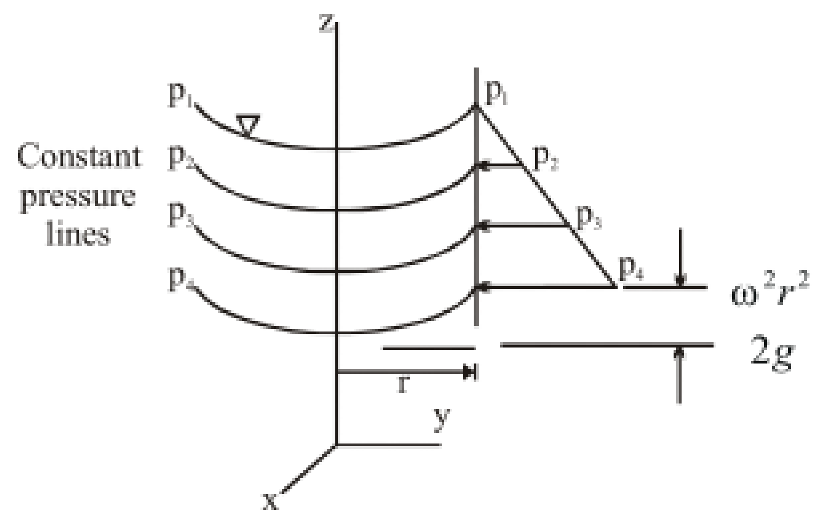
Thus solving we get, $$\frac{dy}{dr} = \frac{r\omega^2}{g}$$

and, therefore, the equation for surfaces of constant pressure is

$$y = \frac{\omega^2 r^2}{2g} + \text{constant}$$

This equation means that these surfaces of constant pressure are parabolic as shown in Fig

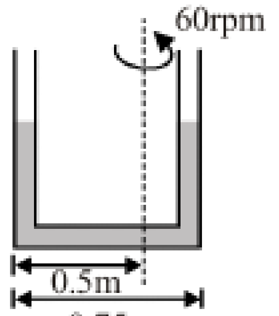




The pressure varies with the distance from the axis of rotation, but at a fixed radius, the pressure varies hydrostatically in the vertical direction as shown in fig.

Illustration :

A vertical U - tube with the two limbs 0.75m apart is filled with water and rotated about a vertical axis 0.5m from the left limb, as shown in the figure. Determine the difference in elevation of the water levels in the two limbs. When speed of rotation is 60 rpm.

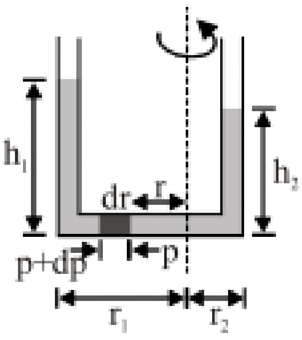


Sol. *consider a small element of length dr at a distance r from the axis of rotration considering the equilibrium of this element.*

$$(p + dp) - p = \rho\omega^2 r dr$$

or $$dp = \rho\omega^2 r\, dr$$

On integration between 1 and 2,



$$p_1 - p_2$$

$$= \rho\omega^2 \int_{-r_2}^{r_1} r dr$$

$$p_1 - p_2 = \frac{\rho\omega^2}{2}(r_1^2 - r_2^2)$$

or $$h_1 - h_2 = \frac{\omega^2}{2g}\left[r_1^2 - r_2^2\right] = \frac{(2\pi)^2}{2(10)}[(0.5)^2 - (0.25)^2] = 0.37\text{m}$$

Pascal's Principle

When you squeeze one end of a tube to get toothpaste out the other end, you are watching **Pascal's principle** in action. This principle is also the basis for the Heimlich maneuver, in which a sharp pressure increase properly applied to the abdomen is transmitted to the throat, forcefully ejecting food lodged there. The principle was first stated clearly in 1652 by Blaise Pascal (for whom the unit of pressure is named):

A change in the pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the walls of its container.

Demonstrating Pascal's Principle

Consider the case in which the incompressible fluid is a liquid contained in a tall cylinder, as in Fig. The cylinder is fitted with a piston on which a container of lead shot rests. The atmosphere, container, and shot exert pressure p_{ext} on the piston and thus on the liquid. The pressure p at any point P in the liquid is then

$$P = P_{ext} + \rho_{gh}$$

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Let us add a little more lead shot to the container to increase P_{ext} by an amount P_{ext}. The quantities P_{ext}, g and h in Eq. are unchanged, so the pressure change at P is

$$\Delta p = \Delta p_{ext}$$

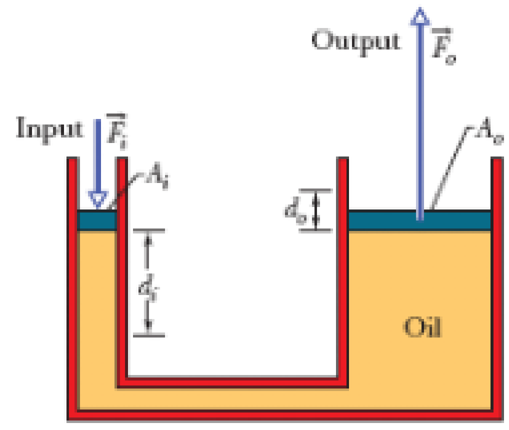
This pressure change is independent of h, so it must hold for all points within the liquid, as Pascal's principle states.

Pascal's Principle and the Hydraulic Lever

Figure shows how Pascal's principle can be made the basis of a hydraulic lever. In operation, let an external force of magnitude F_i be directed downwardon the left-hand (or input) piston, whose surface area is A_i. An incompressible liquid in the device then produces an upward force of magnitude F_0 on the right-hand (or output) piston, whose surface area is A_0. To keep the system in equilibrium, there must be a downward force of magnitude *Fo* on the output piston from an external load (not shown). The force $\vec{F}_i$ applied on the left and the downward force $\vec{F}_0$ from the load on the right produce a change Δp in the pressure of the liquid that is given by

$$\Delta p = \frac{F_i}{A_i} = \frac{F_0}{A_0}$$

$$F_0 = F_i \frac{A_0}{A_i}$$



Equation shows that the output force F_0 on the load must be greater than the input force F_i if $A_0 > A_i$ as is the case in figure.

If we move the input piston downward a distance d_i, the output piston moves upward a distance do, such that the same volume V of the incompressible liquid is displaced at both pistons.Then

$$V = A_i d_i = A_0 d_0$$

which we can write as

$$d_0 = d_i \frac{A_i}{A_0}$$

This shows that, if Ao > Ai (as in Figure), the output piston moves a smaller distance than the input piston moves.

From Eqs. we can write the output work as

$$W = F_0 d_0 = \left(F_i \frac{A_0}{A_i}\right)\left(d_i \frac{A_i}{A_0}\right) = F_i d_i$$

which shows that the work W done on the input piston by the applied force is equal to the work W done by the output piston in lifting the load placed on it.

The advantage of a hydraulic lever is this:

With a hydraulic lever, a given force applied over a given distance can be transformed to a greater force applied over a smaller distance.

The product of force and distance remains unchanged so that the same work is done. However, there is often tremendous advantage in being able to exert the larger force. Most of us, for example, cannot lift an automobile directly but can with a hydraulic jack, even though we have to pump the handle farther than the automobile rises in a series of small strokes.

Illustration :

Find the pressure in the air column at which the piston remains in equilibrium. Assume the piston to be massless and frictionless.

Sol. *Let p_a be the air pressure above the piston.*

Applying pascal's law at points A and B.

$$p_{atm} + \rho_w g(5) = p_a + \rho_k g(1.73)\sin 60^o$$

$$p_a = (10^3)(10)(5) + 10^5 - (800)(10)\frac{\sqrt{3}}{2}$$

$$= 138\ kPa$$

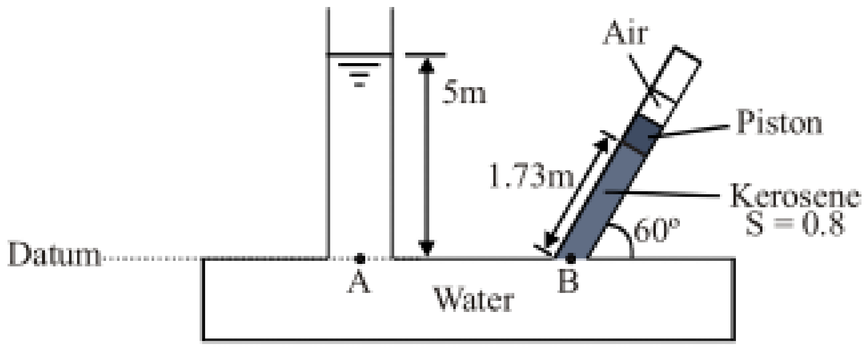
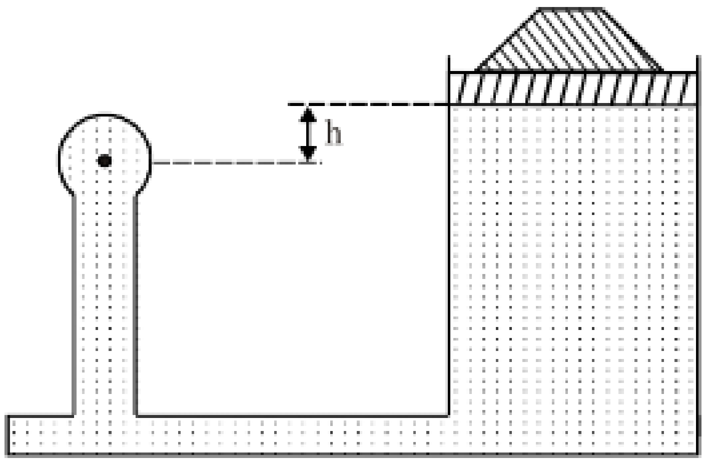


Illustration :

A weighted piston confines a fluid of density ρ in a closed container, as shown in the figure. The combined weight of piston and weight is $W = 200$ N, and the cross-sectional area of the piston is $A = 8\ cm^2$. Find the total pressure at point B if the fluid is mercury and $h = 25$ cm ($\rho_m = 13600\ kg/m^3$). What would an ordinary pressure gauge read at B ?

Sol. *Notice what Pascal's principle tells us about the pressure applied to the fluid by the piston and atmosphere. This added pressure is applied at all points within the fluid. Therefore, the total pressure at B is composed of three parts :*



Pressure of atmosphere = 1.0×10^5 *Pa*

Pressure due to pistion and weight = $\frac{W}{A} = \frac{200N}{8\times10^{-4}m^2} = 2.5\times10^5 Pa$

Pressure due to height h of fluid = $h\rho g = 0.33 \times 10^5$ *Pa*

In this case, the pressure of the fluid itself is relatively small. We have

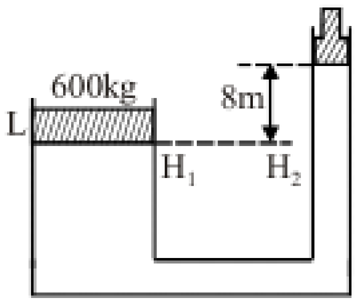
Total pressure at B = 3.8×10^5 *Pa = 380 kPa*

The gauge pressure does not include atmospheric pressure. Therefore,

Gauge pressure at B = 280 kPa

Illustration :

For the system shown in figure, the cylinder on the left, at L, has mass of 600 kg and a cross-sectional area 25 cm² and negligible weight. If the apparatus is filled with oil (ρ = 0.78 g/cm³), find the force F required to hold the system in equilibrium as shown in figure.



Sol. *The pressure at point H_1 and H_2 are equal because they are at the same level in the single connected fluid. Therefore,*

Pressure at H_1 = Pressure at H_2

(Pressure due to left piston) = (Pressure due to F and right piston) + (pressure due to 8 m of oil)

$$\frac{(600)(9.8)N}{0.08m^2} = \frac{F}{25\times10^{-4}m^2} + (8m)(780kg/m^3)(9.8m/s^2)$$

After solving, we get , F = 31 N.

Practice Exercise

Q.1 The passenger are advised to remove the ink from their pens whils going up in an aeroplane. Explain why?

Q.2 A hydraulic press has a ram (weight arm) 12.5cm in diameter and plunger (Force arm) of 1.25 cm diameter what force would be required by plunger to raise a weight of 1 tonn on the ram.

Q.3 Pressure 3 m below free surface of a liquid is 15 KN/m² in excess of atmosphere pressure. Determine its density and specific gravity. [g = 10 m/sec²]

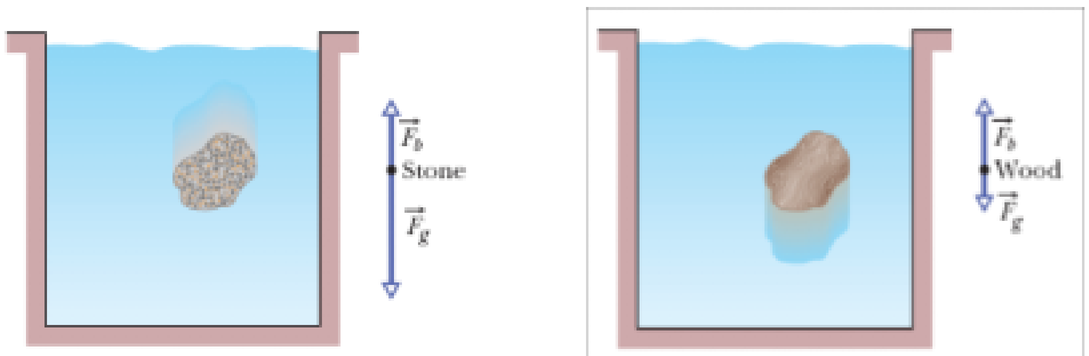
Answers

Q.1 Pressure at heights gets reduced, resulting rising of ink and leakage.

Q.2 10 kg (98.1 N) Q.3 500 kg/m³, 0.5

Archimedes' Principle

Figure shows a student in a swimming pool, manipulating a very thin plastic sack (of negligible mass) that is filled with water. She finds that the sack and its contained water are in static equilibrium, tending neither to rise nor to sink. The downward gravitational force $\vec{F}_g$ on the contained water must be balanced by a net upward force from the water surrounding the sack.



(b) (c)

This net upward force is a **buoyant force** $\vec{F}_b$. It exists because the pressure in the surrounding water increases with depth below the surface. Thus, the pressure near the bottom of the sack is greater than the pressure near the top, which means the forces on the sack due to this pressure are greater in magnitude near the bottom of the sack than near the top. Some of the forces are represented in Fig. (a), where the space occupied by the sack has been left empty. Note that the force vectors drawn near the bottom of that space (with upward components) have longer lengths than those drawn near the top of the sack (with downward components). If we vectorially add all the forces on the sack from the water, the horizontal components cancel and the vertical components add to yield the upward buoyant force $\vec{F}_b$ on the sack. (Force $\vec{F}_b$ is shown to the right of the pool in Fig. (a.) Because the sack of water is in static equilibrium, the magnitude $\vec{F}_b$ of is equal to the magnitude $m_f g$ of the gravitational force $\vec{F}_g$ on the sack of water: $F_b = m_f g$. (Subscript *f* refers to fluid, here the water.) In words, the magnitude of the buoyant force is equal to the weight of the water in the sack.

In Fig. (b), we have replaced the sack of water with a stone that exactly fills the hole in Fig. (a). The stone is said to displace the water, meaning that the stone occupies space that would otherwise be occupied by water. We have changed nothing about the shape of the hole, so the forces at the hole's surface must be the same as when the water-filled sack was in place. Thus, the same upward buoyant force that acted on the water-filled sack now acts on the stone; that is, the magnitude $\vec{F}_b$ of the buoyant force is equal to $m_f g$, the weight of the water displaced by the stone.

Unlike the water-filled sack, the stone is not in static equilibrium. The downward gravitational force $\vec{F}_g$ on the stone is greater in magnitude than the upward buoyant force, as is shown in the free-body diagram in Fig. (*b)*. The stone thus accelerates downward, sinking to the bottom of the pool. Let us next exactly fill the hole in Fig. (a) with a block of lightweight wood, as in Fig. (c*)*. Again, nothing has changed

about the forces at the hole's surface, so the magnitude $\vec{F}_b$ of the buoyant force is still equal to $m_f g$, the weight of the displaced water. Like the stone, the block is not in static equilibrium. However, this time the gravitational force $\vec{F}_g$ is lesser in magnitude than the buoyant force (as shown to the right of the pool), and so the block accelerates upward, rising to the top surface of the water. Our results with the sack, stone, and block apply to all fluids and are summarized in **Archimedes' principle:**

When a body is fully or partially submerged in a fluid, a buoyant force $\vec{F}_b$ from the surrounding fluid acts on the body.The force is directed upward and has a magnitude equal to the weight $m_f g$ of the fluid that has been displaced by the body. The buoyant force on a body in a fluid has the magnitude

$$F_b = m_f g \text{ (buoyant force)}$$

where m_f is the mass of the fluid that is displaced by the body.

Illustration :

Fin the density and specific gravity of gasoline if 51 g occupies 75 cm³ ?

Sol. $Density = \dfrac{\text{mass}}{\text{volume}} = \dfrac{0.051\text{kg}}{75\times10^{-6}\text{m}^3} = 680\,\text{kg/m}^3$

$$Sp.gr = \frac{\text{density of gasoline}}{\text{density of water}} = \frac{680\text{kg/m}^3}{1000\,\text{kg/m}^3} = 0.68$$

$$Sp.\ gravity = \frac{\text{mass of } 75\,\text{cm}^3 \text{ gasoline}}{\text{mass of } 75\text{cm}^3 \text{ water}} = \frac{51\text{kg}}{75\,\text{kg}} = 0.68$$

Illustration :

The mass of a liter of milk is 1.032 kg. The butterfat that it contains has a density of 865 kg/m³ when pure, and it constitutes 4 present of the milk by volume. What is the density of the fat-free skimmend milk?

Sol. *Volume of fat in 1000cm³ of milk* $= 4\% \times 1000\ cm^3 = 40cm^3$

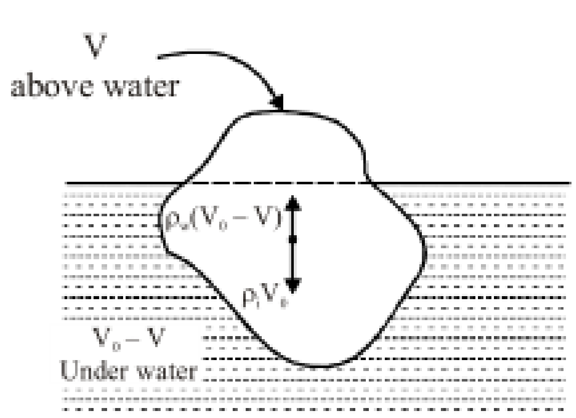
Mass of 40 cm³ fat $= v\rho = (40\times10^{-6}m^3)\ (865\ kg/m^3) = 0.0346\ kg$

$$Denstiy\ of\ skimmed\ milk = \frac{\text{mass}}{\text{volume}} = \frac{(1.032-0.0346)\text{kg}}{(1000-40)\times10^{-6}\text{m}^3} = 1039\text{kg/m}^3$$

Illustration :

An iceberg with a diensity of 920 kg m⁻³. What fraction of the iceberg is visible.

Sol. *Let V be the volume of the iceberg above the water surface, then the volume under water will be* $V_0 - V$.
Under floating conditions, the weight $(\rho_i V_0 g)$ *of the iceberg is balanced by the buoyant force* $\rho_w (V_0 - V)g$. *Thus,*



$$\rho_i V_0 g = \rho_w (V_0 - V)g$$

or $$\rho_w V = (\rho_w - \rho_i)\, V_0$$

or $$\rho_w V = (\rho_w - \rho_i)V_0$$

or $$\frac{V}{V_0} = \left(\frac{\rho_w - \rho_i}{\rho_w}\right)$$

Since $\rho_w = 1025 \text{ kg m}^{-3}$ and $\rho_i = 920 \text{ kg m}^{-3}$, therefore,

$$\frac{V}{V_0} = \frac{1025 - 920}{1025} = 0.10$$

Hence 10% of the total volume is visible.

Illustration :

When a 2.5 kg crown is immersed in water, it has an apparent weight of 22 N. What is the density of the crown ?

Sol. *Let* W = *actual weight of the crown*

W' = *apparent weight of the crown*

ρ = *density of crown*

ρ_0 = *density of water*

The buoyant force is given by

$$F_B = W - W'$$

or $$\rho_0 Vg = W - W'$$

Since $W = \rho Vg$ therefore, $V = \dfrac{W}{\rho g}$

Eliminating V from the above two equation, we get

$$\rho = \frac{\rho_0 W}{W - W'}$$

Here $W = 25N$; $W' = 22$ N; $\rho_0 = 10^3 \text{ kg m}^{-3}$

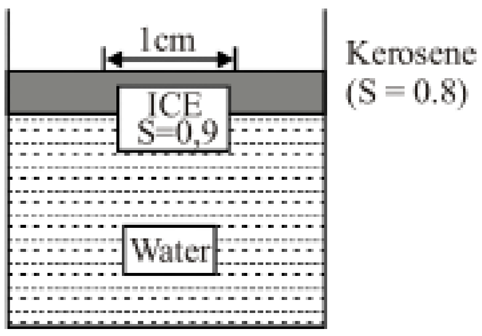
$$\therefore \quad \rho = \frac{(10^3)(25)}{25 - 22} = 8.3 \times 10^3 \text{ kg m}^{-3}$$

Illustration :

An ice cube of side 1 cm is floating at the interface of kerosene and water in a beaker of base area 10 cm². The level of kerosene is just covering the top surface of the ice cube.

(a) Find the depth of submergence in the kerosene and that in the water.

(b) Find the change in the total level of the liquid when the whole ice melts into water.



Sol. *(a) Condition of floating*

$$0.8\, \rho_w g h_k + \rho_w g h_w = 0.9\, \rho_w g h$$

or $$0.8 h_k + h_w = (0.9)\, h \qquad \text{...(i)}$$

Where h_k *and* h_w *be the submerged depth of the ice in the kerosene and water, respectively.*

Also $$h_k + h_w = h \qquad \text{...(ii)}$$

Solving equation (i) and (ii) we get $h_k = 0.5$ cm $h_w = 0.5$ cm

(b) $\underset{\text{(Ice)}}{1\text{cm}^3} \xrightarrow{\text{melts}} \underset{\text{(water)}}{0.9\text{cm}^3}$

Fall in the level of kerosene $\Delta h_k = \dfrac{0.5}{A}$

Rise in the level of water $\Delta h_w = \dfrac{0.9-0.5}{A} = \dfrac{0.4}{A}$

Net fall in the overall level.

$$\Delta h = \frac{0.1}{A} = \frac{0.1}{10} = 0.01\text{cm} = 0.1\text{mm}$$

Illustration :

Find the density and specific gravity of gasoline if 51 g occupies 75 cm³ ?

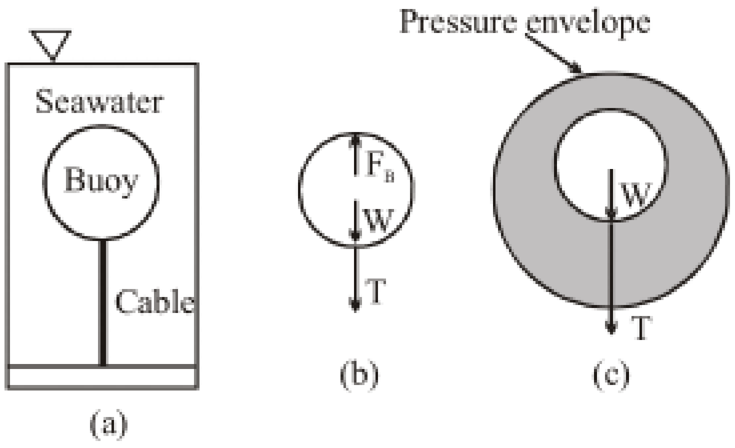
Sol. $$Density = \frac{\text{mass}}{\text{volume}} = \frac{0.51\text{kg}}{75\times10^{-6}\text{m}^3} = 680\text{kg}/\text{m}^3$$

$$Sp.\ gr = \frac{\text{denstiy of gasoline}}{\text{diensity of water}} = \frac{680\text{kg}/\text{m}^3}{1000\text{kg}/\text{m}^3} = 0.68$$

or $$Sp.\ graviy = \frac{\text{mass of } 75\text{cm}^3 \text{ gasoline}}{\text{mass of } 75\text{cm}^3 \text{ water}} = \frac{51\text{g}}{75\text{g}} = 0.68$$

Illustration :

A spherical buoy has a diameter of 2 m, and mass 100 kg, and is anchored to the seafloor with a cable as is shown in Fig.(a). The buoy is completely immersed in water as illustrated. For this condition what is the tension of the cable ?



Sol. *We first draw a free-body diagram of the buoy as is shown in Fig. (b), where F_B is the buoyant force acting on the buoy, W is the weight of the buoy, and T is the tension in the cable.*

For equilibrium it follows that

$$T = F_B - W$$

From Eq. $$F_B = \rho V g$$

and for water with $\rho = 1000$ Kg/m³ and $V = \pi d^3/6$ then

$F_B = (1000\ Kg/m^3)\ [(\pi/6)(2m)^3] = 4188.8\ N$

The tension in the cable can now be calculated as

$T = 4188.8\ N - 1000\ N = 3188.8\ N$ *(Ans.)*

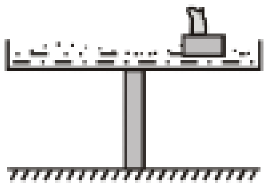
Illustration :

A wooden block floats vertically in a glass filled with water. How will the level of the water in the glass change if the block is kept in a horizontal position ?

Sol. *The level of the water will not change because the quantity of water displaced will remain the same.*

Illustration :

A vessel filled with water is placed exactly in middle of a thin wall(fig.). Will the system topple if a small wooden boat carrying some weight is floated in the vessel?



Sol. *The system will not topple, since according to Pascal's law the pressure on the bottom of the vessel will be the same everywhere thus the body will still remain in rotational equilibirium*

Illustration :

A homogeneous piece of ice floats in a glass filled with water. How will the level of the water in the glass change when the ice melts ?

Sol. *Since the piece of ice floats, the weight of the water displaced by it is equal to the weight of the ice itself or the weight of the water it produces upon melting. For this reason the water formed by the piece of ice will occupy a volume equal to that of the submerged portion, and the level of the water will not change.*

Illustration :

A piece of ice is floating in a tub filled with water. How will the level of the water in the tub change when the ice melts ? Consider the following cases :

(1) a stone is frozen in the ice

(2) the ice contains an air bubble

Sol. *(1) The volume of the submerged portion of the piece with the stone is greater than the sum of the volumes of the stone and the water produced by the melting ice. Therefore, the level of the water in the glass will drop.*

(2) The weight of the displaced water is equal to that of the ice (the weight of the air in the bubble may be neglected). For this reason, as in conceptual eg., the level of the water will not change.

Illustration :

A vessel with a body floating in it is kept in elevator accelerating downwards with acceleration a such that $a < g$. Will the body rise or sink further in the vessel?

Sol. *The force of bouyancy on the body can be written as $F = \rho V_2(g - a)$, where V_2 is the volume of the submerged portion of the body in the lift. As pressure at a point h below the surface will become $\rho(g - a)h$ instead of ρgh. Applying the Newton's second Law, remembering that the body was accelerating upwards at a.*

$$Mg - \rho V_2 (g - a) = Ma$$

Hence, $V_2 = \dfrac{M}{\rho}$ *thus* $V_2 = V$*, as in a stationary vessel,* $V = \dfrac{M}{\rho}$*. Thus the body does not rise to the surface.*

Illustration :

Mercury is poured into two communicating identical cylindrical vessels then equal amount of water is poured in both the vessels above the mercury. The level of the water in both vessels becomes same. Will the level of the water and the mercury be the same if a piece of wood is dropped into one vessel and some water equal in weight to this piece is added to the other ?

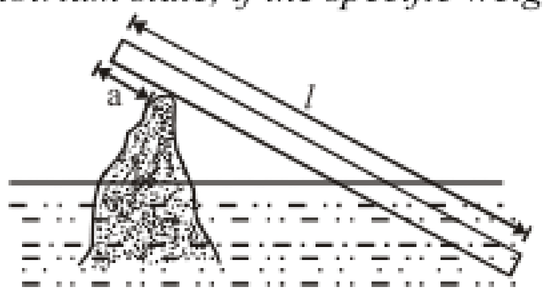
Sol. *Before the piece of wood is droped the level of the mercury was same, as the level of the water in both the vessel is same.*

The piece of wood applies the same force as the water added, on the mercury. So the level of the mercury in both vessels will be same.

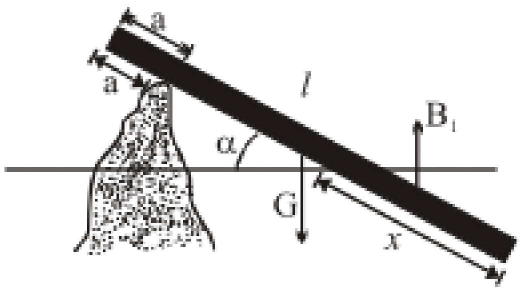
The submerged portion of the piece of wood occupies the same volume as the equal weight of the water. As water that will be displaced by this piece is equal to, its weight which is equal to the amount of water added. Therefore, if the crossections of the vessels are the same, the level of the water in both vessels will coincide.

Illustration :

One end of a board of length l is hinged on top of a stone protruding from water. Length a of the board is above the point of support (Fig.). What part of the board is below the surface of the water in equillibirium state, if the specific weight of wood is γ?



Sol. *We can not solve this problem by equating net force to zero,as the force applied by the stone is unknown. Thus we will balance the torques of the forces acting on the board with respect to point C (Fig.). This will exclude the force applied by the stone, as the moment arm for this force will be zero.*



Here B_l is the bouyant force applied by the water on the submerged portion and isequal to the volume of submerged portion multiplied to density of water.

$B_l = Ax\gamma_0$ *The length of moment arm is* $\left(l-a-\frac{x}{2}\right)\cos\alpha$

Weight of the board is $W = Al\gamma$. Here A is the cross-sectional area of the board and γ_0 the specific weight of the water. Equating torque we get

$$B_l\left(l-a-\frac{x}{2}\right)\cos\alpha = W\left(\frac{1}{2}-a\right)\cos\alpha$$

Hence , $$x = (l-a) \pm \sqrt{(l-a)^2 - \frac{\gamma}{\gamma_0}l(l-2a)}$$

Since $x < l - a$, only one solution is valid :

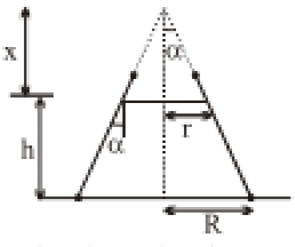
$$x = (l-a) - \sqrt{(l-a)^2 - \frac{\gamma}{\gamma_0}l(l-2a)}$$

Illustration :

A conical vessel without a bottom tightly stands on a table. A liquid is poured into the vessel and as soon as its level reaches the height h, the pressure of the liquid raises the vessel. The radius of the bottom greater base of the vessel is R, the semi vertex angle of the cone is α, and the weight of the vessel is W. What is the density of the liquid ?.

Sol. *We need to calculate the force mentioned above. The direct solution may appear to be taking elemental rings and integrating the force. But there is a faster approach. If we consider the liquid as a system in equilibirium, then only unknown force acting on it will be the one applied by the sidewalls. According to Newton's third law, an identical force acts on the vessel.*

Lets assume when the height of the liquid is h in the vessel the vessel rises.



$$\tan\alpha = \frac{r}{x} = \frac{R}{x+h}$$

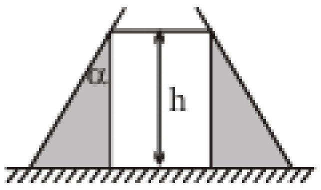
on solving $r = R - h\tan\alpha$ ....(1)

The pressure on the "bottom" of the vessel is ρgh and the force with which the hatched portion (the truncated cone minus the cylinder volume) of the liquid presses on the table is $\rho gh\pi(R^2 - r^2) = \rho gh\pi(2Rh\tan\alpha - h^2\tan^2\alpha)$

According to Newton's third law, an identical force acts on the liquid. For equilibrium of the liquid at the moment when the vessel starts rising

$$W + W_l = \rho gh\pi(2Rh\tan\alpha - h^2\tan^2\alpha)$$

where W_l is the weight of the hatched portion of the liquid



$$W_l = \frac{\rho g h}{3}\{\pi R^2 + \pi(R - h\tan\alpha)^2 + \pi R(R - h\tan\alpha)\} - \rho g h \pi (R - h\tan\alpha)^2$$

Therefore, $$\rho = \frac{W}{\pi g h^2 \tan\alpha\left(R - \frac{h\tan\alpha}{3}\right)}$$

Practice Exercise

Q.1 A boy is carrying a fish in one hand and a bucket full of water in the other hand. He then place the fish in the bucket and thinks that in accordance with Archimedes's prinicple he is now carrying less weight as weight of fish will reduce due to upthrust. Is he thinking right ?

Q.2 Ice flows in water nine tenth of its volume submerged. What is the frctional volume submerged for an iceberg floating on a fresh water lake of a (hypothetical) planet whose gravity is ten times of earth ?

Q.3 If the body is non-homogeneous, then the body rotates in the fluid why ?

Q.4 A cube of wood supporting a 200 gm mass just floats in water. When the mass is removed the cube rise by 2 cm. Find the size of cube

Q.5 A solid ball of density half that of water falls freely under gravity from a height of 19.6 m and then enter water. Upto what depht will the ball go ? How much time will it take to come again to the water surface ? Neglect air resistance and viscosity effects in water.

Q.6 A balloon filled with hydrogen has a volume of 1000 liters and its mass of 1 kg. What would be volume of the block of a very light material which it can just lift ? One litre of the material has a mass of 91.3 gm. (Density of air = 1.3 gm/ litre)

Q.7 An iceberg of density 915 kg/m^3 extends above the surface of sea water of density 1030 kg/m^3. What percent-age of the total volume of iceberg is visibel to an obserber.

Answers

Q.1 it's density is high because of salt Q.2 Same

Q.3 Centre of Buoyancy and centre of gravity are diffrent resulting torque.

Q.4 10 cm Q.5 19.6 m, 4 sec Q.6 3.33 litre. Q.7 11.15%

Floating

When we release a block of lightweight wood just above the water in a pool, the block moves into the water because the gravitational force on it pulls it downward.

As the block displaces more and more water, the magnitude F_b of the upward buoyant force acting on it increases. Eventually, F_b is large enough to equal the magnitude F_g of the downward gravitational force on the block, and the block comes to rest. The block is then in static equilibrium and is said to be floating in the water. In general, When a body floats in a fluid, the magnitude F_b of the buoyant force on the body is equal to the magnitude F_g of the gravitational force on the body.

We can write this statement as

$$F_b = F_g \text{ (Floating)}$$

From Eq. we know that $F_b = m_f g$. Thus,

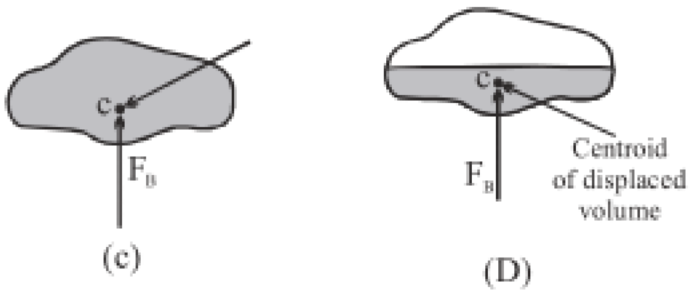
When a body floats in a fluid, the magnitude F_g of the gravitational force on the body is equal to the weight $m_f g$ of the fluid that has been displaced by the body.

We can write this statement as

$$F_g = m_f g$$

In other words, a floating body displaces its own weight of fluid.

The location of the line of action of the buoyant force can be determined by adding torques of the forces due to pressure forces, with respect to some convenient axis. The buoyant force must pass through the center of mass of the desplaced volume, as shown in Fig. (c), as it was in translational and rotational equilibirium. The point through which the buoyant force acts is called the center of buoyancy.



These same results apply to floating bodies which are only partially submerged, as shown in Fig.(d), if the density of the fluid above the liquid surface is very small compared with the liquid in which the body floats. Since the fluid above the surface is usually air, for practial purposes this condition is satisfied.

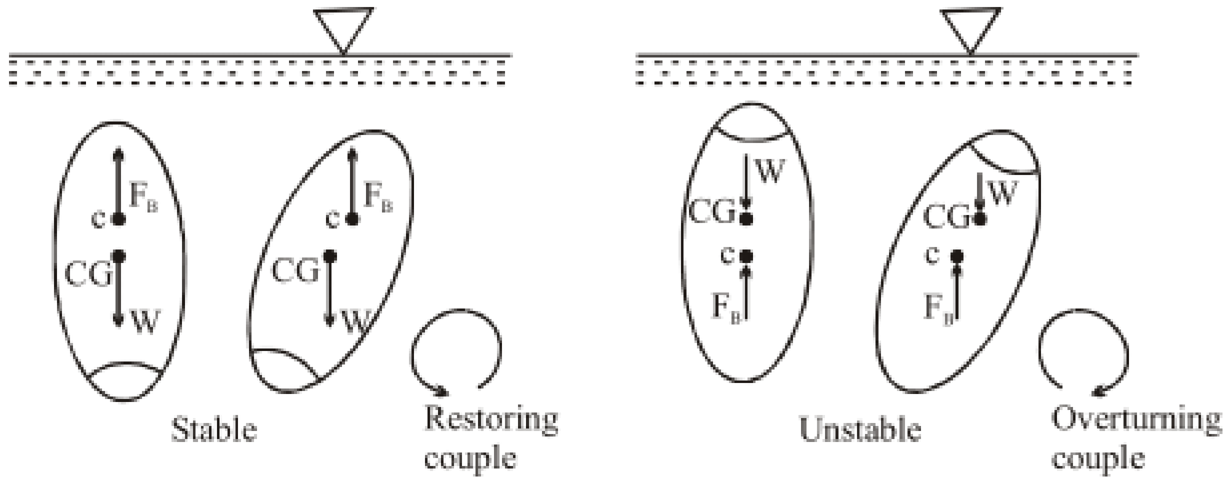
In the above discussion, the fluid is assumed to have a constant density. If a body is immersed in a fluid in which density varies with depth, such as having multiple layers of fluid, the magnitude of the buoyant force remains equal to the weight of the displaced fluid and the buoyant force passes through the center of mass of the displaced volume.

Stability

The center of buoyancy and center of gravity do not necessarily coincide so the floating or submerged body may not be in stable equilibirium. A small rotation can cause the buoyant force to produce either a restoring or overturning torque. For example, for the completely submerged body shown in Fig., which has a center of gravity below the center of buoyancy, a rotation from its equilibrium position will create a restoring torque by the buoyant force, F_B, which causes the body to rotate back to its original position. Thus, if the center of gravity falls below the center of buoyancy, the body is stable.

However, as shown in Fig., if the center of gravity of the completely submerged body is above the center of buoyancy, the resulting torque formed by the weight and the buoyant force will cause the body to overturn and move to a new equilibrium position. Thus, a completely submerged body with its center of gravity above its center of buoyancy is in an unstable equilibrium position.

For floating bodies the stability problem is more complicated, since as the body rotates the location of the center of buoyancy may change.



Fluid Dynamics

Ideal Fluids in Motion

The motion of real fluids is very complicated and not yet fully understood. Instead, we shall discuss the motion of an **ideal fluid,** which is simpler to handle mathematically and yet provides useful results. Here are four assumptions that we make about our ideal fluid; they all are concerned with flow:

1. **Steady flow** In steady (or laminar) flow, the velocity of the moving fluid at any fixed point does not change with time, either in magnitude or in direction. The gentle flow of water near the center of a quiet stream is steady; the flow in a chain of rapids is not. The speed of the smoke particles increases as they rise and, at a certain critical speed, the flow changes from steady to nonsteady.

2. **Incompressible flow** We assume, as for fluids at rest, that our ideal fluid is incompressible; that is, its density has a constant, uniform value.

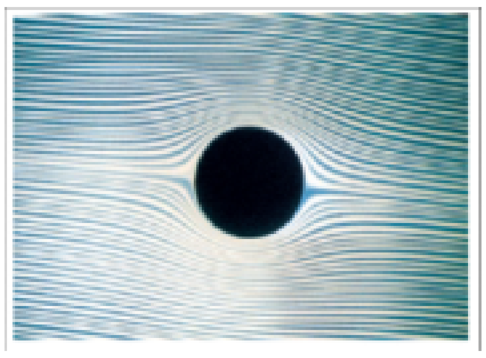
3. **Nonviscous flow** Roughly speaking, the viscosity of a fluid is a measure of how resistive the fluid is to flow. For example, thick honey is more resistive to flow than water, and so honey is said to be more viscous than water. Viscosity is the fluid analog of friction between solids; both are mechanisms by which

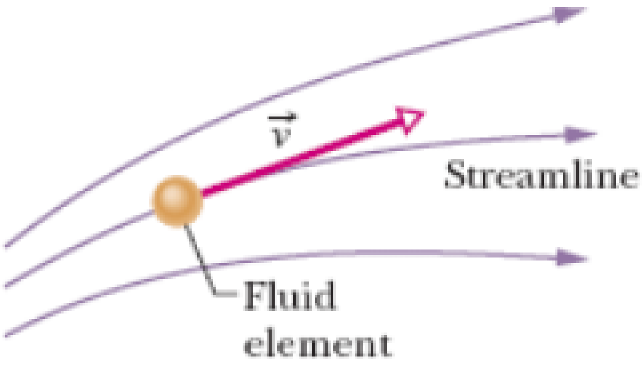
the kinetic energy of moving objects can be transferred to thermal energy. In the absence of friction, a block could glide at constant speed along a horizontal surface. In the same way, an object moving through a nonviscous fluid would experience no viscous drag force-that is, no resistive force due to viscosity; it could move at constant speed through the fluid.

4. **Irrotational flow** : Although it need not concern us further, we also assume that the flow is irrotational. To test for this property, let a tiny grain of dust move with the fluid. Although this test body may (or may not) move in a circular path, in irrotational flow the test body will not rotate about an axis through its own center of mass. For a loose analogy, the motion of a Ferris wheel is rotational; that of its passengers is irrotational. T8hat the velocity of a particle is always tangent to the path taken by the particle. Here the particle is the fluid element, and its velocity is $\vec{v}$ always tangent to a streamline (Figure). For this reason, two streamlines can never intersect; if they did, then an element arriving at their intersection would have two different velocities simultaneously an impossibility.

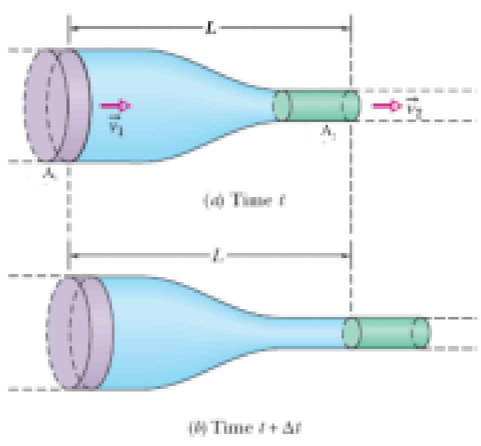
The Equation of Continuity

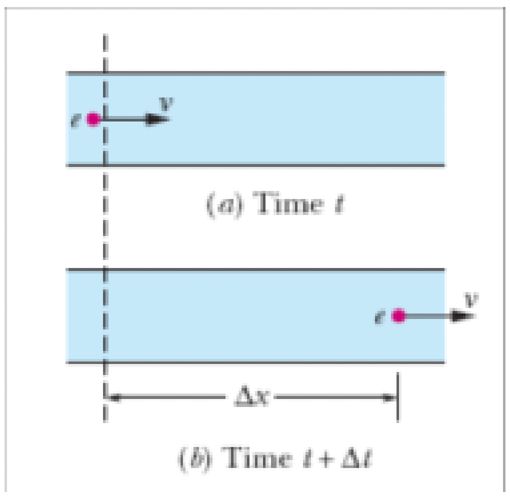
You may have noticed that you can increase the speed of the water emerging from a garden hose by partially closing the hose opening with your thumb. Apparently the speed v of the water depends on the cross-sectional area A through which the water flows.

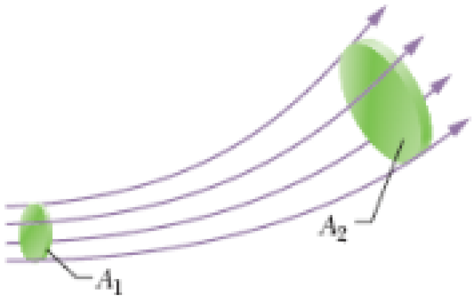




Here we wish to derive an expression that relates v and A for the steady flow of an ideal fluid through a tube with varying cross section, like that in Figure. The flow there is toward the right, and the tube segment shown (part of a longer tube) has length L. The fluid has speeds v_1 at the left end of the segment and v_2 at the right end. The tube has cross-sectional areas A_1 at the left end and A_2 at the right end. Suppose that in a time interval Δt a volume ΔV of fluid enters the tube segment at its left end (that volume is colored purple in Figure. Then, because the fluid is incompressible, an identical volume ΔV must emerge from the right end of the segment (it is colored green in Figure). We can use this common volume ΔV to relate the speeds and areas. To do so, we first consider Fig. , which shows a side view of a tube of uniform crosssectional area A. In Fig.(a), a fluid element e is about to pass through the dashed line drawn across the tube width.The element's speed is v, so during a time interval Δt, the element moves along the tube a distance $\Delta x = v\,\Delta t$.The volume ΔV of fluid that has passed through the dashed line in that time interval Δt is







$$\Delta V = A\,\Delta\,x = Av\,\Delta t.$$

Applying Eq. to both the left and right ends of the tube segment in Fig., we have

$$\Delta V = A_1 v_1\,\Delta t = A_2 v_2\,\Delta t$$

Or $$A_1 v_1 = A_2 v_2 \text{ (equation of continuity)}$$

This relation between speed and cross-sectional area is called the **equation of continuity** for the flow of an ideal fluid. It tells us that the flow speed increases when we decrease the cross-sectional area through which the fluid flows (as when we partially close off a garden hose with a thumb). Equation applies not only to an actual tube but also to any so-called tube of flow, or imaginary tube whose boundary consists of streamlines. Such a tube acts like a real tube because no fluid element can cross a streamline; thus, all the fluid within a tube of flow must remain within its boundary. Figure shows a tube of flow in which the cross-sectional area increases from area A_1 to area A_2 along the flow direction. From Eq. we know that, with the increase in area, the speed must decrease, as is indicated by the greater spacing between streamlines at the right in Fig. . Similarly, you can see that in Fig. the speed of the flow is greatest just above and just below the cylinder. We can rewrite Eq. as

$R_V = Av$ = a constant (volume flow rate, equation of continuity), in which R_V is the **volume flow rate** of the fluid (volume past a given point per unit time). Its SI unit is the cubic meter per second (m^3/s). If the density ρ of the fluid is uniform, we can multiply Eq. by that density to get the **mass flow rate** R_m (mass per unit time):

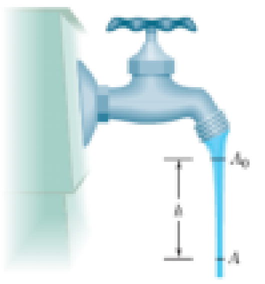
$$R_m = \rho R_V = \rho A v = \text{a constant (mass flow rate).}$$

The SI unit of mass flow rate is the kilogram per second (kg/s). Equation says that the mass that flows into the tube segment of Fig. each second must be equal to the mass that flows out of that segment each second.

Illustration :

Figure shows how the stream of water emerging from a faucet "necks down" as it falls. The indicated cross-sectional areas are A_0 = 1.2 cm2 and A =0.35 cm^2. The two levels are separated by a vertical distance h = 45 mm.What is the volume flow rate from the tap?

The volume flow rate through the higher cross section must be the same as that through the lower cross section.



Sol. *where v_0 and v are the water speeds at the levels corresponding to A_0 and A. From Eq. we can also write, because the water is falling freely with acceleration g,*

$$v_2 = v^2_0 - 2gh.$$

Eliminating v between Eqs. and solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2ghA^2}{A_0^2 - A^2}}$$

$$v_0 = \sqrt{\frac{(2)\times(9.8\text{m}/\text{s}^2)(0.045\text{m})(0.35\text{cm}^2)^2}{(1.2\text{cm}^2)-(0.35\text{cm}^2)^2}}$$

$$v_0 = 0.286 \text{ m/s} = 28.6 \text{ cm/s}.$$

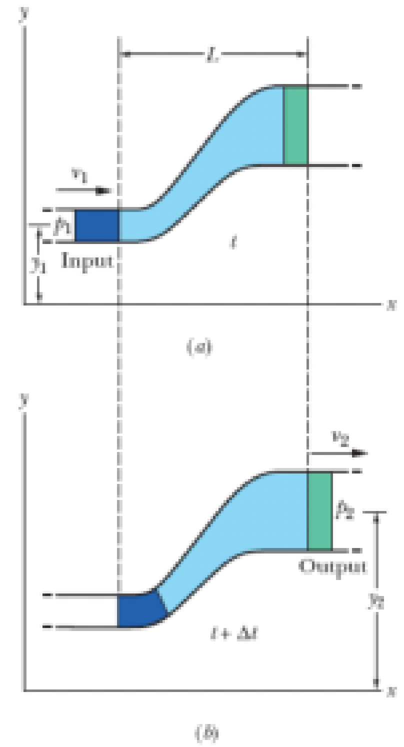
From Eq. , the volume flow rate R_V is then

$$R_V = A_0 v_0 = (1.2 \text{ cm}^2)(28.6 \text{ cm/s})$$
$$= 34 \text{ cm}^3/\text{s}.$$

Bernoulli's Equation

Figure represents a tube through which an ideal fluid is flowing at a steady rate. In a time interval Δt, suppose that a volume of fluid ΔV, colored purple in Fig. , enters the tube at the left (or input) end and an identical volume, colored green in Fig. , emerges at the right (or output) end. The emerging volume must be the same as the entering volume because the fluid is incompressible, with an assumed constant density ρ.

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$



Let y_1, v_1, and p_1 be the elevation, speed, and pressure of the fluid entering at the left, and y_2, v_2, and p_2 be the corresponding quantities for the fluid emerging at the right. By applying the principle of conservation of energy to the fluid, we shall show that these quantities are related by

In general, the term $\frac{1}{2}\rho v^2$ is called the fluid's **kinetic energy density** (kinetic energy per unit volume). We can also write Eq. as

$$p + \frac{1}{2}\rho v^2 + \rho g y \text{ a constant (Bernoulli's equation).}$$

Equations are equivalent forms of **Bernoulli's equation,** after Daniel Bernoulli, who studied fluid flow in the 1700s.* Like the equation of continuity Eq., Bernoulli's equation is not a new principle but simply the reformulation of a familiar principle in a form more suitable to fluid mechanics. As a check, let us apply Bernoulli's equation to fluids at rest, by putting $v_1 = v_2 = 0$ in Eq. The result is

$$p_2 = p_1 + \rho g(y_1 - y_2)$$

Which is equation.

A major prediction of Bernoulli's equation emerges if we take y to be a constant ($y = 0$, say) so that the fluid does not change elevation as it flows. Equation then becomes

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

Which tells us that :

If the speed of a fluid element increases as the element travels along a horizontal streamline, the pressure of the fluid must decrease, and conversely.

Put another way, where the streamlines are relatively close together (where the velocity is relatively great), the pressure is relatively low, and conversely. The link between a change in speed and a change in pressure makes sense if you consider a fluid element. When the element nears a narrow region, the higher pressure behind it accelerates it so that it then has a greater speed in the narrow region. When it nears a wide region, the higher pressure ahead of it decelerates it so that it then has a lesser speed in the wide region. Bernoulli's equation is strictly valid only to the extent that the fluid is ideal. If viscous forces are present, thermal energy will be involved. We take no account of this in the derivation that follows.

Proof of Bernoulli's Equation

Let us take as our system the entire volume of the (ideal) fluid shown in Fig. .We shall apply the principle of conservation of energy to this system as it moves from its initial state to its final state. The fluid lying between the two vertical planes separated by a distance L in Fig. does not change its properties during this process; we need be concerned only with changes that take place at the input and output ends. First, we apply energy conservation in the form of the work–kinetic energy theorem,

$$W = \Delta K,$$

which tells us that the change in the kinetic energy of our system must equal the net work done on the system. The change in kinetic energy results from the change in speed between the ends of the tube and is

$$\Delta K = \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 = \frac{1}{2}\rho \Delta V (v_2^2 - v_1^2)$$

in which Δm ($= \rho\, \Delta V$) is the mass of the fluid that enters at the input end and leaves at the output end during a small time interval Δt.

The work done on the system arises from two sources. The work W_g done by the gravitational force ($\Delta m\, \vec{g}$) on the fluid of mass Δm during the vertical lift of the mass from the input level to the output level is

$$W_g = \Delta m\, g(y_2 - y_1)$$
$$= -\rho\, g \Delta V (y_2 - y_1)$$

This work is negative because the upward displacement and the downward gravitational force have opposite directions.

Work must also be done on the system (at the input end) to push the entering fluid into the tube and by the system (at the output end) to push forward the fluid that is located ahead of the emerging fluid. In general, the work done by a force of magnitude F, acting on a fluid sample contained in a tube of area A to move the fluid through a distance Δx, is

$$F\Delta x = (pA)(\Delta x) = p(A\Delta x) = p\Delta v$$

The work done on the system is then $p_1\, \Delta V$, and the work done by the system is $-p_2\, \Delta V$. Their sum Wp is Wp

$$Wp = p_1\, \Delta V - p_2\, \Delta V$$
$$= -(p_2 - p_1)\, \Delta V.$$

The work–kinetic energy theorem of Eq. now becomes

$$W = Wg + Wp = \Delta K.$$

Substituting from Eqs. yields

$$-\rho g \Delta V (y_2 - y_1) - \Delta V (p_2 - p_1) = \frac{1}{2}\rho \Delta v (v_2^2 - v_1^2)$$

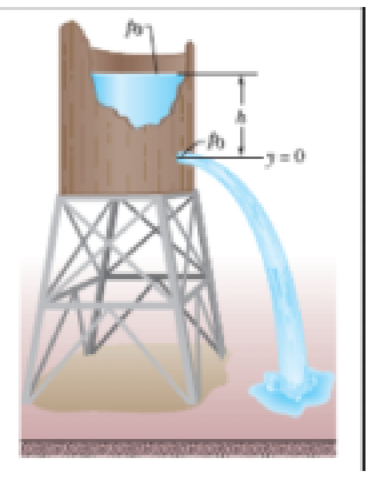
This, after a slight rearrangement, matches Eq. , which we set out to prove.

Illustration :

In the old West, a desperado fires a bullet into an open water tank (Fig.), creating a hole a distance h below the water surface. What is the speed v of the water exiting the tank?

Sol. *From Eq.* $R_V = av = Av_0$ *and thus*

$$v_0 = \frac{a}{A} v$$



Because $a << A$, *we see that* $v_0 << v$. *To apply Bernoulli's equation, we take the level of the hole as our reference level for measuring elevations (and thus gravitational potential energy). Noting that the pressure at the top of the tank and at the bullet hole is the atmospheric pressure* p_0 *(because both places are exposed to the atmosphere), we write Eq. as*

$$p_0 + \frac{1}{2}\rho v_0^2 + \rho g h = p_0 + \frac{1}{2}\rho v^2 + \rho g(0)$$

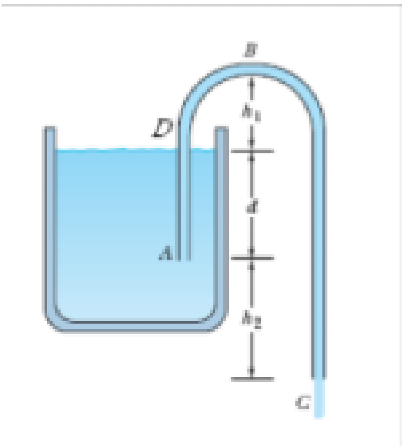
(Here the top of the tank is represented by the left side of the equation and the hole by the right side. The zero on the right indicates that the hole is at our reference level.) Before we solve Eq. for v, we can use our result that $v_0 << v$ *to simplify it: We assume that* v^2_0 *and thus the term* $\frac{1}{2}\rho v_0^2$ *in Eq. , is negligible relative to the other terms, and we drop it. Solving the remaining equation for v then yields*

$$v = \sqrt{2gh} \quad Ans.$$

This is the same speed that an object would have when falling a height h from rest. This is because the work done by atmospheric pressure is cancelling out at open surface and the hole.

Illustration :

Figure shows a siphon, which is a device for removing liquid from a container. Tube ABC must initially be filled, but once this has been done, liquid will flow through the tube until the liquid surface in the container is level with the tube opening at A. The liquid has density 1000 kg/m^3 *and negligible viscosity. The dis-tances shown are* $h_1 = 25$ *cm,* $d = 12$ *cm, and* $h_2 = 40$ *cm. (a) With what speed does the liquid emerge from the tube at C? (b) If the atmospheric pressure is* 1.0×10^5 *Pa, what is the pressure in the liquid at the topmost point B? (c) Theoretically, what is the greatest possible height* h_1 *that a siphon can lift water?*



Sol. *You may have used siphon and you may recollect that lower the exit point of the fluid is, faster the fluid flows out.*

We consider a point D on the surface of the liquid in the container, in the same tube of flow with points A, B and C. Applying Bernoulli's equation to points D and C, we obtain

$$p_D + \frac{1}{2}\rho v_D^2 + \rho g h_D = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$$

$$v_C = \sqrt{\frac{2(p_D - p_C)}{\rho} + 2g(h_D - h_C) + v_D^2}$$

$$\approx \sqrt{2g(d + h_2)}$$

where in the last step we set $p_D = p_C = p_{air}$ *and* $v_D/v_C \approx 0$. *Plugging in the values, we obtain*

$$v_c = \sqrt{2(9.8\text{m}/\text{s}^2)(0.40\text{m} + 0.12\text{m})} = 3.2\ m/s$$

The result confirms our experience.
We now consider points B and C:

$$p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$$

Since $v_B = v_C$ *by equation of continuity, and* $p_C = p_{air}$, *Bernoulli's equation becomes*

$$p_B = p_C + \rho g(h_C - h_g) = p_{air} - \rho g(h_1 + h_2 + d)$$

$$= 1.0 \times 10^5\ Pa\ (1.0 \times 10^3\ kg/m\ )(9.8\ m/s^2\ )(0.25\ m + 0.40\ m + 0.12\ m)$$

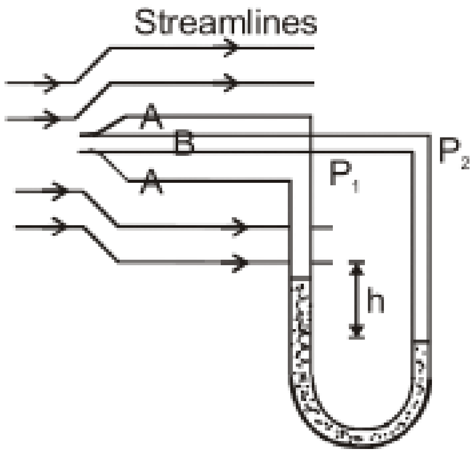
$$= 9.2 \times 10\ Pa.$$

Since $p_B \geq 0$, *we must let* $p_{air} - \rho g\ (h_1 + d + h_2) \geq 0$, *which yields*

$$h_1 \leq h_{1\max} = \frac{p_{air}}{\rho} - d - h_2 \leq \frac{p_{air}}{\rho} = 10.3\text{m}$$

Illustration :

Fig. shows a device called pitot's tube. It measures the velocity of moving fluids. Determine the velocity of the fluid in terms of the density ρ, *the density of the fluid in manometer (U-tube)* σ *and the height 'h'.*



figure

Sol. *The difference in the two tubes is that liquid will flow into the tube B with full Kinetic Energy while it will just pass over the tube A without directly entering into it.*

This problem is based on the use of Bernoulli's principle, on two different situations.

The fluid inside the right tube must be at rest as the fluid exactly at the end is in contact with the fluid in pitot tube, which is at rest.
The velocity υ_1 is the fluid velocity. The velocity υ_2 of the fluid at point B is zero and the pressure in the right arm is P_2 (called stagnation pressure).

Thus using Bernoulli's principle $$P_1 + \frac{1}{2}\sigma\upsilon_1^2 = P_2 + \frac{1}{2}\sigma\upsilon_2^2$$

We get $$P_2 = P_1 + \frac{1}{2}\sigma\upsilon_1^2$$

On the other hand the openings at point A is not along the flow lines, so we dont need to use Bernoulli's eqn. We can simply say that the pressure just outside the opening is same as that within the pitot tube.
Therefore the pressure at the left arm of the manometer is same as the fluid pressre P_f i.e., $P_1 = P_f$

Also $$P_2 = P_1 + (\rho - \sigma)gh \qquad \text{....(3)}$$

Generally $\sigma << \rho$, so it is ignored.

Thus $$P_2 = P_1 + \rho gh \qquad \text{....(4)}$$

From eqns. (4) and (3), $$\frac{1}{2}\sigma\upsilon_f^2 = \rho gh$$

or $$\upsilon_f = \sqrt{\frac{2\rho gh}{\sigma}}$$

Thus we can see that we have measured the fluid velocity as this was the only difference between the two tubes leading to the pressure difference between the tubes.

Illustration :

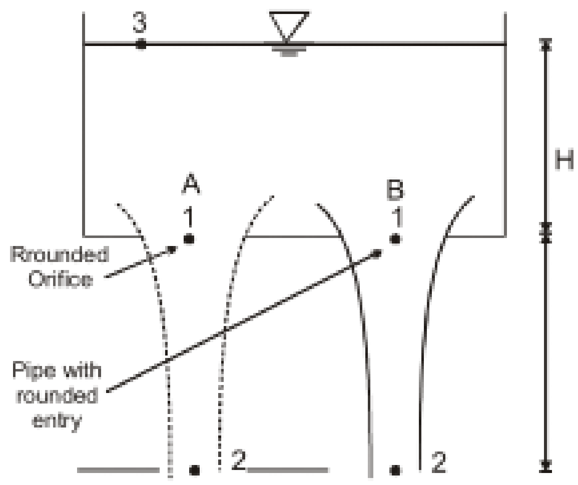
A tank has two oulets (i) a rounded orifice A of diameter D and (ii) a pipe B with well rounded entry and of length L, as shown in fig. For a height of water H in the tank determine the (i) discharge from the outlets A and B, (ii) velocities in the two outlets at levels 1 and 2 indicated in Fig.

Sol. *The difference in the two situations is that in part (i) pressure at the point 1 and all points below it will be atmospheric. On the other hand in part (ii) only at point 2 pressure will be atmospheric.*
This problem is based on the use of Bernoulli's principle, on two different situations.
Part (i)
Rounded orifice A
Applying bernoulli equation to a point on the water surface 3 and point 1.

$$\frac{p_0}{\gamma} + 0 + H = \frac{p_1}{\gamma} + \frac{V_{1a}^2}{2g} + 0$$

and $V_1 = \sqrt{2gH}$. *The discharge* $Q_a = \frac{\pi}{4} D^2 \sqrt{2gH}$

At point 2, the pressure is atmospheric and hence by applying Bernoulli equation between points 3 and 2.



$$\frac{p_0}{\gamma} + 0 + (H + L) = \frac{p_0}{\gamma} + \frac{V_{2a}^2}{2g} + 0$$

or $$V_2 = \sqrt{2g(H + L)}$$

As the discharge is Q_a, the diameter at 2 will be smaller than D.

Part (ii)

Pipe :

by applying Bernoulli equation between points 3 and 2.

$$\frac{p_0}{\gamma} + 0 + (H + L) = \frac{p_0}{\gamma} + \frac{V_{2b}^2}{2g} + 0$$

or $$V_{2b} = \sqrt{2g(H + L)}$$

As the pipe size is uniform from point 1 to 2, by continuity equation

$$V_{1b} = V_{2b} = \sqrt{2g(H + L)}$$

thus the results are :

	Orifice	Pipe
Velocity at 1 =	$\sqrt{2gH}$	$\sqrt{2g(H+L)}$
Velocity at 2 =	$\sqrt{2g(H+L)}$	$\sqrt{2g(H+L)}$
Discharge Q =	$\frac{\pi}{4} D^2 \sqrt{2gH}$	$\frac{\pi}{4} D^2 \sqrt{2g(H+L)}$

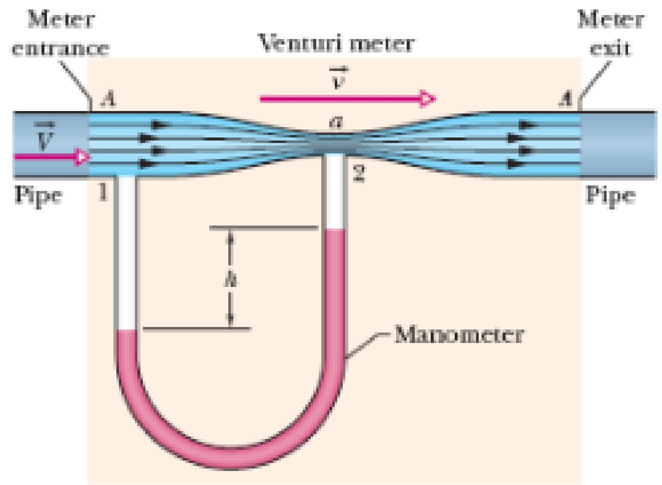
We can see that velocity at point 2 is same in both the cases, this could have been directly concluded by applying Bernoulli equation between points 3 and 2 in both parts.

Conceptual example:A venturi meter is used to measure the flow speed of a fluid in a pipe. The meter is connected between two sections A venturi meter is used to measure the flow speed of a fluid in a pipe. The meter is connected between two sections of the pipe; the cross-sectional area A of the entrance and

exit of the meter matches the pipe's crosssectional area. Between the entrance and exit, the fluid flows from the pipe with speed V and then through a narrow "throat" of cross-sectional area a with speed v A manometer connects the wider portion of the meter to the narrower portion. The change in the fluid's speed is accompanied by a change Dp in the fluid's pressure, which causes a height difference h of the liquid in the two arms of the manometer. (Here Δp means pressure in the throat minus pressure in the pipe.) (a) By applying Bernoulli's equation and the equation of continuity to points 1 and 2 in Fig. , show that

$$v = \sqrt{\frac{2a^2 \Delta p}{\rho(a^2 - A^2)}}$$

where ρ is the density of the fluid. (b) Suppose that the fluid is fresh water, that the cross-sectional areas are 64 cm^2 in the pipe and 32 cm^2 in the throat, and that the pressure is 55 kPa in the pipe and 41 kPa in the throat. What is the rate of water flow in cubic meters per second?



(a) The continuity equation yields AV = aV, and Bernoulli's equation yields $\Delta p + \frac{1}{2}\rho v^2 = \frac{1}{2}\rho V^2$ where $\Delta p = p_1 - p_2$. The first equation gives AV = aV. We use this to substitute for V in the second equation, and obtain $\Delta p + \frac{1}{2}\rho v^2 = \frac{1}{2}\rho (A/a)^2 v^2$ We solve for v. The result is

$$v = \sqrt{\frac{2\Delta p}{\rho((A/a)^2 - 1)}} = \sqrt{\frac{2a^2 \Delta p}{\rho(A^2 - a^2)}}$$

(b) We substitute values to obtain

$$v = \sqrt{\frac{2(32\times10^{-4}\,m^2)^2(55\times10^3\,Pa - 41\times10^3\,Pa)}{(1000 kg/m^3)((64\times10^{-4}\,m^2)^2 - (32\times10^{-4}\,m^2)^2)}} = 30.06\ m/s$$

Consequently, the flow rate is

$$Av = (64 \times 10^{-4} m^2)(3.06\ m/s) = 2.0 \times 10^{-2} m^3/s.$$

Illustration :

A siphon tube discharging a liquid of specific gravity 0.9 from a rbeservoir as shown in the figure.

(a) Find the velocity of the lique through the siphon

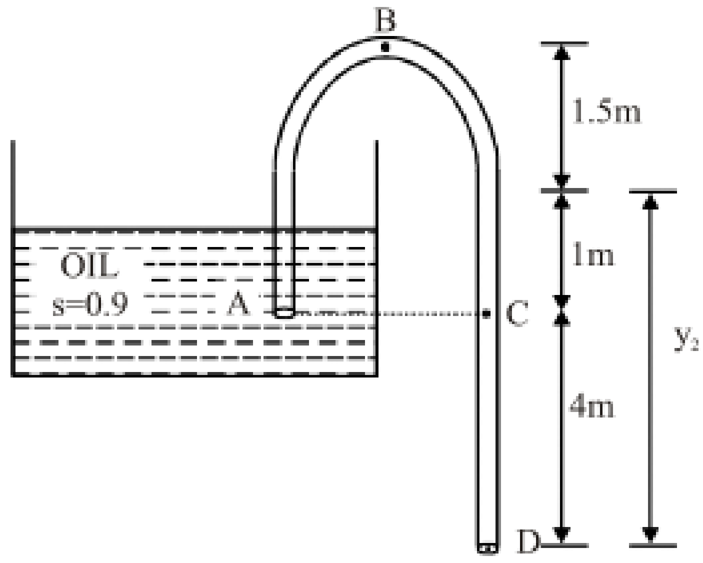
(b) Fin dthe presswure of highest point B.

(c) Find the pressure at the points A (out side the tube) and C.

(d) Would the rate of flow be more, less or the same if the liquid were water ?

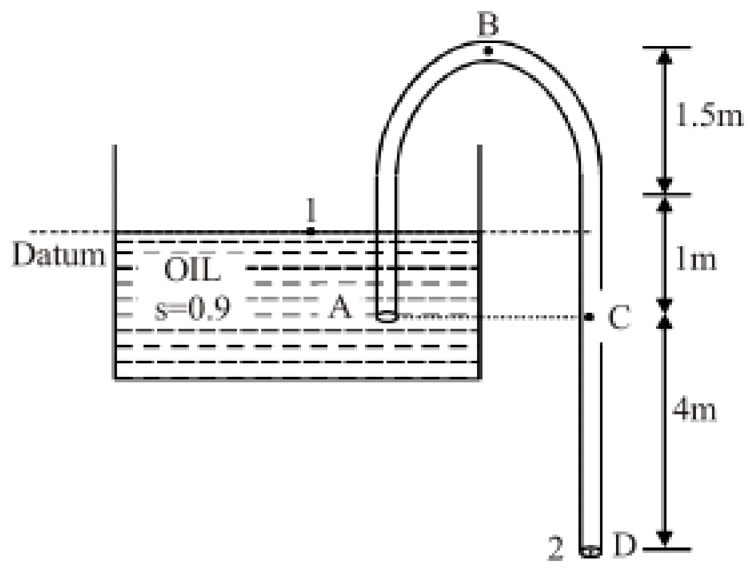
(e) Is there a limit on the maximum height of B above the liquid level in the reservoir.

(f) Is there a limit on the veritcal depth of the right limb of the siphon.



Sol. *Assume datum at the free surface of the liquid.*

(a) Applying Bernoulli's equation on point 1 and 2, as shown in the figure.



$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+y_1=\frac{p_2}{\rho g}+y_2$$

Here $p_1 = p_2 = p_0 = 10^5\ N/m^2$; $y_1 = -5m$

Since area of the tube is very small as compared to the reservoir, therefore,

$$v_1 << v_2.\ thus\ \frac{v_1^2}{2g}\approx 0$$

$\therefore v_2=\sqrt{2g(y_1-y_2)}=\sqrt{2(10)(5)}\ = 10\ m/s$

(b) Applying Bernoulli's equation at 1 and B.

$$\frac{p_B}{\rho g}+\frac{v_B^2}{2g}+y_B=\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+y_1$$

Here, $p_1 = 10^5\ N/m^2$; $\frac{v_1^2}{2g} \approx 0$;

$y_1 = 0,\ v_B = v_2 = 10\ m/s,\ y_B = 1.5\ m$

$\therefore \quad p_B = p_1 - \frac{1}{2}\rho v_2^2 - \rho g y_B$

or $\quad p_B = 10^5 - \frac{1}{2}(900)(10)^2 - (900)(10)(1.5) = 41.5\ kN/m^2$

(c) Applying bernoulli's equation at 1 and A

$p_A = p_1 + \rho g\,(y_1 - y_A)$

or $\quad p_A = 10^5 + (900)(10)(1) = 109\ kN/m^2$

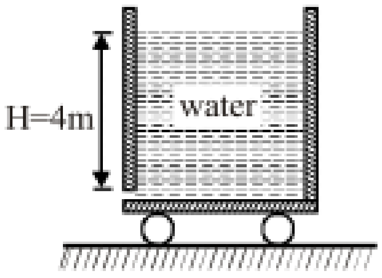
Applying Bernoulli's equation at 1 and C,

$$p_C = p_1 - \frac{1}{2}\rho v_2^2 - \rho g y_C$$

$$= 10^5 - \frac{1}{2}(900)(10)^2 - (900)(10)(-) = 10^5 - 45000 + 9000 = 64\ kN/m^2$$

Illustration :

A tank initaially at rest, is filled with water to a height H = A m. A small orifice is made at the bottom of the wall. Find the velocity attained by the tank when it becomes completely empty. Assume mss of the tank to be negligible. Friction is negligible.



Sol. *Let v be the intantaneous velocity of the tank and c be the instantaneous velocity of efflux with respect to the tank.*

Thrust exrted on the tank is

$$F = \rho a c^2$$

Where a is the crross-sectional area of the orifice.

$$c = \sqrt{2gh}$$

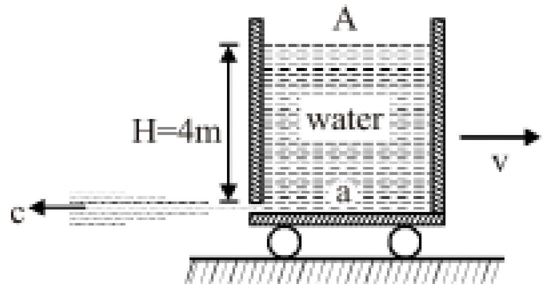
Where h is the instantaneous height of water in the tank.

Mass of the tank at any time t is

$$m = \rho A h$$

A = cross-sectional area of the tank.

Using Newton's second law



$$F = m\frac{dv}{dt} = \rho A h\frac{dv}{dt}$$

$$\therefore \quad \rho A h\frac{dv}{dt} = \rho a c^2 = 2\rho g a h$$

or $$\frac{dv}{dt} = 2g\left(\frac{a}{A}\right) \qquad ...(i)$$

In a time dt if the water level falls by dh, then according to the conservation of mass.

$$-\rho A dh = \rho a c dt \qquad or \qquad \frac{dh}{dt} = -\frac{ac}{A}$$

Equation (i) can be written as

$$\frac{dv}{dh}\frac{dh}{dt}=2g\left(\frac{a}{A}\right) \quad or \quad \frac{dv}{dh}\left(-\frac{ac}{A}\right)=2g\left(\frac{a}{A}\right)$$

or $$\frac{dv}{dh}=-\frac{2g}{c}=-\frac{2g}{\sqrt{2gh}}=-\sqrt{\frac{2g}{h}}$$

On integrating

$$\int_0^v dv=-\sqrt{2g}\int_H^0\frac{dh}{\sqrt{h}}$$

$$v = 2\sqrt{2gH}$$

Since H = 4 m, therefore $v = 2\sqrt{2(10)(4)} = 17.9\ m/s$

Practice Exercise

Q.1 During wind storm. light roofs are blown off. Why ?

Q.2 A man standing on the platform just near the raliway line be sucked in by a fast moving train. Explain.

Q.3 Air is streaming past a horizontal airplane wing such that its speed is 120 ms^{-1} over the upper surface and 90 ms^{-1} at the lower surface. If the density of air is 1.3 kgm^{-3}, find the difference in pressure between the top and bottom of the wing. If the wing is 10 m long and has average width of 2 m. Calculate the gross lift of the wing.

Q.4 A liquid is kept in cylindrical vessel which is rotated along its axis. The liquid rises at the sides. If the radius of the vessel is 0.05 m and the speed of rotation is 2 rev per sec. Find the difference in the height of the liquid at the cntre of the vessel and at its sides.

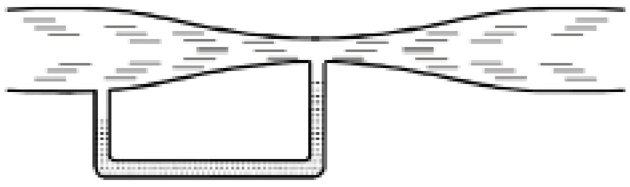
Q.5 The pressures of water in a water pipe when tap is open and closed respectively 3×10^5 N/m^2 and 3.5×10^5 N/m^2. If tap is opened, then find out-

(a) velocity of water flowing (b) rate of volume of water flowing if area of cross-section of tap is 2 cm^2.

Q.6 Water flows through a horizontal tube of variable cross-section (figure). The area of cross-section at A and B are 4 mm^2 and 2 mm^2 respectively. If 1 cc of water eneters persecond through A, find (a) the speed of water at A, (b) the speed of water at B and (c) the pressre differnece $P_A - P_B$.

Q.7 Water flows through the tube shown infigure. The areas of cross-section of the wide and the narrow portion of the tube are 5 cm^2 and 2 cm^2 respectively. The rate of flow of water through the tube is 500 cm^3/s. Find the difference of mercury levels in the u-tube.

(density of mercury = 13.6gm/cm^3)



Answers

Q.1 Due to high velocity of wind above roof, pressure decreases resulting upward force.

Q.2 Due to decreases in air pressure between preson and train.

Q.3 Due to decrease in pressure in between.

Q.4 $h = 2$ cm

Q.5 (a) 10 m/s (b) 2×10^{-3} m^3/s

Q.6 (a) 25 cm/s (b) 50 cm/s (c) 94 N/m^2 Q.7 2.13 cm

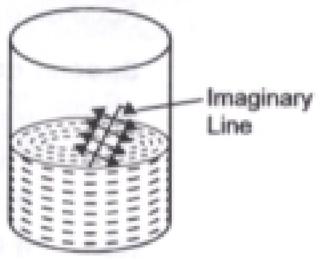
Surface Tension & Viscosity

Surface Tension

Surface Tension is a property of liquid at rest by virtue of which a liquid surface gets contracted to a minimum area and behaves like a stretched membrane.

Surface Tension of a liquid is measured by force per unit length on either side of any imaginary line drawn tangentially over the liquid surface, force being normal to the imaginary line as shown in fig. i.e. Surface tension.

$$(T) = \frac{\text{Total force on either of the imginary line (F)}}{\text{Length of the line } (l)}$$



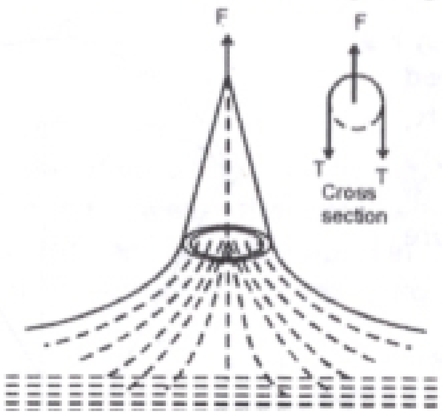
Unit of Surface Tension

In C. G. S. system the unit of surface tension is dyne/cm (dyne cm^{-1}) and SI system its units is Nm^{-1}

Illustration :

A ring is cut form a platinum tube of 8.5 cm internal and 8.7 cm external diameter. It is supported horizontally from a pan of a balance so that it comes in contact with the water in a glass vessel. What is the surface tension of water if an extra 3.97 g weight is required to pull it away from water? ($g = 980\ cm/s^2$).

Sol.



The ring is in contact with water along its inner and outer circumference ; so when pulled out the total force on it due to surface tension will be

$$F = T(2\pi r_1 + 2\pi r_2)$$

So, $$T = \frac{mg}{2\pi(r_1 + r_2)} \qquad [\because F = mg]$$

i.e., $$T = \frac{3.97 \times 980}{3.14 \times (8.5 + 8.7)} = 72.13 \text{ dyne/cm}$$

Explanation of some observed phenomena

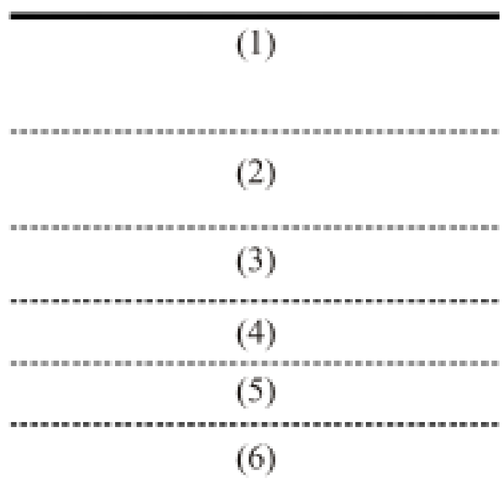
1. Lead balls are spherical in shape.
2. Rain drops and a globule of mercury placed on glass plate are spherical.
3. Hair of a shaving brush/painting brush, when dipped in water spread out, but as soon as it is taken out. Its hair stick together.
4. A greased needle placed gently on the free surface of water in a beaker does not sink.
5. Similarly, insects can walk on the free surface of water without drowning.
6. Bits of Camphor gum move inregularly when placed on water surface.

Surface energy

The course of reasoning given below is usually followed to prove that the molecules of the surface layer of a liquid have surplus potential energy. A molecule inside the liquid is acted upon by the forces of attraction from the other molecules which compensate each other on the average. If a molecule is singled out on the surface, the resulting force of attraction from the other molecule is directed into the liquid. For this reason the molecule tends to move into the liquid, and definite work should be done to bring it to the surface. Therefore, each molecule of the surface layer has excess potential energy equal to this work. The average force that acts on any molecule from the side of all the others, however, is always equal to zero if the liquid is in equilibrium. This is why the work done to move the liquid from a depth to the surface should also be zero. What is the origin, in this case, of the surface energy ?

The forces of attraction acting on a molecule in the surface layer from all the other molecules produce a resultant directed downward. The closest neighbours, however, exert a force of repulsion on the molecule which is therefore in equilibrium.

Owing to the forces of attraction and repulsion, the density of the liquid is smaller in the surface layer than inside. Indeed, molecule 1 (figure) is acted upon by the force of repulsion from molecule 2 and the forces of attraction from all the other molecules (3, 4, ...). Molecule 2 is acted upon by the forces of repulsion from 3 and 1 and the forces of attraction from the molecules in the deep layers. As a result, distance 1-2 should be greater than 2-3, etc.



This course of reasoning is quite approximate (thermal motion, etc. is disregarded), but nevertheless it gives a qualitatively correct result.

An increase in the surface of the liquid causes new sections of the rarefied surface layer to appear. Here work should be performed against the forces of attraction between the molecules. It is this work that constitutes the surface energy.

We know that the molecules on the liquid surface experience net downward force. So to bring a molecule from the interior of the liquid to the free surface, some work is required to the done against the intermolecular

force of attraction, which will be stored as potential energy of the molecule on the surface. The potential energy of surface molecules per unit area of the surface is called surface is called surface energy. Unit of surface energy is erg cm^{-2} in C.G..S. system and Jm^{-2} in SI system. Dimensional formula of surface energy is $[ML^{\circ}T^{-2}]$ surface energy depends on number of surfaces e.g. a liquid drop is having one liquid air surface while bubble is having two liquid air surface.

Relation between surface tension and surface energy

Consider a rectangular frame PQRS of wire, whose arm RS can slide on the arms PR and QS. If this frame is dipped in a soap solution. then a soap film is produced in the frame PQRS in fig. Due to surface tension (T), the film exerts a force on the frame (towards the interior of the film). Let l be the length of the arm RS, then the force acting on the arm RS towards the film is $F = T \times 2l$ [Since soap film has two surfaces, that is way the length is taken twice.]

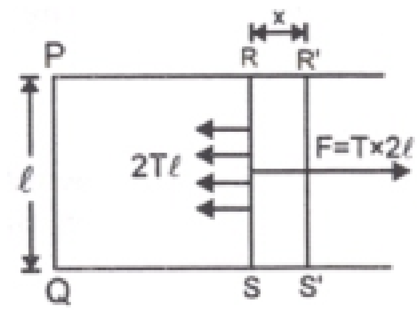
$\therefore$ work done, $W = Fx = 2Tlx$

Increase in potential energy of the soap film.

$= EA = 2Elx =$ work done in increasing the area (ΔW)

where E = surface energy of the soap film per unit area.

According the law of conservation of energy, the work done must be equal to the increase in the potential energy.



$$\therefore \quad 2Tlx = 2Elx \text{ or } T = E = \frac{\Delta W}{A}$$

Thus, surface tension is numerically equal to surface energy or work done per unit increase surface area.

Illustration :

A mercury drop of radius 1 cm is sprayed into 10^6 droplets of equal size. Calculate the energy expanded if surface tension of mercury is 35×10^{-3} N/m.

Sol. *If drop of radius R is sprayed into n droplets of equal radius r, then as a drop has only one surface, the initial surface area will be $4\pi R^2$ while final area is $n\,(4\pi r^2)$. So the increase in area*

$$\Delta S = n(4\pi r^2) - 4\pi R^2$$

So energy expended in the process.

$$W = T\Delta S = 4\pi T\,[nr^2 - R^2] \quad \ldots (1)$$

Now since the total volume of n droplets is the same as that of initial drop, i.e.

$$\frac{4}{3}\pi R^3 = n[(4/3)\,\pi r^3] \text{ or } \quad r = R/n^{1/3} \quad \ldots (2)$$

Putting the value of r from equation (2) in (1)

$$W = 4\pi R^2 T\,[(n)^{1/3} - 1$$

Illustration :

If a number of little droplets of water, each of radius r, coalesce to form a single drop of radius R, show that the rise in temperature will be given by

$$\frac{3T}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$$

where T is the surface tension of water and J is the mechanical equivalent of heat.

Sol. *Let n be the number of little droplets.*

Since volume will remain constant, hence volume of n little droplets = volume of single drop

$$\therefore \quad n \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3 \quad or \quad nr^3 = R^3$$

Decrease in surface area $= n \times 4\pi^3 - 4\pi R^2$

$$or \quad \Delta A = 4\pi\,[nr^2 - R^2] = 4\pi\left[\frac{nr^3}{r} - R^2\right] = 4\pi\left[\frac{R^3}{r} - R^2\right] = 4\pi R^3\left[\frac{1}{r}-\frac{1}{R}\right]$$

Energy evolved $W = T \times$ *decrease in surface area* $= T \times 4\pi R^3\left[\frac{1}{r}-\frac{1}{R}\right]$

Heat produced, $Q = \frac{W}{J} = \frac{4\pi TR^3}{J}\left[\frac{1}{r}-\frac{1}{R}\right]$ *But* $Q = ms\,d\theta$

where m is the mass of big drop, s is the specific heat of water and $d\theta$ *is the rise in temperature.*

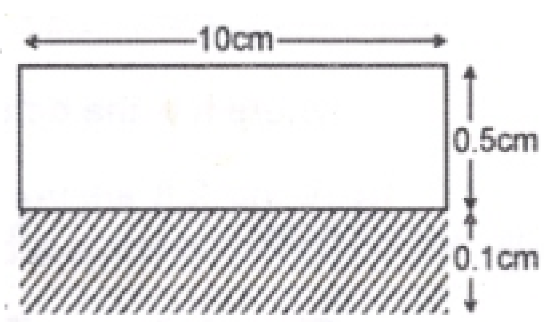
$$\therefore \quad \frac{4\pi TR^3}{J}\left[\frac{1}{r}-\frac{1}{R}\right] = \text{volume of big drop} \times \text{density of water} \times \text{sp. heat of water} \times d\theta$$

$$or, \quad \frac{4}{3}\pi R^3 \times 1 \times 1 \times d\theta = \frac{4\pi TR^3}{J}\left(\frac{1}{r}-\frac{1}{R}\right) \quad or, \quad d\theta = \frac{3T}{J}\left[\frac{1}{r}-\frac{1}{R}\right]$$

Illustration :

A film of water is formed between two straight parallel wires each 10cm long and at a separation 0.5 cm. Calculate the work required to increase 1mm distance between them. Surface tension of water 72×10^{-3} *N/m.*

Sol. *Here the increase in area is shown by shaded portion in the figure.*



Since this a water film, it has two surface, therefore increase in area, $\Delta S = 2 \times 10 \times 0.1 = 2cm^2$

$\therefore$ *Work required to be done*

$W = \Delta S \times T$

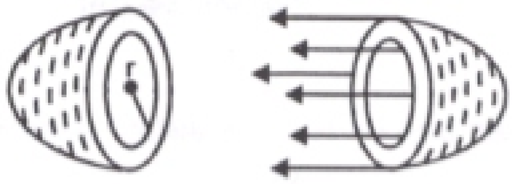
$= 2 \times 10^{-4} \times 72 \times 10^{-3}$

$= 144 \times 10^{-7}$ *joule*

$= 1.44 \times 10^{-5}$ *joule*

Excess pressure inside A liquid drop and a bubble

1. Inside a bubble : Consider a soap bubble of radius r. Let p be the pressure inside the bubble and p_a outside. The excess pressure $= p - p_a$. Imagine the bubble broken into two halves, and consider one half of it as shown in fig. Since there are two surface, inner and outer, so the force due to surface tension is



$F =$ surface tension $\times$ length $= T \times 2$ (circumference of the bubble) $= T \times 2\,(2\pi r)$ ... (1)

The excess pressure $(p - p_a)$ acts on a cross-sectional area πr^2, so the force due to excess pressure is

$\Rightarrow \quad F = (p - p_a)\,\pi r^2$ ... (2)

The surface tension force given by equation (1) must balance the force due to excess pressure given by equation (2) to maintain the equilibrium , i.e. $(p - p_a)\,\pi r^2 = T \times 2(2\pi r)$

or $$(p - p_a) = \frac{4T}{r} = p_{excess}$$

above expression can also be obtained by equation of excess pressure of curve surface by putting $R_1 = R_2$.

2. Inside the drop : In a drop, there is only one surface and hence excess pressure can be written as

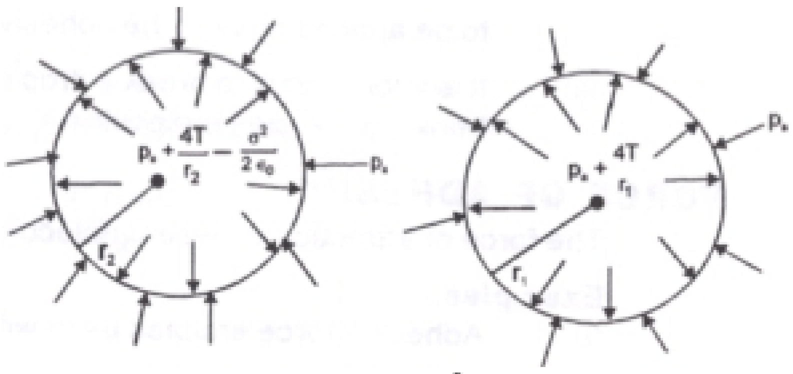
$$(p - p_a) = \frac{2T}{r} = p_{excess}$$

3. Inside air bubble in a liquid :

$$(p - p_a) = \frac{2T}{r} = p_{excess}$$

4. A charged bubble : If bubble is charged, it's radius increases. Bubble has pressure excess due to charge too. Initially pressure inside the bubble

$$= p_a + \frac{4T}{r_1}$$



for charge bubble, pressure inside $= p_a + \dfrac{4T}{r_2} - \dfrac{\sigma^2}{2\in_0}$, where σ surface is surface charge density. Taking temperature remains constant, then from Boyle's law

$$\left(p_a + \frac{4T}{r_1}\right)\frac{4}{3}\pi r_1^{\,3} = \left[p_a + \frac{4T}{r_2} - \frac{\sigma^2}{2\in_0}\right]\frac{4}{3}\pi r_2^{\,3}$$

From above expression the radius of charged drop may be calculated. It can conclude that radius of charged bubble increases, i.e. $r_2 > r_1$.

Illustration :

A minute spherical air bubble is rising slowly through a column of mercury contained in a deep jar. If the radius of the bubble at a depth of 100cm is 0.1 mm, calculate its depth where its radius is 0.126 mm, given that the surface tension of mercury is 567 dyne/cm. Assume that the atmospheric pressure is 76cm of mercury.

Sol. *The total pressure inside the bubble at depth h_1 is (P atmospheric pressure)*

$$= (P + h_1\rho g) + \frac{2T}{r_1} = P_1$$

and the total pressure inside the bubble at depth h_2 is $= (P + h_2\rho g) + \frac{2T}{r_2} = P_2$

Now, according to Boyl's Law

$$P_1V_1 = P_2V_2 \text{ where } V_1 = \frac{4}{3}\pi r_1^3, \text{ and } V_2 = \frac{4}{3}\pi r_2^3$$

Hence we get $$\left[(P+h_1\rho g)+\frac{2T}{r_1}\right]\frac{4}{3}\pi r_1^3 = \left[(P+h_2\rho g)+\frac{2T}{r_2}\right]\frac{4}{3}\pi r_2^3$$

or, $$\left[(P+h_1\rho g)+\frac{2T}{r_1}\right]r_1^3 = \left[(P+h_2\rho g)+\frac{2T}{r_2}\right]r_2^3$$

Given that : $h_1 = 100$ *cm,* $r_1 = 0.1$ *mm* $= 0.01$ *cm,* $r_2 = 0.126$ *cm,* $T = 567$ *dyne / cm,* $P = 76$ *cm of mercury. Substituting all the values, we get*

$$h_2 = 9.48 \text{ cm.}$$

The force of cohesion

The force of attraction between the molecules of the same substance is called cohesion.

In case of solids, the force of cohesion is very large and due to this solids have definite shape and size. On the other hand, the force of cohesion in case of liquids is weaker than that of solids. Hence liquids do not have definite shape but have definite volume. The force of cohesion is negligible in case of gases. Because of this fact, gases have neither fixed shape nor volume.

Example

(i) Two drops of a liquid Two drops of a liquid coalesce into one when brought in mutual contact because of the cohesive force.

(ii) It is difficult to separate two sticky plates of glass wetted with water because a large force has to be applied against the cohesive force between the molecules of water.

(iii) It is very difficult to break a drop of mercury into small droplets because of large cohesive force between mercury molecules.

Force of Adhesion

The force of attraction between molecules of different substance is called adhesion.

Example

(i) Adhesive force enables us to write on the black board with a chalk.

(ii) Adhesive force helps us to write on the paper with ink.

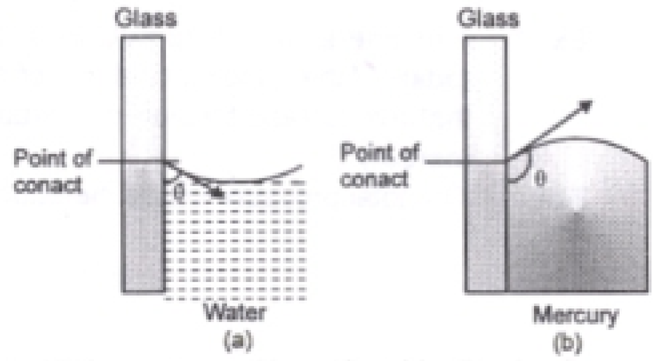
(iii) Large force of adhesion between cement and bricks helps us in construction work.

(iv) Due to force of adhesive, water wets the glass plate.

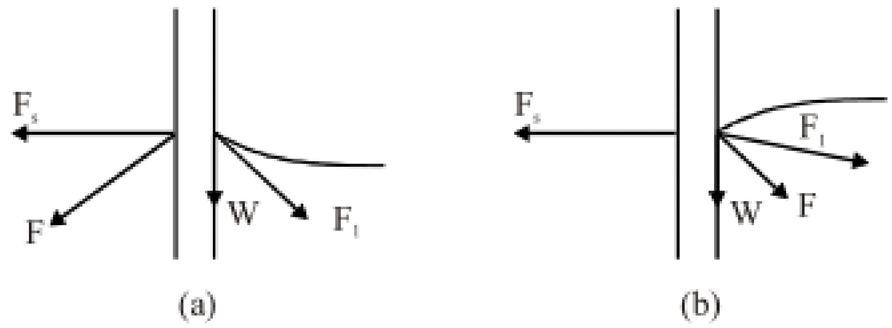
(v) Fevicol and gum are used in gluing two surfaces together because of adhesive force.

Angle of contact

The angle which the tangent to the liquid surface at the point of contact makes with the solid surface inside the liquid is called angle of contact. Those liquids which we the wall of the container (say in case of water and glass) have meniscus concave upwards and their value of angle of contact is less than 90° (also called acute angle). However, those liquids which don't wet the walls of the container (say in case of mercury and glass) have meniscus convex upwards and their value of angle of contact is greater than 90° (also called obtuse angle). The angle of contact of mercury with glass about 140°, whereas the angle of contact of water with glass is about 8°. But, for pure water, the angle of contact θ with glass is taken as 0°.



Let us now see why the liquid surface bands near the contact with a solid. A liqued in equilibrium cannot sustain tangential stress.The resultant force onany small part of the surface layer must be perpendicular to the surface there. Consider a small part of the liquid surface near its contact with the solid



The ;force acting on this part are

(a) F_s, attracting due to the molecules of the solid surface near it,

(b) F_l, the force due to the liquid molecules near this part, and

(b) W, the weight of the part considered.

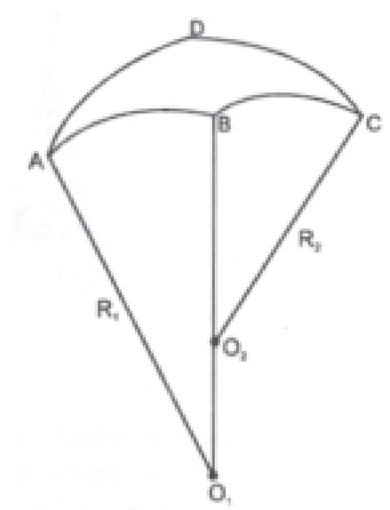
The force between the molecules of the same material is known as cohesive force and the force between the molecules of different kinds of material is called adhesive force. Here F_s is adhesive force and F_l is cohesive force.

As is clear from the figure, the adhesive force F_s is perpendicular to the solid surface and is into the solid. The cohesive force F_l is in the liquid, its direction and magnitude depends on the shape of the liquid surface as this determines the distribution of the molecules attracting the part considered. Of course, F_s and F_l depend on the nature of the substance especialy on their densitites.

The direction of the resultant of F_s, F_l and W decides the shape of the surface near the constact. The liquid rests in such a way that the surface is through the solid, the surface is concave upward and the liquid rises along the solid. If the resultant passes through the liquid, the surface is convex upwards and the liquid is demressed near the solid.

Relation between surface tension, radii of curvature and excess pressure on a curved surface

Let us consider a small element ABCD (fig.) of a curved liquid surface which is convex on the upper side. R_1 and R_2 are the maximum and minimum radii of curvature respectively. They are called the 'principal radii of curvature' of the surface. Let p be the excess pressure on the concave side.



then $p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$. If instead of a liquid surface,

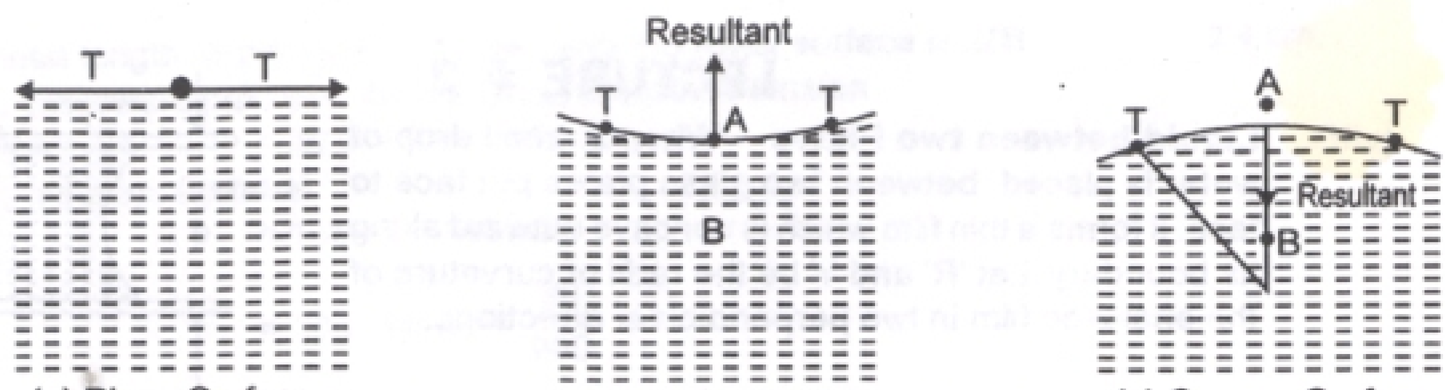
we have a liquid film, the above expression will be

$p = 2T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, because a film has two surface.

Excess of pressure inside a curved surface

1. Plane surface : If the surface of the liquid is plane [as shown in fig. (a)], the molecule on the liquid surface is attracted equally in all directions. The resultant force due to surface tension is zero. The pressure, therefore on the liquid surface is normal.

2. Concave surface : If the surface is concave upward [as shown in fig. (b)], there will be upward resultant force due to surface tension acting on the molecule. Since the molecule on the surface is in equilibrium, there must be an excess of pressure on the concave side in the downward direction to balance the resultant force of surface tension $p_A - p_B = \frac{2T}{r}$.



(a) Plane Surface (a) Concave Surface (c) Convex Surface

3. Convex surface : If the surface is convex [as shown fig.(c)], the resultant force due to surface tension acts in the downward direction. Since the molecule on the surface are in equilibrium, there must be an excess of pressure on the concave side of the surface acting in the upward direction to balance the downward resultant force of surface tension. Hence there is always in excess of pressure on concave side of a curved surface over that on the convex side.

$$P_B - P_A = \frac{2T}{r}$$

Illustration :

A barometer contains two uniform capillaries of radii 1.44×10^{-3} m and 7.2×10^{-4} m. if the height of the liquid in the tube is 0.2m more than that in the wide tube, calculate the true pressure difference. Density of liquid = 10^3 kg/m^3, surface tension = 72×10^{-3} N/m and g = 9.8 m/s^2.

Sol. *Let the pressure in the wide and narrow capillaries of radii r_1 and r_2 respectively be P_1 and P_2. Then pressure just below the mensiscus in the wide and narrow tubes respectively are :*

$$\left(P_1 - \frac{2T}{r_1}\right) \text{ and } \left(P_2 - \frac{2T}{r_2}\right) \qquad \left[\text{excess pressure} = \frac{2T}{r}\right]$$

Difference in these pressure $= \left(P_1 - \frac{2T}{r_1}\right) - \left(P_2 - \frac{2T}{r_2}\right) = h\rho g$

$\therefore$ *True pressure difference* $= P_1 - P_2$

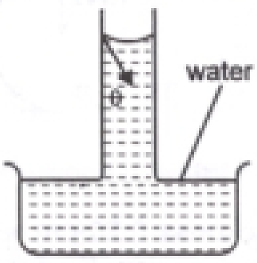
$$= h\rho g + 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= 0.2 \times 10^3 \times 9.8 + 2 \times 72 \times 10^{-3}\left[\frac{1}{1.44 \times 10^{-3}} - \frac{1}{7.2 \times 10^{-4}}\right]$$

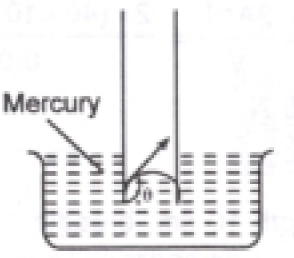
$$= 1.86 \times 10^3 = 1860 \text{ N/m}^2$$

Capillarity

A glass tube of very fine bore throughout the length of the tube is called capillary tube. If the capillary tube is dipped in water, the water wets the inner side of the tube and rises in it [shown in figure (a)]. If the same capillary tube is dipped in the mercury, then the mercury is depressed [shown in figure (b)]. The phenomenon of rise or fall of liquids in a capillary tube is called capillarity.



(a)



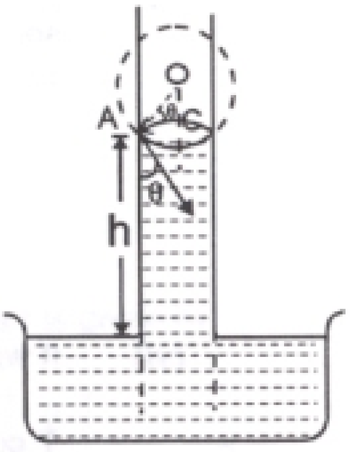
(b)

Practical applications of capillarity

1. The oil in a lamp rises in the wick by capillary action.
2. The tip of nib of a pen is split up, to make a narrow capillary so that the ink rises upto the tin or nib continuously.
3. Sap and water rise upto the top of the leaves of the tree by capillary action.
4. If one end of the towel dips into a bucket of water and the Other end hangs over the bucket the towel soon becomes wet throughout due to capillary action.
5. Ink is absorbed by the blotter due to capillary action.
6. Sandy soil is more dry than clay. It is because the capillaries between sand particles are not so fine as to draw the water up by capillaries.
7. The moisture rises in the capillaries of soil to the surface, where it evaporates. To preserve the moisture m the soil, capillaries must be broken up. This is done by ploughing and leveling the fields.
8. Bricks are porous and behave like capillaries.

Capillary rise (height of a liquid in a capillary tube) ascent formula

consider the liquid which wets the wall of the tube, forms a concave meniscus shown in figure. Consider a capillary tube of radius r dipped in a liquid of surface tension T and density ρ. Let h be the height through which the liquid rises in the tube. Let p be the pressure on the concave side of the meniscus and p_a be the pressure on the convex side of the meniscus. The excess pressure



$(p - p_a)$ is given by $(p - p_a) = \dfrac{2T}{R}$

Where R is the radius of the meniscus. Due to this excess pressure, the liquid will rise in the capillary tube till it becomes equal to the hydrostatic pressure hρg. Thus in equilibrium state.

Excess pressure = Hydrostatic pressure or $\dfrac{2T}{R} = h\rho g$

Let θ be the angle of contact and r be the radius of the capillary tube shown in the fig.

From ΔOAC, $\dfrac{OC}{OA} = \cos\theta$ or $R = \dfrac{r}{\cos\theta}$ $\Rightarrow h = \dfrac{2T\cos\theta}{r\rho g}$

The expression is called Ascent formula.

Discussion.

(i) For liquids which wet the glass tube or capillary tube, angle of contact $\theta < 90^\circ$. Hence $\cos\theta$ = positive $\Rightarrow$ h = positive. It means that these liquids rise in the capillary tube. Hence, the liquids which wet capillary tube rise in the capillary tube. For example, water milk, kerosene oil, patrol etc.

Illustration :

A liquid of specific gravity 1.5 is observed to rise 3.0 cm in a capillary tube of diameter 0.50 mm and the liquid wets the surface of the tube. Calculate th excess pressure inside a spherical bubble of 1.0 cm diameter blown from the same liquid. Angle of contact = 0°.

Sol. *The surface tension of the liquid is*

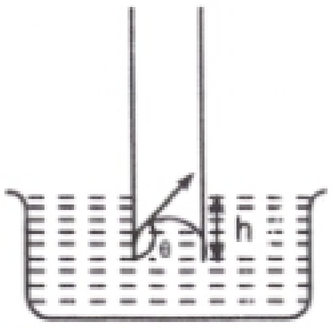
$$T = \frac{rh\rho g}{2} = \frac{(0.025\text{cm})(3.0\text{cm})(1.5\,\text{gm/cm}^3)(980\,\text{cm/sec}^2)}{2}$$

$$= 55 \text{ dyne/cm.}$$

Hence excess pressure inside a spherical bubble

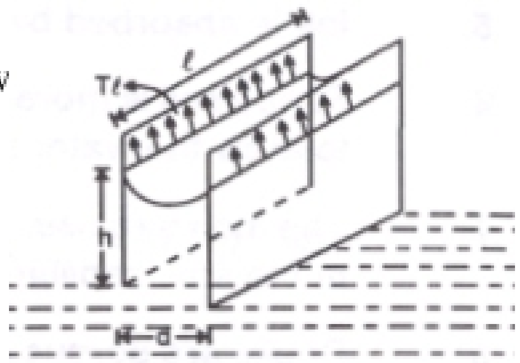
$$p = \frac{4T}{R} = \frac{4\times 55\text{dyne/cm}}{(0.5\text{cm})} = 440 \text{ dyne/cm}^2.$$

(ii) For liquids which do not wet the glass tube or capillary tube, angle of contact $\theta > 90^\circ$. Hence $\cos\theta$ negative $\Rightarrow$ h negative. Hence, the liquids which do not wet capillary tube are depressed in the capillary tube. For example, mercury.



(iii) T, θ, ρ and g are constant and hence $h \propto \frac{1}{r}$. Thus, the liquid rises more in a narrow tube and less in a wider tube. This is called Jurin's Law.

(iv) If two parallel plates with the spacing 'd' are placed in w then height of rise



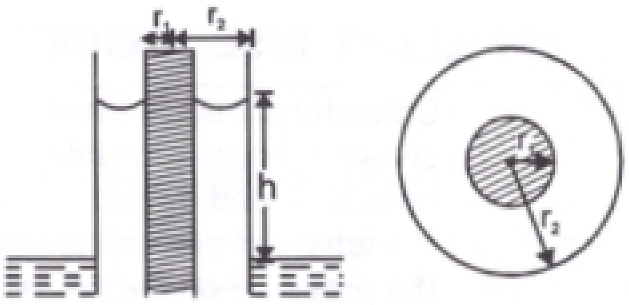
$$\Rightarrow \quad 2Tl = \rho l h d g$$

$$h = \frac{2T}{\rho d g}$$

(v) If two concentric tube of radius 'r_1' and 'r_2' (inner one is solid) are placed in water reservoir, then height of rise

$$\Rightarrow \quad T\,[2\pi r_1 + 2\pi r_2] = [\pi r_2^2 h - \pi r_1^2 h]\,\rho g$$

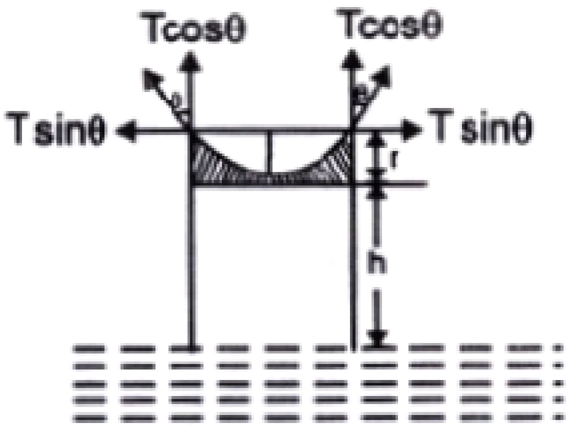
$$h = \frac{2T}{(r_2 - r_1)\rho g}$$



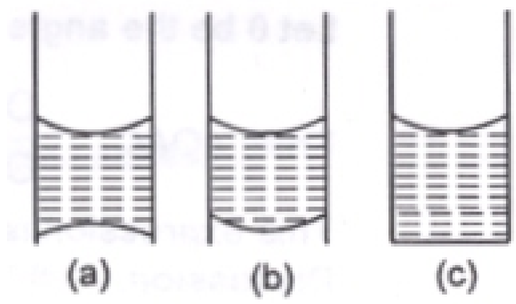
(vi) If weight of the liquid in the menscus is to be consider :

$$T\cos\theta \times 2\pi r = [\pi r^2 h + \frac{1}{3}\,\pi r^2 \times \pi r_1{}^2 h]\,\rho g$$

$$\left[h + \frac{r}{3}\right] = \frac{2T\cos\theta}{r\rho g}$$



(vii) When capillary tube (radius, 'r') is in vertical position, the upper meniscus is concave and pressure due to surface tension is directed vertically upward and is given by $p_1 = 2T / R_1$ where R_1 = radius of curvature of upper meniscus.



The hydrostatic pressure $p_2 = h\rho g$ is always directed downwards.

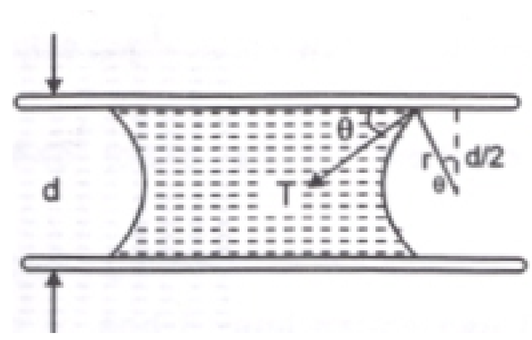
If $p_1 > p_2$ i.e. resulting pressure is directed upward. For equilibrium, the pressure due to lower meniscus should be downward. This makes lower meniscus concave downward (fig. (a)). The radius of lower meniscus R_2 can be given by $\frac{2T}{R_2} = (p_1 - p_2)$.

If $p_1 < p_2$ i.e. resulting pressure is directed downward for equilibrium, the pressure due to lower meniscus should be upward. This makes lower meniscus convex upward (fig. b).

The radius of lower meniscus can be given by $\frac{2T}{R_2} = p_2 - p_1$.

If $p_1 = p_2$, then is no resulting pressure, then $p_1 - p_2 = \frac{2T}{R_2} = 0$ or $R_2 = \infty$ i.e. lower surface will be FLAT. (fig.c).

(viii) **Liquid between two plates :** When a small drop of water is placed between two glass plates put face to face, it forms a thin film which is concave outward along its boundary. Let 'R' and 'r' be the radii of curvature of the enclosed film in two perpendicular directions.



Hence the pressure inside the film is less than the atmospheric pressure outside it by an amount p given by $p = T\left(\frac{1}{r} + \frac{1}{r = \infty}\right)$ and we have, $p = \frac{T}{r}$.

If d be the distance between the two plates and θ the angle of contact for water and glass, then, from the figure, $\cos\theta = \frac{\frac{1}{2}d}{r}$ or $\frac{1}{r} = \frac{2\cos\theta}{d}$.

Substituting for $\frac{1}{r}$ in, we get $p = \frac{2T}{d}\cos\theta$.

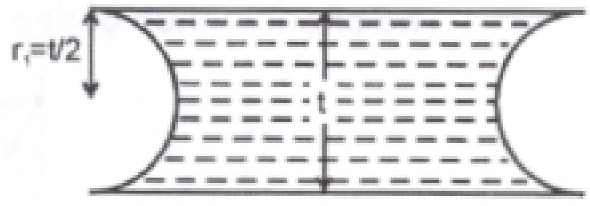
θ can be taken zero for water and glass, i.e. $\cos\theta = 1$. Thus the upper plate is pressed downward by the atmospheric pressure minus $\frac{2T}{d}$. Hence the resultant downward pressure acting on the upper plate is

$\frac{2T}{d}$. If A be the are of the plate wetted by the film, the resultant force F pressing the upper plate downward is given by F = resultant pressure × area = $\frac{2TA}{d}$. For very nearly plane surface, d will be very small and hence the pressing force F very large. Therefore it will be difficult to separate the two plates normally.

Illustration :

A drop of water volume 0.05 cm³ is pressed between two glass-plates, as a consequence of which, it spreads and occupies an area of 40 cm². If the surface tension of water is 70 dyne/cm, find the normal force required to separate out the two glass plates in newton.

Sol. *Pressure inside the film is less than outside by an amount, $P = T\left[\frac{1}{r_1} + \frac{1}{r_2}\right]$, where r_1 and r_2 are the radii of curvature of the meniscus. Here $r_1 = 1/2$ and $r_2 = \infty$, then the force required to separate the two glass plates, between which a liquid film is enclosed (figure) is, $F = P \times A = \frac{2AT}{t}$, where t is the thickness of the film, A = area of film.*



$$F = \frac{2A^2T}{At} = \frac{2A^2T}{V} = \frac{2\times(40\times10^{-4})^2\times(70\times10^{-3})}{0.05\times10^{-6}} = 45\ \text{N}$$

Illustration :

A glass plate of length 10cm, breadth 1.54 cm and thickness 0.20 cm weigh 8.2 gm in air. It is held vertically with the long side horizontal and the lower half under water. Find the apparent weight of the plate. Surface tension of water 73 dyne per cm, g = 980 cm/sec².

Sol. *Volume of the portion of the plate immersed in water is*

$$10 \times \frac{1}{2}(1.54) \times 0.2 = 1.54\ cm^3.$$

Therefore, if the density of water is taken as 1, then upthrust

= wt. of the water displaced

= 1.54 × 1 × 980 = 1509.2 dynes.

Now, the total length of the plate in contact with the water surface is 2 (10 + 0.2) = 20.4 cm,

∴ *downward pull upon the plate due to surface tension*

= 20.4 × 73 = 1489.2 dynes

$\therefore$ resultant upthrust

$= 1509.2 - 1489.2$

$= 20.0 \text{ dynes} = \dfrac{20}{980}$

$= 0.0204$ gm. wt.

$\therefore$ apparent weight of the plate in water

= weight of the plate in air – resultant upthrust

$= 8.2 - 0.0204 = 8.1796$ gm

Illustration :

A glass tube of circular cross-section is closed at one end. This end is weighted and the tube floats vertically in water, heavy end down. How far below the water surface is the end of the tube? Give : Outer radius of the tube 0.14 cm, mass of weighted tube 0.2 gm, surface tension os water 73 dyne/cm and g = 980 cm/sec^2.

Sol. Let l be the length of the tube inside water. The forces acting on the tube are :

(i) Upthrust of water acting upward

$$= \pi r^2 l \times 1 \times 980 \qquad = \frac{22}{7} \times (0.14)^2\, l \times 980 = 60.368\, l \text{ dyne.}$$

(ii) Weight of the system acting downward

$= mg = 0.2 \times 980 = 196$ dyne.

(iii) Force of surface tension acting downward

$$= 2\pi r T$$

$$= 2 \times \frac{22}{7} \times 0.14 \times 73 = 64.24 \text{ dyne.}$$

Since the tube is in equilibrium, the upward force is balanced by the downward forces. That is,

$$60.368\, l = 196 + 64.24 = 260.24.$$

$$\therefore \quad l = \frac{260.24}{60.368} = 4.31 \text{ cm}$$

Illustration:

A glass U-tube is such that the diametre of one limb is 3.0 mm and that of the other is 6.00 mm. The tube is inverted vertically with the open ends below the surface of water in a beaker. What is the difference between the heights to which water rises in the two limbs? Surface tension of water is 0.07 N/m. Assume that the angle of contact between water and glass is 0°.

Sol. Suppose pressure at the points A, B, C and D be P_A, P_B, P_C and P_0 respectively.

The pressure on the concave side of the liquid surface is greater than that on the other side by 2T/R.

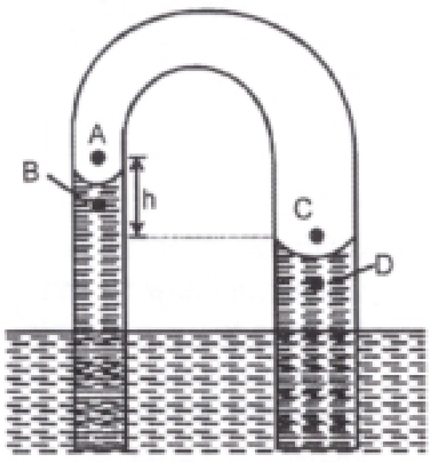
Ang angle of contact θ is given to be 0°, hence $\quad R \cos 0° = r$ or $R = r$

$$\therefore \quad P_A = P_B + 2T/r_1 \quad \text{and} \quad P_C = P_0 + 2T/r_2$$

where r_1 and r_2 are the radii of the two limbs

But $\quad P_A = P_C$

$$\therefore \quad P_B + \frac{2T}{r_1} = P_D + \frac{2T}{r_2}$$



$$\text{or} \quad P_D - P_B = 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where h is the difference in water levels in the two limbs

$$\text{Now,} \quad h = \frac{2T}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Given that $\quad T = 0.07\ Nm^{-1},\ \rho = 1000\ kgm^{-3}$

$$r_1 = \frac{3}{2}\ mm = \frac{3}{20}\ cm = \frac{3}{20 \times 100}\ m = 1.5 \times 10^{-3}\ m,\ r_2 = 3 \times 10^{-3}\ m$$

$$\therefore \quad h = \frac{2 \times 0.07}{1000 \times 9.8}\left(\frac{1}{1.5 \times 10^{-3}} - \frac{1}{3 \times 10^{-3}}\right) m = 4.76 \times 10^{-3}\ m = 4.76\ mm$$

Illustration :

Two narrow bores of diameters 3.0 mm and 6.0 mm are joined together to form a U-shaped tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube? Surface tension of water at the temperature of the experiment is $7.3 \times 10^{-2}\ Nm^{-1}$. Take the angle of contact to be zero.

Sol. Given that $r_1 = \frac{3.0}{2} = 1.5\ mm = 1.5 \times 10^{-3}\ m,\ \ r_2 = \frac{6.0}{2} = 3.0 \times 10^{-3}\ m,$

$T = 7.3 \times 10^{-3}\ Nm^{-1},\ \ \theta = 0° \ \ \rho = 1.0 \times 10^3\ kg\ m^{-1},\ g = 9.8\ ms^{-2}$

When angle of contact is zero degree, the radius of the meniscus equals radius of bore.

Excess pressure in the first bore, $P_1 = \frac{2T}{r_1} = \frac{2 \times 7.3 \times 10^{-3}}{1.5 \times 10^{-3}} = 97.3$ *Pascal*

Excess pressure in the second bore, $P_2 = \frac{2T}{r_2} = \frac{2 \times 7.3 \times 10^{-2}}{3 \times 10^{-3}} = 48.7$ *Pascal*

Hence, pressure difference in the two limbs of the tube

$$\Delta P = P_1 - P_2 = h\rho g$$

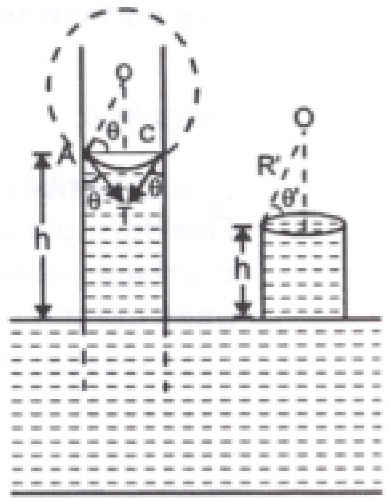
$$\text{or} \quad h = \frac{P_1 - P_2}{\rho g} = \frac{97.3 - 48.7}{1.0 \times 10^3 \times 9.8} = 5.0\ mm.$$

Capillary rise in a tube of insufficient length

We know, the height through which a liquid rises in the capillary tube of radius r is given by

$$\therefore \quad h = \frac{2T}{R\rho g} \text{ or } h\,R = \frac{2T}{\rho g} = \text{constant}$$

When the capillary tube is cut and its length is less then h (i.e. h′). then the liquid rises upto the top of the tube and spreads in such a way that the radius (R′) of the liquid meniscus increases and it becomes more flat so that hR = h′R′ = Constant. Hence the liquid does not overflow.



$$\text{If} \quad h' < h \text{ then } R' > R \quad \text{or} \quad \frac{r}{\cos\theta'} > \frac{r}{\cos\theta}$$

$$\Rightarrow \quad \cos\theta' < \cos\theta \quad \Rightarrow \quad \theta' > \theta$$

Illustration:

If a 5cm long capillary tube with 0.1 mm internal diameter open at both ends is slightly dipped in water having surface tension 75 dyne cm^{-1}, state whether (i) water will rise half way in the capillary. (ii) Water will rise up to the upper end of capillary (iii) What will overflow out of the upper end of capillary. Explain your answer.

Sol. *Given that surface tension of water, T = 75 dyne/cm*

$$\text{Radius } r = \frac{0.1}{2} \text{ mm} = 0.05 \text{ mm} = 0.005 \text{ cm}$$

density $\rho = 1$ gm/cm^3, angle of contact, $\theta = 0°$

Let h be the height to which water rise in the capillary tube. Then

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2 \times 75 \times \cos 0°}{0.005 \times 1 \times 981} \text{ cm} = 30.58 \text{ cm}$$

But length of capillary tube, h′ = 5cm

(i) *Because $h > \frac{h'}{2}$ therefore the first possibility does not exist.*

(ii) *Because the tube is of insufficient length therefore the water will rise upto the upper end of the tube.*

(iii) *The water will not overflow out of the upper end of the capillary. It will rise only upto the upper end of the capillary.*

The liquid meniscus will adjust its radius of curvature R′ in such a way that

$$R'h' = Rh \qquad \left[\because hR = \frac{2T}{\rho g} = \text{constant}\right]$$

where R is the radius of curvature that the liquid meniscus would possess if the capillary tube were of sufficient length.

$$\therefore \quad R' = \frac{Rh}{h'} = \frac{rh}{h'} \qquad \left[\because R = \frac{r}{\cos\theta} = \frac{r}{\cos 0°} = r\right] = \frac{0.005 \times 30.58}{5} = 0.0306 \text{ cm}$$

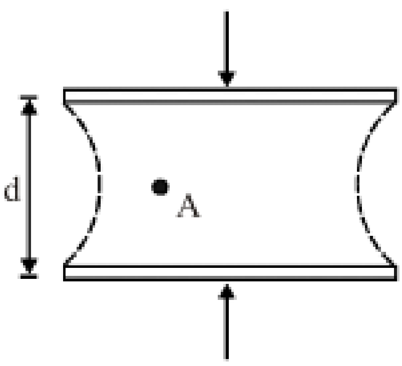
Illustration :

A drop of water of mass m = 0.2 g is placed between two clean glass plates, the distance between which is 0.01 cm. Find the force of attraction between the plates. Surface tension of water = 0.07 Nm^{-1}.

Sol. *Let R be the radius of the circular layer of water. Then* $\pi R^2 d \times \rho = m$

Pressure at A $= p_0 - \dfrac{2T}{d}$ *($\because$ meniscus is cylindrical in shape)*

Thus pressure between the plates is less than the atmospheric pressure and so the plates are pressed together as though attracted towards each other.



F, force of attaraction = $\Delta p \times$ area $\Rightarrow \quad F = \dfrac{2T}{d} \times \pi R^2$

$$\Rightarrow \quad F = \frac{2T}{d} \times \frac{m}{d.\rho} = \frac{2Tm}{d^2.\rho} = \frac{2 \times 0.2 \times 10^{-3} \times 0.07}{0.01^2 \times 10^{-4} \times 100} = 2.8\text{N}$$

Illustration :

A glass capillary sealed at the upper end is of length 0.11 m and internal diameter 2×10^{-5} m. The tube is immersed vertically into a liquid of surface tension 5.0×10^{-2} N/m. To what length has the capillary to be immersed so that liquid level inside and outside the capillary becomes same ? What will happen to the water level inside the capillary if the seal is now broken ?

Sol. *If A is the cross-sectional area of the tube and L its length, the inital volume of air inside it will be $V_1 = AL$ While pressure $p_1 = p_0$ = atmospheric pressure.*

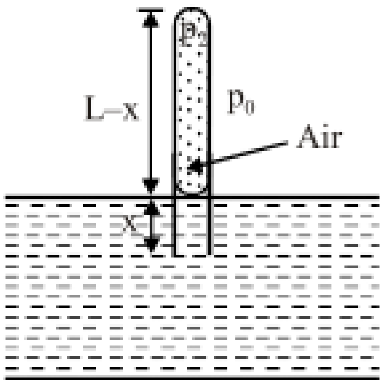
Now when the tube is immersed in water with its length x in water, the level of water inside and outside is same, so the volume of air in the tube will be $V_2 = A(L - x)$. Futher if p_2 is the pressure of gas in the tube,

$$p_2 - \frac{2T}{r} = p_0, \qquad \text{i.e. } p_2 = p_0 + \frac{2T}{r}$$

Now if temperature is constant

$$P_1 V_1 = P_2 V_2$$

$$p_0 AL = \left[p_0 + \frac{2T}{r}\right] A(L - x) \quad or \quad x\left[1 + \frac{rp_0}{2T}\right] = L$$



p₂–2T/t

i.e. $\quad x\left[1 + \dfrac{1.012 \times 10^5 \times 1 \times 10^{-5}}{2 \times 5.06 \times 10^{-2}}\right] = 0.11 \quad or \quad x = \dfrac{0.11}{11} = 0.01\text{m}$

If the seal is broken the pressure inside the capillary become atmospheric, i.e. p_0 while capillarity will take place and the rise will be

$$h = \frac{2T}{r\rho g} = \frac{2 \times 5.06 \times 10^{-2}}{10^{-5} \times 10^3 \times 9.8} = 1.03\,\text{m}$$

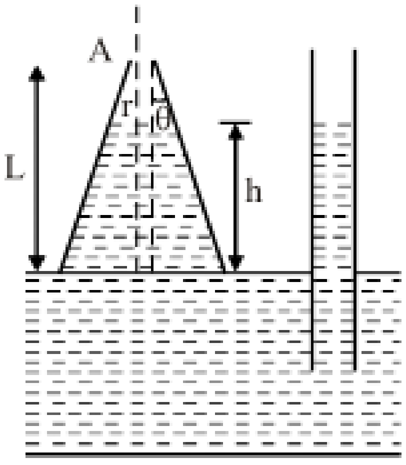
However, the length of the tube outside the water is 0.11 – 0.01 = 0.1 m; so the tube will be of insufficient length and so the liquid will rise to the top of the tube and will stay with radius of meniscus,

$$r = \frac{h}{L} = \frac{1.03 \times 10^{-3}}{0.1} = 1.03 \times 10^{-4}\,\text{m}$$

Illustration:

A conical glass capillary tube of length 0.1 m has diameters 10^{-3} and 5×10^{-4} m at the ends. When it is just immersed in a liquid at 0ºC with larger diameter in contact with it, the liquid rises to 8×10^{-2} m in the tube. If another cylindrical glass capillary tube B is immersed inthe same liquid at 0ºC, the liquid rises to 6×10^{-2} m height. The rise of liquid in the tube B is only 5.5×10^{-2} m when the linquid is at 50ºC. Find the rate at which the surface tension changes with temperature considering the change to be linear. The density of the liquid is $(1/14) \times 10^4$ kg/m³ and angle of contact is zero Effect of temperature on density of liquid and glass is negligible.

Sol. *If r is the radius of the meniscus in the conical tube, then as shown in figure.*



$$\tan\theta = \frac{r - r_1}{L - h} = \frac{r_2 - r_1}{L}$$

i.e. $$\frac{r - 2.5\times10^{-4}}{0.1 - 0.08} = \frac{(5.2.5)\times10^{-4}}{0.1}$$

i.e., $r\times10^4 - 2.5 = 0.2 \times 2.5$ *i.e.,* $r = 3 \times 10^4$ m

Now as capillarity is independent of the shape of tube so at same temperature $\theta = 0^o$ C.

$$h_A r_A = h_B r_B = (2T_0/\rho g) = \text{constant}$$

so $r_B = (0.08\times3\times10^{-4})/(6\times10^{-2}) = 4 \times 10^{-4}$ m

Now as from $h = (2T/r\rho g)$ *for cylindrical tube,*

$$T_0 = \frac{h_0\rho gr}{2} = \frac{1}{2}\left[6\times10^{-2}\times\frac{1}{14}\times10^4\times9.8\times4\times10^{-4}\right]$$

$$= 8.4 \times10^{-2}\ \text{N/m}$$

Now as for a given tube and liquid $T \propto h$ *(as* $T = h\rho gr/2$*)*

$$\frac{T}{T_0} = \frac{h_{50}}{h_0} \qquad so \qquad T_{50} = \frac{5.5\times10^{-2}}{6\times10^{-2}}\times8.4\times10^{-2}\times10^{-2} = 7.7\times10^{-2}\,\text{N/m}$$

So rate of change of surface tension with temperature assuming lineartiy,

$$\frac{\Delta T}{\Delta\theta} = \frac{T_{50} - T_0}{50 - 0} = \frac{(7.7 - 8.4)\times10^{-2}}{50} = 1.4\times10^{-2}\,\text{N/m}^\circ\text{C}$$

Negative sign shows that with with rise in temperature surface tension decreases.

Applications of surface tension

(i) The wetting property is made use of in detergents and waterproofing. When the detergent materials are added to liquids, the angle of contact decreases and hence the wettability increases. On the other hand, when water proofing material is added to a fabric, it increases the angle of contact, making the fabric water-repellant.

(ii) The antiseptics have very low value of surface tension. The low value of surface tension prevents the formation of drops that may otherwise block the entrance to skin or a wound. Due to low surface tension the antiseptics spreads properly over the wound. The lubricating oils and paints also have low surface tension. So they can spread properly.

(iii) Surface tension of all lubricating oils and paints is kept low so that they spread over a large area.

(iv) Oil spreads over the surface of water because the surface tension of oil is less than the surface tension of cold water.

(v) A rough sea can be calmed by pouring oil on its surface.

Effect of temperature and impurities on surface tension

The surface tension of a liquid decreases with the rise in temperature and vice versa. According to Ferguson, $T = T_0\left(1-\frac{\theta}{\theta_C}\right)^n$ where T_0 is surface tension at 0°C, θ is absolute temperature of the liquid, θ_C is the critical temperature and n is a constant varies slightly from liquid and has means value 1.21. This formula shows that the surface tension becomes zero at the critical temperature, that is why machinery parts get jammed in winter.

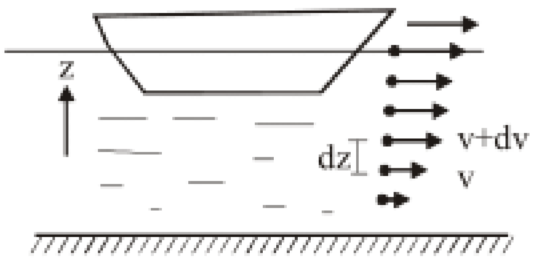
The surface tension of a liquid change appreciably with addition of impurities. For example, surface tension of water increases with addition of highly soluble substances like NaCl, $ZnSO_4$ etc. On the other hand surface tension of water gets reduced with addition of sparingly soluble substances like phenol, soap etc.

Viscosity

When a layer of a fluid slips or tens to slip on another layer in contact, the two layers exert tangential forces on each other. The directions are such that the relative motion between the layers is opposed. this property of a fluid to oppose relative motion between its layers is called viscosity. The forces between the layers opposing relative motion between them are known as the forces of viscosity. Thus, viscosity may be thought of as the internal friction of a fluid in motion.

If a solid surface is kept in contact with a fluid and is moved, forces of viscosity appear between the solid surface and the fluid layer in contact. the fluid in contact is dragged with the solid. If the viscosity is sufficient, the layer moves with the solid and there is no relative slipping. When a boat moves slowly on

the water of a calm river, the water in contact with the boat is dragged with it, whereas the water in contact with the bed of the river remains at rest. Velocities of different layers are different. Let v be the velocity of the layer at a distance z from the bed and v +dv be the velocity at a distance z +dz (figure).



Thus, the velocity differs by dv in going through a distance dz perpendicular to it. The quantity dv/dz is called the velocity gradient.

The force of viscosity between two layers of a fluid is proportional to the velocity gradient in the direction perpendicular to the layers. Also the force is proportional to the area of the layer.

Thus, if F is the force exerted by a layer of area A on a layer in contact,

$$F \propto A \text{ and } F \propto dv/dz$$

or, $$F = -\eta A\, dv/dz$$

The negative sign is included as the force is frictional in nature and opposes relative motion. The constant of proportionality η is called the coefficient of viscosity.

The SI unit of viscosity can be easily worked out from equation. It is N-s/m^2 . However, the corresponding CGS unit dyne-s/cm^2 is in common use and is called a poise in honour of the French scientist Poiseuille. We have

$$1 \text{ poise} = 0.1 \text{ N-s/m}^2$$

Terminal velocity

The viscous force on a solid moving through a fluid is proportional to its velocity. When a solid is dropped in a fluid, the forces acting on it are

(a) weight W acting vertically downward,

(b) the viscous force F acting vertically upward and

(c) the buoyancy force B acting vertically upward.

The weight W and the buoyancy B are constant but the force F is proportional to the velocity v, initially, the velocity and hence the viscous force F is zero and the solid is accelerated due to the force W – B. Because of the acceleration, the velocity increases. Accordingly, the viscous force also increases. At a certain instant the viscous force becomes equal to W-B. the net force then becomes zero and the solid falls with constant velocity. This constant velocity is known as the terminal velocity.

Consider a spherical body falling through a liquid. Suppose the density of the body = ρ, density of the liquid = σ, radius of the sphere = r and the terminal velocity = v_0 . The viscous force is

$$F = 6\pi\eta r v_0$$

the weight $$W = \frac{4}{3}\pi r^3 \rho g$$

and the buoyancy force $$B = \frac{4}{3}\pi r^3 \sigma g$$

We have

$$6\pi\eta r v_0 = W - B = \frac{4}{3}\pi r^3 \rho g - \frac{4}{3}\pi r^3 \sigma g$$

or

$$v_0 = \frac{2r^2(\rho-\sigma)g}{9\eta}$$

Illustration:

A large wooden plate of area 10 m^2 floating on the surface of a river is made to move horizontally with a speed of 2 m/s by applying a tangential force. If the river is 1 m deep and the water in contact with the bed is stationary, find the tangential force needed to keep the plate moving. Coefficient of viscosity of water at the temperature of the river = 10^{-2} poise.

Sol. *The velocity decreases from 2 m/s to zero in 1 m of perpendicular length. Hence, velocity gradient.*

$$= dv/dx = 2\ s^{-1}$$

Now,

$$\eta = \left|\frac{F/A}{dv/dx}\right|$$

or,

$$10^{-3}\ \frac{N-s}{m^2} = \frac{F}{(10m)^2(2s^{-1})}$$

or,

$$F = 0.02\ N.$$

Illustration :

The velocity of water in a river is 18 km/hr near the surface. If the river is 5 m deep, find the shearing stress between the horizontal layers of water. The coefficient of viscosity of water = 10^{-2} poise.

Sol. *The velocity gradient in vertical direction is*

$$\frac{dv}{dx} = \frac{18\,km/hr}{5m} = 1.0\,s^{-1}$$

The magnitude of the force of viscosity is

$$F = \eta A \frac{dv}{dx}.$$

The shearing stress is

$$F/A = \eta \frac{dv}{dx} = (10^{-2}\ poise)\ (1.0\ s^{-1}) = 10^{-3}\ N/m^2$$

Solved Example

Q.1 A conical glass capillary tube A of length 0.1 m has diameters 10^{-3} m and 5×10^{-4} m at the ends. When it is just immersed in a liquid at 0ºC with larger in contact with it, te liquid rises to 8×10^{-2} m in the tube. In another cylindrical capillary tube B, when immersed in the ame liquid at 0ºC, the liquid rises to 6×10^{-2} m height. The rise of liquid in tube B is only 5.5×10^{-2} m when the liquid is at 50ºC. Find the rate at which the surface tension chages with temperature considering the chage to be linear. The density of liquid is $(1/14)\times10^4$ kg/m³ and the angle of contact is zero. Effect of temperature on the density of liquid and glass is negligible.

Sol. The situation is shown if figure.

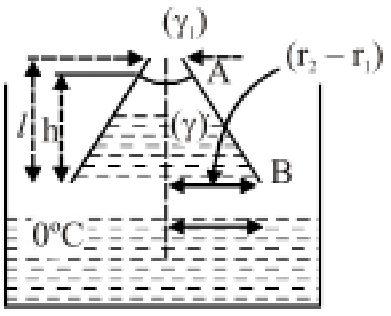
Let r_1 and r_2 be radii of upper and lower ends of the conical capillary tube. The radius r at the meniscus is given by

$$r = r_1 + (r_2 - r_1)\left(\frac{l - h}{l}\right)$$

$$= (2.5\times10^{-4}) + (2.5\times10^{-4})\left(\frac{0.1 - 0.08}{0.1}\right)$$

$$= 3.0\times10^{-4}\ \text{m}$$

The surface tension at 0ºC is given by



$$T_0 = \frac{rh\rho g}{2}$$

$$= \frac{(3.0\times10^{-4})(8\times10^{-2})(1/4\times10^{4})(9.8)}{2}$$

$$= 0.084\ \text{N/m}$$

For tube B

$$\frac{T_0}{T_{50}} = \frac{h_0}{h_{50}} = \frac{6\times10^{-3}}{5.5\times10^{-3}} = \frac{12}{11}$$

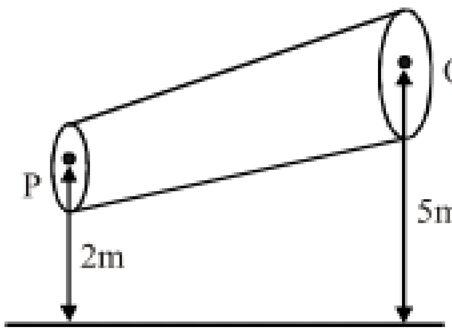
or $$T_{50} = \frac{11}{12}\times T_0 = \frac{11}{12}\times0.084 = 0.077\ \text{N/m}$$

Considering the change in surface tension as linear, the change ; in surface tension with temperature is given by

$$\alpha = \frac{T_{50} - T_0}{T_0 T_{50}} = \frac{0.077 - (0.084)}{0.084\times0.077}$$

$$= \frac{1}{60}\ \text{per K.}$$

Q.2 A non-viscous liquid of constant density 1000 kg/m^3 flows in a streanline motion along a tube of variable cross-section. The tube is kept inclined inthe vertical plane as shown in the figure.the area of cross-sectionof the tube at two points P and Q at heights of 2 mater and 5 meter are respectively 4×10^{-3} m^2 and 8×10^{-3} m^2. The velocity of the liquid at point P is 1 m/s. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point P and Q.



Sol. As gravitational field is conservative i.e., W = – U

So, $\left(\frac{dW}{dV}\right)_g = -\frac{dU}{dV} = -\frac{mg(h_2 - h_1)}{V} = -\rho g(h_2 - h_1)$

So work done by the force of gravity per unit volume

$$\left(\frac{dW}{dV}\right)_g = \rho g\,(h_2 - h_1) = -\,10^3 \times 9.8\,(5-2) = -\,2.94 \times 10^4\ \frac{J}{m^3} \qquad ...(i)$$

Now in case of ideal fluid motion by conservation of mass, i.e.

$$\left(\frac{dm}{dt}\right)_1 = \left(\frac{dm}{dt}\right)_2 \qquad \text{and} \qquad (\rho A\upsilon)_1 = (\rho A\upsilon)_2$$

or $(\rho A\upsilon)_1 = (\rho A\upsilon)_2$ [as ρ = constant (given)]

so $$\upsilon_2 = \frac{A_1\upsilon_1}{A_2} = \frac{4\times 10^{-3}\times 1}{8\times 10^{-3}} = \frac{1}{2}\,m/s \qquad ...(ii)$$

Now as work done per unit volume by pressure,

$$\left(\frac{dW}{dV}\right)_P = \frac{PdV}{dV} = P = (p_1 - p_2) \qquad [\text{as } dW = PdV]$$

But by Bernoulli's theorem,

$$p_1 + \rho g h_1 + \frac{1}{2}\rho\upsilon_1^2 = p_2 + rgh_2 + \frac{1}{2}\rho\upsilon_2^2$$

so $$\left(\frac{dW}{dV}\right)_P = (p_1 - p_2) = \rho g(h_2 - h_1) + \frac{1}{2}\rho(\upsilon_2^2 - \upsilon_1^2) \qquad ...(iii)$$

Using (i), (ii) and (iii) we get

$$\left(\frac{dW}{dV}\right)_P = 2.94\times 10^4 + \frac{1}{2} - 10^3[(0.5)^2 - 1^2] = 29025\,J$$

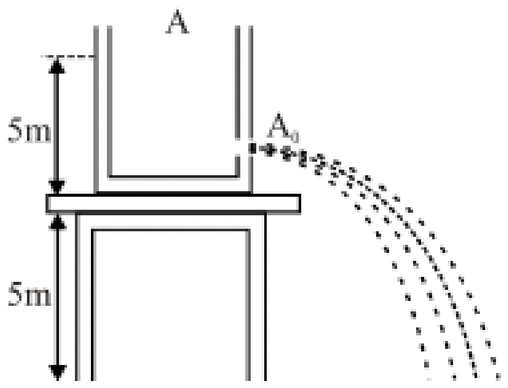
Q.3 A cylindrical tank I m in radius rests on a platform 5 m high. Initially the tank is filled with water up to a height of 5 m. A plug whose area is 10^{-4} m^2 is removed from an orifice on the side of the tank at the bottom. Calculate (a) initial speed with which the water flows from the orifice
(b) initial speed with which the water strikes the ground (c) time taken to empty the tank to half its original value (d) Does the time to empty the tank depend upon the height of stand.

Sol. (a) As speed of efflux is given by

$$\upsilon_H = \sqrt{(2gh)} \text{ so here } u = \sqrt{2\times10\times5} = 10 \text{ m/s}$$

(b) As vertical speed with which water strikes the gournd,

$$\upsilon_v = \sqrt{\upsilon_H^2 + \upsilon_v^2} = 10\sqrt{2} = 14.1\,\text{m/s}$$



(c) When the height of water level above the hole is y,

velocity of flow will be $\upsilon = \sqrt{2gy}$ and so rate flow

$$\frac{dV}{dt} = A_0\upsilon = A_0\sqrt{2gy}$$

or $-Ady = (\sqrt{2gu})\,A_0\,dt$ [as $dV = -Ady$]

Which on integration gives

$$t = \frac{A}{A_0}\sqrt{\frac{2}{g}}[\sqrt{H} - \sqrt{H'}]$$

So

$$t = \frac{\pi\times1^2}{10^{-4}}\sqrt{\frac{2}{10}}[\sqrt{5} - \sqrt{(5/2)}] = 9.2\times10^3\text{s} = 2.5\text{h}$$

(d) No, as expression of t is independent of height of stand.

Q.4 Under isothermal condition two soap bubbles of radii a and b coalesce to from a single bubble of radius c. If the external pressure is p_0 show that surface tension,

$$T = \frac{p_0(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}$$

Sol. As excess pressure for a soap bubble is (4T/r) and exteranal pressure p_0,

$$p_i = p_0 + (4T/r)$$

so $p_a = \left[p_0 + \frac{4T}{a}\right]$, $p_b = \left[p_0 + \frac{4T}{b}\right]$ and $p_c = \left[p_0 + \frac{4T}{c}\right]$ ...(i)

and $V_a = \frac{4}{3}\pi a^3$, $V_b = \frac{4}{3}\pi b^3$ and $V_c = \frac{4}{3}\pi c^3$ ...(ii)

Now as mass is conserved ,

i.e., $\dfrac{p_a V_a}{Rt_a} + \dfrac{p_b V_b}{RT_b} = \dfrac{p_c V_c}{RT_c}$ $\left[\text{as } PV = \mu RT, \text{i.e.,} \mu = \dfrac{pV}{RT}\right]$

As temp, is constant, i.e., $T_a = T_b = T_c$, the above expression reduces to

$$p_a V_a + p_b V_b = p_c V_c$$

Which in the light of Eqn. (i) and (ii) becomes

$$\left[p_0 + \frac{4T}{a}\right]\left[\frac{4}{3}\pi a^3\right] + \left[p_0 + \frac{4T}{b}\right]\left[\frac{4}{3}\pi b^3\right] = \left[p_0 + \frac{4T}{c}\right]\left[\frac{4}{3}\pi c^3\right]$$

i.e., $4T\,(a^2+b^2-c^2) = p_0(c^3-a^3-b^3)$

i.e., $T = \dfrac{p_0(c^3 - a^3 - b^3)}{4(a^2 + b^2 - c^2)}$.

Q.5 The fresh water behind a reservoir dam is 15 m deep. A horizontal pipe 4.0 cm in diameter passed through the dam 6.0 m below the water surface as shown in figure. A plug secures the pipe opening. (a) Find the friction force between theplug and pipe wall. (b) The plug is remove. What volume of water flows out of the pipe in 3.0 hour ?

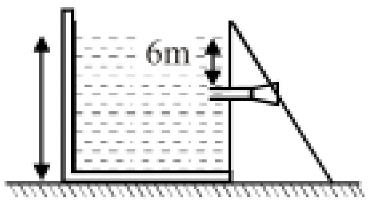
Sol. (a) As the plug secures the pipe opening, the force of friction between plug and pipe wall.

$F = A\,(p_2 - p_1)$

But $p_1 = p_0$ and $p_2 = p_0 + h\rho g$

so $F = Ah\rho g$

i.e., $F = \pi \times (2\times10^{-2})^2 \times 6\ 5\ 10^3 \times 9.8 = 74$ N



(b) As the velocity of efflux.

$$\upsilon = \sqrt{2gh} = \sqrt{2\times10\times6} = 11 \text{m/s}$$

so assuming the level of water in the tank to be constant (i.e., area = ∞) as it is not given the volume coming out per second will be

$$R = \frac{dV}{dt} = A_0\upsilon = \pi(2\times10^{-2})^2 \times 11 \text{ m}^3/\text{s}$$

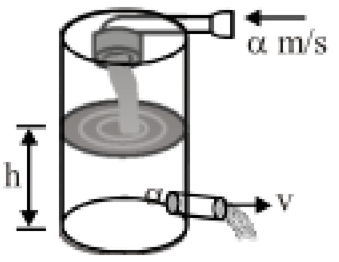
so the volume of the water flowing through the pipe in 3 hours

$$V = R \times t = 44 \times 3.14 \times 10^{-4} \times 3 \times (60 \times 60) = 150 \text{ m}^3.$$

Q.6 A cylindrical vessel of base are A has a small hole of cross-section 'a' punched near its base. At time t = 0, water is supplied into the vessel at a constant rate 'α' m³/s. Find

(a) The maximum water level h_{max} in the vessel

(b) The time 't' when water level becomes h (< h_{max}).



Sol. (a) Water level will have maximum height when inflow rate = outflow rate and there will be no futher change in level.

$\therefore \quad \alpha = av$

or, $\quad \alpha = a\sqrt{2gh_{max}} \qquad [\because v = \sqrt{2gh}\,]$

or, $\quad h_{max} = a^2/2ga^2$

(b) Let the water level by y at time t.

$$\therefore \quad A\left(\frac{dy}{dt}\right) = \alpha - av = \alpha - \sqrt{2gy}$$

Here $\frac{dy}{dt}$ is positive as y increase with time instantaneous effux velocity $v = \sqrt{2gy}$.

Rearranging the above equation

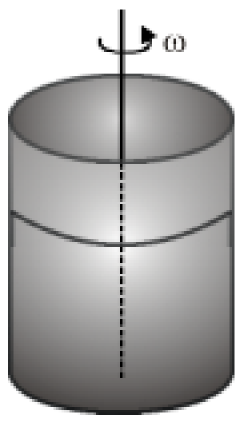
$$\int_0^h \frac{dy}{\alpha - \alpha\sqrt{2gy}} = \frac{1}{A}\int_0^t dt$$

Integration under the given limits, we get the required time

$$t = \frac{A}{ag}\left[\frac{\alpha}{a}\,\mathrm{In}\left(\frac{\alpha - a\sqrt{agh}}{\alpha}\right) - \sqrt{2gh}\right]$$

This gives the time t as a function of h. For any valume $h(\leq h_{max})$, the corresponding time t and be evaluated.

Q.7 A cylindrical vessel of (radius r) containing a liquid spins continously with constant angular velocity ω as shown in the figure. Shown that the pressure at a radial distance r from the axis is



$$P = P_0 + \frac{1}{2}\rho\omega^2 r^2,$$

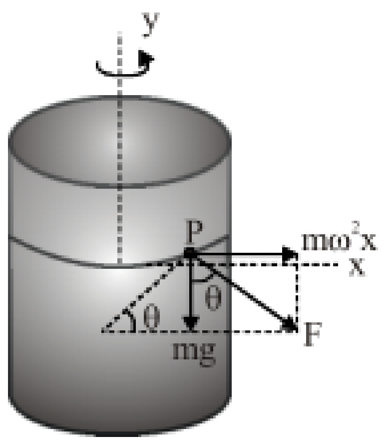
where P_0 = atmospheric pressure.

Sol. Consider a particel of the fluid at a point P (x, y) w.r.t. the corrdinate axes as shown in the figure. The force acting on This particle are $m\omega^2 x$ (the centrifugal force) and the weight mg.

The net force F acting at P should be perpendicular to the free surface, so that

$$\tan\theta = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g}$$

or, $$\frac{dy}{dx} = \frac{m\omega^2}{g} \qquad \left[\because \text{slope} = \tan\theta = \frac{dy}{dx}\right]$$



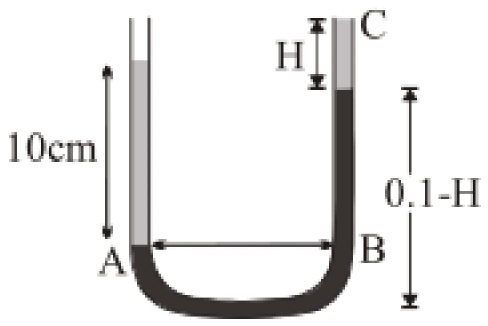
or, $$y = \frac{\omega^2}{2g}x^2.$$

This equation represents a parabola; for which the elevation from origin at $x = r$ will be $y = \frac{\omega^2}{2g}r^2$.

$\therefore$ $$\text{Pressure } P(r) = P_0 + \rho g y = P_0 + \frac{\rho\omega^2 r^2}{2}$$

Q.8 A vertical U-tube of uniform cross-section contains mercury in both arms. A glycerine (relative density = 1.3) column of length 10 cm is introduced into one of the arms. Oil of density 800 kg m^{-3} is poured into the other arm until the upper surface of the oil and glycerine are at the same horizontal level. Find the length of the oil column. Density of mercury is 13.6×10^3 kgm^{-3}.

Sol. Pressure at A and B must be same



Pressure at A = $p_0 + 0.1 \times (1.3 \times 1000) \times g$

where p_0 = atmospheric pressure

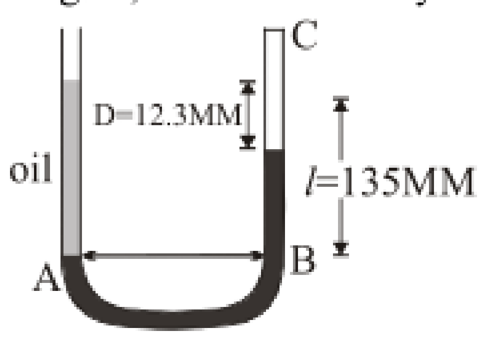
Pressure at B = $p_0 + h \times 800 \times g + (0.1 - h) \times 13.6 \times 1000\, g$

$\therefore$ $p_0 + 0.1 \times 1300 \times g$

$= p_0 + 800\, gh + 1360\, g - 13600 \times g \times h$

or $h = 9.6$ cm

Q.9 For the arrangement shown in the figure, what is the density of oil?



Sol. $p_{surface} = p_0 + \rho_w . g l$

$p_{surface} = p_0 + \rho_{oil}(l + d)\, g$

$$\Rightarrow \rho_{oil} = \frac{\rho_\omega \cdot l}{(l + d)} = \frac{1000(135)}{(135 + 12.3)} = 916 \text{ kg/m}^3$$

Q.10 A pipe of copper having an internal cavity weighs 264 gm in air and 221 gm in water. Find the volume of the cavity. [Density of copper is 8.8 gm/cc.]

Sol. The buoyant force on the copper piece, $F = V\rho g$

Hence, volume of the copper piece $V = \frac{F}{\sigma g} = \frac{(264-221)g}{1 \times g} = 43\,\text{cc}$

The volume of the material of the copper piece

$$V_0 = \frac{\text{mass of copper piece}}{\text{density of material}} = \frac{264}{8.8} = 30\text{ cc}$$

Hence, volume of the cavity $= V - V_0 = 43 - 30 = 13$ cc

Q.11 A piece of brass (alloy of copper and zinc) weighs 12.9 gm in air. When completely immersed in water, it weighs 11.3 gm. What is the mass of copper contained in the alloy?

[Specific gravities of copper and zinc are 8.9 and 7.1, respectively.]

Sol. Let the mass of copper in alloy = x gm.

$\therefore$ Amount of zinc = (12.9 – x) gm

Volume of copper, $V_{Cu} = \frac{x}{\rho_{Cu}} = \frac{x}{8.9}$ and

Volume of zinc $V_{Zn} = \frac{(12.9-x)}{7.1}$

$\therefore$ Total volume of the alloy, $V = V_{Cu} + V_{Zn}$

or $V = \frac{(12.9-x)}{7.1} + \frac{x}{8.9}$ ... (i)

Buoyant force $F = V_{\rho}g$ = loss of weight

$= (12.9 - 11.3)g = 1.6\text{ g}$

Substituting the value of V in equation (i), we get

$x = 7.6$ gm

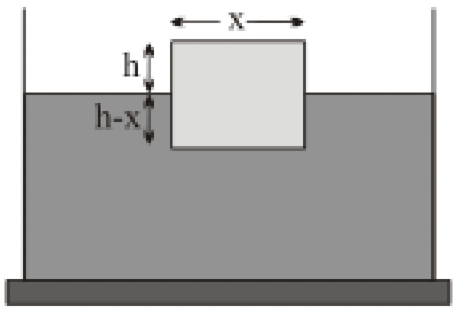
Q.12 A cubical block of iron of edge 5 cm is floating on mercury in a vessel.

(a) What is the height of the block above mercury level?

(b) Water is poured into the vessel so that it just covers the iron block. What is the height of the water column?

[Relative density of Hg = 13.6 and that of Fe = 7.2]

Sol. (a) Let h be the height of the iron block above mercury.



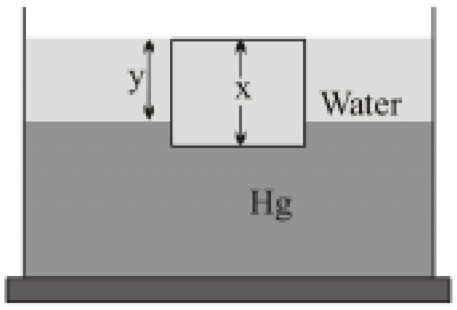
In case of flotation,

Weight of the block = buoyant force

i.e., $x^3 \rho g = [(x - h) \sigma g]\, x^2$

or $h = x\left(1 - \frac{\rho}{\sigma}\right) = 5\left(1 - \frac{7.2}{13.6}\right) = 2.35$ cm

(b) Let y be the height of the water level.



For equilibrium of the block,

$x^3\rho g = [\sigma_w g y + \sigma_{Hg}\, g(x - y)]\, x^2$

$x\sigma = (x - y)\,\sigma_{Hg} + y\sigma_w$

or $y = x\left(\frac{\sigma_{Hg} - \sigma}{\sigma_{Hg} - \sigma_w}\right) = 5\left(\frac{13.6 - 7.2}{13.6 - 1}\right) = 2.54$ cm

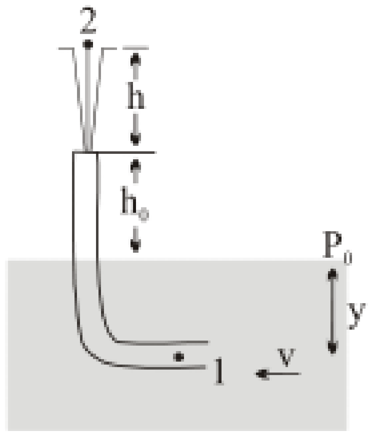
Q.13 The 'tip of the iceberg' in popular speech has come to mean a small visible fraction of something that is mostly hidden. For real icebergs, what is this fraction? ($\rho_{ice} = 917$ kg/m^3, $\rho_{sea\ water} = 1024$ kg/m^3)

Sol. $W_{ice} = \rho_i V_i g$, $W_{sea\ water} = \rho_w V_w g$

For floatation, $\rho_i V_i g = \rho_w V_w g$

Fraction of the volume submerged $= \frac{V_i - V_w}{V_i} = 1 - \frac{917}{1024} = \frac{107}{1024} = 10.45\%$

Q.14 A bent tube is lowered into the stream as shown. The velocity of the stream relative to the tube is equal to V. The closed upper end of the tube is located at height h_0. To what height h will the water jet spurt?



Sol. Let tube's entrance be a depth 'y' below the surface. Take point 1 at entry and point 2 at the maximum height of the fountain. This is a tube of flow. Now let's apply Bernoulli's theorem,

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

Taking, $h_1 = 0$, $h_2 = (y + h_0 + h)$, $V_1 = V$, $V_2 = 0$

$P_1 = P_0 + \rho gy$, $P_2 = P_0$,

Substituting, $P_0 + \rho gy + \rho g \times 0 + \frac{1}{2}\rho V^2 = P_0 + \rho g\,(y + h_0 + h) + \frac{1}{2}\rho \times 0^2$

$\Rightarrow \frac{1}{2}\rho V^2 = \rho g\,(h_0 + h)$ or $h = \left(\frac{V^2}{2g} - h_0\right)$

Q.15 Water enters a house through a pipe with inlet diameter of 2.0 cm at an absolute pressure of 4.0×10^5 Pa (about 4 atm). A 1.0 cm diameter pipe leads to the second floor bathroom 5.0 above. When flow speed at the inlet pipe is 1.6 m/s, find the flow speed, pressure and volume flow rate in the bathroom.

[Sol. Let point 1 and 2 be at the inlet pipe and the bathroom, then from continuity equation

$a_1 v_1 = a_2 v_2 \quad \Rightarrow v_2 = 6.0$ m/s

Now, applying Bernoulli's equation at the inlet ($y = 0$) and at the bathroom ($y_2 = 5.0$ m).

As $p + \frac{1}{2}\sigma v^2 + \sigma gy$ = constant

Hence, $p_2 - p_1 = \frac{1}{2} p\left(v_2^2 - v_1^2\right) - \rho g\,(y_2 - y_1)$

Which gives $p_2 = 3.3 \times 10^5$ Pa

The volume flow rate $= A_2 v_2 = A_1 v_1 = \frac{\pi}{4}(0.1)^2\, 6 = 4.7 \times 10^{-4}$ m^3/s.]

Q.16 Water coming out the jet having across sectional area a, with a speed v strikes a stationary plate and stops after striking. Find the force exerted by the water jet on the plate.

Sol. The change of momentum of water in time $dt = 0 - \rho av^2 dt\ \hat{i} = -\rho av^2 dt\ \hat{i}$ where $\hat{i}$ is a unit vector in the direction of the velocity of the jet. The rate of change of momentum of water jet $= -\rho av^2\ \hat{i}$

Thus the force exerted on the water jet by the plate $= -\rho av^2\ \hat{i}$

The force exerted on the plate by the water jet $= \rho av^2\ \hat{i}$.

Q.17 What is the surface energy of an air bubble inside a soap solution?

Sol. $E = T \times A = 4\pi r^2 T$, as it has only one surface.

Q.18 A metal plate 0.04 m^2 in area is lying on liquid layer of thickness 10^{-3} m and co-efficient of viscosity 140 poise. Calculate the horizontal force needed to move the plate with a speed of 0.040 m/s.

Sol. Area of the plate, $A = 0.04\ m^2$

Thickness, $\Delta x = 10^{-3}$ m

Δx is the distance of the free surface with respect to the fixed surface.

Velocity gradient, $\frac{\Delta v}{\Delta x} = \frac{0.04}{10^{-3}} = 40\,s^{-1}$

Co-efficient of viscosity, $\eta = 14\ kg\ ms^{-1}\ s^{-1}$

Let F be the required force.

Then, $F = \eta A \frac{\Delta v}{\Delta x} = 22.4\,N$

Electrostatics

Introduction

Electromagnetism is, almost unarguably, the most important basic technology in the world today. Almost every modern device, from cars to kitchen appliances to computers, is dependent upon it. Life, for most of us, would be almost unimaginable without electromagnetism. In fact, electromagnetism cuts such a wide path through modern life that the teaching of electromagnetism has developed into several different specialties.Initially electricity and magnetism were classified as independent phenomena, but after some experiments (we will discuss later) it was found they are interrelated so we use the name **Electromagnetism** . In electromagnetism we have to study basic properties of electromagnetic force and field (the term field will be introduced in later section).The electromagnetic force between charged particles is one of the fundamental forces of nature. We begin this chapter by describing some of the basic properties of one manifestation of the electromagnetic force, the electrostatic force between charges (the force between two charges when they are at rest) under the heading **electrostatics.**

Electric Charge

A number of simple experiments demonstrate the existence of electric forces and charges. For example, after running a comb through your hair on a dry day, you will find that the comb attracts bits of paper. The attractive force is often strong enough to suspend the paper. The same effect occurs when certain materials are rubbed together, such as glass rubbed with silk or rubber with fur. When materials behave in this way, they are said to be *electrified*, or to have become electrically charged.A neutral body can get charged only by transfer of electrons, thus the lowest unit of free charge that may appear on a body is charge of electron whose magnitude is e. When a body gets n electrons from other body charge on it becomes – ne while charge on body loosing n electrons becomes + ne.

Unit of charge :

SI unit : coulomb (C). c.g.s. unit : e.s.u (electrostatic unit) or stat coulomb.

$1C = 2.998 \times 10^9$ esu

Basic properties of electric charge

(i) There exist two types of charges in nature : positive and negative

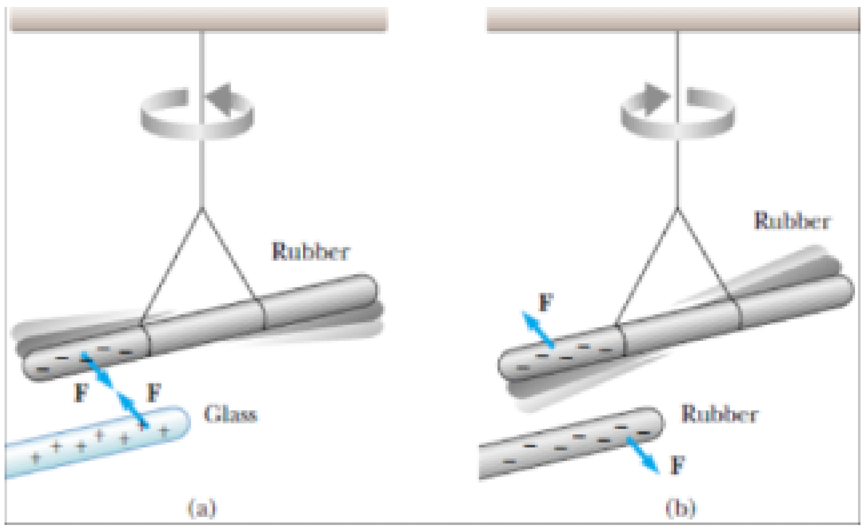
Experimentally, it was found that there are two kinds of electric charges, which were given the names positive and negative by Benjamin Franklin (1706–1790). We identify negative charge as that type possessed by electrons and positive charge as that possessed by protons.

Secondly, it should also be noted that naming one charge as (+)ve and the other as (–)ve is a matter of convention; there is no intrinsically compelling reason for this choice.

(ii) Like charges repel and unlike charges attract.

To verify this, suppose a hard rubber rod that has been rubbed with fur is suspended by a sewing thread, as shown in Figure . When a glass rod that has been rubbed with silk is brought near the rubber rod, the

two attract each other (Fig. a). On the other hand, if two charged rubber rods (or two charged glass rods) are brought near each other, as shown in Figure b, the two repel each other. This observation shows that the rubber and glass have two different types of charge on them. On the basis of these observations, we conclude that charges of the same sign repel one another and charges with opposite signs attract one another.



(iii) Charges are additive i.e. the charges add algebraically despite the fact that the words (+)ve and (–)ve don't have any algebraic meaning due to the property of the charges that equal amount of two types of charges present at a point neutralize the effect of one another and hence, the presence of none can be felt i.e., they behave like an uncharged state.

(iv) Charge is quantized : Charge exists in discrete units equal to the integral multiple of electronic charge (Charge on one electron)

i.e., $Q = ne$

Where $e(>0)$ is the lowest possible magnitude of charge and n belongs to the set of integers :

$n = 0, \pm 1, \pm 2, \pm 3, ...$ and

e = magnitude of charge on one electron = 1.6×10^{-19} C

Our advanced nuclear research, however, suggests that the elementary particles of Hadron family, like, protons and Neutrons have internal structures. They are composed of basic units called "Quarks" having charges $-\frac{1}{3}e$ (down quark 'd') and $+\frac{2}{3}e$ (up quark 'u'). Proton is made up of three quarks, two up quarks and one down quark and its structure is 'uud'. Similarly the structure of Neutrons is 'udd'.

Despite the overwhelming evidence of quarks having fractional electronic charges, we have sufficient theoretical grounds to state that the liberation of a single quark is a physical impossibility i.e. quarks don't have independent existence. They always exist in such groups that the net charge of that group is equal to the integral multiple of electronic charge and we still state the principle of quantization of charge as

$Q = ne$

Since loss or gain of electron is responsible for creating charge on a body and electron is a particle with mass, every charged body will have mass also.

Illustration:

A copper sphere contains about 2×10^{22} atoms. The charge on the nucleus of each atom is 29e. what fraction of the electrons must be removed from the sphere to give it a charge of $+2\mu$ C ?

Sol. The total number of electrons is $29(2 \times 10^{22}) = 5.8 \times 10^{23}$.
Electrons removed $= (2 \times 10^{-6}C)/(1.6 \times 10^{-19}C) = 1.25 \times 10^{13}$,
So the fraction removed = electrons removed / total number electrons $= 2.16 \times 10^{-11}$.

(v) Charge is conserved : The total charge of universe remains constant. It may alternatively be stated as follows : "The total charge of an Isolated system remains constant i.e. for a closed system of particles.

$$\sum_i e_i^+ - \sum_i e_i^- = \text{Const.}$$

The above principle suggests that :

(a) Charge can neither be created nor be destroyed.

(b) Only (+)ve or only (–)ve charge can never be created.

(c) Simultaneous production of equal and opposite charges or simultaneous annihilation of equal and opposite charge don't violate the principle of conservation of charge.

Illustration:

Three metallic spheres say X, Y and Z have charges 10C, –10C, 10C respectively. X, Y, Z are brought in contact such that charge on each of A and B becomes 3C what is charge on Z.

Sol. Net charge initially on X, Y and Z = (+10 –10 +10) = 10C
= Final net charge on X, Y and Z = $q_X + q_Y + q_Z$
$= 3+3+q_c = 10$ C $\therefore$ $q_c = 4$C.

Classification of Substance on the Basis of Electrical Passage

We can classify materials generally according to the ability of charge to move through them. Conductors are materials through which charge can move rather freely; examples include metals (such as copper in common lamp wire), the human body, and tap water. Nonconductors-also called insulators -are materials through which charge cannot move freely; examples include rubber (such as the insulation on common lamp wire), plastic, glass, and chemically pure water. Semiconductors are materials that are intermediate between conductors and insulators; examples include silicon and germanium in computer chips. Superconductors are materials that are perfecl conductors, allowing charge to move without any hindrance.

Charging of body

Mainly there are following three methods of charging a body:

(i) Charging by rubbing

The simplest way to experience electric charges is to rub certain bodies against each other. When a glass rod is rubbed with a silk cloth the glass rod acquires some positive charge and the silk cloth acquires negative charge by the same amount. The explanation of appearance of electric charge on rubbing is simple. All material bodies contain large number of electrons and equal number of protons in their normal state. When rubbed against each other, some electrons from one body pass onto the other body. The body that donates the electrons becomes positively charged while that which receives the electrons becomes negatively charged. For example when glass rod is rubbed with silk cloth, glass rod becomes positively charged because it donates the electrons while the silk cloth becomes negatively charged because it receives electrons. Electricity so obtained by rubbing two objects is also known as **frictional electricity.** The other places where the frictional electricity can be observed are when amber is rubbed with wool or a comb is passed through a dry hair. Clouds also become charged by friction.

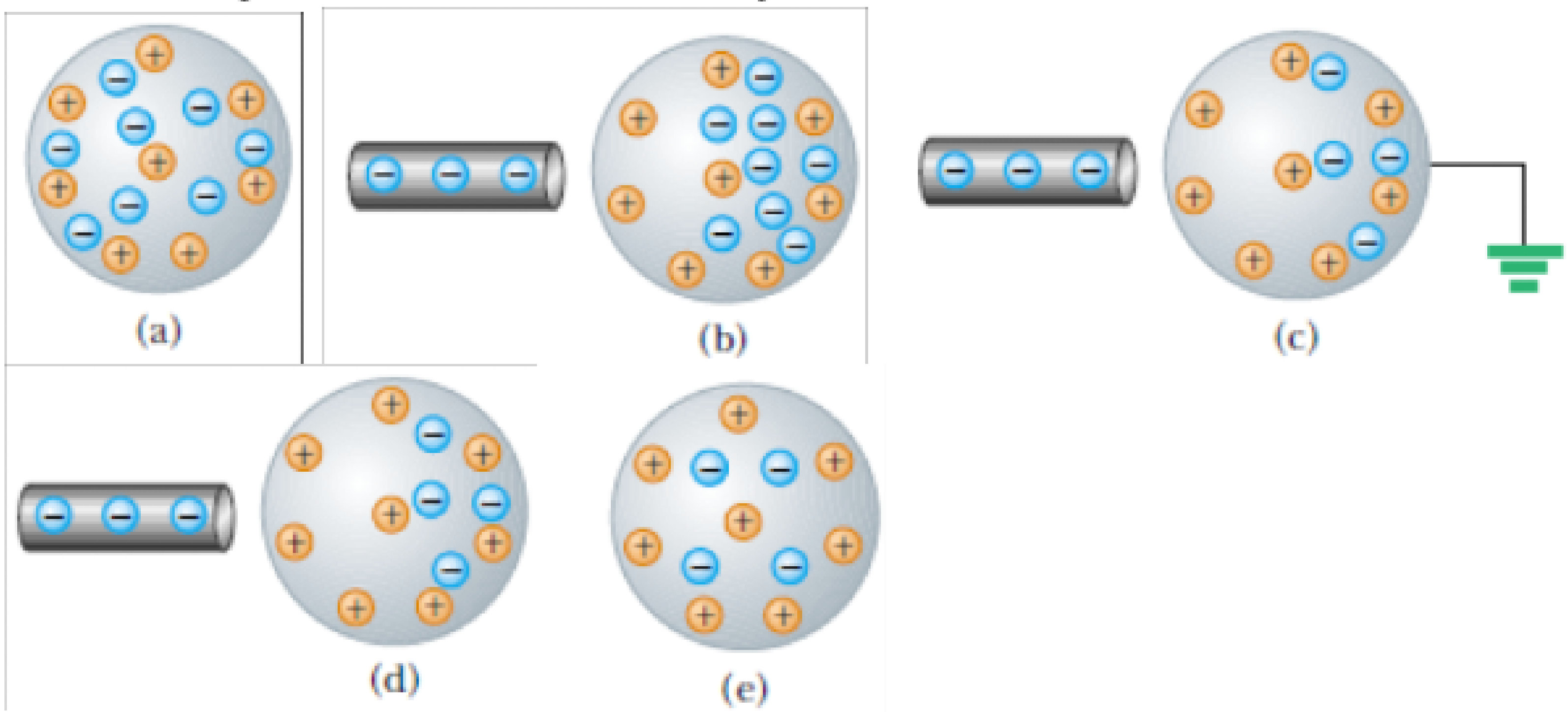
(ii) Charging by contact

When a negatively charged ebonite rod is rubbed on a metal object, such as a sphere, some of the excess electrons from the rod are transferred to the sphere. Once the electrons are on the metal sphere, where they can move readily, they repel one another and spread out over the sphere's surface. The insulated stand prevents them from flowing to the earth. When the rod is removed the sphere is left with a negative charge distributed over its surface. In a similar manner the sphere will be left with a positive charge after being rubbed with a positively charged rod. In this case, electrons from the sphere would be transferred to the rod. The process of giving one object a net electric charge by placing it in contact with another object that is already charged is known as **charging** by **contact.**

(iii) Charging by induction

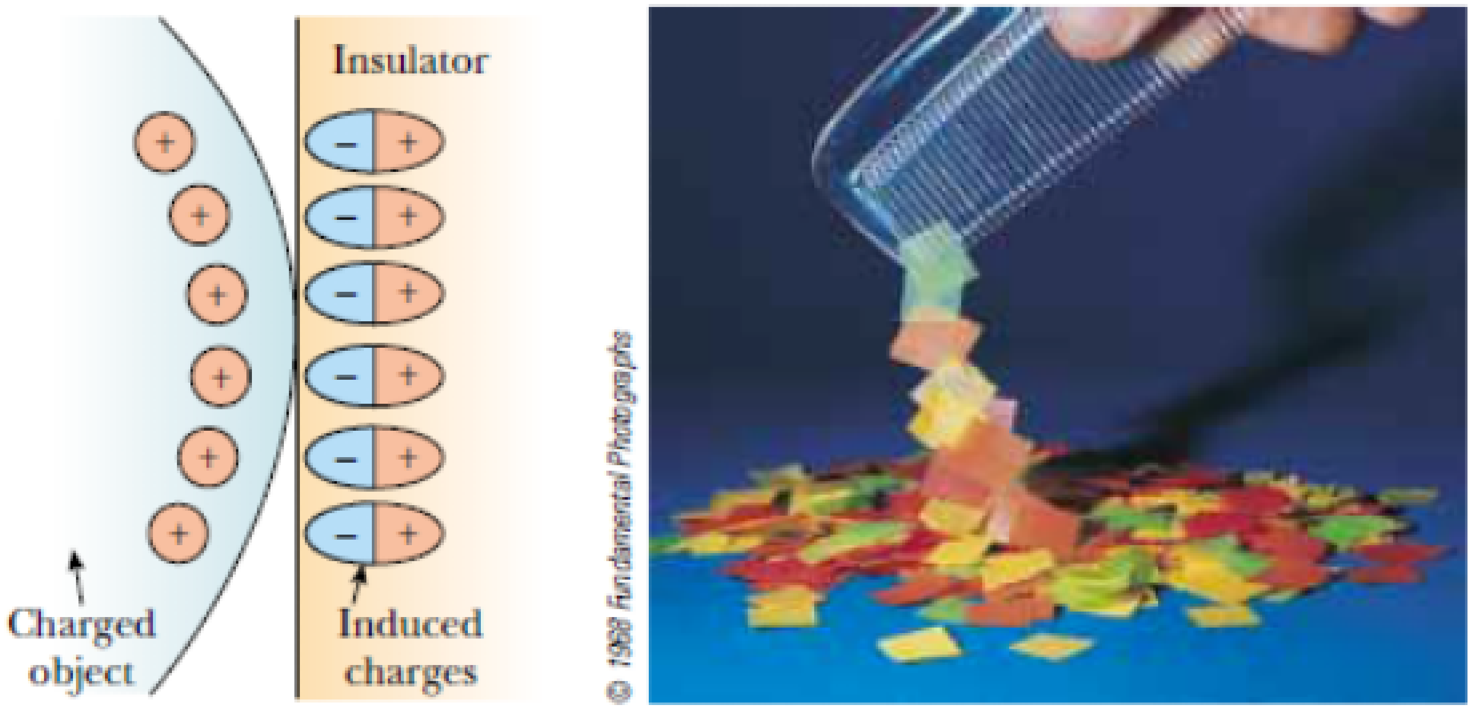
To understand how to charge a conductor by a process known as induction, consider a neutral (uncharged) conducting sphere insulated from the ground, as shown in Fig. a. There are an equal number of electrons and protons in the sphere if the charge on the sphere is exactly zero. When a negatively charged rubber rod is brought near the sphere, electrons in the region nearest the rod experience a repulsive force and migrate to the opposite side of the sphere. This leaves the side of the sphere near the rod with an effective positive charge because of the diminished number of electrons, as in Fig b. (The left side of the sphere in Fig b is positively charged *as if* positive charges moved into this region, but remember that it is only electrons that are free to move.) This occurs even if the rod never actually touches the sphere. If the same experiment is performed with a conducting wire connected from the sphere to the Earth (Fig. c), some of the electrons in the conductor are so strongly repelled by the presence of the negative charge in the rod that they move out of the sphere through the wire and into the Earth. The symbol at the end of the wire in Fig c indicates that the wire is connected to ground, which means a reservoir, such as the Earth, that can accept or provide electrons freely with negligible effect on its electrical characteristics. If the wire to ground is then removed (Fig.d), the conducting sphere contains an excess of *induced* positive charge because it has fewer electrons than it needs to cancel out the positive charge of the protons. When the rubber rod is removed from the vicinity of the sphere (Fig.e), this induced positive charge remains on the ungrounded sphere. Note that the rubber rod loses none of its negative charge during this process. Charging an object by induction requires no contact with the

object inducing the charge. This is in contrast to charging an object by rubbing (that is, by *conduction*), which does require contact between the two objects.



Now similarly you can explain why **a comb rubbed on hair attracts bits of paper.**

A process similar to induction in conductors takes place in insulators. In most neutral molecules, the center of positive charge coincides with the center of negative charge. However, in the presence of a charged object, these centers inside each molecule in an insulator may shift slightly, resulting in more positive charge on one side of the molecule than on the other. This realignment of charge within individual molecules produces a layer of charge on the surface of the insulator, as shown in Figure 23.5a. Knowing about induction in insulators, you should be able to explain why a comb that has been rubbed through hair attracts bits of electrically neutral paper and why a balloon that has been rubbed against your clothing is able to stick to an electrically neutral wall.



(a) (b)

Note :

1. If the charge on one ball is large as compared to the similar charge on the other, the ball with large charge will induce a large charge of opposite kind on the other ball. As a result, **attraction will result inspite of repulsion**.

2. Repulsion is the sure test for electrificaiton.

Coulomb's Law

On the basis of "Torsion balance" experiment "Charles Augustine Coulomb" put a quantitative law for the force of attraction or repulsion on the charges which states that–

"The force of attraction or repulsion on one charge q_2 placed at some separation from another charge q_1 (whose dimensions are small compared to their distance of separation) in infite homogeneous medium, is directly proportional to the product of the magnitude of charges and inversely proportional to the square of the distance between them."

$$F \propto |q_1||q_2|$$

$$\propto \frac{1}{r^2}$$

$$F \propto \frac{|q_1||q_2|}{r^2} \Rightarrow F = k\,\frac{q_1 q_2}{r^2}$$

Where k is a constant of proportionality. In C.G.S. Unit i.e. if F is measured in dyne, q_1 and q_2 in stat coulomb and r in cm, then k = 1

In c.g.s. unit $\qquad F = \frac{q_1 q_2}{r^2}$

But if the force is measured in newton, q_1 and q_2 in coulomb and r in metre then

Then $\quad k = \frac{1}{4\pi \epsilon_0}$ in air or vacuum and $k = \frac{1}{4\pi \epsilon}$ in an infinite homogeneous material medium other than air.

So if the intervening medium between the charges is air or it is vacuum then in S.I. Units

$$F = \frac{1}{4\pi \epsilon_0}\,\frac{q_1 q_2}{r^2}$$

ϵ_0 is called permittivity of free space.

ϵ is called absolute permittivity of the given material medium.

The ratio of the absolute permittivity of a given medium and that of the permittivity of free space is called relative permittivity of that medium or its dielectric constant (represented by symbol ϵ_r or K)

ϵ_r or $K = \frac{\epsilon}{\epsilon_0}$ So, $\epsilon = \epsilon_0 \epsilon_r$ or $K\,\epsilon_0$

Hence, $\quad F_{medium} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2} = \frac{1}{4\pi\varepsilon_0 K}\frac{q_1 q_2}{r^2} = \frac{F}{K}$

Note

(i) K, the dielectric constant of the medium (also called relative permittivity) being ratio of two like quantities, is a dimensionless constant.

(ii) Air or vacuum has minimum relative permittivity ($K = \varepsilon_0/\varepsilon_0 = 1$). The relative permittivity of all other media is greater than 1 usually and K = infinite for a conducting medium.

(iii) When the charges are placed in infite dielectric medium then dielectric medium is getting polarized and force on q_1 or q_2 is not simply due to q_1 or q_{2_i} but due to polarized charges also and net force on q_i or q_j becomes $\frac{1}{\epsilon_r}$ times. (See next Illustration)

Illustration :

Two equal point charges (10^{-3}C)are placed 1 cm apart in medium of dielectric constant K = 5
(a) Find the interaction force between the point charges.
(b) Net force on any of the charge.

Sol.

(a) Interaction force between point charges

$$F = \frac{1}{4\pi \in_0} \frac{q_1 q_2}{r^2} = 9 \times 10^9 \frac{(10^{-3})^2}{(10^{-2})^2}$$

$$= 9 \times 10^7 \text{ N}$$

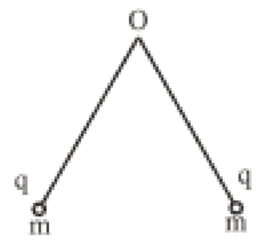
(b) Net force

$$F' = \frac{1}{4\pi \in_0 k} \frac{q_1 q_2}{r^2} = \frac{9 \times 10^9}{5} \frac{(10^{-3})^2}{(10^{-2})^2}$$

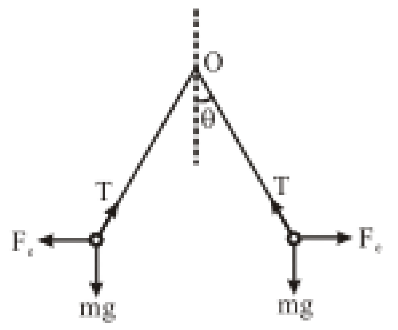
$$= 18 \times 10^6 \text{ N}$$

Illustration :

Two small balls each of mass m and charge q on each of them are suspended through two light insulating string of length l from a point. Find the expression for angle θ made by any of the string with vertical when under static equilibrium.



Sol. *Let angle of any string with vertical be θ as shown*



$$F_e = \frac{1}{4\pi \in_0} \frac{q^2}{(2\ell \sin\theta)^2} = T\sin\theta \qquad \text{... (i) for horizontal direction}$$

$$T\cos\theta = mg \qquad \text{... (ii) for vertical direction}$$

Dividing (i) by (ii)

$$\tan\theta = \frac{F_e}{mg}$$

(i) The force of electrostatic interaction between two charges is operative along the line joining the charges.

(ii) The force obeys inverse square law

(iii) If $q_1q_2 > 0$ (it means the product of the two charges is positive) this implies that charges are similar, i.e., either both positive or both negative. Hence, repulsion will result.

(iv) If $q_1q_2 < 0$, it means the product of the magnitude of the charges is negative. In other words, these are unlike charges, i.e., one charge is positive and the other charge is negative. Hence, the electrostatic force between them is attractive.

Like charges repel each other unlike charges attract

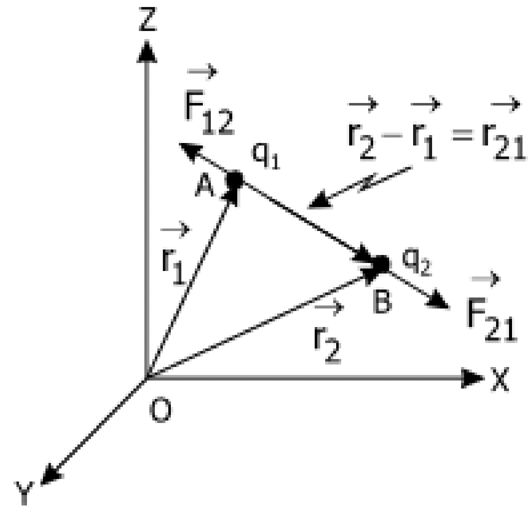
(vi) The force on q_1 due to q_2 is equal and opposite to the force on q_2 due to q_1

$$\vec{F}_{12} = -\vec{F}_{21}$$

i.e. The force of electrostatic interaction between two charges obey Newton's 3rd law. It should, however, be noted that the equality $\vec{F}_{12} = -\vec{F}_{21}$ breaks down when one charge is accelerated towards the other i.e., Newton's 3rd law doesn't hold . This is why Newton's 3rd law is supposed to be a weak law of physics.

Force Between Two Charges in Terms of Their Position Vectors :

Consider two like charge q_1 and q_2 located in vacuum at positions A and B respectively. Let the positions of A and B with reference tothe orign O of the coordinate frame be given by position vectors



$\vec{r}_1$ and $\vec{r}_2$ respectively, i.e., $\vec{OA} = \vec{r}_1$ and $\vec{OB} = \vec{r}_2$

Now, $\vec{OA} + \vec{AB} = \vec{OB}$ (triangle law of vectors)

$$\therefore \qquad \vec{AB} = \vec{OB} - \vec{OA} = \vec{r}_2 - \vec{r}_1 = \vec{r}_{21}$$

According to Coulomb's law, force on q_2 due to q_1 is

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{21}^2}\hat{r}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{21}^3}\vec{r}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1)$$

Similarly, $$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$$

It should be noted that :

(i) For coulomb's law to give precise result the spatial extent of charges should be very small in comparison to their separation.

(ii) Coulomb's law is valid for wide range of distance .

Illustration :

Two point charges A and B have charges respectively $\frac{1}{2}$ C and 2 C with their position vectors respectively as $(\hat{i}+\hat{j}+\hat{k})$ and $(-\vec{i}-\vec{j}+3\vec{k})$. *Find the force on charge at A due to B.*

Sol.

$$q_A = \frac{1}{2}C \qquad \vec{r}_A = \hat{i}+\hat{j}+\hat{k}$$

$$q_B = 2C \qquad \vec{r}_B = -\hat{i}-\hat{j}+3\hat{k}$$

$$\vec{F}_{AB} = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{|\overrightarrow{r_{AB}}|^2}\hat{r}_{AB}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{\left|\overrightarrow{r_{AB}}\right|^3}\left(\overrightarrow{r_A} - \overrightarrow{r_B}\right) = (9\times10^9)\times\frac{\frac{1}{2}\times1}{\left|2\hat{i}+2\hat{j}-2\hat{k}\right|^3}(2\hat{i}+2\hat{j}-2\hat{k})$$

$$= \frac{9\times10^9\times(\hat{i}+\hat{j}-k)}{24\sqrt{3}}N.$$

SUPERPOSITION OF ELECTROSTATIC FORCES

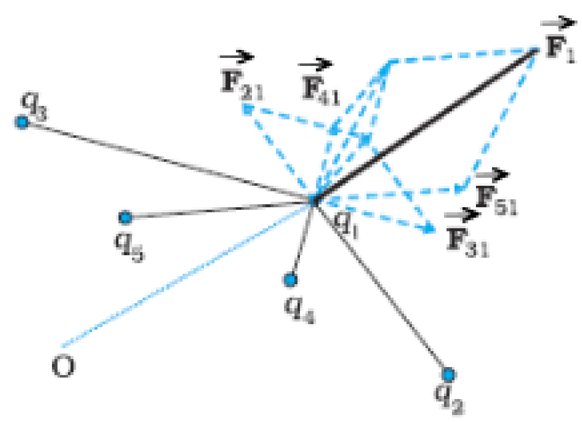
Experimentally it is verified that force on any charge due to a number of other charges is the vector sum of all the forces on that charge due to the other charges, taken one at a time. The individual forces are unaffected due to the presence of other charges. This is termed as the principle of superposition.

The principle of superposition says that in a system of charges $q_1, q_2, ..., q_n$, the force on q_1 due to q_2 is the same as given by Coulomb's law, i.e., it is unaffected by the presence of the other charges $q_3, q_4, ..., q_n$. The total force $\vec{F}_1$ on the charge q_1, due to all other charges, is then given by the vector sum of the forces $\vec{F}_{12}, \vec{F}_{13}, ..., \vec{F}_{1n}$:

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + ... + \vec{F}_{1n}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} + \frac{q_1 q_3}{r_{13}^2}\hat{r}_{13} + ... + \frac{q_1 q_n}{r_{1n}^2}\hat{r}_{1n}\right]$$

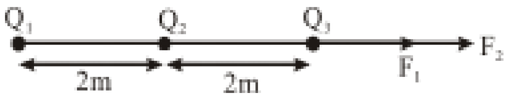
$$= \frac{q_1}{4\pi\varepsilon_0}\sum_{j=2}^{n}\frac{q_j}{r_{1j}^2}\hat{r}_{1j}$$



The vector sum is obtained as usual by the parallelogram law of addition of vectors. All of electrostatics is basically a consequence of Coulomb's law and the superposition principle.

Illustration :

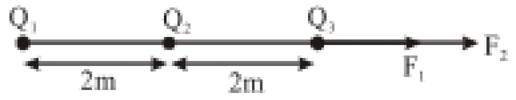
Three charges each of $20\mu C$ *are placed along a straight line, successive charges being 2 m apart as shown in Figure. Calculate the force on the charge on the right end.*



Sol.

$$F = F_1 + F_2$$

$$F_1 = \frac{kQ_1Q_2}{r^2} = \frac{(9\text{x}10^9)(20\text{x}10^{-6})^2}{4^2} = 0.225\text{N}$$



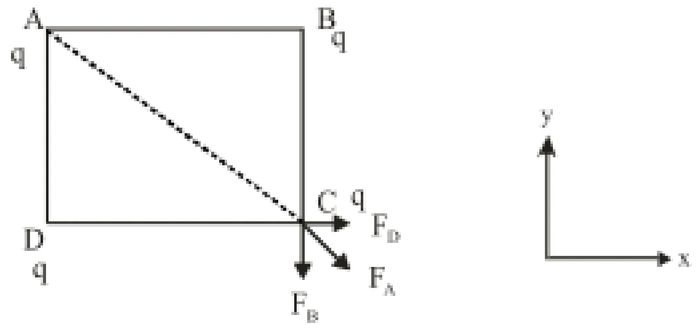
$$F_2 = \frac{kQ_2Q_3}{r^2} = \frac{(9\text{x}10^9)(20\text{x}10^{-6})^2}{2^2} = 0.9\,\text{N}$$

F = 0.225+0.9 = 1.125 N to the right

Illustration:

Four identical point charges each of magnitude q are placed at the corners of a square of side a. Find the net electrostatic force on any of the charge.

Sol.



Let the concerned charge be at C then charge at C will experience the force due to charges at A, B and D. Let these forces respectively be $\vec{F}_A$, $\vec{F}_B$ and $\vec{F}_D$ *thus forces are given as*

$$\vec{F}_A = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{AC^2} \text{ along } AC = \frac{q^2}{4\pi\varepsilon_0 2a^2}\left(\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}\right)$$

$$\vec{F}_B = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{BC^2} \text{ along } BC = \frac{q^2}{4\pi\varepsilon_0 a^2}(-\hat{j})$$

$$\vec{F}_D = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{DC^2} \text{ along } DC = \frac{q^2}{4\pi\varepsilon_0 a^2}(\hat{i})$$

$$\vec{F}_{net} = \vec{F}_A + \vec{F}_B + \vec{F}_D$$

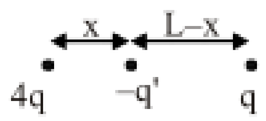
$$= \frac{q^2}{4\pi\varepsilon_0 a^2}\left[\hat{i}\left(\frac{1}{2\sqrt{2}}+1\right) - \hat{j}\left(\frac{1}{2\sqrt{2}}+1\right)\right]$$

$$\left|\vec{F}_{net}\right| = \sqrt{2}\left(\frac{1}{2\sqrt{2}}+1\right)\frac{q^2}{4\pi\varepsilon_0 a^2} \qquad = \left(\frac{1}{2}+\sqrt{2}\right)\frac{q^2}{4\pi\varepsilon_0 a^2}.$$

Illustration:

Two point charges +4q and +q are placed at a distance L apart. A third charge is so placed that all the three charges are in equilibrium. Find the location, magnitude and nature of third charge. Discuss also, whether the equilibrium of the system is stable, unstable or neutral.

Sol.



Third charge should be placed between 4q and q so that force on third charge to be zero (let at distance x from 4q). Third charge should be –ve (let –q') for the equilibrium of other charges
For equilibrium of third charge

$$\frac{K(4q)(q')}{x^2} = \frac{K(q)(q')}{(L-x)^2} \Rightarrow x = \frac{2L}{3}$$

for equilibrium 4q

$$\frac{K(4q)(q')}{\left(\frac{2L}{3}\right)^2} = \frac{K(4q)(q)}{L^2} \qquad \Rightarrow \qquad q' = \frac{4q}{9}$$

Practice Exercise

Q.1 Three particles, each of mass 1 g and carrying a charge q, are suspended from a common point by insulating massless strings, each 100 cm long. If the particles are in equilibrium and are located at the corners of an equilateral triangle of side 3 cm, calculate the charge q on each particle. (g= 10m/ s^2).

Q.2 Five point charges, each of value +q are placed on five vertices of a regular hexagon of side L m. What is the magnitude of the force on a point charge of value –q coulomb placed at the centre of hexagon?

Q.3 Two equal point charges q are fixed at x = – a and x = a along the x-axis. A particle of mass m and charge q/2 is brought to the origin and given a small displacement along the (a) x-axis and (b) y-axis. Describe the motion in the two cases.

Answers

Q.1 3.16×10^{-9} C Q.2 $\frac{1}{4\pi \in_0}\left[\frac{q}{R}\right]^2$ Q.3 (a) Accelerated motion (b) SHM

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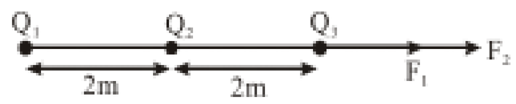

Sol.

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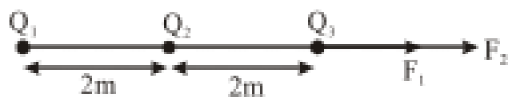

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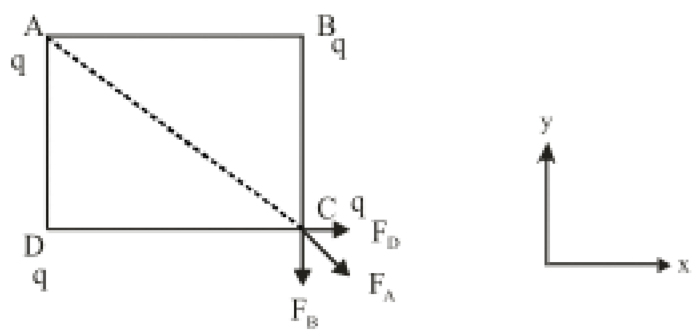

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$$\vec{F}_A = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{AC^2} \text{ along } AC = \frac{q^2}{4\pi\varepsilon_0 2a^2}\left(\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}\right)$$

$$\vec{F}_B = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{BC^2} \text{ along } BC = \frac{q^2}{4\pi\varepsilon_0 a^2}(-\hat{j})$$

$$\vec{F}_D = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{DC^2} \text{ along } DC = \frac{q^2}{4\pi\varepsilon_0 a^2}(\hat{i})$$

$$\vec{F}_{net} = \vec{F}_A + \vec{F}_B + \vec{F}_D$$

$$= \frac{q^2}{4\pi\varepsilon_0 a^2}\left[\hat{i}\left(\frac{1}{2\sqrt{2}}+1\right) - \hat{j}\left(\frac{1}{2\sqrt{2}}+1\right)\right]$$

$$\left|\vec{F}_{net}\right| = \sqrt{2}\left(\frac{1}{2\sqrt{2}}+1\right)\frac{q^2}{4\pi\varepsilon_0 a^2} \qquad = \left(\frac{1}{2}+\sqrt{2}\right)\frac{q^2}{4\pi\varepsilon_0 a^2}\;.$$

Practice Exercise

Q.1 A particle having a charge of 2.0 10 6 C and a mass of 100 g is placed at the bottom of a smooth inclined plane of inclination 30 . Where should another particle B, having same charge and mass, be placed on the incline so that it may remain in equilibrium?

Q.2 A copper atom consists of copper nucleus surrounded by 29 electrons. The atomic weight of copper is 63.5 g/mol. Let us now take two pieces of copper each weighing 10 g. Let us transfer one electron from one piece to another for every 1000 atoms in a piece. What will be the coulomb force between the two pieces after the transfer of electrons if they are 10 cm apart? [e= 1.6×10^{-19}C, $(1/4\pi\epsilon_0) = 9\times10^9$ and Avogadro's number $=6\times10^{23}$ per mol]

Q.3 Two spherical conductors B and C having equal radii and carrying equal charges with them repel each other with a force F when kept apart at some distance. A third spherical conductor having same radius as that of B but uncharged is brought in contact with B, then brought in contact with C and finally removed away from both. What is the new force of repulsion between B and C ? (When two conductors of identical geometry having charges q_1 and q_2 if touched have final charges $\frac{q_1+q_2}{2}$ on each conductor)

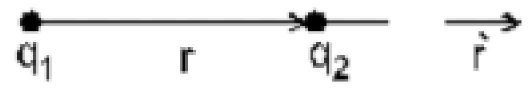
Q.4 A ring of radius 0.1 m is made our of a metallic wire of area of cross– section 10^{-6} m^2. The ring has a uniform charge of π coulomb. Find the change in the radius of the ring when a charge of 10^{-8} coulomb is placed at the centre of the ring.
Young's modulus of the metal is 2×10^{11} N/ m^2.

Answers

Q.1	27 cm from the bottom	Q.2	2.08×10^{14}N	Q.3	3F/8
Q.4	2.25×10^{-13} m				

Vector form of coulomb's law :

By stating coulomb's law in vector form more information can be packed in it.



Let the position vector of charge q_2 relative to charge q_1 be $\vec{r}$ and $\hat{r}$ is a unit vector in the direction of $\vec{r}$

so, $\vec{r} = |\vec{r}|\,\hat{r} = r\,\hat{r}$ hence, $\hat{r} = \frac{\vec{r}}{r}$

Coulomb's law, in vector form, may be written as

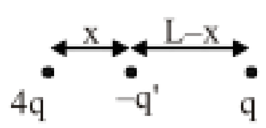
$$\vec{F} = \frac{1}{4\pi\in}\frac{q_1 q_2}{r^2}\hat{r} = \frac{1}{4\pi\in}\frac{q_1 q_2}{r^3}\vec{r}$$

From the above form of the coulomb's law, It may be justified that,

Illustration:

Two point charges +4q and +q are placed at a distance L apart. A third charge is so placed that all the three charges are in equilibrium. Find the location, magnitude and nature of third charge. Discuss also, whether the equilibrium of the system is stable, unstable or neutral.

Sol.



Third charge should be placed between 4q and q so that force on third charge to be zero (let at distance x from 4q). Third charge should be –ve (let –q') for the equilibrium of other charges
For equilibrium of third charge

$$\frac{K(4q)(q')}{x^2} = \frac{K(q)(q')}{(L-x)^2} \Rightarrow x = \frac{2L}{3}$$

for equilibrium 4q

$$\frac{K(4q)(q')}{\left(\frac{2L}{3}\right)^2} = \frac{K(4q)(q)}{L^2} \qquad \Rightarrow \qquad q' = \frac{4q}{9}$$

Practice Exercise

Q.1 Three particles, each of mass 1 g and carrying a charge q, are suspended from a common point by insulating massless strings, each 100 cm long. If the particles are in equilibrium and are located at the corners of an equilateral triangle of side 3 cm, calculate the charge q on each particle. ($g = 10 m/s^2$).

Q.2 Five point charges, each of value +q are placed on five vertices of a regular hexagon of side L m. What is the magnitude of the force on a point charge of value –q coulomb placed at the centre of hexagon?

Q.3 Two equal point charges q are fixed at x = – a and x = a along the x-axis. A particle of mass m and charge q/2 is brought to the origin and given a small displacement along the (a) x-axis and (b) y-axis. Describe the motion in the two cases.

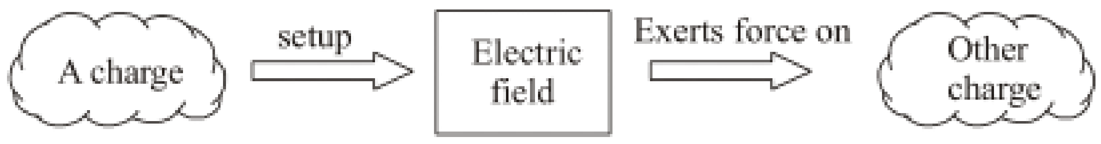
Answers

Q.1 3.16×10^{-9} C Q.2 $\frac{1}{4\pi \epsilon_0}\left[\frac{q}{R}\right]^2$ Q.3 (a) Accelerated motion (b) SHM

ELECTRIC FIELD

Introduction

Now a question arises how a charge q_1 exerts force on another charge q_2. The answer is q_1 influence its surrounding electrically and this electrically infuluenced surrounding exerts force on q_2. Here we say that q_1 set up its electric field and this electric field exerts force q_2. This electric influence of a charge is measured by a vector called intensity of electric field (or loosely speaking electric field) and represented by $\vec{E}$.



To define electric field of a charge q (called source charge) at a point P in its surrounding place a small positive charge q_0 (called test charge) at point P. If q exerts force $\vec{F}$ on q_0 then inesity of electric field due of q at P is defined as

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$$

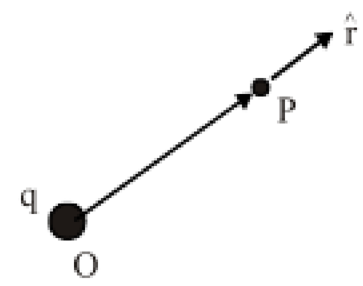
q_0 is taken positive so that $\vec{F}$ give direction of $\vec{E}$. q_0 is taken small because if it is taken large it can disturb q. It should be also kept is mind that $\vec{E}$ is property of q i.e. it will still present if test charge is absent.

Electric field due to a point charge

A point charge q is placed at point O. We have to express its electric field at point P whose position vector with respect to point P is $\vec{r}$.

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{\frac{kqq_0}{r^2}\hat{r}}{q_0}$$

$$\vec{E} = \frac{kq}{r^2}\cdot\hat{r} = \frac{kq}{r^3}\vec{r}$$

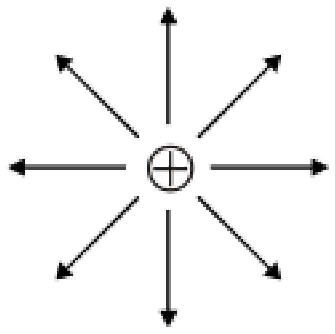


magnitude of electric field is given by

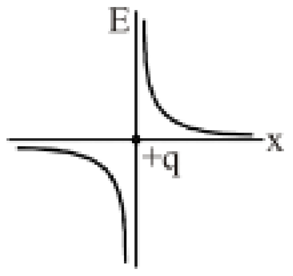
$$E = \frac{k\,|q|}{r^2}$$

Its direction will be radially outward is q is positive and will be radially inward if q is negative

Electric field of a positive point charge

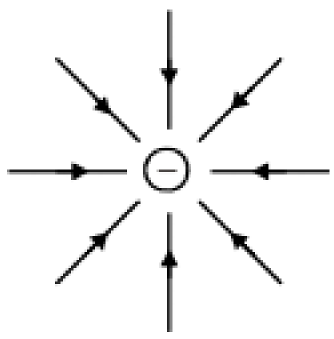


plot of electric field of a positive point charge at different points on x-axis

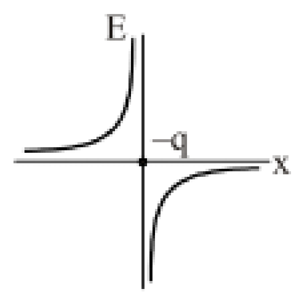


Electric field due to a point positive charge is radially outward

Electric field of a negative point charge



plot of electric field of a negative point charge at different points on x-axis



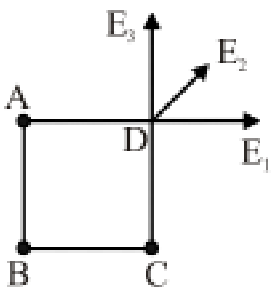
Electric field due to a point negative charge is radially inward

Illustration :

Three point charge $q_1 = +q$, $q_2 = -2q$ and $q_3 = +q$ are placed at corners A , B and C of square ABCD of side 'a' find electric field at D.

Sol.

$$E_1 = \frac{kq}{a^2} \text{ (alone } \overrightarrow{AD}\text{)}$$



$$E_2 = \frac{k\,|-2q\,|}{(\sqrt{2}a)^2} = \frac{2kq}{2a^2}$$

$$\frac{kq}{a^2}(\text{a long}\overrightarrow{DB})$$

$$E_3 = \frac{kq}{a^2} \text{ (a long } \overrightarrow{CD}\text{)}$$

$$\therefore \quad E_{resultant} = \frac{kq}{a^2} \times \sqrt{2} - \frac{kq}{a^2} = \frac{kq}{a^2}\left(\sqrt{2}-1\right)$$

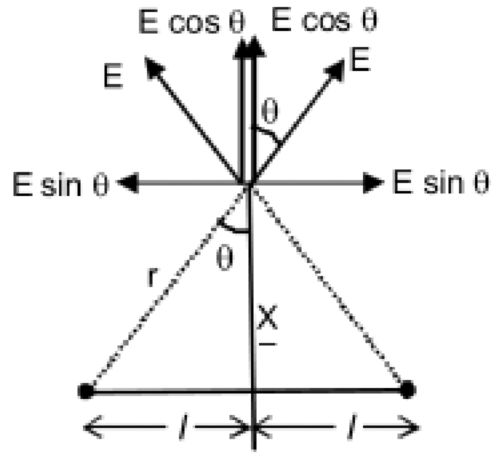
Illustration :

Two point charges each having charge q are placed at separation 2l find the electric field at a point on the perpendicular bisector at a distance x from its mid point.

Sol. *Electric field due to any of the charge will be*

$$E = \frac{kq}{r^2}$$

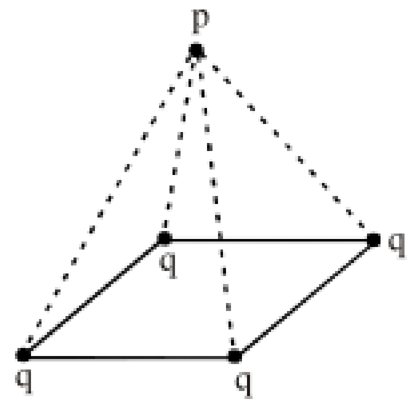
Resolve the electric field into two components. E cos θ and E sin θ (fig.). Horizontal components of electric field of both charges nullifies each other. Hence the net electric field will be.



$$E_{net} = 2E\cos\theta = \frac{2kq}{r^2} \times \frac{z}{r} = \frac{2kqz}{(\ell^2 + z^2)^{3/2}}$$

Illustration :

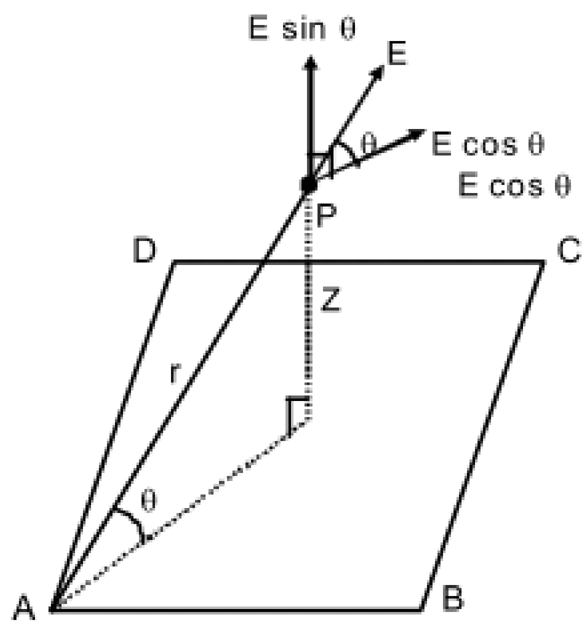
Four identical charges are fixed at the corners of a square of side a. Find electric field at point p which is at a distance z lying on the line perpendicular to the plane of the square passing through the centre of square.



Sol.

Let us first calculate electric field due to point charge present at A.

$$E_A = \frac{kq}{r^2}$$



This electric field can be resolved into two components E cos θ (horizontal) and E sin θ (vertical). From symmetry it is clear that sum of horizontal components of electric field will be zero.

Hence the net electric field will be

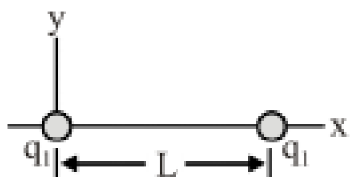
$$E_{net} = 4E \sin\theta \text{ (vertical)}$$

$$= 4 \times \frac{kq}{r^2} \times \frac{z}{r} = 4 \cdot \frac{1}{4\pi\epsilon_0} \frac{qz}{\left(\frac{a^2}{2} + z^2\right)^{3/2}}$$

Illustration :

Two charged particles lie along the x-axis as shown in figure. The particle with charge $q_2 = +8\mu C$ is at $x = 6.00$ m, and the particle with charge $q_1 = +2\mu C$ is at the origin. Locate the point whre the resultant electric field is zero.

Sol.



Two charged particles kept on the x-axis.

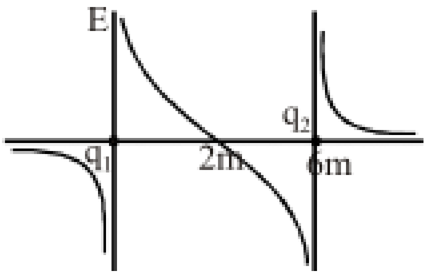
Before calculating, let us physically see the location of the point where the electric field can be zero. At points other than the x-axis, say above the x-axis, both the charges will have a component of the electric field in the positive direction. This y component of the electric field does not cancel out. So the net electric field at that point will not be zero. The same statement also holds true for points which are not in xy-plane.

On the x-axis also we can see that on the points beyond x = 6 m and points on the negative x-axis, both the electric fields will be in the same direction. So the net electric field cannot be zero. At

some point between the two charges, the electric field due to both f them will be in opposite direction.
So the electric field will be zero at a point between x = 0 and x = 6m.
To find the electric field at any point, we can apply the superposition principal as in equation.

$$E_x = \frac{k \times 2 \times 10^{-6}}{x^2} - \frac{k \times 8 \times 10^{-6}}{(6-x)^2} = 0$$

Solving we get x = 2m. We are ignoring the other trivial solution.
To analyze this situation more deeply, let us draw a graph of the net electric field at different points on the x-axis versus the position on the x-axis.



Electric field at various points of x-axis due to q_1 and q_2.

This point where the electric field is zero is called a neutral point. We can also say that if the electric field at this point is zero, a charged particle kept at this point will not experience any force.

$$\vec{E} = \frac{\vec{F}}{q}$$

Then $\vec{F} = q\vec{E}$. *Hence, at this point any charged particle will be in equilibrium.*

Practice Exercise

Q.1 Two particles A and B having charges of + 2.00 10 6 C of 4.00 10 6 C respectively are held fixed at a separation of 20.0 cm. Locate the point(s) on the line AB where the electric field is zero

Ans 48.3 cm from A along BA

Q.2 Three identical charges, each having a value 1.0 10 8 C, are placed at the corners of an equilateral triangle of side 20 cm. Find the electric field at the centre of the triangle.

Ans zero,

Answers

Q.1 48.3 cm from A along BA Q.2 zero,

Continuous charge distribution

We have so far dealt with charge configurations involving discrete charges $q_1, q_2, ..., q_n$. One reason why we restricted to discrete charges is that the mathematical treatment is simpler and does not involve calculus. For many purposes, however, it is impractical to work in terms of discrete charges and we need to work with continuous charge distributions. For example, on the surface of a charged conductor, it is impractical to specify the charge distribution in terms of the locations of the microscopic charged constituents. It is more feasible to consider an area element ΔS on the surface of the conductor (which is very small on the macroscopic scale but big enough to include a very large number of electrons) and specify the charge ΔQ on that element. We then define a surface charge density σ at the area element by

$$\sigma = \operatorname*{Lim}_{\Delta S \to 0} \frac{\Delta Q}{\Delta S} = \frac{dQ}{dS}$$

We can do this at different points on the conductor and thus arrive at a continuous function σ called the surface charge density. The surface charge density σ so defined ignores the quantisation of charge and the discontinuity in charge distribution at the microscopic level. σ represents macroscopic surface charge density, which in a sense, is a smoothed out average of the microscopic charge density over an area element ΔS represents macroscopic surface charge density, which in a sense, is a smoothed out average of the microscopic charge density over an area element σ are C/m^2.

Similar considerations apply for a line charge distribution and a volume charge distribution. The linear charge density λ of a wire is defined by

$$\lambda = \operatorname*{Lim}_{\Delta l \to 0} \frac{\Delta Q}{\Delta l} = \frac{dQ}{dl}$$

Where Δl is a small line element of wire on the macroscopic scale that, however, includes a large number of microscopic charged constituents, and ΔQ is the charge contained in that line element. The units for λ are C/m. The volume charge density (sometimes simply called charge density) is defined in a similar manner:

$$\rho = \operatorname*{Lim}_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} = \frac{dQ}{dV}$$

Where ΔQ is the charge included in the macroscopically small volume element ΔV that includes a large number of microscopic charged constituents. The units for ρ are C/m^3. The notion of continuous charge distribution is similar to that we adopt for continuous mass distribution in mechanics. When we refer to the density of a liquid, we are referring to its macroscopic density. We regard it as a continuous fluid and ignore its discrete molecular constitution.

Electric field of a continuous charge distribution

The total electric field at P due to all elements in the charge distribution is approximately.

$$\vec{E} \approx k_e \sum_i \frac{\Delta q_i}{r_i^2} \hat{r}_i$$

Considering the charge distribution as continuous, the total field at P in the limit $\Delta q_i \to 0$ is

$$\vec{E} = k_e \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i}{r_i^2} \hat{r}_i = k_e \int \frac{dq}{r^2} \hat{r}$$

Steps for calculating the electric field for continuous charge distributions

1. Identify the type of charge distribution and compute the charge density λ, σ or ρ.
2. Divide the charge distribution into infinitesimal charges dq, each of which will act as a tiny point charge.
3. The amount of charge dq, i.e., within a small element d*l*, dA or dV is
   $dq = \lambda dl$ (charge distributed in length)
   $dq = \sigma dA$ (charge distributed over a surface)
   $dq = \rho\, dV$ (charge distributed throughout a volume)
4. Draw at point P the dE vector produced by the charge dq. The magnitude of dE is

   $$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$

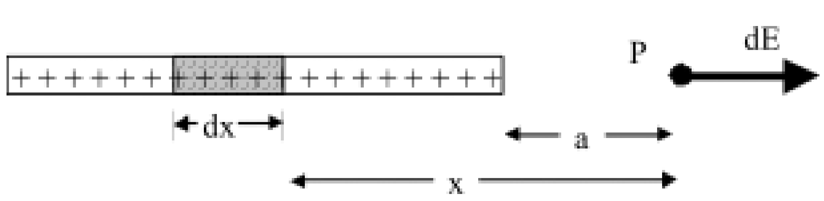
   Vector dE is along radial line joining dq to P, dE is directed away for positive charge dq while directed towards dq for negative dq.
5. Resolve the dE vector into its components. Identify any special symmetry features to show whether any component(s) of the field that are not cancelled by other components.
6. Write the distance r and any trigonometric factors in terms of given coordinates and parameters.
7. The electric field is obtained by summing over all the infinitesimal contributions.

   $$\vec{E} = \int d\vec{E} = \int \frac{dq}{4\pi\varepsilon_0 r^2}$$

8. Perform the indicated integration over limit of integration that include all the source charges.

Electric field intensity at any point due to a uniformly charged rod (of linear charge density λ) at a point on its axis

Consider an element, dx at a distance, x from the point, P, where we have to find the electric field. The elemental charge, $dq = \lambda dx$



Now, electric field due to elementry part will be

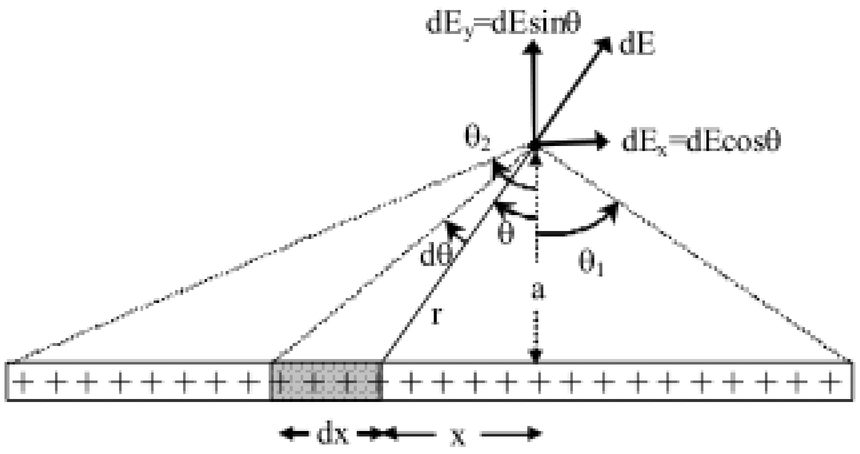
$$dE = k.\frac{\lambda dx}{x^2}$$

Then, electric field due to entire rod will be

$$E = k\lambda\int_a^{a+L}\frac{1}{x^2}dx = k\lambda\left[-\frac{1}{x}\right]_a^{a+L} = k\lambda\left[\frac{-1}{a+L}+\frac{1}{a}\right]$$

Thus $$E = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{L+a}\right]$$

Electric field intensity at any point due to a uniformly charged rod (of linear charge density λ) at a genaral point



Consider an element, dx at a distance, x from the point, P, from normal. The elemental charge, dq $=\lambda$dx
Now, electric field due to elementry part will be

$$dE = k.\frac{\lambda dx}{r^2}$$

This electric field has two components $dE_x = dE\cos\theta$ and $dE_y = dE\sin\theta$
The x-component of resultant electric field will be

$$E_x = \int dE_x = \int dE\sin\theta = \int\left(k.\frac{\lambda dx}{r^2}\right)\left(\frac{x}{r}\right) = k\lambda\int\frac{xdx}{r^3} = k\lambda\int\frac{xdx}{\left(a^2+x^2\right)^{3/2}}$$

substitute

$$x = a\tan\theta \qquad \Rightarrow \qquad dx = a\sec^2\theta d\theta$$

and

$$\left(a^2+x^2\right)^{3/2} = \left(a^2+a^2\tan^2\theta\right)^{3/2} = a^3\sec^3\theta$$

Then, x-component of electric field due to entire rod will be

$$E_x = \int dE_x = k\lambda\int\frac{xdx}{\left(a^2+x^2\right)^{3/2}} = k\lambda\int_{-\theta_1}^{+\theta_2}\frac{\left(a\tan\theta\right)\left(a\sec^2\theta d\theta\right)}{a^3\sec^3\theta} = \frac{k\lambda}{a}\int_{-\theta_1}^{+\theta_2}\sin\theta d\theta = \frac{k\lambda}{a}\left[-\cos\theta\right]_{-\theta_1}^{+\theta_2} = \frac{k\lambda}{a}\left[\cos\theta\right]_{+\theta_2}^{-\theta_1}$$

$$E_x = \frac{k\lambda}{a}\left(\cos\theta_1 - \cos\theta_2\right)$$

The y-component of resultant electric field will be

$$E_y = \int dE_y = \int dE\cos\theta = \int\left(k.\frac{\lambda dx}{r^2}\right)\left(\frac{a}{r}\right) = k\lambda a\int\frac{dx}{r^3} = k\lambda a\int\frac{dx}{\left(a^2+x^2\right)^{3/2}}$$

Again using, $dx = a\sec^2\theta d\theta$ and $\left(a^2+x^2\right)^{3/2} = \left(a^2+a^2\tan^2\theta\right)^{3/2} = a^3\sec^3\theta$

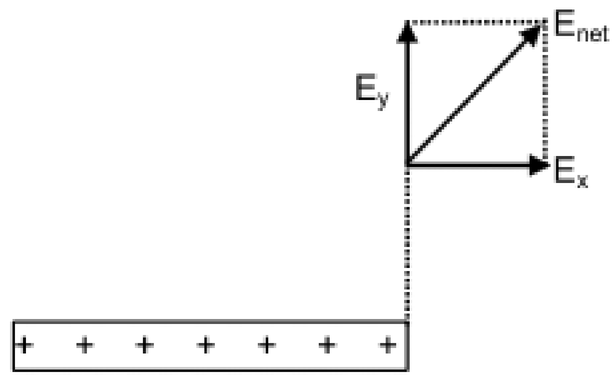
Y-component of electric field due to entire rod will be

$$E_y = \int dE_y = k\lambda a\int\frac{dx}{\left(a^2+x^2\right)^{3/2}} = k\lambda a\int_{-\theta_1}^{+\theta_2}\frac{a\sec^2\theta d\theta}{a^3\sec^3\theta} = \frac{k\lambda}{a}\int_{-\theta_1}^{+\theta_2}\cos\theta d\theta = \frac{k\lambda}{a}\left[\sin\theta\right]_{-\theta_1}^{+\theta_2}$$

$$E_y = \frac{k\lambda}{a}(\sin\theta_1 + \sin\theta_2)$$

Special case 1

If the wire is semi infinite



$$\theta_1 = 0 \quad \theta_2 = \frac{\pi}{2}$$

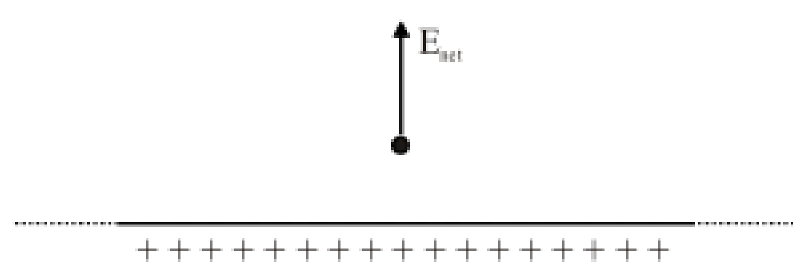
$$E_x = \frac{k\lambda}{a} \text{ and } E_y = \frac{k\lambda}{a}$$

$$\therefore E = \sqrt{E_x^2 + E_y^2} = \frac{\sqrt{2}k\lambda}{a}$$

$$\tan\theta = \frac{E_y}{E_x} = 1 \Rightarrow \theta = 45°$$

Special case-2

If the wire is infinite



$$\theta_1 = \theta_2 = \frac{\pi}{2}$$

$$E_x = 0 \quad E_y = \frac{2k\lambda}{a}$$

$$\therefore E_{net} = \frac{2k\lambda}{a}$$

Electric field due to uniformly charged circular arc (of linear charge density λ) at its centre

Consider an element at an angle ϕ from bisctor and haivng angle $d\phi$ at centre.

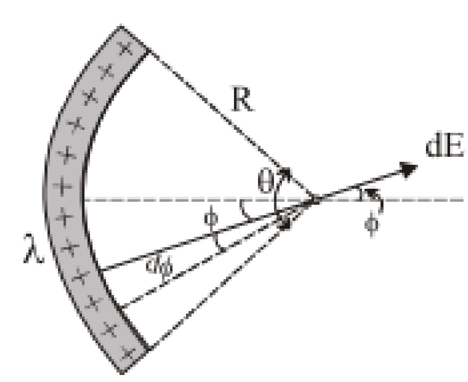
Length of the elementry part will be

$$dl = Rd\phi.$$

Elementry charge

$$dq = \lambda R d\phi$$

Electric field due to this elementry part will be



$$dE = \frac{k(\lambda R d\phi)}{R^2} = \frac{k\lambda}{R}d\phi$$

This electric field has two componets $dE_x = dE\cos\phi$ and $dE_y = dE\sin\phi$. From symmetry it is clear that sum of dE_y will be zero

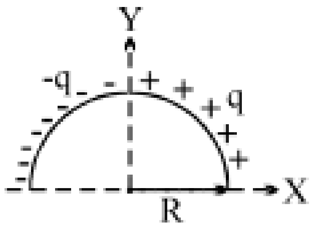
Hence the net electric field will be sum of dE_x, i.e.

$$E_{net} = \int dE_x = \int dE\cos\phi$$

$$= \int_{-\theta/2}^{+\theta/2} \frac{k\lambda}{R}\cos\phi\, d\phi = \frac{2k\lambda}{R}\sin\frac{\theta}{2}$$

Practice Exercise

Q.1 Find the electric field at centre of semicircular ring shown in figure.

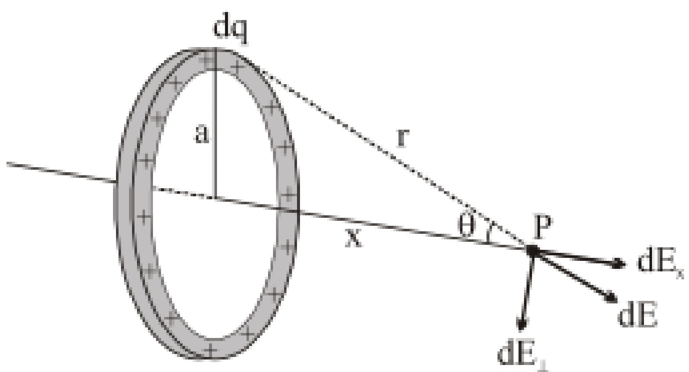


Answers

Q.1 $\frac{4kq}{\pi R^2}\hat{i}$

Electric field due to uniformly charged circular ring at a point on its axis

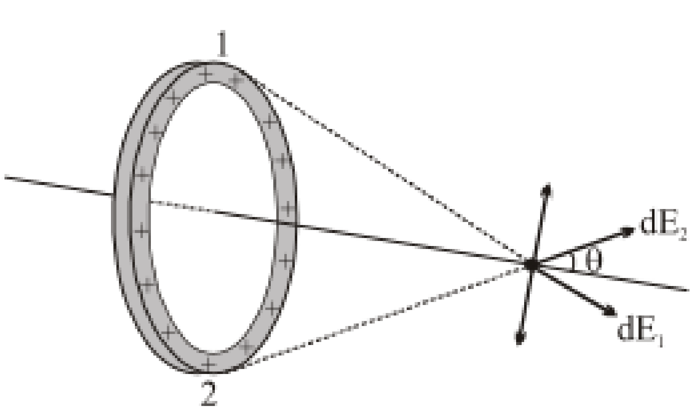
(at a point P lying a distance x from its center along the central axis perpendicular to the plane of the ring)



The magnitude of the electric field at P due to the segment of charge dq is

$$dE = k_e \frac{dq}{r^2}$$

This field has an x component $dE_x = dE\cos\theta$ along the x-axis and a component $dE_\perp$ perpendicular to the x-axis. The resultant field at P must lie along the x-axis because the perpendicular components of the field created by any charge element is canceled by the perpendicular component created by an element on the opposite side of the ring.

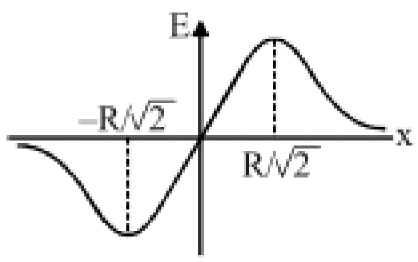


$$dE_x = dE\cos\theta = \left(k_e \frac{dq}{r^2}\right)\frac{x}{r} = \frac{k_e x}{(x^2+a^2)^{3/2}}dq$$

All elements of the ring make the same contribution to the field at P because they are all equidistant from this point. Thus, we can integrate to obtain the total field at P

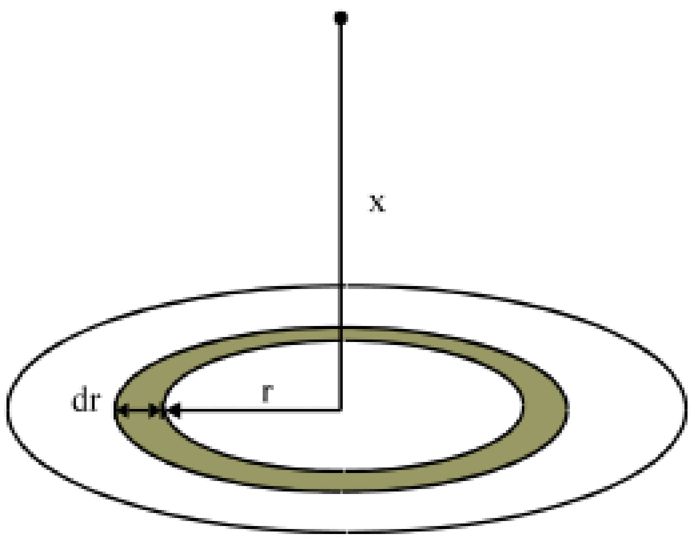
$$E_x = \int \frac{k_e x}{(x^2+a^2)^{3/2}}dq = \frac{k_e x}{(x^2+a^2)^{3/2}}Q$$

Plot of electric field a different points on x-axis



Electric field on the axis of a uniformly charged disc (of surface charge density σ)

Consider a disc of uniform surface charge density 'σ'.Let us calculate the electric field due to a ring of charge situated at a distance r, from the centre and having a width, dr.



$dE = \frac{kx\,dq}{(x^2+r^2)^{3/2}}$, directed along the line OP.

[Here we are using the expression for electric field intensity for a charged ring of radius r at a point on the axis at a distance x from the centre, $E = \frac{kxq}{(x^2+r^2)^{3/2}}$ directed along the axis outwards from the centre.]

Now, the area of the ring, $dS = 2\pi r.\,dr$, $\Rightarrow dq = \sigma 2\pi r dr$,

Thus,

$$|E|_P = \int dE = kx\int_0^R \frac{\sigma 2\pi r dr}{(x^2+r^2)^{3/2}} = \frac{1}{4\pi\epsilon_o}x.\sigma\pi\int_0^R \frac{2r}{(x^2+r^2)^{3/2}}dr$$

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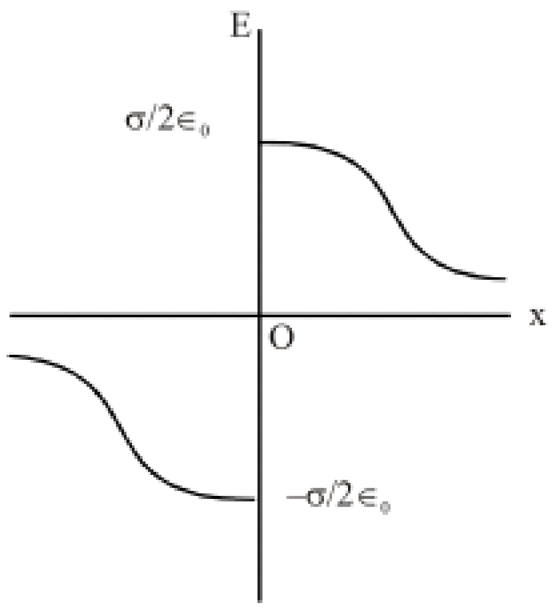
$$E = \frac{\sigma}{2\in_0}\left[1 - \frac{x}{\left(x^2 + R^2\right)^{1/2}}\right] = \frac{\sigma}{2\varepsilon_0}(1 - \cos\theta),$$

where θ = semi vertical plane angle subtend by the disc at P.

Plot of electric field due to uniformly charged disc on its axis



Also, as $R \to \infty$ $\Rightarrow$ $E = \frac{\sigma}{2\in_0}$

which is the electric field in front of an infinite plane sheet of charge.

Electric field due to non uniform charge distribution

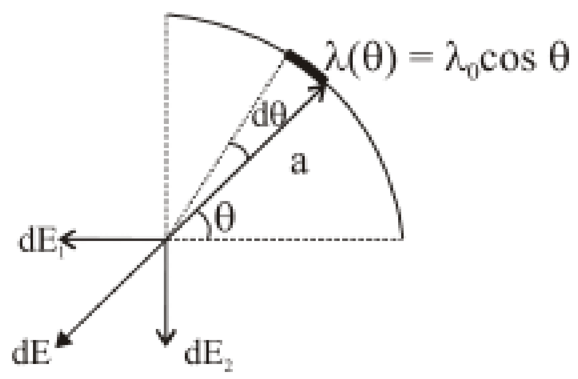
Illustration :

Figure shows circular arc which is non uniformly charged. The linear charge density on the arc is given by

$$\lambda = \lambda_0 \cos\theta$$

where θ is measured from x-axis. Find the electric field at centre of arc

Sol.



let us take an elemetry charge at location θ from x-axis and subtending angle $d\theta$ at centre of the arc

elementry charge is

$$dQ = \lambda dl = (\lambda_0 \cos\theta)(Rd\theta)$$

Electric field due to this elementry charge will be

$$dE = \frac{kdQ}{R^2} = \frac{k\lambda_0 R\cos\theta\, d\theta}{R^2} = \frac{k\lambda_0 \cos\theta\, d\theta}{R}$$

This electric field can be resolved into two components $dE_1 = dE\cos\theta$ (along x-axis) and $dE_2 = dE\sin\theta$ (along y-axis)

The net electric field along x-axis will be

$$E_1 = \int dE_1 = \int dE\cos\theta = \int_0^{\pi/2} \frac{k\lambda_0 \cos\theta\, d\theta \cos\theta}{R} = \frac{k\lambda_0}{R}\cdot\frac{\pi}{4}$$

The net electric field along y-axis will be

$$E_2 = \int dE_2 = \int dE\sin\theta = \int_0^{\pi/2} \frac{k\lambda_0 \cos\theta\; d\theta \sin\theta}{R} = \frac{k\lambda_0}{R}\cdot\frac{1}{2}$$

Practice Exercise

Q.1 A rod of length L has a total charge Q distributed uniformly along its length. It is bent in the shape of a semicircle. Find the magnitude of the electric field at the centre of curvature of the semicircle. Find the magnitude of the electric field at the centre of curvature of the semicircle.

Ans. $\frac{Q}{2\varepsilon_0 L^2}$

Q.2 A circular wire-loop of radius carries a total charge Q distributed uniformly over its length. A small length dL of the wire is cut off. Find the electric field at the centre due to the remaining wire.

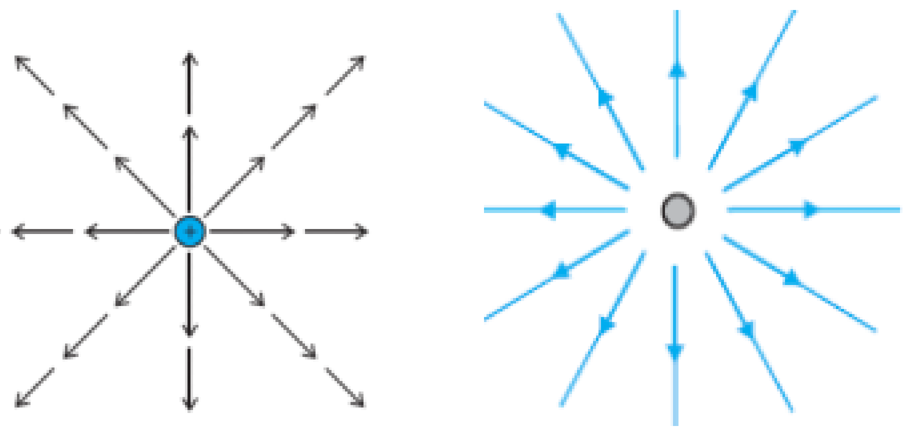
Ans. $\frac{QdL}{8\pi^2\varepsilon_0\alpha^2}$

Answers

Q.1 $\frac{Q}{2\varepsilon_0 L^2}$ Q.2 $\frac{QdL}{8\pi^2\varepsilon_0\alpha^2}$

Electric filed lines

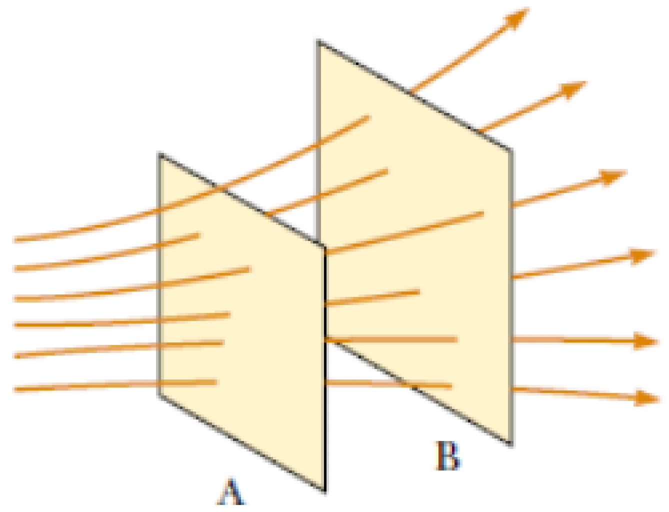
According to Michael Faraday we can visualize electric field with the help of lines. These imaginary lines or curves are termed as electri field lines or electric lines of forces. They are drawn such that tangent at any point on an electric line of force gives the direction of the field at that point.



The first figure represents direction of electric field (force on unit positive test charge) due to a positive charge at some points whereas second figure represents corresponding electric field lines.

The number density of lines (number lines crssing unit area normally) represent the relative strength of the field.In other language at points where the intensity is low, the lines of forces will be widely separated and where the intensity is higher, the lines of force will be closely packed.

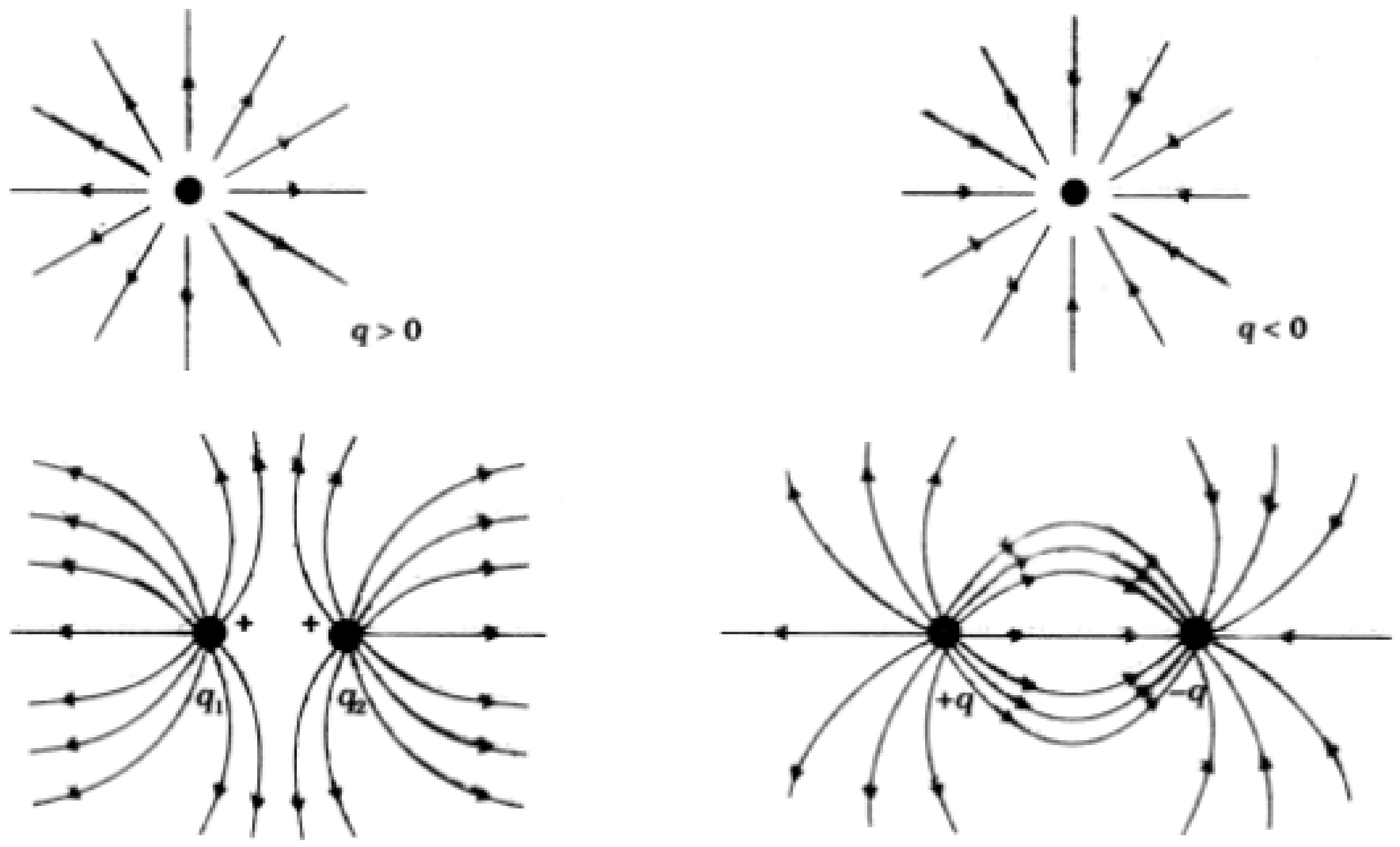
Another person may draw more lines. But the number of lines is not important. In fact, an infinite number of lines can be drawn in any region. It is the relative density of lines in different regions which is important.



Electric field lines penetrating two surfaces. The magnitude of the field is greater on surface A than on surface B.

Properties of electric lines of force

(i) Electric lines of forces originate from (+)ve charge and terminate (end) on negative charge i.e. (+)ve charges are sources of electric lines of forces and (–) ve charges are sinks for them.



(ii) Electric lines of forces are open curves (not closed curves like magnetic lines of forces). This property of electric lines of forces signifies the fact that electric field is a conservative force field.

(iii) Two electric lines of force never cross each other because if they do cross then there will be two directions of field at same point.

(iv) The number of lines drawn leaving a positive charge or approaching a negative charge is proportional to the magnitude of the charge.

If magnitude as well as direction of electric field is same then electric field is said to be uniform otherwise nonuniform

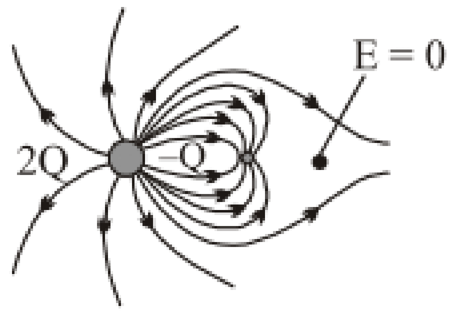
In the first figure the magnitude as well as direction is uniform. In the second figure magnitude of the electric field is uniform where as direction is non uniform. In the third figure direction of the electric field is uniform where as magnitude is non uniform. In the fourth figure the magnitude as well as direction is non uniform.

The first figure represent uniform electric field where as the remaining three figures represent non uniform electric field

Illustration

Figure shows the sketch of field lines for two point charges 2Q and – Q.

Sol.



The pattern of field lines can be deduced by considering the following points:

(a) Symmetry : For every point above the line joining the two charges there is an equivalent point below it. Therefore, the pattern must be symmetrical about the line joining the two charges.

(b) Near field : Very close to a charge, its own field predominates. Therefore, the lines are radial and spherically symmetric.

(c) Far field : Far from the system of charges, the pattern should look like that of a single point charge of value (2Q – Q) = + Q, i.e., the lines should be radially outward.

(d) Null point : There is one point at which E = 0. No lines should pass through this point.

(e) Number of lines : Twice as many lines leave + 2Q as entre – Q.

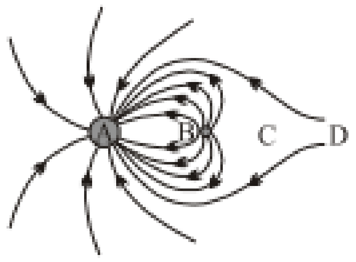
Practice Exercise

Q.1 The electric field of two point charges separated by 4.14 cm is as shown in **figure**.

(a) What is the nature of charges?

(b) What is the ratio of magnitude of charges?

(c) Is the field uniform?



(d) Apart from infinity where is the neutral point ?

(e) Will a positive charge follow the line of force if free to move?

(f) Where do the extra lines end?

Answers

Q.1 (a) A is –ve and B is +ve (b) $|q_A| = 2|q_B|$ (c) Feild is not uniform (d) BC = 10 cm (e) No (f) at infinity

Equilibrium and Motion of Charged Particles in the presence of Electric Field

When a particle of charge q and mass m is placed in an electric field E, the electric force exerted on the charge is $q\vec{E}$ according to Equation $\vec{E} = \frac{\vec{F}}{q}$. If this is the only force exerted on the particle, it must be the net force and causes the particle to accelerate according to Newton's second law. Thus,

$$\vec{a} = \frac{q\vec{E}}{m}$$

If the particle has a positive charge, its force is in the direction of the electric field. If the particle has a negative charge, its acceleration is in the direction opposite the electric field.

Illustration:

A positive point charge q of mass m is released from rest in a uniform electric field $\vec{E}$ directed along the x-axis, as shown in figure. Describe its motion.

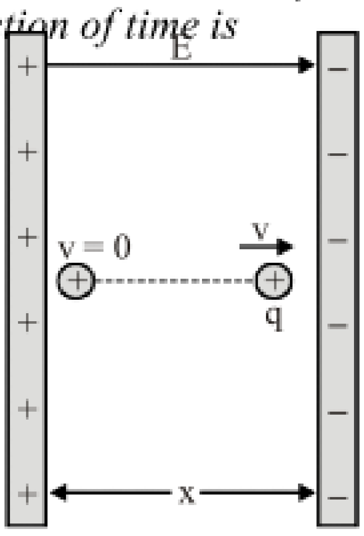
Sol. *The acceleration is constant and is given by q**E**/m. The motion is simple linear motion along the x-axis. Therefore, we can apply the equations of kinematics in one dimension*

$$x_f = x_i + v_i t + \frac{1}{2}at^2$$

$$v_f = v_i + at$$

$$v_f^2 = v_i^2 + 2a(x_f - x_i)$$

Choosing the initial position of the charge as $x_i = 0$ and assigning $v_i = 0$ because the particle starts from rest, the position of the particle as a function of time is



$$x_f = \frac{1}{2}at^2 = \frac{qE}{2m}t^2$$

The speed of the particle is given by

$$v_f = at = \frac{qE}{m}t$$

The third kinematics equation gives us

$$v_f^2 = 2ax_f = \left(\frac{2qE}{m}\right)x_f$$

from which we can find the kinetic energy of the charge after it has moved a distance $\Delta x = x_f - x_i$

$$K = \frac{1}{2}mv_f^2 = \frac{1}{2}m\left(\frac{2qE}{m}\right)\Delta x = qE\Delta x$$

We can also obtain this result from the work-kinetic energy theorem because the work done by the electric force is $F_l\Delta x = qE\Delta x$ and $W = \Delta K$.

Illustration:

A particle of charge $1\mu C$ and mass 1 gram is suspended in air near surface of the earth such that weight of particle is balanced by electrostatic force on particle. Find the electric field at position of the particle.

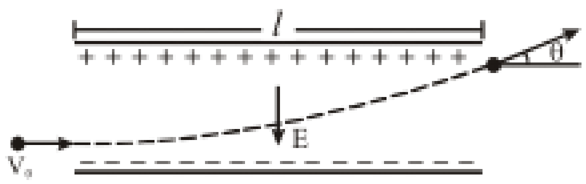
Sol. *Electric force is equal and opposite to weight such that* $\vec{F_e} + \vec{W} = 0$

$$\Rightarrow \quad q\vec{E} + mg(-\hat{j}) = 0$$

$$\Rightarrow \quad \vec{E} = \frac{mg}{q}\hat{j} = 10^4 \text{ N/C vertically upward.}$$

Illustration:

A uniform electric field E is created between two parallel, charged plates as shown in figure. An electron enters the field symetrically between the plates with a speed u_0. The length of each plate is ℓ. Find the angle of deviation of the path of the electron as it comes out of the field.



Sol. *The acceleration of the electron is $a = \frac{eE}{m}$ in the upward direction. The horizontal velocity remains u_0 as there is no acceleration in this direction. Thus, the time taken in crossing the field is :*

$$t = \frac{l}{u_0}$$

The upward component of the velocity of the electron as it emerges from the field region is

$$u_y = at = \frac{eEl}{mu_0}$$

The horizontal component of the velocity remains

$$u_x = u_0.$$

The angle θ made by the resultant velocity with the original direction is given by

$$\tan\theta = \frac{u_y}{u_x} = \frac{eEl}{mu_0^2}$$

Thus, the electron deviates by an angle

$$\theta = \tan^{-1}\frac{u_y}{u_x} = \frac{eEl}{mu_0^2}.$$

Practice Exercise

Q.1 An electron and a proton are situated in a uniform electric field. What is the ratio of their acceleration?

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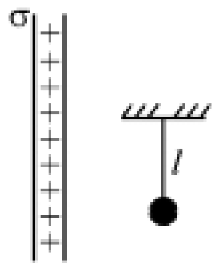
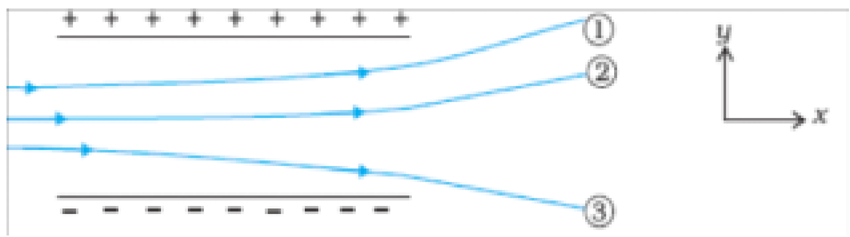
Q.2 An infinite plane of positive charge has a surface charge density σ. A metal ball B of mass m and charge q is attached to a thread and tied to a point A on the sheet PQ. Find the angle θ which AB makes with the plane PQ.

Q.3 A simple pendulum of length l and bob mass m is hanging in front of a large nonconducting sheet having surface charge density σ. If suddenly a charge +q is given to the bob & it is released from the position shown in figure. Find the maximum angle through which the string is deflected from vertical.



Q.4 Figure shows tracks of three charged particles in a uniform electrostatic field. Give the signs of the three charges. Which particle has the highest charge to mass ratio?



Q.5 A simple pendulum consists of a small sphere of mass m suspended by a thread of length l. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. With what period will the pendulum oscillate if the electrostatic force acting on the sphere is less that the gravitational force? Assume the oscillations to be small.

Q.6 An inclined plane makes an angle of 30^0 with a horizontal is placed in a uniform horizontal electric field $E = 100\ Vm^{-1}$. A particle of mass 1 kg and charge 0.01 C, is allowed to slide down from a height of 1 m. If the coefficient of friction is 0.2, find the time taken by the body to reach the bottom of the inclined plane.

Q.7 A pendulum bob of mass 80 milligrams and carrying a charge of 2×10^{-8} coulomb is at equilibrium. in a horizontal uniform electric field of 20,000 Vm^{-1}. Find the tension in the thread of the pendulum at equilibrium.

Q.8 A uniform electric field of strength 10^6 V/m is directed vertically downwards. A particle of mass 0.01 kg and charge 10^{-6} coulomb is suspended by an inextensible thread of length 1 m. The particle is displaced slightly from its mean position and released. Calculate the time period of its oscillation. What minimum velocity should be given to the particle at rest so that it completes full circle in a vertical plane without the thread getting slack? Calculate the maximum and minimum tensions in the thread in this situation.

Answers

Q.1 1836 Q.2 $\tan^{-1}\left[\dfrac{q\sigma}{2\epsilon_0\ mg}\right]$ Q.3 $2\tan^{-1}\left(\dfrac{\sigma q_0}{2\varepsilon_0 mg}\right)$ Q.4

Q.5 $\left[T = 2\pi\left[\dfrac{l}{\left(g - \dfrac{qE}{m}\right)}\right]^{1/2}\right]$ Q.6 1.3 sec Q.7 8.8×10^{-4} N

Q.8 0.6 s; $v_{min} = 23.43$ m/s; $T_{min} = 0$ & $T_m = 6.59$ N

GAUSS'S LAW

Area as a Vector

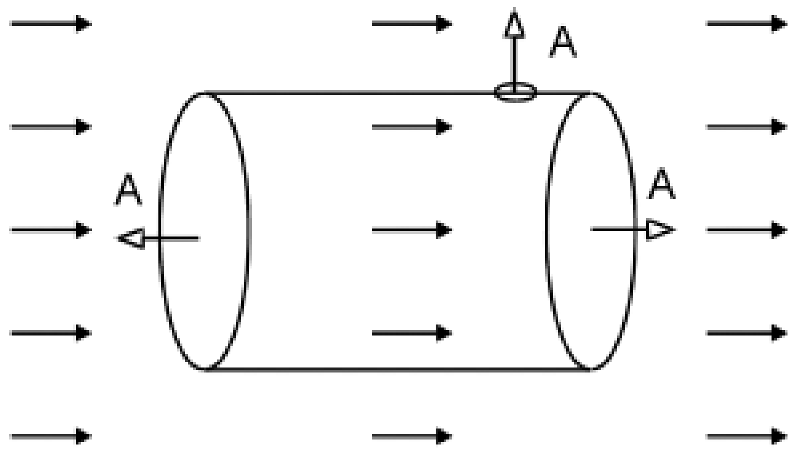
In some calculations it is useful to use area as a vector. Plane suface area or small elemental area of a cuved surface can be treated as vector quantities. Its direction is taken along the outward normal drawn on the surface and magnitude is equal to its area. It should be noted here that the direction of outward normal at a given point of a closed surface is unique but in case of open surfaces the terms outward and inward are not well defined and so the choice of outward normal is purely arbitrary.

Flux of a vector field

When a planel area represented by an area vector $\vec{A}$ is placed in any uniform vector field (let $\vec{X}$) then a flux(represented by the symbol Φ) of that vector field is said to be linked through that area and defined as

$$\Phi = \vec{X} \cdot \vec{A}$$

To understand its meaning clearly let us take example of velocity field. We imagine a hypothhetical cylindrical having cross-sectional area A, Lowered in a flowing river in which velocity of water is uniform.



flux of the vector velocty vector on left cross section

$$\Phi_1 = \vec{v} \cdot \vec{A} = vA\cos 180^0 = -vA \ \text{m}^3/\text{sec}.$$

flux of the vector velocty vector on right cross section

$$\Phi_2 = \vec{v} \cdot \vec{A} = vA\cos 0^0 = +vA \ \text{m}^3/\text{sec}.$$

flux of the vector velocty vector on on any elemental area of curved surface

$$\Phi_3 = \vec{v} \cdot \vec{A} = vA\cos 90^0 = 0 \ \text{m}^3/\text{sec}.$$

As the unit suggests the flux of $\vec{v}$ through $\vec{A}$ at left end gives the net inflow of water across the left cross-section of the pipe per-sec .Similarly the flux of $\vec{v}$ through $\vec{A}$ at right end gives the net outflow of water across the right cross-section of the pipe per-sec . Zero flux on curved surface represents there is no flow through curved surface.

Overall we can say that flux of any vector field through an area vector placed in that field defines the net inflow or net outflow of something across that area. (-)ve flux indicates inflow, (+)ve flux indicates outflow.

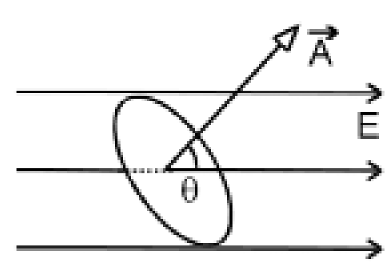
Flux of electric field

Flux of electric field through an area placed in that field represents the part of the electric field intercepted by that area or the net inflow or outflow of electric lines of forces normally through that area.

When a small area represented by an area vector $d\vec{A}$ is placed in any vector field then a flux of that field is said to be linked through that area.

Considering a small surface represented by area vector $d\vec{A}$, placed in an electric field $\vec{E}$ in such a way that angle between $\vec{E}$ and $d\vec{A}$ is θ then the flux of electric field $\vec{E}$ through the area $d\vec{A}$ is defined as the dot product of the field vector and that of the area vector and is given by

$$d\Phi = \vec{E}\cdot d\vec{A} = EdA\cos\theta$$



$\Rightarrow \quad d\Phi = (E\cos\theta)\, dA =$ Product of the area and that of the component of $\vec{E}$ along $d\vec{A}$ i.e., along the normal to the area or,

$d\Phi = E(dA\cos\theta) =$ Product of E and the projection of the area vector normal to the direction of $\vec{E}$.

To calculate the flux of nonuniform electric field through a surface of any arbitrary shape we divide the whole surface into little patches which are so small that over any one patch the surface is practically flat and the electric field doesn't change appreciably from one part of a patch to another. The area of a patch has a certain magnitude and the outward pointing normal to its surface gives its direction. We calculate the fluxes of electric field through each such patches and the algebraic sum of the fluxes through individual patches gives the total flux of electric field through the given surface of arbitrary shape because flux is a scalar quantity.

The flux of nonuniform electric field over an arbitrary surface A is given by the integral

$$\Phi = \int d\Phi = \int_S \vec{E}\cdot d\vec{A} = \int_S E\,dA\,\hat{n}$$

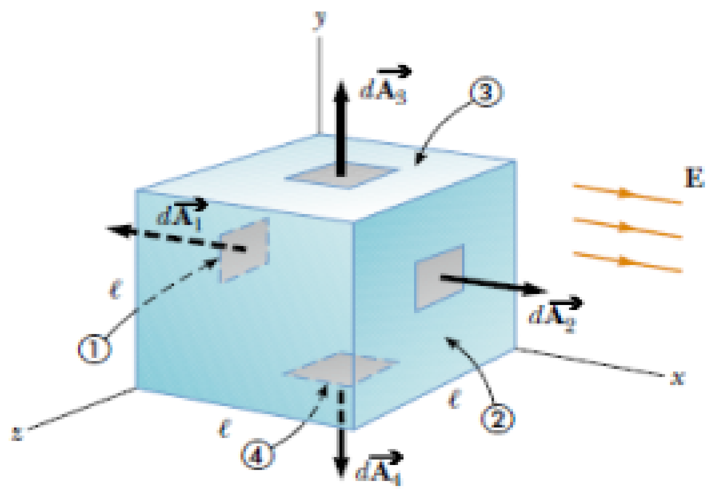
where $\hat{n}$ is the unit vector in the direction of normal to the surface

In case of closed surface flux is written by special sumbol

$$\Phi = \int d\Phi = \oint \vec{E}\cdot d\vec{A} = \oint E\,dA\,\hat{n}$$

Illuatration:

Consider a uniform electric field $\vec{E}$ *oriented in the x direction.Find the net electric flux through the surface of a cube of edge length* ℓ*, oriented as shown in Figure*



Sol. *The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (3, 4 and the unnumbered ones) is zero because* $\vec{E}$ *is perpendicular to* $d\vec{A}$ *on these faces.*

The net flux through faces 1 and 2 is

$$\phi_E = \int_1 \vec{E}\cdot d\vec{A} + \int_2 \vec{E}\cdot d\vec{A}$$

For face 1, $\vec{E}$ *is constant and directed inward but* $d\vec{A}_1$ *is directed outward (θ= 180°); thus, the flux through this face is*

$$\phi_1 = \int_1 \vec{E}\cdot d\vec{A} = \int_1 EdA\cos 180^0 = -EA = -E\ell^2$$

For face 2, $\vec{E}$ *is constant and outward and in the same direction as* $d\vec{A}_2$ *(θ= 0°); hence, the flux through this face is*

$$\phi_2 = \int_1 \vec{E}\cdot d\vec{A} = \int_1 EdA\cos 0^0 = EA = E\ell^2$$

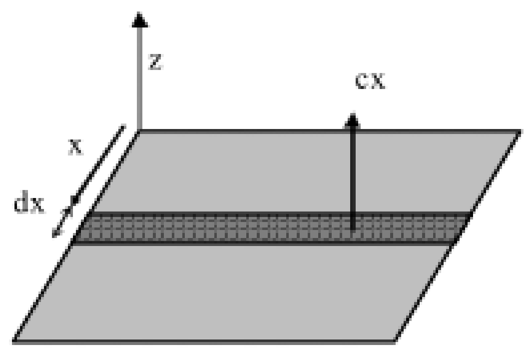
Therefore, the net flux over all six faces is

$$\phi_{total} = E\ell^2 + \left(-E\ell^2\right) + 0 + 0 + 0 + 0 = 0$$

Illuatration:

A nonuniform electric field is given by the expression $\vec{E} = ay\hat{i} + bz\hat{j} + cx\hat{k}$*, where a, b, and c are constants. Determine the electric flux through a rectangular surface in the xy plane, extending from* $x = 0$ *to* $x = w$ *and from* $y = 0$ *to* $y = h$.

Sol. *The direction of area vector is along z axis hence there will be no flux due to x and y component of electric field. So let us calculate flux due to z component of electric field.*

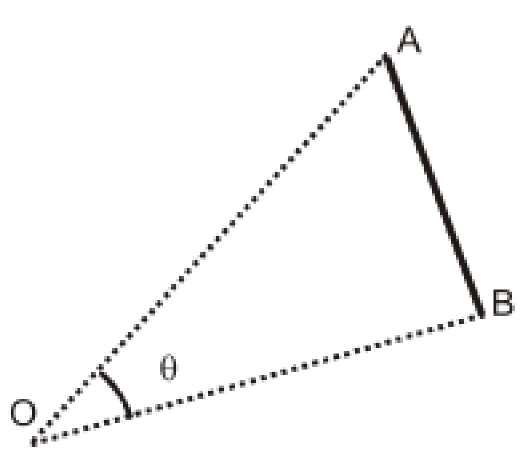


$$\phi = \int d\phi = \int EdA = \int_0^w (cx)(hdx) = \frac{chw^2}{2}$$

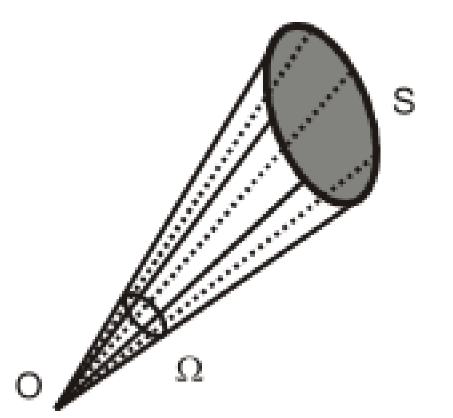
Solid Angle

A plane angle is a two dimensional concept. Solid angle is a three dimensional generalisation of the two dimensional concept of plane angle. A plane angle is formed at a point by a line segment or an arc but a surface is responsible for the formation of solid angle at a given point. The S.I. unit of plane angle is radian whereas that of solid angle is steradian (Sr).

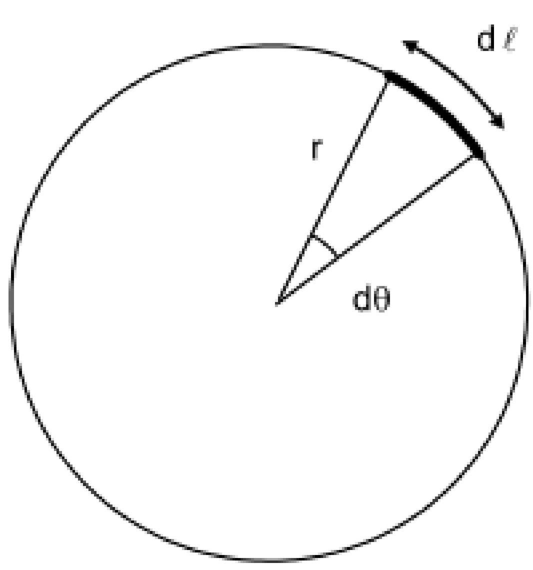
AB is the line segment forming plane angle θ at O



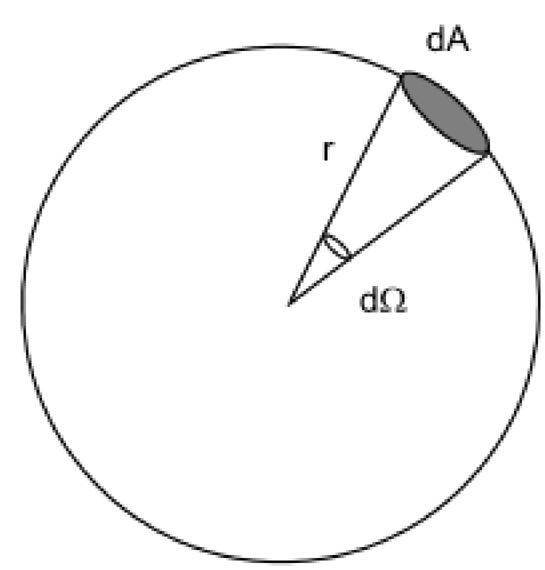
S is the surface segment forming solid angle Ω at O



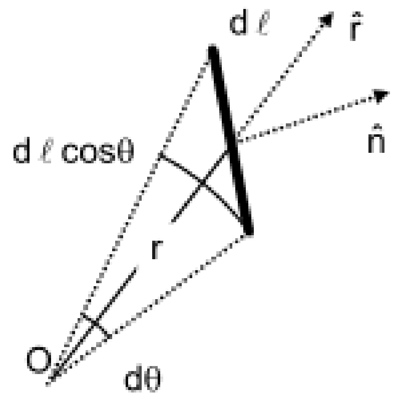
Plane angle formed by a small arc of a circle at its centre is defined as $d\theta = \frac{d\ell}{r}$



solid angle formed by a small area of a sphere at its centre is defined as $d\theta = \frac{d\ell}{r}$



If a small length element $d\ell$ is oblique such that normal to the length element ($\hat{n}$) is making an angle θ with radial direction ($\hat{r}$) then the plane angle is defined as $d\theta = \frac{d\ell\cos\theta}{r}$



If a small surface element dA is oblique such that normal to the surface element ($\hat{n}$) is making an angle θ with radial direction ($\hat{r}$) then the plane angle is defined as $d\Omega = \frac{dA\cos\theta}{r^2}$

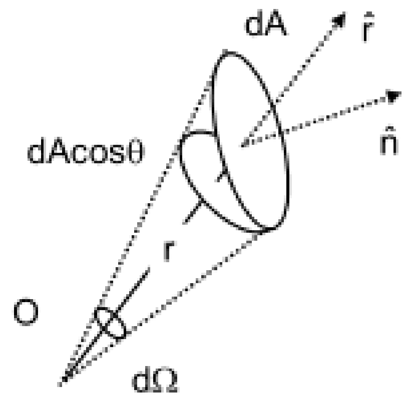


Illustration:

Find the solid angle formed by a disc at a point on its axis where its radius forms plane angle θ

Sol. *Let us first calculate the solid angle formed by elementry ring of radius r and width dr*

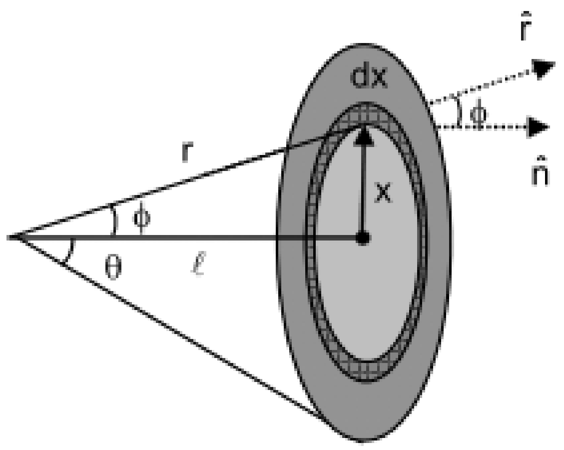
$$d\Omega = \frac{dA\cos\phi}{r^2} = \frac{(2\pi x dx)\cos\phi}{r^2}$$

substitute $x = \ell\tan\phi \Rightarrow dx = \ell\sec^2\phi d\phi$ *and*

$$r = \sqrt{\ell^2 + x^2} = \sqrt{\ell^2 + (\ell\tan\phi)^2} = \ell\sec\phi$$

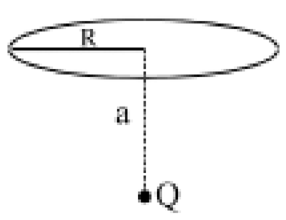
$$d\Omega = \frac{(2\pi\ell\tan\phi\,\ell\sec^2\phi d\phi)\cos\phi}{(\ell\sec\phi)^2} = 2\pi\cos\phi d\phi$$

$$\therefore\ \Omega = \int d\Omega = \int_0^\theta 2\pi\cos\phi d\phi = 2\pi(1-\cos\theta)$$



Practice Exercise

Q.1 A point charge Q is located on the axis of a disc of radius R at a distance a from the plane of the disc. If the soild angle subtended by disc at the point charge is $\pi/2$, then find the relation between a & R.

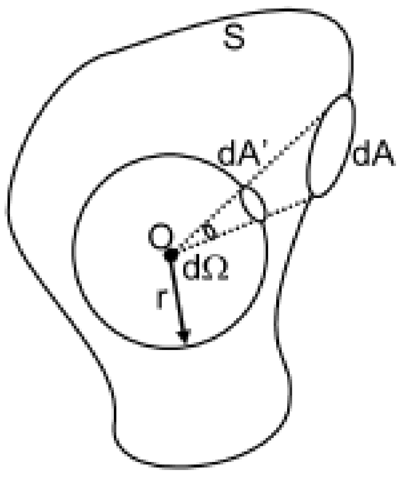


Answers

Q.1 $a = \frac{R}{\sqrt{3}}$

Some important results regarding solid angle

(i) ***A closed surface subtends a solid angle of 4π(Sr) at an internal point.***



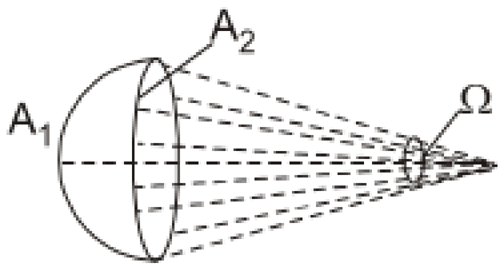
Considering a three dimensional closed surface S of any arbitrary shape. Let O be an interior point. Again, consider a spherical surface of radius 'r' having its centre at 'O'. Let a small part of the surface, having surface area 'dA' is contained inside a solid angle dΩ. This cone emerging from point O encloses an area dA' of the spherical surface.

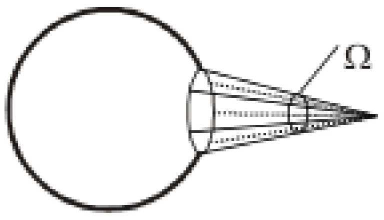
So Solid angle subtended by dA at point O = dΩ = solid angle subtended by dA' at point O = $\frac{dA'}{r^2}$

Every small area, on the surface S of any arbitrary shape, may be shown in correspondence with an area on the spherical surface. So, total solid angle subtended by a closed surface of any arbitrary shape at an internal point.

$$\Omega = \int d\Omega = \frac{1}{r^2}\int dA' = \frac{1}{r^2}4\pi(r^2)$$
$$= 4\pi(Sr)$$

(ii) ***A closed surface subtends a solid angle of 0 (Sr) at an external point.***

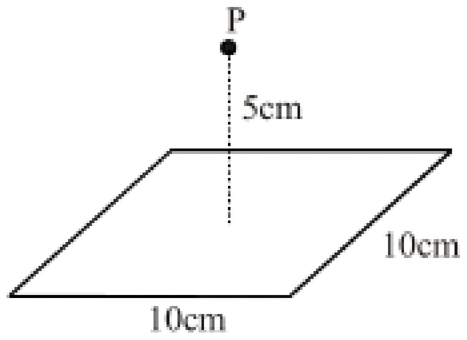




If the boundary of a hemispherical surface is gradually closed then the solid angle Ω start decreasing, when it takes the shape of a closed surface, the solid angle subtended at the external point becomes zero.

Illustration

What is the solid angle subtended by square at the point P.

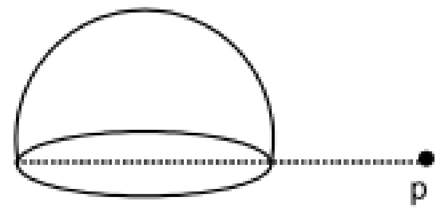


Sol. *If a consider a cube of side 10 cm point P will be at its centre. The total solid angle subtended by all the six surface of the cube at P is 4π staradian. Also from symmetry it is obvious that solid angle formed by each surface at P is equal hence equal to*

$$\Omega = \frac{1}{6}(4\pi) = \frac{2\pi}{3} \text{ staradian}$$

Practice Exercise

Q.1 What is the solid angle subtended by hemisphere (shown in figure) at point P.



Q.2 Find the solid angle subtended by a cube at its corner

Answers

Q.1 Zero. Q.2 $\frac{\pi}{2}$

Illustration

Find the flux of electric field of a point charge over a closed surface of arbitrary shape if point shape point charge is placed

(i) inside closed surface
(ii) outside closed surface

Sol. *Flux on elementary area will be*

$$d\phi = \vec{E}\cdot d\vec{A} = EdA\cos\theta = \frac{q}{4\pi\epsilon_0 r^2} dA\cos\theta = \frac{q}{4\pi\epsilon_0}\frac{dA\cos\theta}{r^2} = \frac{q}{4\pi\epsilon_0} d\Omega$$

Where $d\Omega$ is the solid angle subtended by elementary area at the location of point charge

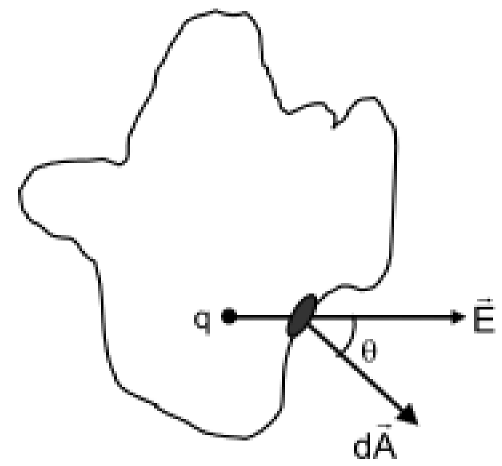
$\therefore$ *Total flux* $\phi = \int d\phi = \frac{q}{4\pi\epsilon_0}\int d\Omega = \frac{q}{4\pi\epsilon_0}\Omega$

Where Ω is the total solid angle subtended by area at the location of point charge.
If the point charge is placed inside the surface

$$\Omega = 4\pi \Rightarrow \phi = \frac{q}{\epsilon_0}$$

If the point charge is placed outside the surface

$$\Omega = 0 \Rightarrow \phi = 0$$



Here we get a result

Electric flux due to a point charge on any surface $= \frac{\text{charge}}{4\pi\epsilon_0}$ *(solid angle subtended by the surface at the location of point charge)*

Practice Exercise

Q.1 A point charge q is placed on the axis of a disc (imaginary) of radius R at a distance x from the centre of the disc. Find the flux of electric field of the charge q on disc.

Ans. flux $\phi = \frac{q}{4\pi \epsilon_0}\Omega = \frac{q}{2\epsilon_0}(1-\cos\theta) = \frac{q}{2\epsilon_0}\left(1-\frac{x}{\sqrt{R^2+y^2}}\right)$

Answers

Q.1 flux $\phi = \frac{q}{4\pi \epsilon_0}\Omega = \frac{q}{2\epsilon_0}(1-\cos\theta) = \frac{q}{2\epsilon_0}\left(1-\frac{x}{\sqrt{R^2+y^2}}\right)$

Gauss theorem

The flux of net electric field (due to the charges enclosed by the surface as well as the charges outside it) through any imaginary closed surface(called gaussian surface) of any arbitrary shape is equal to the total charge enclosed by the surface divided by ϵ_0**.**

$$\oint \vec{E}_{res}.d\vec{A} = \frac{\Sigma q_{enclosed}}{\epsilon_0}$$

<u>Some points to be emphasized about the gauss law :</u>

(i) It is true for any closed surface no matter what its shape and size.

(ii) The q includes algebraic sum of all charges enclosed by the surface.

(iii) In situations when the surface is so chosen that there are some charges inside and some outside, the $\vec{E}$ (whose flux appear in the equation) is due to all charges, just term 'q' in the law represents only total charge inside.

(iv) *The gaussian surface should not pass through any discrete charge however, it can pass through a continuous charge distribution.*
Gaussian surface should not contain any finite non zero charge.

Illuatration:

A charge Q is placed at the centre of an imaginary hemispherical surface. Using symmetry arguments and the Gauss's law, find the flux of the electric field due to this charge through the surface of the hemisphere

Sol. *Let us imagine another hemispherical surface over identical given one.*
Both being symmetric with respect to Q, hence flux will be same through both the hemisphere $(\phi_1 = \phi_2)$.

Net flux $\phi_1 + \phi_2 = \frac{q_{nc}}{\epsilon_0} = \frac{Q}{\epsilon_0}$

$\Rightarrow$ $\phi_1 = \frac{Q}{2\epsilon_0}$.

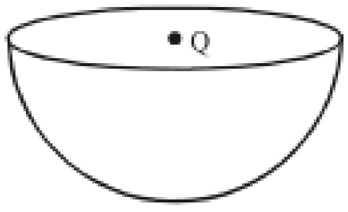
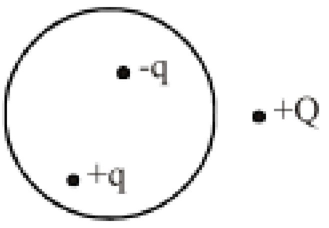


Illustration:

Three point charges –q, +q and Q are placed in a region as shown. P is a point on imaginary Gaussian sphere enclosing –q and +q and Q is outside
Will it be correct to say that flux at every part of sphere due to Q is zero.

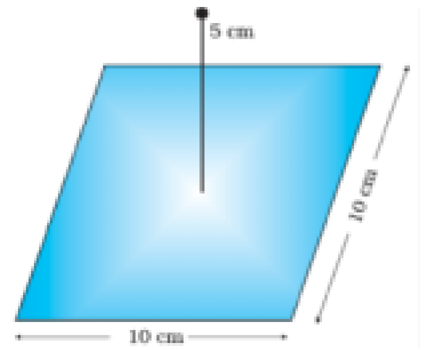


Sol. *No flux at every part of sphere due to Q is not zero because electric field is intercepted by some area*

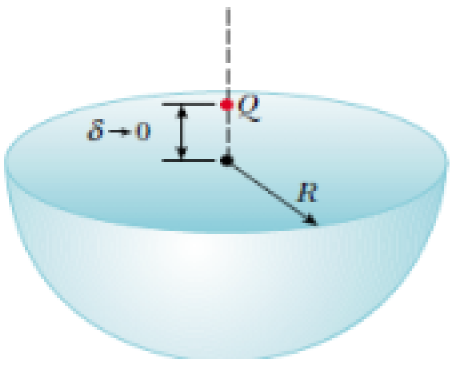
Q.1 A point charge q is placed at one corner of a cube of edge a. What is the flux through each of the cube faces?

Q.2 A point charge q is placed at near infinite plane. What is the flux through the plane?

Q.3 A point charge q is a distance 5 cm directly above the centre of a square of side 10 cm, as shown in Fig. What is the magnitude of the electric flux through the square?



Q.4 A point charge Q is located just above the center of the flat face of a hemisphere of radius R as shown in Figure . What is the electric flux (a) through the curved surface and (b) through the flat face?



Q.1 $\frac{1}{24}\frac{q}{\epsilon_0}$ Q.2 $\frac{q}{2\epsilon_0}$ Q.3 $\frac{q}{6\epsilon_0}$ Q.4 (a) $\frac{q}{2\epsilon_0}$ (b) $-\frac{q}{2\epsilon_0}$

Applications of Gauss Theorem

Choice of gaussian surface in evaluating electric field.

Definition of a Gaussian surface :

While applying Gauss's law we are interested in evaluating the integral

$$\phi_E = \oint E \cdot dA$$

The closed surface for which the flux is calculated is generally an imaginary or hypothetical surface, called a Gaussian surface. Whenever we apply Gauss's law we may choose a surface of any size and shape as our Gaussian surface. But selecting a proper size and shape for a Gaussian surface is a key factor for determining flux and electric field. Here are the list of different types of the Gaussian surfaces to be chosen for a given charge distribution.

Charge distribution	Gaussian surface	Electric field
Point charge	Spherical	Radial
Spherical charge distribution	Spherical	Radial
Line of charge	Cylindrical	Radial
Planar charge	Cylindrical	Normal to surface

(i) Electric field due to a point charge

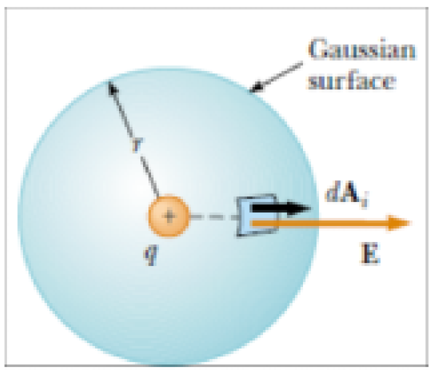
$$\oint \vec{E}.d\vec{A} = \oint E\,dA = \oint dA = E4\pi r^2$$

$$\sum q_{en} = Q$$

$\therefore$ from Gauss' law

$$E4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \quad E = \frac{Q}{4\pi\epsilon_0 r^2}$$

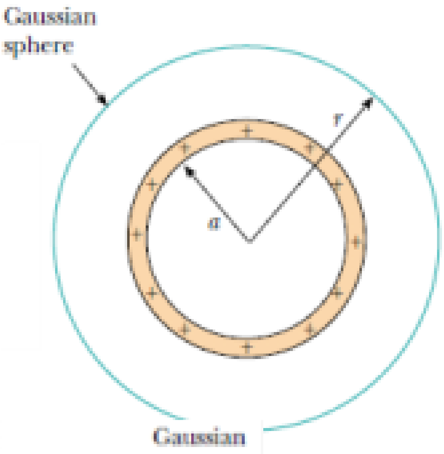


(ii) Evaluation of el. fd. due to uniformly charged spherical shell (charge Q):

Case 1 **At external point (r>R)**

from Gauss' law $\quad E4\pi r^2 = \frac{Q}{\epsilon_0}$

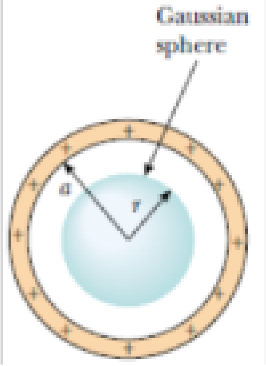
$$\Rightarrow \quad E = \frac{Q}{4\pi\epsilon_0 r^2}$$



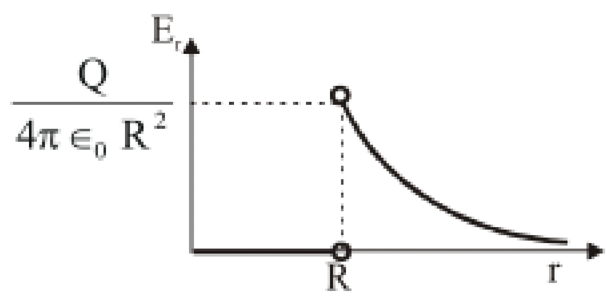
Case 2 **At internal point (r<R)**

$$E4\pi r^2 = \frac{0}{\epsilon_0}$$

$$\Rightarrow \quad E = 0$$



Plot of variation of electric field



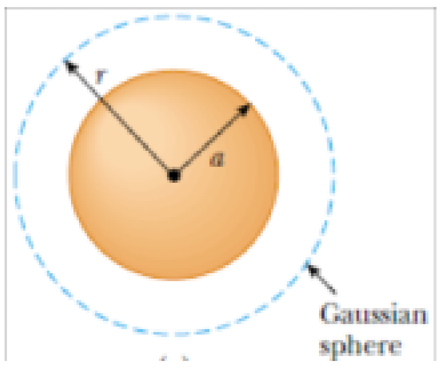
(iii) Evaluation of el. fd. due to uniformly charged spherical volume (charge Q)

Case1 At external point (r>R)

from Gauss' law

$$E4\pi r^2 = \frac{Q}{\in_0}$$

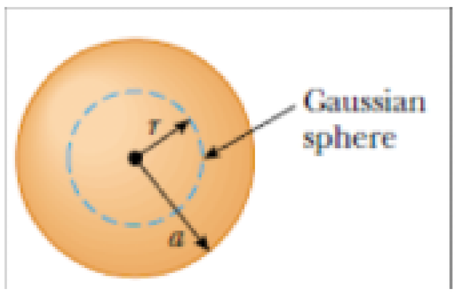
$$\Rightarrow \quad E = \frac{Q}{4\pi \in_0 r^2}$$



Case 2:- At internal point (r<R)

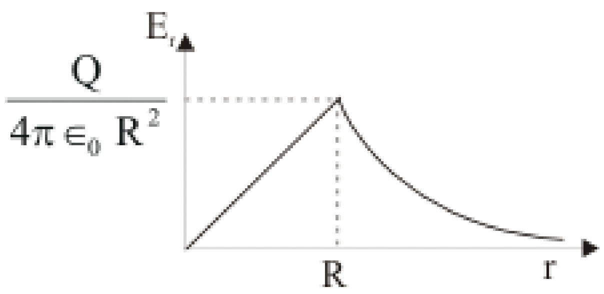
from Gauss' law

$$E(4\pi r^2) = \frac{\frac{Q}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi r^3\right)}{\varepsilon_0}$$



$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_0 R^3} r = \frac{\rho}{3\varepsilon_0} r$$

Plot of variation of electric field

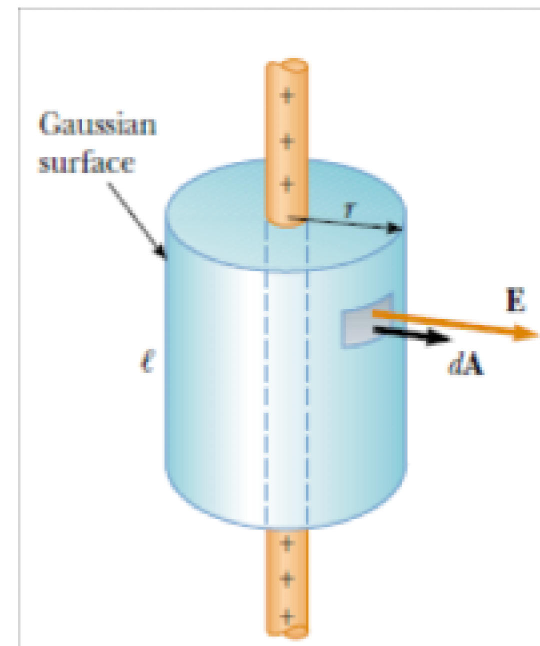


(iv) Electric field at a point near an infinitely long uniform linear charge distribution:

$$\oint \vec{E}.d\vec{A} = \int_{\substack{\text{upper}\\\text{Surface}}} \vec{E}.d\vec{A} + \int_{\substack{\text{curved}\\\text{surface}}} \vec{E}.d\vec{A} + \int_{\substack{\text{lower}\\\text{surface}}} \vec{E}.d\vec{A}$$

$$= \int_{\substack{\text{upper}\\ \text{surface}}} EdA\cos 90^0 + \int_{\substack{\text{curved}\\ \text{surface}}} EdA\cos 0^0 + \int_{\substack{\text{lower}\\ \text{surface}}} EdA\cos 90^0$$

$$= 0 + \oint E\, dA + 0 = E2\pi r l$$



$$\& \sum q_{\text{enclosed}} = \lambda l$$

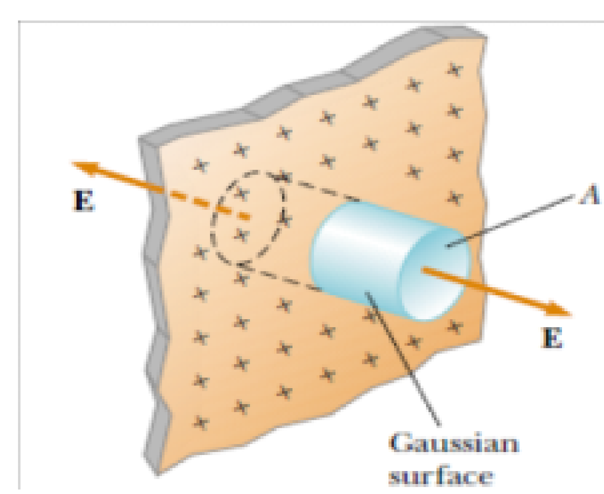
$\therefore$ from Gauss's law

$$E 2\pi r l = \lambda l / \epsilon_o$$

$$\therefore\ E = \frac{\lambda}{2\pi\epsilon_o r}$$

(v) Electric field intensity at a point near an infinitely charged plane sheet having surface charge density

$$\oint \vec{E}.d\vec{A} = \int_{\substack{\text{front}\\ \text{surface}}} \vec{E}.d\vec{A} + \int_{\substack{\text{curved}\\ \text{surface}}} \vec{E}.d\vec{A} + \int_{\substack{\text{rear}\\ \text{surface}}} \vec{E}.d\vec{A}$$



$$= \int_{\substack{\text{upper}\\ \text{surface}}} EdA\cos 0^0 + \int_{\substack{\text{curved}\\ \text{surface}}} EdA\cos 90^0 + \int_{\substack{\text{lower}\\ \text{surface}}} EdA\cos 0^0$$

$$= \oint E\, dA + 0 + \oint E\, dA = EA + 0 + EA = 2EA$$

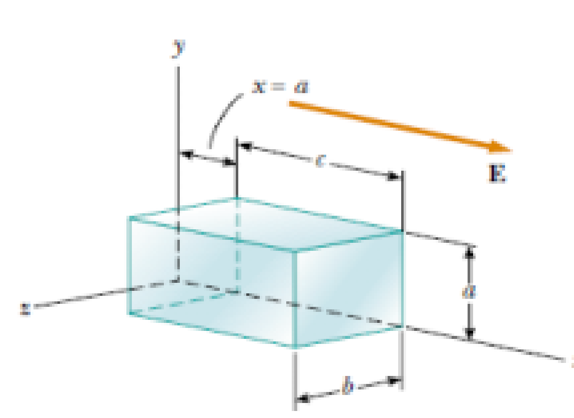
$$\sum q_{en} = \sigma S$$

$\therefore$ from Gauss' law $\quad 2ES = \dfrac{\sigma S}{\epsilon_o}$

$$\Rightarrow \quad E = \frac{\sigma}{2\epsilon_o}$$

Illuatration:

A closed surface with dimensions $a = b = 0.400$ m and $c = 0.600$ m is located as in Figure. The left edge of the closed surface is located at position $x = a$. The electric field throughout the region is nonuniform and given by $\vec{E} = (3 + 2x^2)\hat{i}$ N/C, where x is in meters. Calculate the net electric flux leaving the closed surface. What net charge is enclosed by the surface?



Sol. *The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (parallel to y-z and x-z plane) is zero because $\vec{E}$ is perpendicular to $d\vec{A}$ on these faces.*

For face parallel to y-z plane lying at x=a , $\vec{E}$ *is constant and directed inward but* $d\vec{A}_1$ *is* directed outward (*θ= 180°); thus, the flux through this face is*

$$\phi_1 = \oint \vec{E}.d\vec{A} = \int EdA\cos 180^o = -EA = -[3+2a^2]ab$$

For face parallel to y-z plane lying at x=a+c , $\vec{E}$ *is constant aand outward and in the same direction as* $d\vec{A}_2$ *(θ= 0°); thus, the flux through this face is*

$$\phi_2 = \int \vec{E}.d\vec{A} = \int EdA\cos 0^0 = +EA = +\left[3+2(a+c)^2\right]ab$$

Therefore, the net flux over all six faces is

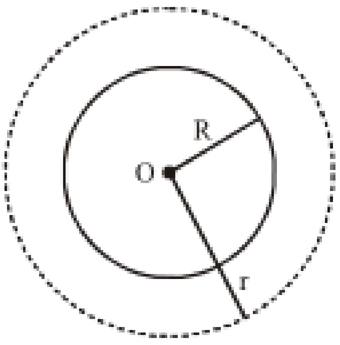
$$\phi = -\left[3+2a^2\right]ab + \left[3+2(a+c)^2\right]ab + 0 + 0 + 0 + 0$$

Illuatration:

A system consists of a ball of radius R carrying a spherically symmetric charge and the surrounding space filled with a charge of volume density $\rho = a/r$ *where a is a constant, r is the distance from the centre of ball. Find the ball's charge at which the magnitude of the electric field is independent of r outside the ball. How high is this strength?*

Sol. *Let us consider a spherical surface of radius* $r(r>R)$ *concentric with the ball and apply Gauss's Law.*

$$\oint \vec{E}.d\vec{A} = \frac{q}{\varepsilon_0}$$



Let Q= total charge of the ball

$$\varepsilon_0 E(4\pi r^2) = Q + \int_R^r \rho 4\pi x^2\,dx$$

$$\varepsilon_0 E(4\pi r^2) = Q + 4\pi\int_R^r \frac{a}{x} x^2\,dx$$

$$\varepsilon_0 E(4\pi r^2) = Q + 2\pi a(r^2 - R^2)$$

$$\Rightarrow \quad E = \left(\frac{Q - 2\pi a R^2}{4\pi\varepsilon_0}\right)\frac{1}{r^2} + \frac{2\pi a}{4\pi\varepsilon_0}$$

For E to be independent of r,

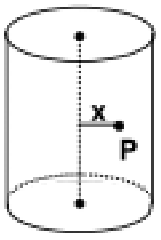
$$Q = 2\pi a R^2$$

and the value of E is

$$E = \frac{a}{2\varepsilon_0}.$$

Practice Exercise

Q.1 A long cylindrical volume contains a uniformly distributed charge of density ρ. Find the electric field at a point P inside the cylindrical volume at a distance x from its axis in the figure.



Q.2 A non conducting sheet of large surface area and thickness d contains uniform charge distribution of density ρ. Find the electric field at a point P inside the plate, at a distance x from the central plane.

Q.3 Consider the classical–model of an atom such that a nucleus of charge + e is uniformly distributed within a sphere of radius 2Å. An electron of charge (–e) at a radial distance 1Å moves inside the sphere. Find the force attracting the electron to the centre of the sphere. Calculate the frequency with which the electron would oscillate about the centre of the sphere. [* This is the earliest model of atom proposed by J.J. Thomson.]

Answers

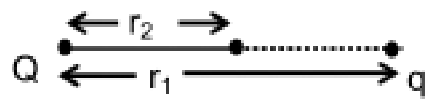
Q.1 $\frac{\rho x}{2\varepsilon_0}$ Q.2 $\frac{\rho x}{\varepsilon_0}$ Q.3 9×10^{14} Hz.

ELECTRIC POTENTIAL ENERGY

Whenever a charge is moved in an electrostatic field, work is done by electrostatic forces. An electrostatic field is a conservative force field. Therefore, any work done against the field is stored as potential energy. Change in electric potential energy will be $\Delta U = -W_{el}$, where W_{el} is the work done by electrostatic forces.

Electric potential energy between two point charges

Consider two charges Q and q separated by a distance r_1. If the charge q is moved along the line joining the charge and the final separation becomes r_2, then the work done by the electric force during the process is



$$W_{el} = \int_{r_1}^{r_2} \frac{kQq}{r^2} dr = kQq\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$\therefore$ The change in potential energy is defined as

$$U_2 - U_1 = -W_{el} = kQq\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

The potential energy of a two-charge system is taken to be zero, when the distance between the charges is infinity. i.e. $U = 0$ if $r = \infty$

Now, the potential energy of a two charge system when their separation is r, is

$$U(r) - 0 = kQq\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$U(r) = \frac{kQq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r}$$

For like charges U is +ve & for unlike charges U is-ve.

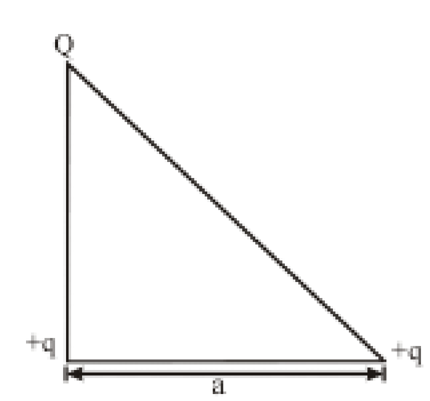
Electric potential energy between more than two point charges

The above equation gives the potential energy of a pair of charges. In case of three charges (say q_1, q_2 and q_3) there are three pairs (q_1, q_2), $(q_2,, q_3)$ and (q_3, q_1). Thus the total potential energy of the system will have three terms.

$$U = \frac{kq_1q_2}{r_{12}} + \frac{kq_2q_3}{r_{23}} + \frac{kq_3q_1}{r_{31}}$$

Illustration:

Three charges Q, +q and +q are placed at the vertices of a right angled isosceles triangle as shown in the figure. The net electrostatic energy of
the configuration is zero., if Q is equal to :



Sol. *Net electrostatic energy*

$$U = \frac{kQq}{a} + \frac{kQq}{\sqrt{2}a} + \frac{kqq}{a}$$

For U = 0; $\frac{Qq}{a} + \frac{Qq}{\sqrt{2}\,a} + \frac{qq}{a} = 0$

$$\Rightarrow \quad Qq\left(\frac{1}{a} + \frac{1}{\sqrt{2}\,a}\right) = \frac{-q^2}{a}$$

$$\Rightarrow \quad Q\left(\frac{\sqrt{2}+1}{\sqrt{2}\;a}\right) = \frac{-q}{a}$$

$$\Rightarrow \quad Q = -q\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right)$$

$$= -q\left(\frac{2}{2+\sqrt{2}}\right).$$

Illustration:

A particle of mass 100 gm and charge 2 μC is released from a distance of 50 cm from a fixed charge of 5μC. Find the speed of the particle when its distance from the fixed charge becomes 3 m. Neglect any other force.

Sol. *Loss of potential energy = gain in kinetic energy*

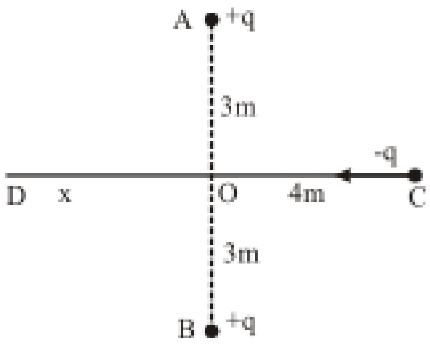
$$U_1 - U_2 = \Delta K.$$

$$kQq\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \tfrac{1}{2} mv^2 - 0$$

$$v = \sqrt{\frac{2kQq}{m}\left[\frac{r_2 - r_1}{r_1\,r_2}\right]} = \sqrt{\frac{2\times 9\times 10^9 \times 5\times 10^{-6} \times 2\times 10^{-6} \times 2.5}{0.1\times 3\times 0.5}} = \sqrt{3}\ m/s.$$

Illustration:

Two fixed positive charges, each of magnitude $5\times 10^{-5}\,C$ are located at points A and B, separated by a distance of 6 m. An equal and opposite charge moves towards them along the line COD, the perpendicular bisector of lien AB. The moving charge, when reaches the point C at a distance of 4 m from O, has a kinetic energy of 4 joules. Calculate the distance of the farthest point D which the negative charge will reach before returning towards C.



Sol. *The kinetic energy is lost and converted to electrostatic potential energy of the system as the negative charge goes from C to D and comes to rest at D instantaneously.*

Loss of KE = gain in PE

$$4 = U_f - U_i$$

$$4 = \left[\frac{qq}{4\pi\varepsilon_0(6)} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9+x^2}}\right] - \left[\frac{qq}{4\pi\varepsilon_0(6)} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9+16}}\right]$$

$$For\ U = 0;\ \frac{Qq}{a} + \frac{Qq}{\sqrt{2}\,a} + \frac{qq}{a} = 0$$

$$\Rightarrow \quad Qq\left(\frac{1}{a} + \frac{1}{\sqrt{2}\,a}\right) = \frac{-q^2}{a}$$

$$\Rightarrow \quad Q\left(\frac{\sqrt{2}+1}{\sqrt{2}\ a}\right) = \frac{-q}{a}$$

$$\Rightarrow \quad Q = -q\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right)$$

$$= -q\left(\frac{2}{2+\sqrt{2}}\right).$$

Illustration:

A particle of mass 100 gm and charge 2 μC is released from a distance of 50 cm from a fixed charge of 5μC. Find the speed of the particle when its distance from the fixed charge becomes 3 m. Neglect any other force.

Sol. *Loss of potential energy = gain in kinetic energy*

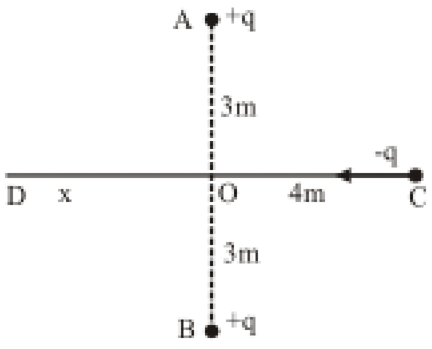
$$U_1 - U_2 = \Delta K.$$

$$kQq\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \tfrac{1}{2}\, mv^2 - 0$$

$$v = \sqrt{\frac{2kQq}{m}\left[\frac{r_2 - r_1}{r_1\, r_2}\right]} = \sqrt{\frac{2\times 9\times 10^9 \times 5\times 10^{-6}\times 2\times 10^{-6}\times 2.5}{0.1\times 3\times 0.5}} = \sqrt{3}\ m/s.$$

Illustration:

Two fixed positive charges, each of magnitude 5×10^{-5} C are located at points A and B, separated by a distance of 6 m. An equal and opposite charge moves towards them along the line COD, the perpendicular bisector of lien AB. The moving charge, when reaches the point C at a distance of 4 m from O, has a kinetic energy of 4 joules. Calculate the distance of the farthest point D which the negative charge will reach before returning towards C.



Sol. *The kinetic energy is lost and converted to electrostatic potential energy of the system as the negative charge goes from C to D and comes to rest at D instantaneously.*

Loss of KE = gain in PE

$$4 = U_f - U_i$$

$$4 = \left[\frac{qq}{4\pi\varepsilon_0(6)} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9+x^2}}\right] - \left[\frac{qq}{4\pi\varepsilon_0(6)} + \frac{2q(-q)}{4\pi\varepsilon_0\sqrt{9+16}}\right]$$

$$4=\frac{2q^2}{4\pi\varepsilon_0}\left[\frac{1}{5}-\frac{1}{\sqrt{9+x^2}}\right]$$

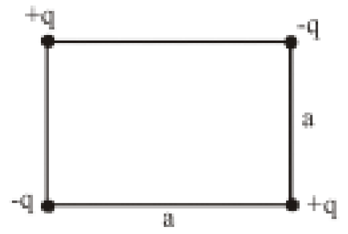
$$4=2(5\times10^{-5})^2(9\times10^9)\left(\frac{1}{5}-\frac{1}{\sqrt{9+x^2}}\right)$$

$$4=9-\frac{45}{\sqrt{9+x^2}}$$

$$x=\sqrt{72}=8.48\text{ m}.$$

Illustration :

What is work done by the electrostatic field when we put the four charges together, as shown in the figure. Each side of the square has a length a. Initially charges were at infinity.



Sol. $U_i=0$ *[Where charges are separated by infinite distance]*

$$U_f=\frac{1}{4\pi\varepsilon_0}\left(\frac{-4q^2}{a}+\frac{q^2}{\sqrt{2}a}+\frac{(-q)^2}{\sqrt{2}a}\right)$$ *[for 6 pairs of charges]*

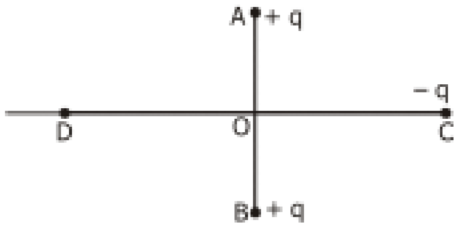
Work done by field

$$-\Delta U=U_i-U_f=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left(4-\frac{\sqrt{2}}{a}\right)$$

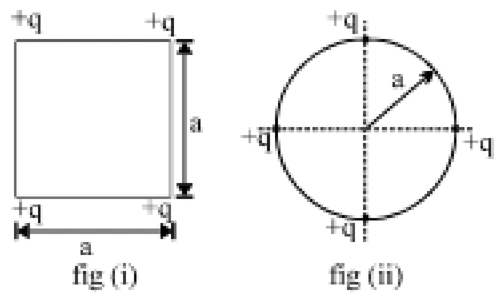
Practice Exercise

Q.1 Three point charges of 0.1 C each are placed at the corners of an equilateral triangle with side L = 1 m. If this system is supplied energy at the rate of 1 kW, how much time will be required to move one of the charges on to the mid point of the line joining the two others?

Q.2 Two fixed, equal, positive charges, each of magnitude 5×10^{-5} C, are located at points A and B separated by a distance of 6 m. An equal and opposite charge moves towards them along the line COD, the perpendicular bisector of the line AB. The moving charge, when it reaches the point C at a distance of 4 m from O, has kinetic energy of 4 J. Calculate the distance of the farthest point D which the negative charge will reach before returning towards C.



Q.3 Consider the configuration of a system of four charges each of value +q. Find the work done by external agent in changing the configuration of the system from figure (i) to fig (ii).



Answers

Q.1 50 hrs. Q.2 8.485 m Q.3 $-\dfrac{kq^2}{a}\left(3-\sqrt{2}\right)$

Electric potential

Electric potential (represented by symbol V) due to a point charge or a charge configuration at a point is defined as

$$V = \frac{U}{q} = \text{P.E. per unit test charge.}$$

In other words it is amount of work done by external agency to bring a unit positive charge from reference point (usually taken at infinity) to given point

Electric potential due a point charge

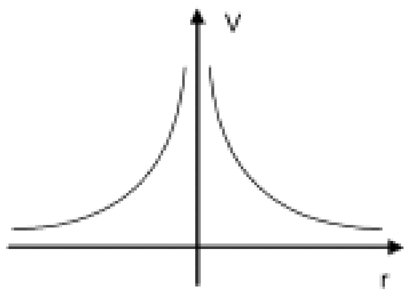
In the electric field of a point charge Q electric potential at a pont will be

$$V(\vec{r}) = \frac{\frac{kQq}{r}}{q} = \frac{kQ}{r}$$

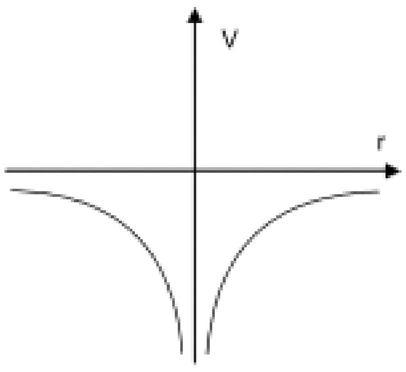
Potential is a scalar quantity.Electric potential due to a positive charge is taken to be positive and that due to a negative charge is taken to be negative. The potential at a point due to more than one charge can be found simply by adding the potentials due each charge separately.

plot of electric potential

electric potential due to positive charge



electric potential due to negative charge



Practice Exercise

Q.1 The electric potential at point A is 20V and at B is -40V. Find the work done by an external force and electrostatic force in moving an electron slowly from B to A.

Q.2 Find the work done by some external force in moving a charge q=2 μC from infinity to a point where electric potential is 10^4 V.

Q.3 Find out the points on the line joining two charges +q and -3q (kept at a distance of 1.0m) where electric potential is zero.

Q.4 An infinite number of charges each equal to q are placed along the x-axis at x = 1, x = 2, x = 8, ... and so on. Find the potential and the electric field at the point x = 0 due to this set of charges. What will be the potential and electric field if, in the above set up, the consecutive charges have opposite sign?

Q.5 Four point charges + q, –q, + q, and – q are placed respectively at the corners A, B, C and D of a square of side a. (a) Calculate the electric potential at O, the centre of the square. (b) if E and F are the midpoints of sides BC and CD respectively, what is the work that will be done in carrying an electron from : (i) O to E, and (ii) O to F? Given $q = 1.0 \times 10^{-6}$ C, a = 0.10 m and charge on an electron $= -1.6 \times 10^{-19}$ C.

Answers

Q.1 $-9.6\times10^{-18}\,J, 9.6\times10^{-18}\,J$ Q.2 $2\times10^{-2}\,J$

Q.3 The potential will be zero at point P on the axis which is either 0.5 m to the left or 0.25m to the right of charge +q.

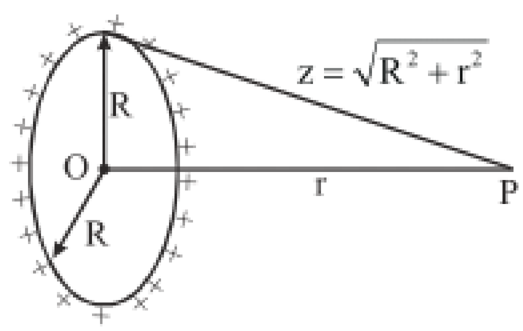
Q.4 $\frac{2q}{4\pi\epsilon_0}, \frac{1}{4\pi\epsilon_0}\left[\frac{4}{3}q\right]; \left[\frac{2}{3}q\right], \frac{1}{4\pi\varepsilon_0}\left[\frac{4}{5}q\right]$

Q.5 (a) 0 (b) (i) 0 (ii) 0

Electric Potential due to a Charged Ring

A charge Q is uniformly distributed over the circumference of a ring. Let us calculate the electric potential at an axial point at a distance r from the centre of the ring.

The electric potential at P due to the charge element dq of the ring is given by



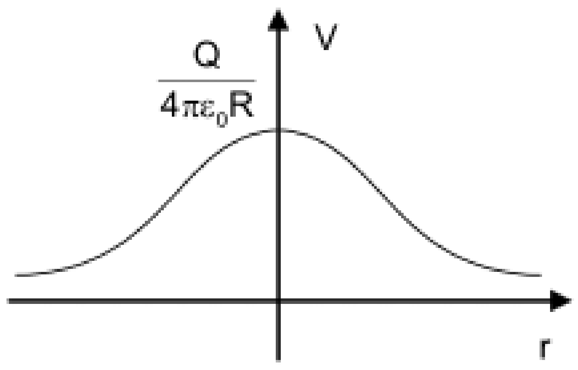
$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{Z} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(R^2 + r^2)^{1/2}}$$

Hence, the electric potential at P due to the uniformly charged ring is given by

$$V = \int \frac{1}{4\pi\varepsilon_0} \frac{dq}{(R^2 + r^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0} \frac{1}{(R^2 + r^2)^{1/2}} \int dq$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{(R^2 + r^2)}}.$$

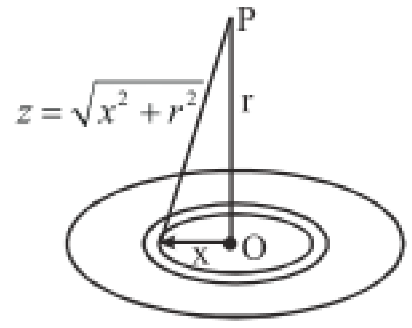
plot of electric potential



Electric Potential Due to a Charged Disc at a Point on the Axis:

A non-conducting disc of radius 'R' has a uniform surface charge density $\sigma\ C/m^2$. Let us calcuate the potential at a point on the axis of the disc at a distance 'r' from its centre. The symemtry of the disc tells us that the appropriate choice of element is a ring of radius x and thickness dx. All points on this ring are at the same distance $Z = \sqrt{x^2 + r^2}$, from the point P. The charge on the ring is $dq = \sigma dA = \sigma(2\pi x\ dx)$ and so the potential due to the ring is

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{dq}{Z} = \frac{1}{4\pi\varepsilon_0} \frac{\sigma(2\pi\ x\ dx)}{\sqrt{x^2 + r^2}}$$



Since potential is scalar

The potential due to the whole disc is given by

$$V = \frac{\sigma}{2\varepsilon_0} \int_0^R \frac{x\ dx}{\sqrt{x^2 + r^2}}$$

$$= \frac{\sigma}{2\varepsilon_0} \left[(x^2 + r^2)^{1/2} \right]_0^R$$

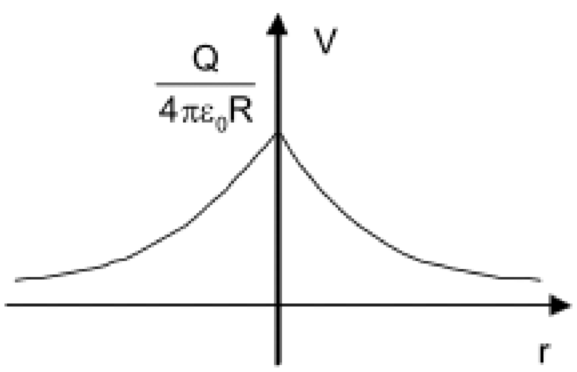
$$= \frac{\sigma}{2\varepsilon_0} \left[(R^2 + r^2)^{1/2} - r \right] \qquad ...(x\ B)$$

Let us see this expression at large distance when $r >> R$.

$V = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}$, where $Q = \pi r^2 \sigma$ is the total charge on the disc.

Thus, we conclude that at large distance, the potential due to the disc is the same as that of a point charge Q.

plot of electric potential



Relationship between electric field and electric Potential

Place a charge q in electric field $\vec{E}$ .This field exerts a force $\vec{F}=q\vec{E}$ on the charge.If you displace the charge by by $d\vec{r}$ then field will do some work $dW=\vec{F}\cdot d\vec{r}=q\vec{E}\cdot d\vec{r}$ on the charge.

Change in electric potential energy during this displacement will be given by

$$dU=-q\,\vec{E}\cdot d\vec{r}$$

Therfore potential difference between initial and final point will be given by

$$dV=-\vec{E}\cdot\vec{dr}$$

Calculation of electric potential difference from electric field

$$dV=-\vec{E}\cdot\vec{dr}$$

$$\Rightarrow\int_{V(\vec{r}_1)}^{V(\vec{r}_2)}dV=-\int_{\vec{r}_1}^{\vec{r}_2}\vec{E}\cdot d\vec{r}$$

$$\Rightarrow V(\vec{r}_2)-V(\vec{r}_1)=\int dV=-\int_{\vec{r}_1}^{\vec{r}_2}\vec{E}\cdot d\vec{r}$$

In cartesian coordinate system

$$d\vec{r}=dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}$$

$$\Rightarrow\vec{E}\cdot d\vec{r}=E_x dx+E_y dy+E_z dz$$

$$\Rightarrow V(\vec{r}_2)-V(\vec{r}_1)=-\int_{r_1}^{r_2}\vec{E}.d\vec{r}=-\int_{x_1}^{x_2}E_x dx-\int_{y_1}^{y_2}E_y dy-\int_{z_1}^{z_2}E_z dz$$

Illuatratrion:

Electric field in a region is given by $\vec{E}=(2\hat{i}+3\hat{j}-4\hat{k})$ *V/m. Find the potential difference between points (0, 0, 0) and (1,2,3) in this region*

Sol. *p.d. across the points*

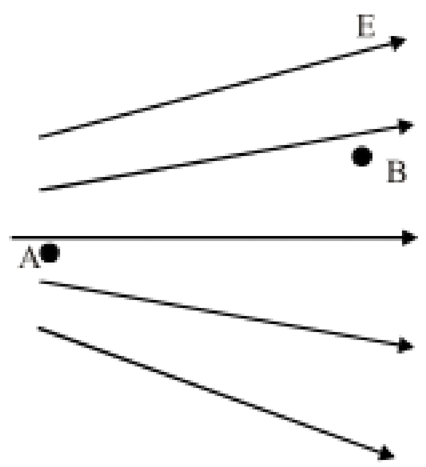
$$V=\int_{\vec{r}_1}^{\vec{r}_2}\vec{E}.d\vec{r}=-\int_{x_1}^{x_2}\vec{E}_x dx-\int_{y_1}^{y_2}E_y dy-\int_{z_1}^{z_2}E_z dz=-\int_0^1 2dx+\int_0^2 3dy+\int_0^3 -4dx$$

$= -2 - 6 + 12 = 4$ *volts.*

Practice Exercise

Q.1 An electric field of 20 N/C exists along the X-axis in space. Calculate the potential difference $V_A - V_B$ where the points A and B are given by, A = (0, 0); B = (4 m, 2 m) .

Q.2 In figure, two points A and B are located within a region in which there is an electric field.
(i) The sign of potential difference $\Delta V = V_B - V_A$ is
(ii) A negative charge is placed at A and then moved to B. The sign of change in potential energy of the charge-field system for this process is



Q.3 Two points A and B are 2cm apart and a uniform electric field E acts along the straight line AB directed from A to B with E=200 N/C. A particle of charge $+10^{-6}$ C is taken from A to B along AB. Calculate (a) the force on the charge (b) the potential difference $V_A - V_B$ and (c) the work done on the charge by $\vec{E}$.

Answers

Q.1 –80 V Q.2 (i) -ve (ii) +ve Q.3 2×10^{-4} N, 4volt and 4×10^{-6} J

Calculation of electric potential from electric field

For our convenience we select potential of a point to be zero.This point is called reference point.Usually reference point is taken at infinity. In the above equation let us take $\vec{r}_1 = \infty$ and $\vec{r}_2 = \vec{r}$,then the potential at a point can be written as

$$V\left(\vec{r}\right)=-\int_{\infty}^{\vec{r}}\vec{E}\cdot d\vec{r}$$

Illuatratrion:

The electric field in a region is given by $\vec{E} = (A/x)^3 \hat{i}$. *Write a suitable SI unit for A. Write an expression for the potential in the region assuming the potential at infinity to be zero.*

Sol. *The SI unit of electric field is N/C or V/m. Thus,*

The unit of A is $\frac{N-m^3}{C}$ *or V-m*^{2}.

$$V(x, y, z) = -\int_{\infty}^{(x,y,z)} \vec{E}.d\vec{r}$$

$$= -\int_{\infty}^{(x,\,y,\,z)} \frac{A\,dx}{x^3} = \frac{A}{2x^2}.$$

Practice Exercise

Q.1 An electric field $\vec{E} = \vec{i}\,Ax$ exists in the space, where A = 10 V/m^2. Take the potential at (10 m, 20 m) to be zero. Find the potential at the origin.

Answers

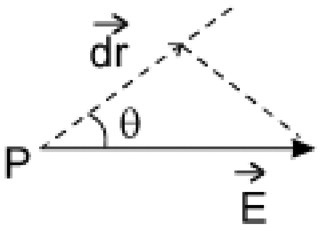
Q.1 500 V

Calculation of electric field from electric potential

$$dV = -\vec{E}\cdot d\vec{r}$$

$$dV = -(E\cos\theta)\,dr$$

$$-\frac{dV}{dr} = E\cos\theta$$



(i.e., Rate of decrease of potential)

Where E cos θ is component of field in the direction of displacement. From the above expression.

Potential decreases maximum in the direction of field, $\theta = 0^\circ$. It is also clear that, $-\frac{dV}{dr}$ is maximum in the direction of the field, so, we may conclude that, the electric potential decreases at maximum rate in the direction of the field.

The cartesian component of electric field can be written as

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k};$$

and an infinitesimal displacement is $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Thus,

$$dV = -\vec{E}.d\vec{r}$$

$$= -[E_x\ dx + E_y\ dy + E_z\ dz]$$

for a displacement in the x-direction,

$dy = dz = 0$ and so

$dV = -E_x\ dx$. Therefore,

$$E_x = -\left(\frac{dV}{dx}\right)_{y,\ z\ constant}$$

A derivative in which all variables except one are held constant is called partial derivative and is written with ∂ instead of d. The electric field is, therefore,

$$\left.\begin{aligned} E_x &= -\frac{\partial V}{\partial x} \\ E_y &= -\frac{\partial V}{\partial y} \\ E_z &= -\frac{\partial V}{\partial z} \end{aligned}\right\}$$

i.e., if scalar potential function is given field can be calculated taking help of the above relations as

$$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right] \qquad |\vec{E}| = \sqrt{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}$$

Illustration :

Potential in the x-y plane is given as $V = 5(x^2 + xy)$ volts. Find the electric field at the point (1, -2).

Sol. $E_x = -\frac{\partial V}{\partial x} = -(10x + 5y) = -10 + 10 = 0,\ E_y = -\frac{\partial V}{\partial y} = -5x = -5\ V/m$

$\therefore \vec{E} = (-5\hat{j})V/m$

Practice Exercise

Q.1 The electric potential existing in space is $V(x, y, z) = A(xy, + yz + zx)$. (a) Write the dimensional formula of A. (b) Find the expression for the electric field. (c) If A is 10 IS units, find the magnitude of the electric field at (1 m, 1 m, 1 m).

Answers

Q.1 (a) $MT^{-3}I^{-1}$ (b) $-A\{\vec{i}(y+z)+\vec{j}(z+x)+\vec{k}(x+y)\}$ (c) 35 N/C

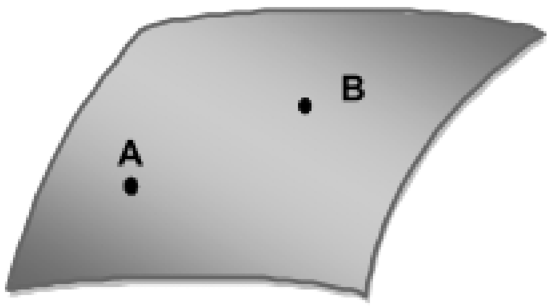
Equipotential surfaces

A locus of points in space that all have the same potential is called an equipotential surface.

Properties of equipotential surfaces :

(i) Work done in moving a charge over an equaipotential surface is zero. Thus the work done in moving a charge +q from one point A to another point B on a equipotential surface is given by;

$$W_{AB} = -q(V_B - V_A) = -q(0) = 0$$



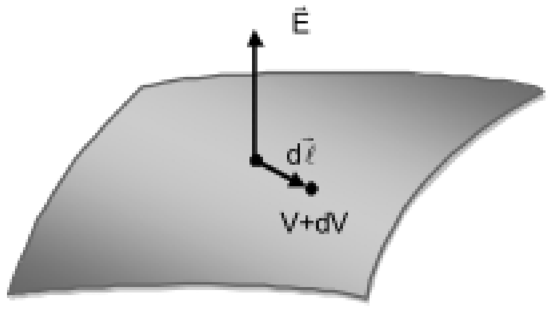
(ii) The electric field is always perpendicular to an equipotential surface. Referring to figure , we have,

$$dV = -\vec{E}\cdot d\vec{l}$$

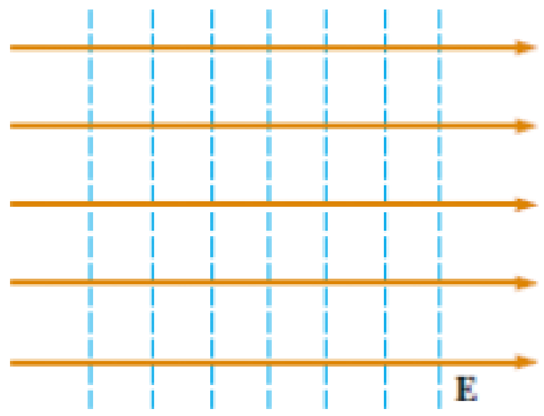
Since $dV = 0$ for an equipotential surface,

$$\therefore \quad \vec{E}\cdot d\vec{l} = 0\cdot$$

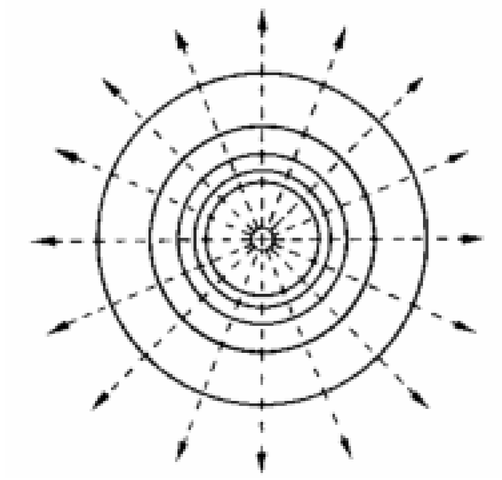
This means that $\vec{E}$ is perpendicular to $d\vec{l}$.



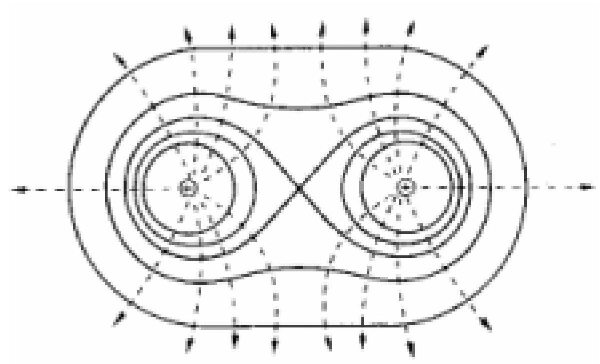
In other words, electric field (or electric lines of force) are perpendicular to the equipotential surface.



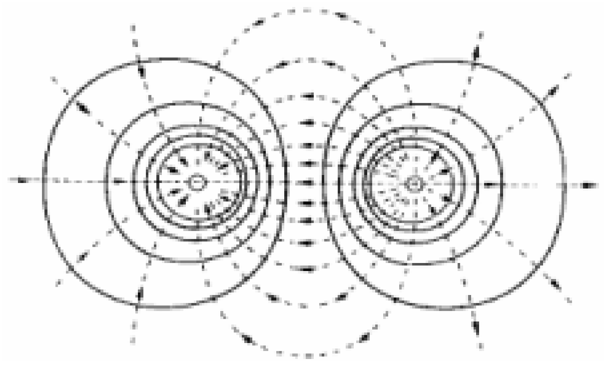
equipotential surface of uniform electric field



equipotential surface of electric field of a charge +q



equipotential surface of electric field of charges +q and +q



equipotential surface of electric field of charges +q and +q

(iii) The spacing between equipotential surfaces enables us to identify regions of strong and weak field.

We know that $E = -\frac{dV}{dr}$

For a given dV (i.e., constant dV), $E \propto 1/dr$. This means that where the equipotential surfaces are crowded, the electric field intensity is greater and vice-versa.

In the above figure equipotential surfaces having constant potrntial difference between two consecutive surfaces are shown. Near the point charge equipotential surfaces are crowded, the electric field intensity is greater

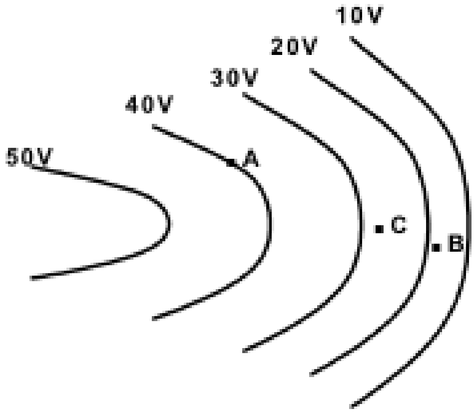
(iv) Two equipotential surfaces can be never intersect. If two equipotential surfaces could intersect, then at the point of intersection there would be two values of electric potential which is not possible.

Example:

(i) In the field of a point charge, the equipotential surfaces are spheres centered on the point charge.
(ii) In a uniform electric field, the equipotential surfaces are planes which are perpendicular to the fields lines.
(iii) In the fields of an infinite line charge, the equipotential surfaces are co-axial cylinders having their axes at the line charge.

Illustration :

The adjoining figure shows the lines of constant potential in a region in which an electric field is present. The value of potentials are shown. At which of the points A, B and C is the magnitude of the electric field the greatest?



Sol. $\vec{E} = -\frac{dV}{dr}\hat{n}$

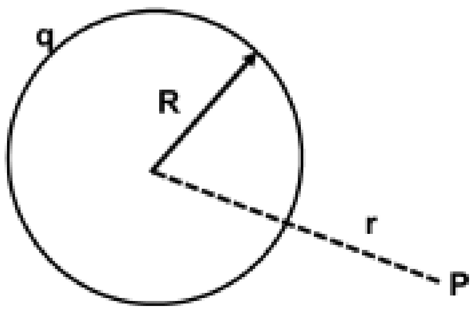
The potential difference between any two connective line $dV = V_1 - V_2 = 10v$ =constant and hence E will be maximum where the distance between the lines minimum. i.e. at B where the lines are closest.

Electric Potential due to a uniformly charged spherical shell

If the charge on the shell $= q$

(i) for $r > R$

$$E = \frac{kq}{r^2}$$

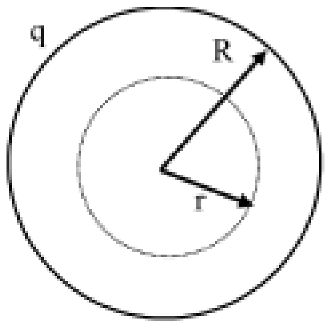


$$\Rightarrow \quad V = -\int_{\infty}^{r} E dr = -\int_{\infty}^{r} \frac{kq}{r^2} dr = \frac{kq}{r}$$

(ii) for $r = R$, $V = \frac{kq}{R}$

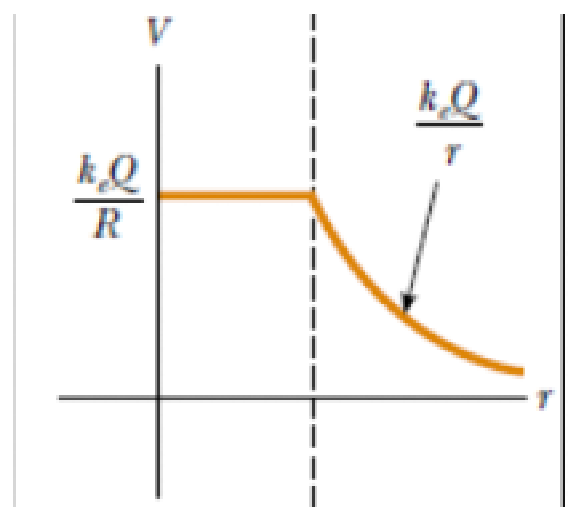
(iii) for $r < R$

$E = 0 (r<R)$



$$\Rightarrow \quad V = -\int_{\infty}^{r} E\,dr = -\left[\int_{\infty}^{R} E\,dr + \int_{R}^{r} E\,dr\right] = -\int_{\infty}^{R} \frac{kq}{r^2}\,dr - \int_{R}^{r} 0\,dr = \frac{kq}{R}$$

plot of electric potential



Electric Potential due to a uniformly charged spherical volume

If the total charge = Q

(i) for $r > R$,

Volume charge density $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$

$$E(r > R) = \frac{kQ}{r^2}$$

$$V = -\int_{\infty}^{r} \frac{kQ}{r^2}\, dr = \frac{kQ}{r}$$

(ii) for $r = R$, $V = \dfrac{kQ}{R}$

(iii) $r < R$, $E(r < R) = \dfrac{\rho . r}{3\varepsilon_o} = \dfrac{Q.r}{\frac{4}{3}\pi R^3 . 3\varepsilon_o}$

$$\Rightarrow \quad E(r < R) = \frac{Q.r}{4\pi\varepsilon_o R^3} = \frac{kQ.r}{R^3}$$

$$V = -\left[\int_{\infty}^{R} E\, dr + \int_{R}^{r} E dr\right] = -\int_{\infty}^{R} \frac{kQ}{r^2}\, dr - \int_{R}^{r} \frac{kQr}{R^3}\, dr = \frac{kQ}{2R}\left[3 - \frac{r^2}{R^2}\right]$$

plot of electric potential

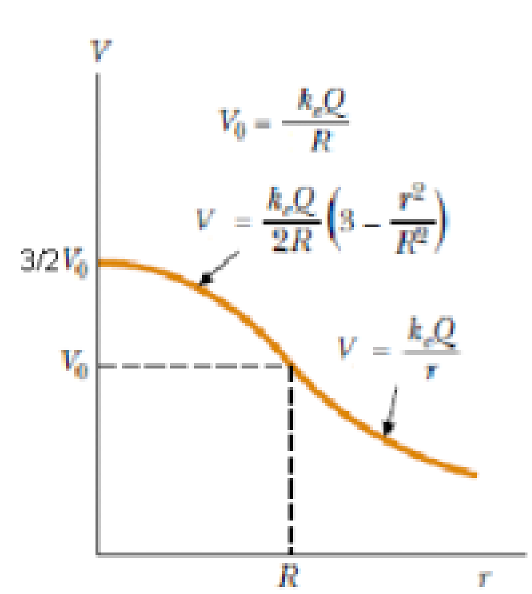


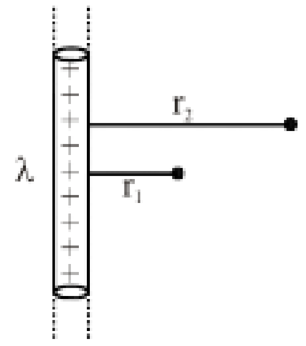
Illustration :

Find the potential difference between to points for a infinite line charge having linear charge density λ

Sol.

$$V_{12} = \int_1^2 \vec{E}.d\vec{r} = \int_{r_1}^{r_2} E\,dr$$

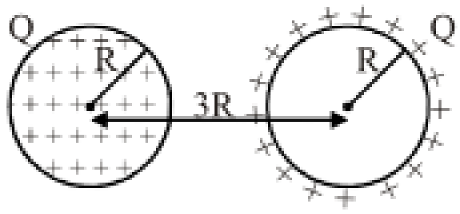
$$= \int_{r_1}^{r_2} \frac{\lambda}{2\pi\varepsilon_0 r}dr = \frac{\lambda}{2\pi\varepsilon_0}\ell n\frac{r_2}{r_1}$$



Practice Exercise

Q.1 Find the potential difference between to points for a infinite sheet having uniform surface charge density σ

Q.2 Find the work done by external agent to bring a charge q from centre of shell to centre of spherical charged volume



Q.3 A circular ring of radius R with uniform positive charge density λ per unit length is located in the y–z plane with its centre at the origin O. A particle of mass m and positive charge q is projected from the point P [$\sqrt{3}$ R, 0, 0] on the positive x–axis directly towards O, with initial speed v. Find the smallest (non zero) value of the speed such that the particle does not return to P.

Answers

Q.1 $V_{12} = \frac{\sigma}{2\varepsilon_0}(r_2 - r_1)$ Q.2 $\frac{Qq}{8\pi\varepsilon_0 R}$ Q.3 $\sqrt{\frac{\lambda q}{2\in_0 m}}$

Expression for interaction energy of n charged particle system in terms of potential

$$U = \frac{1}{2}\sum_{i=1}^{N} V_i q_i$$

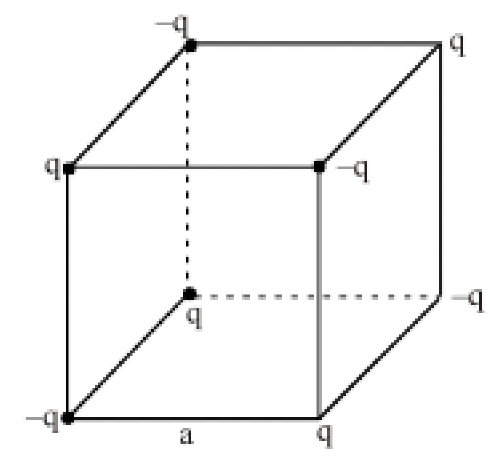
Where

q_i = charge on i-th particle and

V_i = potential at the location of i-th particle due to remaining charge particles

Illustration :

Find interaction energy of. the system



Sol. $U = \frac{1}{2}\sum_{i=1}^{N} V_i q_i = \frac{1}{2}\left[\frac{3k(-q)}{a} + \frac{3kq}{a\sqrt{2}} + \frac{k(-q)}{a\sqrt{3}}\right] \times 8q$

as because of (–q) is also same.

Energy Density of an Electric Field :

Work is done in creating any electrostatic system. This work is stored as energy in the field. The energy per unit volume or the energy density U, of the field is given as

$$U = \frac{1}{2}K\varepsilon_0 E^2$$

Illustration

*A charge Q is uniformly distributed over a spherical surface of radius R. Obtain an expression for the energy of the shell **(termed as self energy)** .*

Sol. *In this case, the electric field exists from surface of the sphere to infinity (i.e only outside the shell). Potential energy is stored in electric field with energy density.*

Electric field at a distance r is (r ≥ R)

$$E = \frac{1}{4\pi\varepsilon_0}.\frac{Q}{r^2}$$

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{\varepsilon_0}{2}\left\{\frac{1}{4\pi\varepsilon_0}.\frac{Q}{r^2}\right\}^2$$

$$dU = udV = \left[\frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}.\frac{Q}{r^2}\right)^2\right](4\pi r^2\, dr)$$

$$dU = \frac{Q^2}{8\pi\varepsilon_0}.\frac{dr}{r^2}$$

$$U = \int_R^\infty dV = \frac{Q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$U = \frac{Q^2}{8\pi\varepsilon_0 R}$$

Practice Exercise

Q.1 Two uniformly charged spherical shells have charges q_1 and q_2 and radii r_1 and r_2. Find the potential energy of the system if their cerntres are separated by a distnace r.

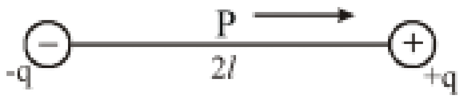
Answers

Q.1 $\frac{q_1^2}{8\pi\varepsilon_0 r_1} + \frac{q_2^2}{8\pi\varepsilon_0 r_2} + \frac{q_1 q_2}{4\pi\varepsilon_0 r}$

Electric Dipole

An arrangement of two equal and opposite charges separated by a small distance is known as an electric dipole.

Let q and –q be two charges separated by distance $2l$. The dipole moment of the dipole is :



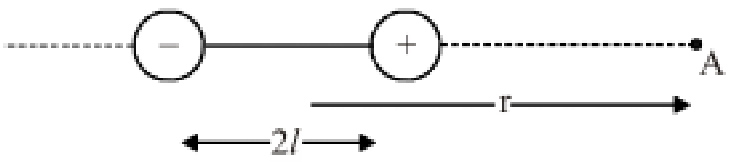
$\vec{p} = q\,2\vec{\ell}$

It is a vector quantity and is directed from –ve charge towards the +ve charge. The line joing -q to +q is known as *the axis of the dipole.*

Electric field due to a dipole

Electric field at axis : (Line joining the charges)

Electric field due to a short dipole on its axis at a point A at a distance r from dipole (ℓ <<r) :



$$E_A = \frac{q}{4\pi\epsilon_0 (r-\ell)^2} - \frac{q}{4\pi\epsilon_0 (r+\ell)^2}$$

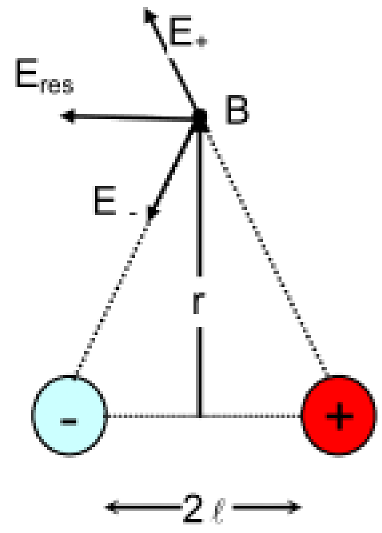
$$E_A = \frac{4q\ell r}{4\pi\epsilon_0 (r^2-\ell^2)^2}$$

after using dipole approximation $\ell << r$ we get

$$\vec{E}_A = \frac{2\vec{P}}{4\pi\epsilon_0 r^3}.$$

Electric field at equator : (Line perpendicular to axis passing through centre)

Electric field at a point distance r from the centre of the short dipole (ℓ << r)



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$$E_B = 2\left[\left\{\frac{q}{4\pi\varepsilon_0\left(r^2+\ell^2\right)}\right\}\cos\theta\right]$$

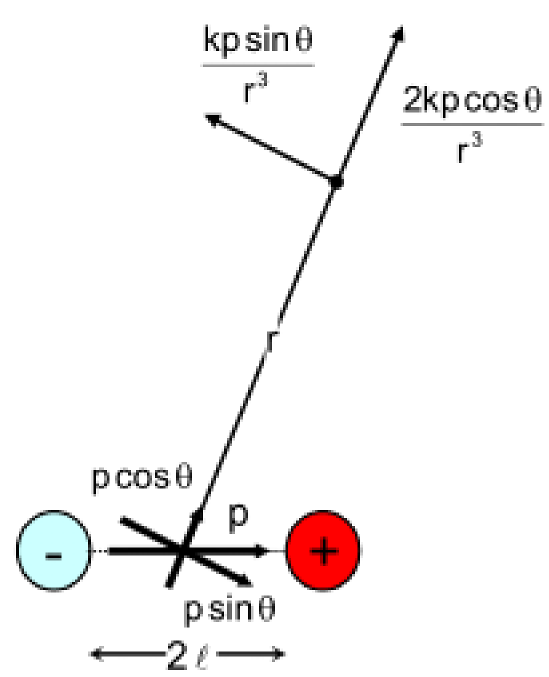
$$E_B = \frac{2q}{4\pi\in_0\left(r+\ell\right)^2}\frac{\ell}{\sqrt{\ell^2+r^2}} = \frac{2q\ell}{4\pi\in_0\left(r^2+\ell^2\right)^{3/2}}$$

after using dipole approximation $\ell << r$ we get

$$\vec{E}_B = \frac{-\vec{p}}{4\pi\in_0 r^3}$$ (-ve sign indicate that field is oppositly directed to dipole direction)

Electric field at any point A (r, θ) due to dipole

Let A be a point at a distance r from the mid-point O of the dipole. Let θ be the angle between OA and the dipole moment p. Since dipole moment is a vector so we can resolve its components $p\cos\theta$ and $p\sin\theta$ along and perpendicular to OA. Due to $p\cos\theta$ (axial point) electric field will be in the direction of $p\cos\theta$ and due to $p\sin\theta$ (equatorial position) electric field will be opposite to $p\sin\theta$



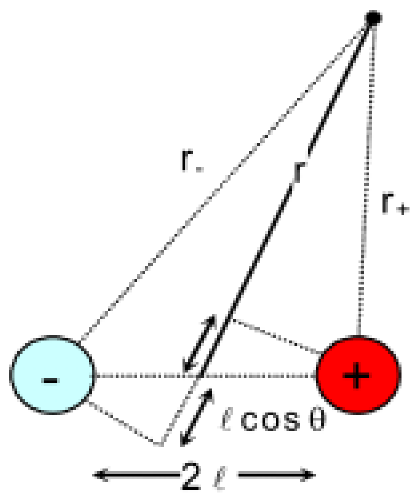
Electric field at A

$$E = \sqrt{\left(\frac{2kp\cos\theta}{r^3}\right)^2 + \left(\frac{kp\sin\theta}{r^3}\right)^2} = \frac{kp}{r^3}\sqrt{1+3\cos^2\theta}$$

$$\& \quad \tan\alpha = \frac{E_\theta}{E_r} = \frac{\sin\theta}{2\cos\theta} = \frac{\tan\theta}{2}$$

Electric potential due to dipole

Electric field at any point A (r, θ) due to dipole :



$$V = V_+ + V_- = \frac{k(+q)}{r_+} + \frac{k(-q)}{r_-} = \frac{k(+q)}{r - l\cos\theta} + \frac{k(-q)}{r + l\cos\theta} = \frac{k(2lq)}{r^2 - l^2\cos^2\theta}$$

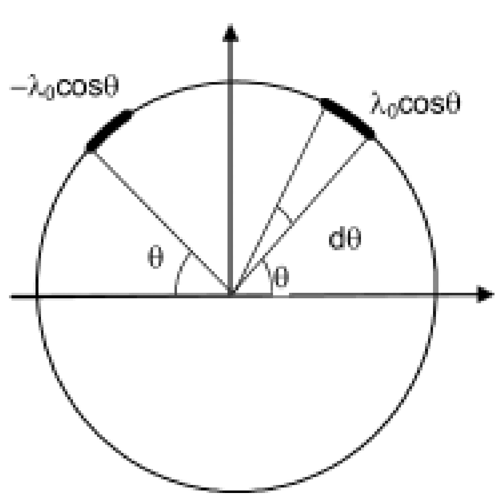
after using dipole approximation $\ell << r$ we get

$$V = \frac{kp}{r^2}$$

Illustration:

A nonunifom charge given on the ring according to the equation $\lambda=\lambda_0\cos\theta$ (where θ is measured from x-axis).Find its dipole moment.

Sol.



In the figure shown the two symmetric elements are equal and opposite charge whose dipole moment will be

$$dp = \{(\lambda_0\cos\theta)(Rd\theta)\}\{2R\cos\theta\} = 2\lambda_0 R^2\cos^2\theta d\theta$$

$$\therefore p = \int dp = 2\lambda_0 R^2 \int_{-\pi/2}^{+\pi/2} \cos^2\theta d\theta = \pi\lambda_0 R^2$$

Illustration:

For a given dipole at a point (away from the center of dipole) intensity of the electric field is E. Charges of the dipole are brought closer such that distance between point charges is half, and magnitude of charges are also halved. Find the intensity of the field now at the same point

Sol. $P_i = 2ql$

$$P_f = 2\frac{q}{2}\frac{l}{2} = \frac{P_i}{4}$$

$$r_f = r_i \quad \theta_f = \theta_i$$

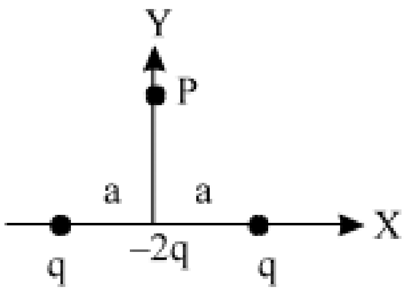
$$E = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{1+3\cos^2\theta}$$

$$\Rightarrow \quad E_f = \frac{E_i}{4}.$$

Practice Exercise

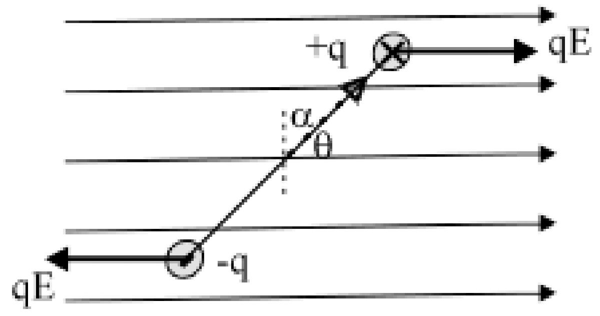
Q.1 Three point charges q, –2q and q are located along the x-axis as shown in figure. Show that the electric field at P (y >> a) along the y-xis is ,

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{3qa^2}{y^4}\hat{j}$$



Dipole in an external uniform electric field

If a dipole is placed in a uniform electric field E,
Force on the dipole is zero.
Torque on the dipole is given as



Here the net force is zero, but there can be a torque.

$$\tau = (qE)(2l\sin\theta) = pE\sin\theta$$

In vector form $\vec{\tau} = \vec{p}\times\vec{E}$

This torque has a tendency to orient dipole moment vector in the direction of field

Potential energy due to action of electric field

$$\tau = pE\cos\alpha$$

$$\therefore W_{fd} = \int dW = pE\int_0^{90-\theta}\cos\alpha\, d\alpha = pE[\sin\alpha]_0^{90-\theta} = pE\cos\theta$$

$$\therefore U_\theta = -pE\cos\theta = -\vec{p}.\vec{E} \quad (\text{taking } U = 0 \text{ at } \theta = 90^0)$$

When $\theta = 0^0$, the dipole moment p is in the direction of the field E and the dipole is in *stable equilibrium.* If it is slightly displaced, it performs oscillations.
When $\theta = 180^0$, the dipole moment p is opposite to the direction of the field E and the dipole is in *unstable equilibrium.*

Illustration:

A dipole of dipole moment P lies in a uniform electric field E such that dipole direction is along field. If dipole is rotated through 180° such that dipole direction becomes opposite to the field direction. Find the work done by the electrostatic field.

Sol.

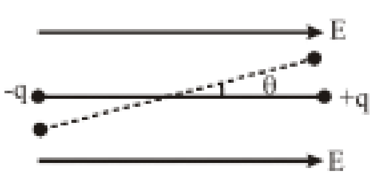
$$U_i = -\vec{P}.\vec{E} = -PE\cos O = -PE$$

$$U_f = -P.E.\cos(180^o) = PE$$

$$\textit{work done by the field} = -\Delta U = U_i - U_f = -2PE$$

Illustration:

Figure shows an electric dipole formed by two particles fixed at the ends of a light rod of length l. The mass of each particle is m and the chargers are $-q$ and $+q$. The system is placed in such a way that the dipole axis is parallel to a uniform electric field E that exists in the region. The dipole is slightly rotated about its centre and released. Show that for small angular displacement, the motion is angular simple harmonic and find its time period.



Sol.

Suppose, the dipole axis makes an angle θ with the electric field at an instant. The magnitude of the torque on it is

$$|\vec{\tau}| = |\vec{P} \times \vec{E}|$$

$$= q\,l\,E\sin\theta$$

This torque will be restoring & tend to rotate the elipole back towards the electric field. Also, for small angular displacement $\sin\theta = \theta$ so that

$$\tau = -q\,l\,E\theta$$

It the moment of inertia of the body about OA is I, the angular acceleration becomes.

$$\alpha = \frac{\tau}{I} = -\frac{qlE}{I}\theta \qquad \alpha = -\omega^2\theta$$

where $\omega^2 = \dfrac{qlE}{I}$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{qlE}}$$

Now, moment of inertia of the system about the axis of rotation is

$$I = 2m\left(\frac{l}{2}\right)^2 - \frac{ml^2}{2}$$

So, $\quad T = 2\pi\sqrt{\dfrac{ml}{2qE}}$.

Practice Exercise

Q.1 A dipole of dipole moment p is placed at origin along x-axis. Another dipole of dipole moment p is kept at (0, 1, 0) along y-axis. Find the resultant potential and electric field at (1, 0, 0)

Q.2 Due to electric dipole, electric field at a distance r on axial position is $\vec{E}_1$ and at distance r on equatorial position is $\vec{E}_2$. What is the relation between $\vec{E}_2$ and $\vec{E}_1$.

Q.3 Three charges Q, Q and -2Q are placed at the three corners of an equilateral triangle of side a. Find the dipole moment of the combination.

Answers

Q.1 $v = kp(\frac{2\sqrt{2}-1}{2\sqrt{2}})$, $\vec{E} = kp[\frac{(8\sqrt{2}-3)}{4\sqrt{2}}\hat{i} + \frac{1}{4\sqrt{2}}\hat{j}]$ Q.2 $\vec{E}_1 = -2\vec{E}_2$

Q.3 $p = \sqrt{3}qa$

Dipole in an external non uniform electric field :

If a dipole is placed in non uniform electric field then let electric field at the location of positive charge will be $\vec{E}_1$ and at the location of –ve charge be $\vec{E}_2$. Usually for non uniform electric field $\vec{E}_1 + \vec{E}_2$ hence dipole will experience a net force (usually) which is equal to

$$\vec{F}_{net} = q\vec{E}_1 + (-q)\vec{E}_2$$

$$= q(\vec{E}_1 - \vec{E}_2)$$

$$= -\frac{q(\vec{E}_2 - \vec{E}_1)}{\ell} \qquad = -p\frac{\partial\vec{E}}{\partial\ell}$$

Where $\frac{\partial\vec{E}}{\partial\ell}$ = rate of change of electric field in the direction of dipole moment.

Torque of this electric field about geometric centre of dipole is still given by

$$\vec{\tau} = \vec{p} \times \vec{E}$$

CONDUCTOR

In a metal, the outer (valence) electrons part away from their atoms and are free to move. These electrons are free within the metal but not free to leave the metal. The free electrons form a kind of 'gas'; they collide with each other and with the ions, and move randomly in different directions. The positive ions made up of the nuclei and the bound electrons remain held in their fixed positions.

Inside a conductor, electrostatic field is zero

Consider a conductor, neutral or charged. There may also be an external electrostatic field. In the static situation, when there is no current inside or on the surface of the conductor, the electric field is zero everywhere inside the conductor. *This fact can be taken as the defining property of a conductor*. A conductor has free electrons. As long as electric field is not zero, the free charge carriers would experience force and drift. In the static situation, the free charges have so distributed themselves that the electric field is zero everywhere inside. ***Electrostatic field is zero inside a conductor.***

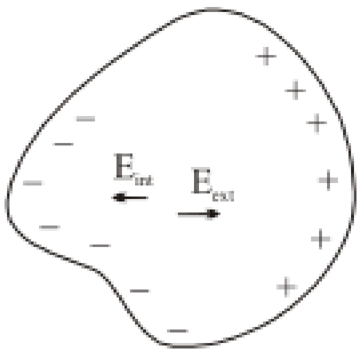
Further Explanation

What happens if conductor is placed in an extermal field.

Lets keep a positive charge q near a neutral or a charged conductor then the electrons goes close to q.

The redistribution of electrons inside conductor takes place which generates an internal electric field $\vec{E}_{int}$.

$$\vec{E}_{net} = \vec{E}_{ext} + \vec{E}_{int}$$



neutral conductor

So an e^- experience $\vec{E}_{net}$

If $E_{int} \neq E_{ext}$, then the e^- move such that they will create a stronger $\vec{E}_{int}$ which will tend to cancel E_{ext}. This constitutes current and therefore energy conservation is not valid.

$\vec{E}_{net}$ has to be O instantaneously.

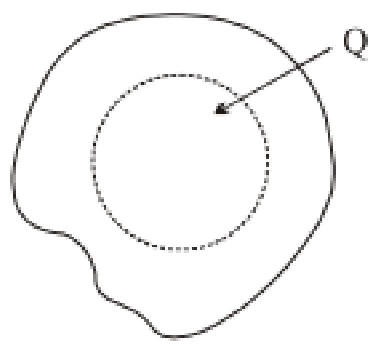
$$\vec{E}_{net} = \vec{E}_{ext} + \vec{E}_{int} = 0$$

The interior of a conductor can have no excess charge in the static situation

A neutral conductor has equal amounts of positive and negative charges in every small volume or surface element. When the conductor is charged, the excess charge can reside only on the surface in the static situation.

Explanation

This follows from the Gauss's law. Consider any arbitrary volume element *v* inside a conductor. If we consider any small gaussian surface inside



$$\oint \vec{E} \cdot d\vec{A} = 0 \qquad [\text{as } E = 0]$$

On the closed surface *S* bounding the volume element *v*, electrostatic field is zero. Thus the total electric flux through *S* is zero. Hence, by Gauss's law, there is no net charge enclosed by *S*.

$\Rightarrow \quad q_{enclosed} = 0$

Since the surface *S* can be made as small as you like, i.e., the volume *v* can be made vanishingly small. This means *there is no net charge at any point inside the conductor*, and *any excess charge must reside at the surface.*

> **Note : Thus Solid " conducting" sphere is same as a shell.**
> *Note : You may emphasise again but q_{in} = 0 does not imply that E = 0 from gauss law.*

At the surface of a charged conductor, electrostatic field must be normal to the surface at every point

If **E** were not normal to the surface, it would have some non-zero component along the surface. Free charges on the surface of the conductor would then experience force and move. In the static situation, therefore, **E** should have no tangential component. Thus *electrostatic field at the surface of a charged conductor must be normal to the surface at every point*. (For a conductor without any surface charge density, field is zero even at the surface.)

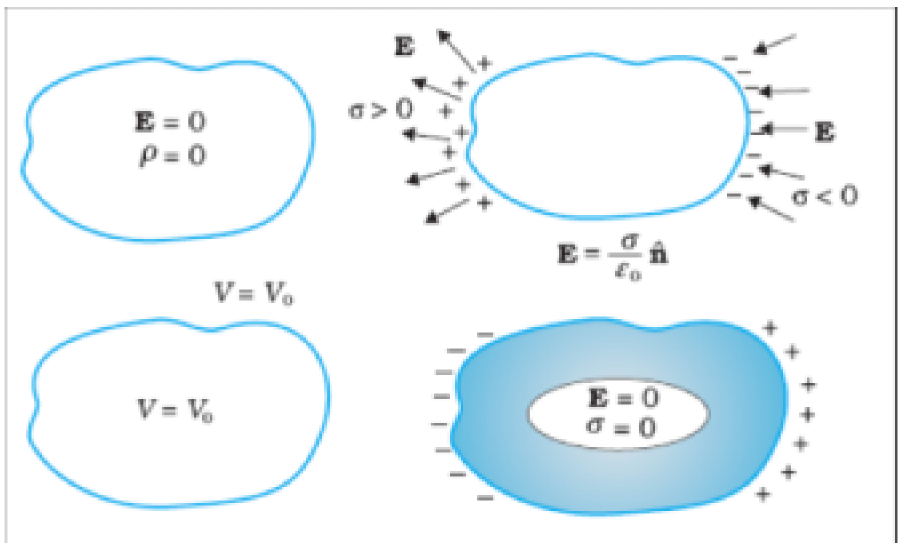
Cavity inside a conductor containing no charge inside it

Cavity is a place surrounded from all sides by the conductor such that without touching the body we can't reach cavity.

Electrostatic shielding

Consider a conductor with a cavity, with no charges inside the cavity. A remarkable result is that the electric field inside the cavity is zero, whatever be the size and shape of the cavity and whatever be the charge on the conductor and the external fields in which it might be placed. We have proved a simple case of this result already: the electric field inside a charged spherical shell is zero. The proof of the result for the shell makes use ofthe spherical symmetry of the shell. But the vanishing of electric field in the (charge-free) cavity of a conductor is, as mentioned above, is a general result. A related result is that even if the conductor is charged or charges are induced on a neutral conductor by an external field, all charges reside only on the outer surface of a conductor with cavity.

Whatever be the charge and field configuration outside, any cavity in a conductor remains shielded from outside electric influence: thefield inside the cavity is always zero. This is known as electrostatic shielding. The effect can be made use of in protecting sensitive instruments from outside electrical influence. Figure gives a summary of the important electrostatic propertiesof a conductor.



If there is no charge present inside the cavity then field inside it is zero.

(This does not mean that we are implying "no net charge". If there is some charge present here and there in cavity such that there sum total is zero then the following disscussion will not be valid)

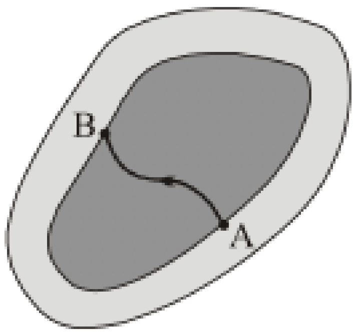
Let us consider a gauesion surface just near to cavity.

$$\therefore \qquad \oint \vec{E}\cdot d\vec{A} = 0$$

$$\Rightarrow \qquad q_{in} = 0$$

From this we don't have proved that no charge resides on cavity but have proved that net charge on cavity surface is O.

Now suppose a conductor of arbitrary shape contains a cavity as shown in figure.



Let us assume that no charges are inside the cavity. In this case, the electric field inside the cavity must be zero regardless of the charge distribution on the outside surface of the conductor. Furthermore, the field in the cavity is zero even if an electric field exists outside the conductor.

To prove this point, we use the fact that every point on the conductor is at the same electric potential, and therefore any two points A and B on the surface of the cavity must be at the same potential. Now imagine that a field E exists in the cavity and evaluate the potential difference $V_B - V_A$ defined by equation.

$$\Delta V \equiv \frac{\Delta U}{q_0} = -\int_A^B E\cdot ds$$

$$V_B - V_A = -\int_A^B E\cdot ds$$

Because $V_B - V_A = 0$, the integral of $E \cdot ds$ must be zero for all paths between any two points A and B on the conductor. The only way that this can be true for all paths is if E is zero everywhere in the cavity.

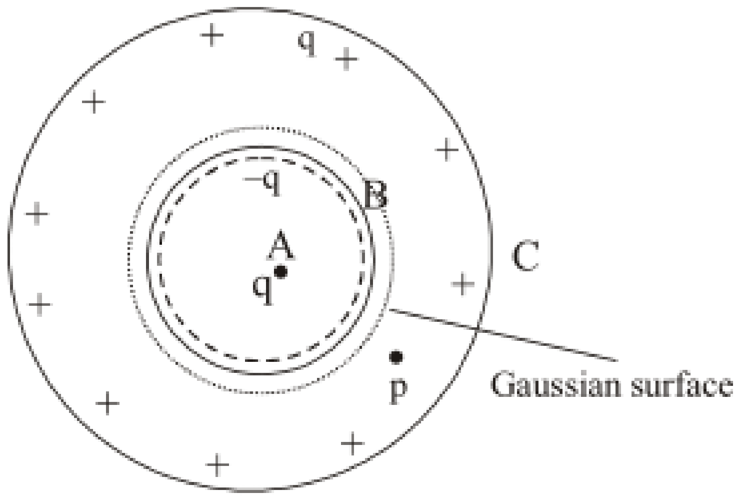
Thus, we conclude that a cavity surrounded by conducting walls is a field-free region as along no charges are inside the cavity.

Cavity inside a conductor containing charge inside it

1. Equal and opposite charge is induced on the inner surface of cavity.

Figure shows a conductor with a cavity inside it. A charge q is placed inside cavity.
In electrostatic equilibrium charge distribution will be as shown in figure

charge on inner surface of cavity is – q since material of conductor is initially neutral, equal and opposite charge appears on outer surface.



Let us consider a gaussian surface just outside cavity inside material of conductor. As $\vec{E}$ in material of conductor is zero.

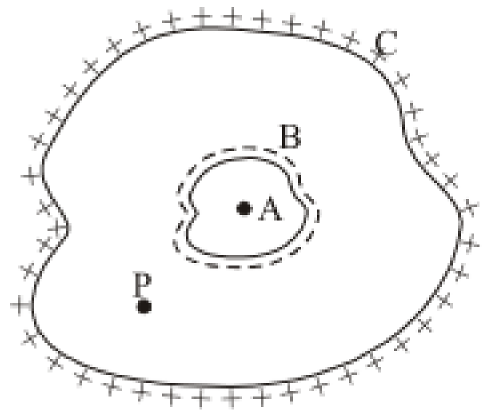
$$\oint E \cdot dA = 0 \qquad \therefore \qquad q_{enclosed} = 0$$

Thus equal and opposite charge is induced on the inner surface of cavity.

2. The electric field due to charges on the inner surface of conductor nullifies the electric field of point charge all the points outside the inner surface

3. The electric field due to charges on the outer surface of conductor is zero for all the points inside the outer surface seperately

Consider a charged conductor having charge $+q_1$ and Q is kept inside the cavity. Lets call charge Q inside cavity as A, the induced charge -Q on the surface of the cavity as B and the charge on the surface of the conductor $Q + q_1$ at C.

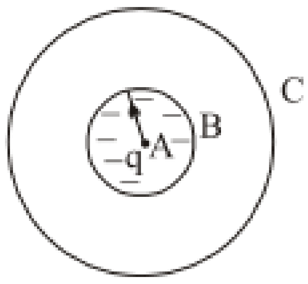


Now field inside the conductor is, $\vec{E}_{net}$ and $\vec{E}_A$, $\vec{E}_B$ and $\vec{E}_C$ are fields due to charge A,B and C inside the conductor.

and, $\vec{E}_{P,net} = \vec{E}_A + \vec{E}_B + \vec{E}_C = 0$

Now the electric field due to charges on the outer surface of conductor is zero for all the points inside the conductor seperately and the $\vec{E}_B + \vec{E}_A$ *is zero seperately.*

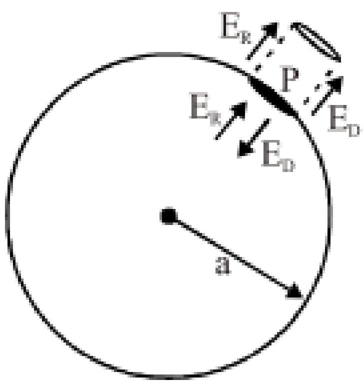
Special case : When spherical cavity in present inside spherical conductor and charge is at the centre. Then field inside the cavity is just due to charge A only as new field lines are already $\perp$ to B and E due to B = 0. [Symmetrical distribution]



Note : When if we displace the charge q inside the cavity, it will only affect the charge distribution at B the charge distribution at C will remain unaffected.

Electrostatic Pressure

If a small piece of radius b is removed from a charged spherical shell of radius a (>>b), calculate electric intensity at the midpoint of the aperture, assuming the density of charge to be σ.



Consider the shell to be made up of a disc of radius b and the remainder. If E_D and E_R are the intensities due to disc and the remainder respectively at P, then for a charged spherical shell (or conductor)

$$E_{out} = \frac{\sigma}{\varepsilon_0} \text{ and } E_{in} = 0$$

Now as for outside the shell both E_D and E_R will be directed outwards while inside E_R will be outwards while E_D inwards so that :

$$E_{out} = E_R + E_D \text{ and } E_{in} = E_R - E_D \qquad \text{........ (2)}$$

And hence equating Eqs. (1) and (2),

$$E_R + E_D = \frac{\sigma}{\varepsilon_0} \text{ and } E_R - E_D = 0$$

Solving these for E_R and E_D :

$$E_R = E_D = \frac{\sigma}{2\varepsilon_0}$$

i.e., field at the aperture will be $(\sigma/2\varepsilon_0)$ directed outwards.

As intensity on the disc (element) the to remainder is $(\sigma/2\varepsilon_0)$, electric force on it will be,

$$dF = dq\, E = (\sigma\, ds)\left[\frac{\sigma}{2\varepsilon_0}\right] = \left[\frac{\sigma^2}{2\varepsilon_0}\right] ds$$

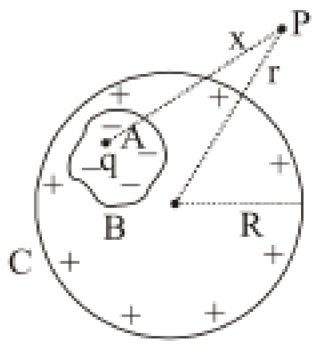
So force per unit area on a charged conductor due to its own charge

$$\frac{dF}{ds} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0 E^2 \qquad \text{[as for a conductor } E = \frac{\sigma}{\varepsilon_0}\text{]}$$

This force is called or electrostatic pressure.]

Illustration :

Find electric field at P, the conductor is neutral. and cavityis having a charge q inside the cavity.

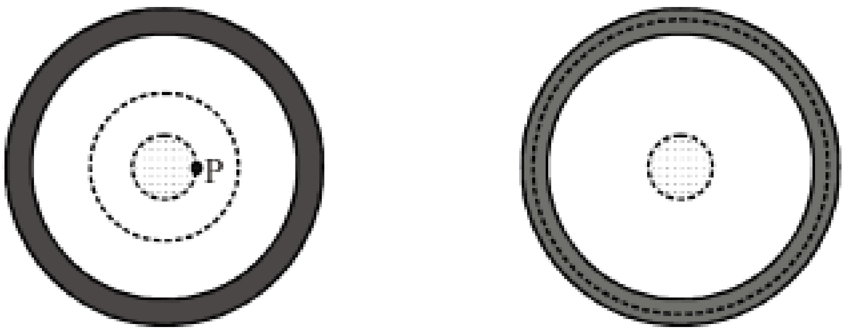


Sol. $\therefore \quad E_p = \dfrac{kQ}{r^2} \qquad [as \quad E_A + E_B = O]$

Illustration :

A charge of 4×10^{-8} C is distributed uniformly on the surface of a sphere of radius 1 cm. It is covered by a concentric, hollow conducting sphere of radius 5 cm. (a) Find the electric field at a point 2 cm away from the centre. (b) A charge of 6×10^{-8} C is placed on the hollow sphere. Find the surface charge density on the outer surface of the hollow sphere.

Sol.



(a) (b)

(a) Let us consider figure. Suppose, we have to find the field at the point P. Draw a concentric spherical surface through P. All the points on this surface are equivalent and by symmetry,the field at all these points will be equal in magnitude and radial in direction.

The flux through this surface $= \oint \vec{E}.d\vec{S}$

$$= \oint E.dS = E \oint dS$$

$$= 4\pi x^2 E.$$

Where $x = 2\ cm = 2 \times 10^{-12}\ m.$

From Gauss's law, this flux is equal to the charge q contained inside the surface divided by ε_0. Thus,

$$4\pi x^2 E = q/\varepsilon_0$$

or $\quad E = \dfrac{q}{4\pi\varepsilon_0 x^2}$

$$= \left(9\times10^{9}\,\frac{N-m^{2}}{C^{2}}\right)\times\frac{4\times10^{-8}C}{4\times10^{-4}m^{2}}$$

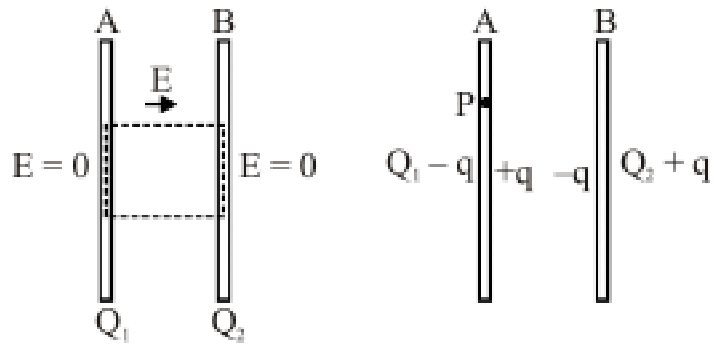
$= 9\times10^{5}$ *N/C.*

(b) See figure. Take a Gaussian surface through the material of the hollow sphere. As the electric field in a conducting material is zero the flux $\oint\vec{E}.d\vec{S}$ *through this Gaussian surface is zero. Using Gauss's law the total charge enclosed must be zero. Hence, the charge on the inner surface of the hollow sphere is* -4×10^{-8} *C. But the total charge given to this hollow sphere is* 6×10^{-8} *C. Hence, the charge on the outer surface will be* 10×10^{-8} *C.*

Illustration :

Two conducting plates A and B are placed parallel to each other. A is given a charge Q_1 *and B a charge* Q_2 *Find the distribution of charge on the four surface.*

Sol. *Consider a Gaussion surface as shown in figure. Two faces of this closed surface lie completely indie the conductor where the lectric field is zero. The flux through these face is, therfore, zero. The flux through these is, therfore zero. The other parts of the closed surface which are outside the conductor are parallel to the electric field and hence the flux on these parts is also zero. The total fulx of the electric field through the cloesed surface is, therefore, zero. From Gauss's law, the total charge inside thsi closed surface should be zero. The charge on the inner surface of A should be equal and oposite to that on the inner surface of B.*



The distribution should be like the one shown in figure. To find the value of q , consider the field at a point P inside the plate A. Suppose, the surface are of the plate (one side) is A. Using the equation $E = \sigma/(2\varepsilon_0)$*, the electric field at P*

due to the charge $Q_1 - q = \dfrac{Q_1 - q}{2A\varepsilon_0}$ *(right)*

due to the charge $+ q = \dfrac{q}{2A\varepsilon_0}$ *(left)*

due to the charge $- q = \dfrac{q}{2A\varepsilon_0}$ *(right)*

and due to the charge $Q_2 + q = \dfrac{Q_2 + q}{2A\varepsilon_0}$ *(left)*

The net electric field at P due the all the four charged surface is (in the right direction)

$$\frac{Q_2 - q}{2A\varepsilon_0} - \frac{q}{2A\varepsilon_0} + \frac{q}{2A\varepsilon_0} - \frac{Q_2 + q}{2A\varepsilon_0}$$

As the point P is inside the conductor, this field should be zero. Hence,

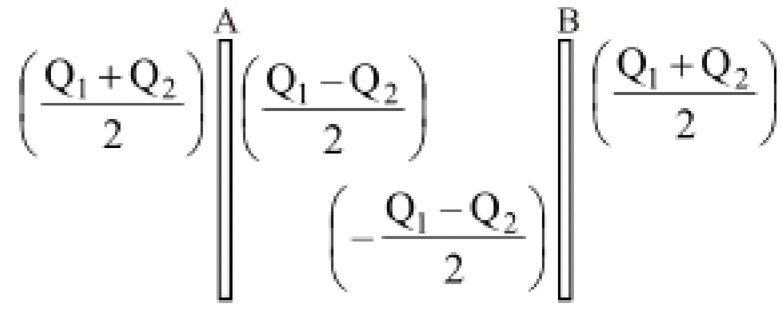
$$Q_1 - q - Q_2 - q = 0$$

or $$q = \frac{Q_1 - Q_2}{2} \qquad \text{...(i)}$$

Thus, $$Q_1 - q = \frac{Q_1 + Q_2}{2} \qquad \text{...(ii)}$$

and $$Q_2 + q = \frac{Q_1 + Q_2}{2}$$

Using these equations, the distribution shown in the figure can be redrawn as in figure



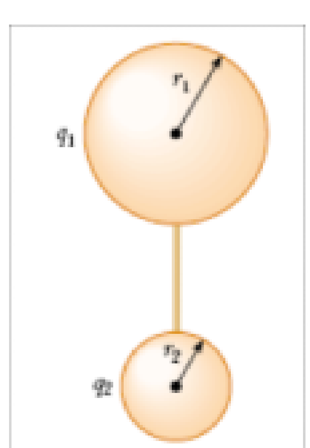
Important :

(i) Thus facing surfaces have equal and opposite charges.

(ii) Using this also prove that outer faces of the two last plates have equal charges.

Illustration:

Two spherical conductors of radii r_1 and r_2 are separated by a distance much greater than the radius of ither sphere. The spheres are connected by a conducting wire, as shown in figure . The charges on the spheres in equilibrium are q_1 and q_2, respectively, and they are uniformly charged. Find the ratio of the magnitudes of the electric fields at the surfaces of the spheres.



Sol. *Because the spheres are connected by a conducting wire, they must both be at the same electric potential:*

$$V = \frac{kq_1}{r_1} = \frac{kq_2}{r_2}$$

Therefore, the ratio of charges is

$$\frac{q_1}{q_2} = \frac{r_1}{r_2} \quad \text{.... (1)}$$

Because the spheres are very far apart and their surfaces uniformly charged, we can express the magnitude of the electric fields at their surfaces as

$$E_1 = \frac{kq_1}{r_1^2} \text{ and } E_2 = \frac{kq_2}{r_2^2}$$

Taking the ratio of these two fields and making use of Equation (1), we find that

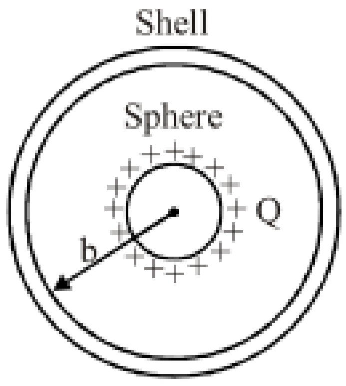
$$\frac{E_1}{E_2} = \frac{r_2}{r_1}$$

Hence, the field is more intense in the vicinity of the smaller sphere even though the electric potentials of both spheres are the same.

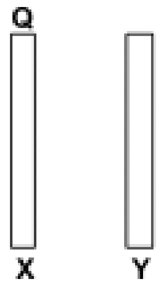
Practice Exercise

Q.1 A hollow, uncharged spherical conductor has inner radius a and outer radius b. A positive point charge +q is in the cavity at the centre of the sphere. Make the graph E and potential V(r) everywhere, assuming that $V = 0$ at $r = \infty$.

Q.2 A solid conducting sphere having a charge Q is surrounded by an uncharged concentric conducting hollow spherical shell. Let the potential difference between the surface of the solid sphere and that of the outer surface of hollow shell be V. What will be the new potential difference between the same two surfaces if the shell given a charge –3Q ?

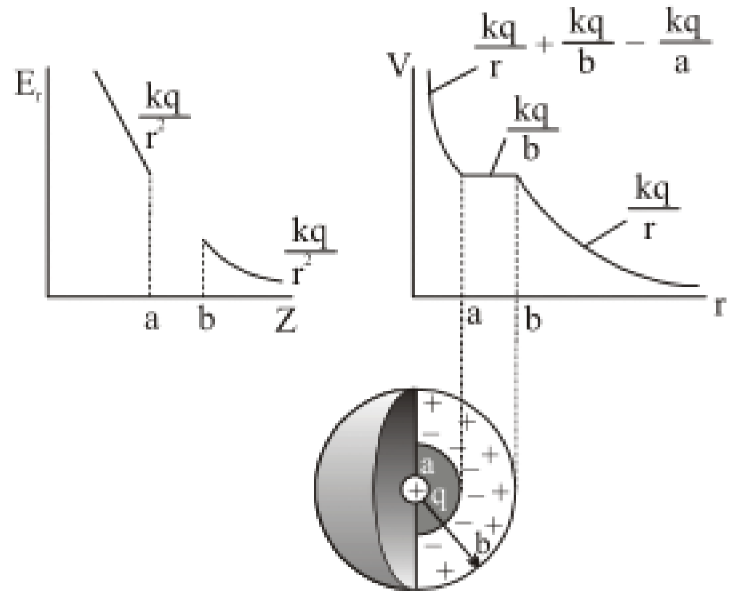


Q.3 Two conducting plates X and Y, each having large surface area A (on one side), are placed parallel to each other as shown in the figure. The plate X is given a charge Q whereas the other is neutral. Find (a) the surface charge density at the inner surface of the plate X, (b) the electric field at a point to the left of the plates, (c) the electric field at a point in between the plates and (d) the electric filed at a point to the right of the plates.



Answers

Q.1



Q.2 V

Q.3 (a) $\frac{Q}{2A}$ (b) $\frac{Q}{2A\varepsilon_0}$ towards left (c) $\frac{Q}{2A\varepsilon_0}$ towards right (d) $\frac{Q}{2A\varepsilon_0}$ towards right

Solved Example

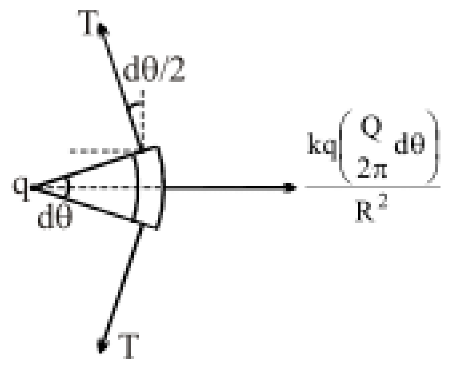
Q.1 A ring has charge Q and radius R. If a charge q is placed at its center, then calculate the increase in tension in the ring.

Sol. Let us take an elementry part subtending an angle dθ at centre.

Charge on the elementry part will be

$$dQ = \frac{Q}{2\pi R}(Rd\theta) = \frac{Q}{2\pi}d\theta$$

Free body diagram will be

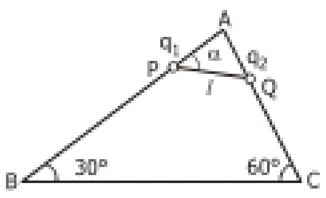


For qeuilibrium $2T\sin\frac{d\theta}{2} = \frac{kQq}{2\pi R^2}d\theta$

$$\Rightarrow 2T\frac{d\theta}{2} = \frac{kQq}{2\pi R^2}d\theta \qquad [\because \text{ for small } \theta, \sin\theta \approx \theta]$$

$$\therefore T = \frac{kQq}{2\pi R^2} \quad \text{Ans.}$$

Q.2 A rigid insulated wire frame in the form of a right-angled triangle ABC, is set in a vertical plane as shown in **Fig**. Two beads of equal masses m each and carrying charges q_1 and q_2 are connected by a cord of length l and can slide without friction on the wires



Considering the case when the beads are stationary, determine

(a) the angle α (b) the tension in the cord and (c) the normal reaction on the beads.

If the cord is now cut what are the value of the charges for which the beads continue to remain stationary?

Sol. Tension and electrostatic force are in opposite direction and along the string. Now each bead is in equilibrium under three concurrent forces.

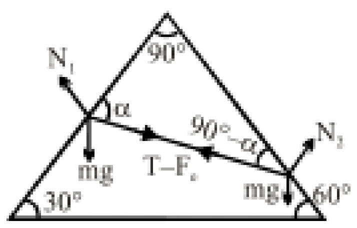
(i) Normal reaction (N)

(ii) Weight (mg) and

(iii) $T - F_e$

where $F_e = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{l^2}$

Applying Lami's theorem for both beads.



$$\frac{N_1}{\sin(120-\alpha)} = \frac{mg}{\cos\alpha} = \frac{T - F_e}{\cos 60^\circ} \qquad \text{..... (i)}$$

$$\frac{N_2}{\sin(60°+\alpha)} = \frac{mg}{\sin\alpha} = \frac{T-F_e}{\cos 30°} \quad ......(ii)$$

Dividing equation (i) by (ii), we have

$$\tan\alpha = \frac{\cos 30°}{\cos 60°} = \sqrt{3} \qquad \therefore \alpha = 60°$$

$$T = F_e + mg = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_1q_2}{l^2} + mg \quad ....(iii)$$

$N_1 = \sqrt{3}\,mg$ and $N_2 = mg$

From Eq. (iii) $T = 0$ when string is cut

or $\quad q_1q_2 = -(4\pi\varepsilon_0)mg\,l^2$

Q.3 A positive charge Q is distributed uniformly over the circumference of a thin circular ring of radius a metres. Calculate the force on a positive point charge q

(i) if it was placed at a distance x from the centre of the ring along its axis.

(ii) if it was placed at the centre of ring.

(iii) At what point on the axis will the force be maximum.

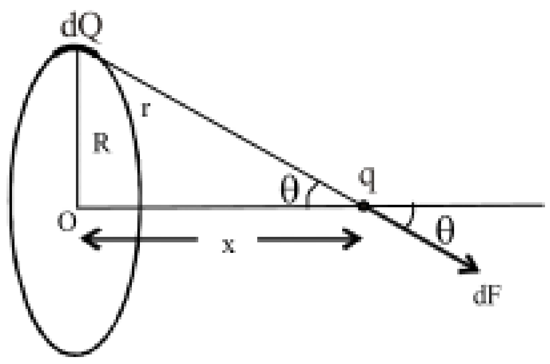
(iv) Draw a qualitative graph of force v/s the distance of the charge from the centre of the ring.

(v) Calculate the time period of that S.H.M for small oscillation if q is replaced by – q and ring is fixed.

Sol. (i) Take an elementry charge dQ on the ring

Force due to this elementary charge on the point charge will be $dF = \frac{qdQ}{4\pi\epsilon_0 r^2}$

This force can be resolved into two rectangular components one along the axis (dF cos θ) and other along perpendicular to the axis (dF sin θ). If we take another elementary charge diametrically opposite to dQ we observe that the vector sum of dF sin θ for these two charge become zero. Hence vector sum of dF sin θ for all the charges result to zero. So the resultant force on the charged particle will be only sum of dFcos θ. Therefore total force on the point charge due to entire ring will be given by



$$F = \int dF\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{\ell^3}\int dq = -\frac{1}{4\pi\varepsilon_0}\frac{Qqx}{(R^2+x^2)^{3/2}}$$

(ii) for centre $x = 0 \qquad \Rightarrow F = 0$

(iii) $F = \frac{Qq}{4\pi\epsilon_0}\frac{x}{(R^2+x^2)^{3/2}}$

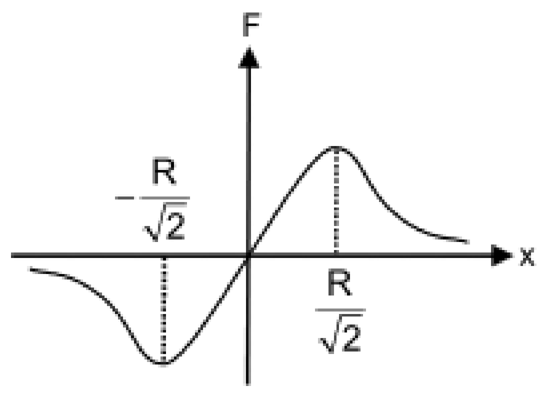
$$\frac{dF}{dx} = \frac{Qq}{4\pi\epsilon_0}\left[\frac{(R^2+x^2)^{3/2}\frac{d}{dx}x - x\frac{d}{dx}(R^2+x^2)^{3/2}}{\{(R^2+x^2)^{3/2}\}^2}\right] = \frac{Qq}{4\pi\epsilon_0}\left[\frac{(R^2+x^2)^{3/2}(1) - x\frac{3}{2}(R^2+x^2)^{1/2}(2x)}{(R^2+x^2)^3}\right]$$

$$= \left(\frac{Qq\sqrt{R^2 + x^2}}{4\pi \in_0} \right) (R^2 - 2x^2)$$

For force to be maximum or minimum

$$\frac{dF}{dx} = 0 \Rightarrow R^2 - 2x^2 = 0 \ \Rightarrow x = \pm \frac{R}{\sqrt{2}}$$

(iv)



(v) $$F = \frac{Q(-q)x}{4\pi \in_0 (R^2 + x^2)^{3/2}}$$

if $x << R$ then

$$F = -\frac{Qqx}{4\pi\varepsilon_0 R^3}$$

Now $$ma = -\frac{Qqx}{4\pi\varepsilon_0 R^3}$$

$$\frac{d^2x}{dt^2} = -\frac{Qq}{4\pi\varepsilon_0 R^3 m} x = -kx$$

$\therefore$ $$k = \frac{Qq}{4\pi\varepsilon_0 R^3 m}$$

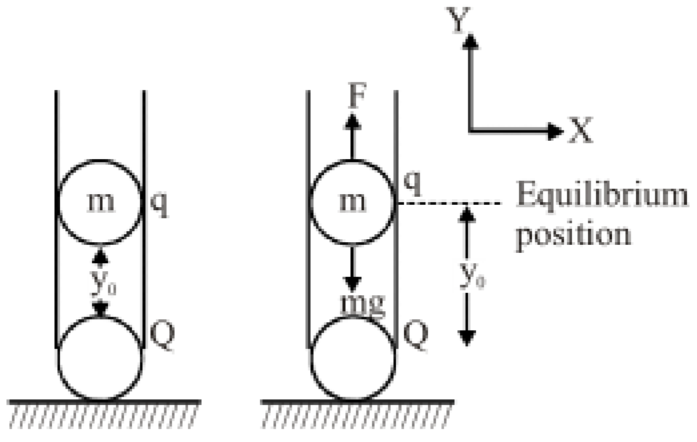
$$T = 2\pi\sqrt{\frac{4\pi\varepsilon_0 m R^3}{qQ}}$$

Q.4 A small point mass m has a charge q, which is constrained to move inside a narrow frictionless cylinder. At the base of the cylinder is a point mass of charge Q having the same sign as q. Show that if he mass m is displaced by a small amount from its equilibrium position and released, it will exhibit simple harmonic motion with angular frequency $\omega = (2g/y_0)^{1/2}$ where y_0 is the equilibrium position of charge q.

Sol. In equilibrium position, gravitational force is balanced by coulombic repulsive force

$$mg = \frac{Qq}{4\pi\varepsilon_0 y_0^2}$$

If charge q is displaced in positive y-direction, such that $y << y_0$ from Newton's second law,



$$\frac{Qq}{4\pi\varepsilon_0(y_0+y)^2}-mg=ma$$

$$\frac{Qq}{4\pi\varepsilon_0 y_0^2}\left[\frac{1}{(1+y/y_0)^2}\right]-mg=ma$$

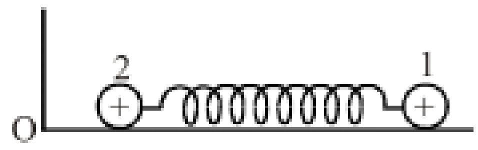
or $$mg\left[1-\frac{2y}{y_0}\right]-mg=ma$$

or $$a=-\frac{2gy}{y_0}$$

$$\frac{d^2y}{dt^2}+\frac{2g}{y_0}=0$$

Which is equation for SHM with $\omega=\sqrt{\frac{2g}{y_0}}$

Q.5 Two small identical balls lying on a horizontal plane are connected by a weightless spring. One ball (ball 2) is fixed at O and the other (ball 1) is free. The balls are changed identically as a result of which the spring length increases η =2 times Determine the change in frequency.



Sol. When the balls are uncharged, $\upsilon_0=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where k is force constant of the spring and m = mass of the oscillating ball (ball 1). When charged we have in the equilibrium position of ball 1,

$$\frac{1}{4\pi\varepsilon_0(\eta l)^2}=k\,(nl-l)=kl\,(\eta-1)$$

$$\Rightarrow \quad l^3=\frac{q^2}{4\pi\varepsilon_0\eta^2(\eta-1)k}$$

When the ball 1 is displaced by a small distance from the equilibrium position to the right, the unbalanced force the right is given by
Resultant force to the right

$$=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(\eta l-x)^2}-k(\eta l+x-l)$$

From Newton's law, we have

$$m\frac{d^2x}{dt^2}=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2 l^2}\left(1+\frac{x}{\eta l}\right)^{-2}-kl\,(\eta-1)-kx$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2 l^2}\left(1-\frac{2x}{\eta l}\right)-kl\,(\eta-1)-kx$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2 l^2} - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2 l^2}\cdot\frac{2x}{\eta l} - kl(\eta-1) - kx$$

$$= -\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{2q^2}{\eta^3 l^3} + k\right)x$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{\eta^3\dfrac{q^2}{4\pi\varepsilon_0\eta^2(\eta-1)k}} + k\right)x$$

$$m\frac{d^2x}{dt^2} = -\left(\frac{2(\eta-1)}{\eta}k + k\right)x = \frac{3\eta-2}{\eta}kx$$

$$\frac{d^2x}{dt^2} = -\frac{3\eta-2}{\eta}\frac{k}{m}x$$

or $\qquad \omega^2 = \dfrac{3\eta-2}{\eta}\dfrac{k}{m}$

$\Rightarrow \qquad \upsilon = \dfrac{1}{2\pi}\sqrt{\dfrac{3\eta-2}{\eta}\dfrac{k}{m}}$

or $\qquad \dfrac{\upsilon}{\upsilon_0} = \sqrt{\dfrac{3\eta-2}{\eta}}$

Thus the frequency is increased $\sqrt{\dfrac{3\eta-2}{\eta}}$ times.

Hence h = 2 and so frequency increases $\sqrt{2}$ times.

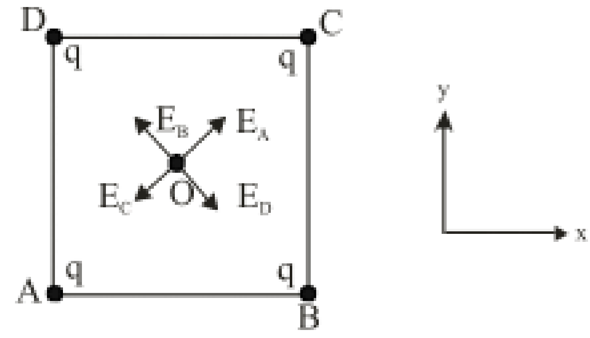
Q.6 Four point charges (each having charge q) are placed at corners of a square of side a. Find the intensity of the field at the (a) Centre of the square (b) Mid point of any side.

Sol. (a) Let position of charges be A, B, C and D and O is centre of square

Let field at O due to A, B, C, D respectively be $\vec{E}_A$, $\vec{E}_B$, $\vec{E}_C$ and $\vec{E}_D$. At O magnitude of field due to each point charge are same as OA = OB = OC = OD thus

$$E_A = E_B = E_C = E_D = \frac{1}{4\pi\varepsilon_0}\frac{q}{OA^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(a/\sqrt{2}\right)^2} = \frac{1}{2\pi\varepsilon_0}\frac{q}{a^2}$$

$$\vec{E}_A = \frac{1}{2\pi\varepsilon_0}\frac{q}{a^2}\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right)$$



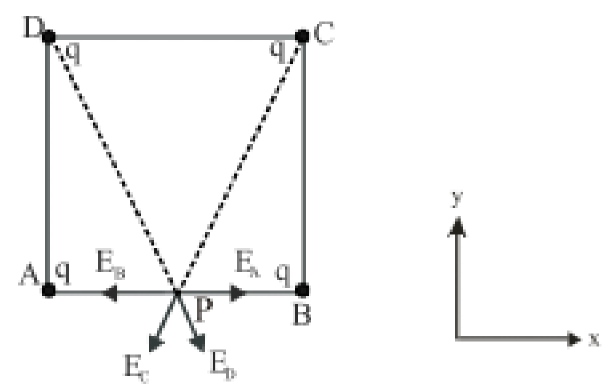
$$\vec{E}_B = \frac{1}{2\pi\varepsilon_0}\frac{q}{a^2}\left(\frac{-\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right)$$

$$\vec{E}_C = \frac{1}{2\pi\varepsilon_0}\frac{q}{a^2}\left(\frac{-\hat{i}}{\sqrt{2}}-\frac{\hat{j}}{\sqrt{2}}\right)$$

$$\vec{E}_D = \frac{1}{2\pi\varepsilon_0}\frac{q}{a^2}\left(\frac{\hat{i}}{\sqrt{2}}-\frac{\hat{j}}{\sqrt{2}}\right)$$

$$\vec{E} = \vec{E}_A + \vec{E}_B + \vec{E}_C + \vec{E}_D \quad = 0$$

(b) Let concerned point be P



$$\left|\vec{E}_A\right| = \left|\vec{E}_B\right| = \frac{1}{4\pi\in_0}\frac{q}{(AP)^2}$$

$$\vec{E}_A = \frac{q}{\pi\in_0 a^2}\hat{i} \qquad \vec{E}_B = \frac{q}{\pi\in_0 a^2}\left(-\hat{i}\right)$$

$$|\vec{E}_D| = |\vec{E}_C| = \frac{1}{4\pi\in_0}\frac{q}{(DP)^2} = \frac{1}{4\pi\in_0}\frac{q}{\left(\frac{\sqrt{5}}{2}a\right)^2} = \frac{q}{\pi\in_0 5a^2}$$

$$\vec{E}_C = \frac{q}{\pi\in_0 5a^2}\left[\frac{-2}{\sqrt{5}}\hat{i}-\left(\frac{1}{\sqrt{5}}\right)\hat{j}\right]$$

$$\vec{E}_D = \frac{q}{\pi\in_0 5a^2}\left[\frac{2}{\sqrt{5}}\hat{i}-\left(\frac{1}{\sqrt{5}}\right)\hat{j}\right]$$

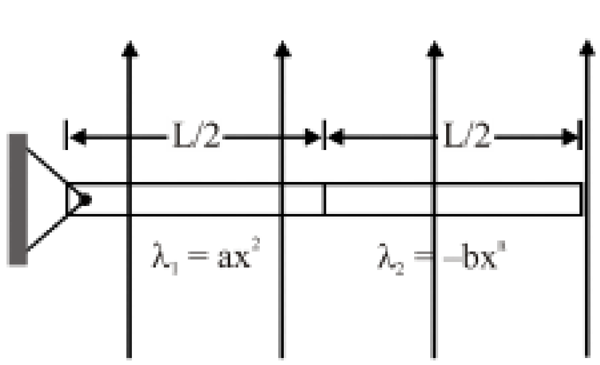
$$\Rightarrow \quad \vec{E} = \vec{E}_A + \vec{E}_B + \vec{E}_C + \vec{E}_D = -\frac{q}{\pi\varepsilon_0 5a^2\sqrt{5}}(\hat{j})$$

Q.7 A thin insulating rod is hinged about one of its ends; it can rotate on a smooth surface in a horizontal plane. The charge density on the rod is defined as

$$\lambda_1 = ax^2, \qquad 0 < x \le \frac{L}{2}$$

$$\lambda_2 = -bx^n, \qquad \frac{L}{2} \le x < L$$

where a and b are positive constants. An electric field E_0, in the horizontal plane and perpendicular to the rod is switched on. Find the value of b and n, if the rod has to remain stationary.



Sol. According to given charge distribution the rod is positively charge in $0 < x \le L/2$ and negatively charge

in $\frac{L}{2} \le x \le L$. The electric force acting on the rod produce torque about hinge; for equilibrium ne torque should zero.

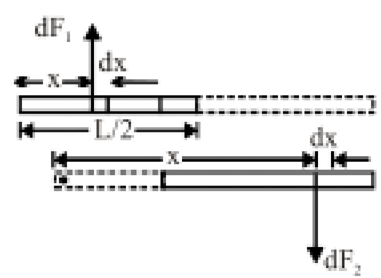
Torque on differential element of left half of the rod.

$$d\tau_1 = E_0\,(\lambda_1\,dx)\,x$$

$$\tau_1 = \int_0^{l/2} E_0(ax^2dx)\,x$$

$$= \frac{E_0aL^4}{2^6}$$

Torque on the right half of the rod.



$$d\tau_2 = -E_0\,(\lambda_2\,dx)\,x$$

$$\tau_2 = -\int_{L/2}^{L} E_0(bx^n dx)\,x$$

$$= \frac{E_0b}{n+2}\,\frac{2^{n+2}-1}{2^{n+2}}\;L^{n+2}$$

According to condition of the problem,

$$|\tau_1| = |\tau_2|$$

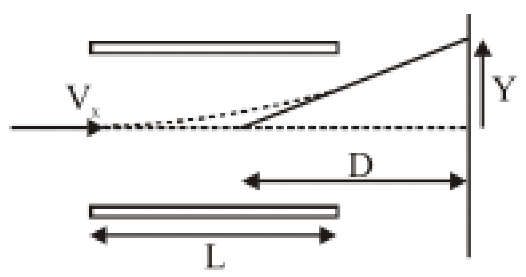
$$\frac{E_0aL^4}{2^6} = \frac{E_0b}{(n+2)}\left(\frac{2^{n+2}-1}{2^{n+2}}\right)L^{n+2}$$

On comparing coefficient of L, we get $n+2=4$ or $n=2$ and

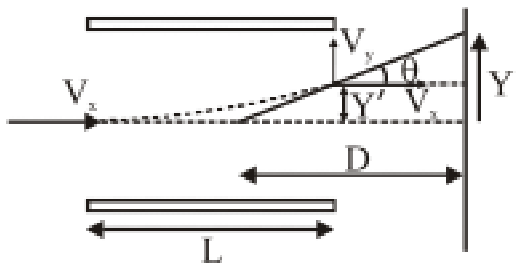
$$\frac{a}{2^6} = \frac{b}{(n+2)}\left(\frac{2^{n+1}-1}{2^{n+2}}\right)$$

or $\quad b = \frac{a}{15}$

Q.8 In an electron ray tube, an electron enters an electric field E between the two plates with a velocity V_x as shown in figure and emerges from the field with a velocity V so as to strike the screen. The separation of the screen from the centre of the plate is D and length of the plate is L. If the charge of the electron is e, then the deflection Y on the screen is (m is the mass of electron)



Sol.



$$V_y = \frac{eE}{m}t = \frac{eE}{m}\,\frac{L}{V_x}$$

$$\tan\theta = \frac{V_y}{V_x} = \frac{eE}{m}\frac{L}{V_x^2} \qquad \ldots(i)$$

Again,

$$\tan\theta = \frac{Y - Y'}{D - L/2}$$

or
$$\tan\theta = \frac{Y - \frac{1}{2}\left(\frac{eE}{m}\right)\left(\frac{L}{V_x}\right)^2}{D - L/2} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{eE}{m}\frac{L}{V_x^2} = \frac{Y - \frac{1}{2}\frac{eE}{m}\frac{L^2}{V_x^2}}{D - L/2}$$

or
$$Y = \frac{eELD}{mV_x^2}.$$

Q.9 An electron of mass m_e initially at rest moves through a certain distance in a uniform electric field in time t_1. A proton of mass m_p also initially at rest takes time t_2 to move through an equal distance in this uniform electric field. Neglecting the effect of gravity find the ratio of $\frac{t_2}{t_1}$.

Sol. Force on a charge particle in a uniform electric field

$$F = qE$$

The acceleration imparted to the particle is

$$a = \frac{qE}{m}$$

The distance traveled by the particle in time t is

$$d = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{qE}{m}\right)t^2$$

For the given problem

$$\frac{t_p^2}{m_p} = \frac{t_e^2}{m_e}$$

$$\frac{t_p^2}{t_e^2} = \frac{m_p}{m_e}$$

$$\Rightarrow \quad \frac{t_p}{t_e} = \sqrt{\frac{m_p}{m_e}}$$

Q.10

A particle of charge q and mass m moves rectilinearly under the action of electric field $E = A - Bx$, where B is positive constant and x is distance from the point where particle was initially at rest then find the distance traveled by the particle before coming to rest first time and acceleration of particle at that moment

Sol.

$$F = qE = q(A - Bx)$$
$$ma = q(A\text{-}Bx)$$
$$a = \frac{q}{m}(A\text{-}Bx) \qquad ...(1)$$
$$\frac{vdv}{dx} = q(A - Bx)$$
$$vdv = \frac{q}{m}(A - Bx)dx$$
$$\int_0^0 vdv = \frac{q}{m}\int_0^x (A - Bx)dx$$
$$Ax - \frac{Bx^2}{2} = 0$$
$$x = 0, x = \frac{2A}{B} \qquad ...(2)$$

From eq. (1) and (2)

$$\frac{q}{m}(A - Bx) = \frac{q}{m}\left(A - B \times \frac{2A}{B}\right)$$
$$= \frac{q}{m}(A - 2A) = \frac{-qA}{m}.$$

Q.11 On a long smooth horizontal plane, over which a horizontal electric field E exists, a charged ball with mass m and charge q is dropped from a height h over the plane. Coefficient of restitution for collision between plane and ball is e. Find the ratio of maximum height attained and horizontal distance moved during the interval of nth collision to (n+l)th collision.

Sol. Time taken to first collision drop.

$$t_1 = \sqrt{\frac{2h}{g}}$$

Velocity on hitting

$$V = \sqrt{2gh}$$

Vertical velocity upward after 1st and 2nd collisions,

$$\Delta t_1 = \frac{2V_1}{g} = \frac{2eV}{g}$$

Vertical velocity upward after 2nd and 3rd collisions,

$$V_2 = eV_1 = e^2V$$

Time gap between 2nd and 3rd collisions,

$$\Delta t_2 = \frac{2V_1}{g} = \frac{2eV}{g}$$

Vertical velocity after nth collision,

$$V_n = e^n V$$

Maximum height attained during time gap between nth to (n + 1)th collision,

$$h = \frac{(e^n V)^2}{2g}$$

$$\Delta t_n = \frac{2e^n V}{g}$$

During these collisions, the ball is moving with acceleration $\frac{qE}{m}$ horizontally.

Distance moved from nth collisions T_n to (n+1)th
Collision T_{n+1},

$$x = \frac{1}{2}a\left(T_{n+1}^2 - T_n^2\right) = \frac{1}{2}\left(T_{n+1}^2 + T_n^2\right)\left(T_{n+1}^2 - T_n^2\right)$$

$$T_n = t_1 + (\Delta t_1 + \Delta t_2 + ...\Delta t_{n+1})$$

$$= \sqrt{\frac{2h}{g}} + \frac{2V}{g}(e + e^2 + ...e^{n-1})$$

$$= \sqrt{\frac{2h}{g}} + \frac{2V}{g}e\left(\frac{1-e^{n-1}}{1-e}\right)$$

$$T_{n+1} = t_1 + (\Delta t_1 + \Delta t_2 + ...\Delta t_n)$$

$$= \sqrt{\frac{2h}{g}} + \frac{2V}{g}e\left(\frac{1-e^n}{1-e}\right)$$

$$\Rightarrow \qquad x = \frac{1}{2}\frac{eE}{m}\left[2\sqrt{\frac{2h}{g}} + \frac{2V}{g}\frac{e}{1-e}(2-e^n-e^{n-1})\right] \times \left[\frac{2V}{g}\frac{e}{1-e}(e^{n-1}-e^n)\right]$$

$$\frac{h}{x} = \frac{e^{2n}V^2}{2g}\frac{2m}{eE\left[2\sqrt{\frac{2h}{g}} + \frac{2V}{g}\frac{e}{1-e}(2-e^n-e^{n-1})\right]} \cdot \frac{g}{2Ve^n}$$

$$= \frac{mVe^{n-1}}{2E\left[2\sqrt{\frac{2h}{g}} + \frac{2V}{g}\frac{e}{1-e}(2-e^n-e^{n-1})\right]}$$

$$= \frac{mge^{n-1}}{4E(1+e-e^n-e^{n+1})}$$

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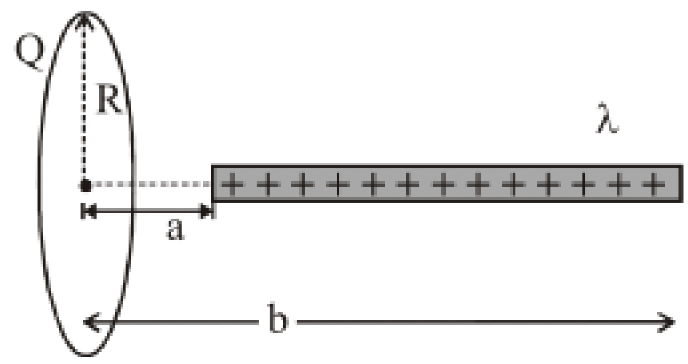
Q.12 Find the interaction force between ring of uniform charge Q and rod having uniform linear charge density λ.



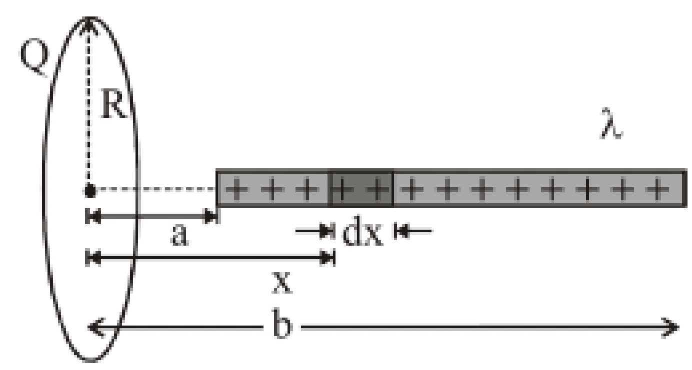
Sol.

Let us take an elementry charge on the line charge of length dx and at distance x from centre of ring. charge of elementry part will be

$$dq = \lambda\, dx$$

Electric field due to ring at the location of elementry charge will be

$$E = \frac{kQx}{(R^2 + x^2)^{3/2}}$$



Force due to ring on elementry charge will be

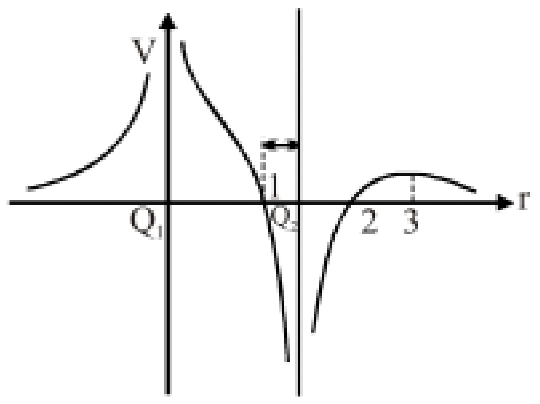
$$dF = Edq = \frac{kQx\ \lambda dx}{(R^2 + x^2)^{3/2}}$$

Therefore the total force on the rod due to ring will be

$$F = \int_a^b \frac{kQx\ \lambda dx}{\left[R^2 + x^2\right]^{3/2}}$$

$$= kQ\lambda\left[\frac{1}{\sqrt{R^2 + a^2}} - \frac{1}{\sqrt{R^2 + b^2}}\right]$$

Q.13 Two point charge Q_1 and Q_2 lie along a line, at a distance from each other. Figure shows the potential variation along the line of charge. At which points 1, 2 and 3 is the electric field zero ? What are the sings of the charges Q_1 and Q_2 and which of the two charges is greater in magnitude?



Sol. The electric field vector is zero at point 3. As $-\dfrac{dV}{dr} = E_r$, the negative of slop of V vs r curve represents component of electric field along r. Slope of curve is zero only at 3.

Near positive charge net potential is positive and negative near a negative charge. Thus charge Q_1 is positive and Q_2 negative. From the graph it can be seen that net potential due to two charge is positive in the region left of charge Q_1 is greater than due to Q_2. Therefore absolute value of charge Q_1 is greater

than due to Q_2. Secondly, the point 1 where potential due to two charge is zero, is nearer to charge Q_2 thereby implying the Q_1 has greater absolute value.

Q.14 In the figure shown find the flux of electric field on curve surface. Assume that electric field is uniform

Sol. As the field is uniform ne flux on the total surface is zero.

$\Rightarrow \quad \phi_{\text{curved surface}} + \phi_{\text{Plane surface}} = 0$

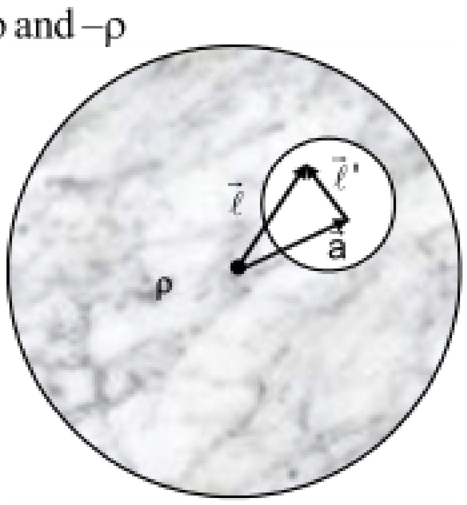
$\Rightarrow \quad \phi_{\text{curved surface}} = -\phi_{\text{Plane surface}} = -E\,\pi r^2 \cos 180^\circ = E\,\pi r^2$

Q.15 A sphere of radius R is made of a nonconducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius ($r<R$) is now removed from the sphere such that centre of the cavity is at position $\vec{a}$ with respect to the centre of sphere. Show that the electric field within the cavity is uniform and is given by $\vec{E} = \dfrac{\rho}{3\varepsilon_0}\vec{a}$

Sol. The cavity can be considered as superposition of two charge density $+\rho$ and $-\rho$

Electric field due to positive charge

$$\vec{E}_+ = \frac{\rho}{3\epsilon_0}\vec{\ell}$$

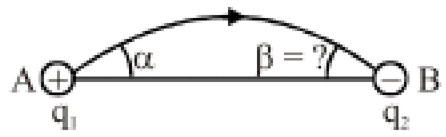


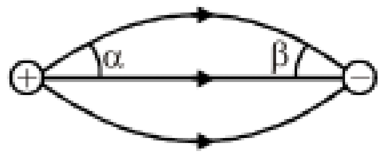
Electric field due to negative charge

$$\vec{E}_- = \frac{(-\rho)}{3\epsilon_0}\vec{\ell}'$$

$$\therefore \vec{E}_{net} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\epsilon_0}\vec{\ell} + \frac{(-\rho)}{3\epsilon_0}\vec{\ell}' = \frac{\rho}{3\epsilon_0}(\vec{\ell}-\vec{\ell}') = \frac{\rho}{3\epsilon_0}\vec{a}$$

Q.16 Two charge $+q_1$ and q_2 are placed at A and B respectively. A line of force emanates from q_1 at angle α with the line AB. At what angle will it terminate at $-q_2$?





Sol. It is the property of the line of force that their number within a tube remains unchanged and the number of line of force is equal to the charge. The line of force emanating from q_1 spreads out equally in all directions. Hence lines of force per unit solid angle are $\dfrac{q_1}{4\pi}$ and the number of lines through cone of half-angle α is $\dfrac{q_1}{4\pi}.2\pi(1-\cos\alpha)$ because solid angle of a cone is $2\pi(1-\cos\alpha)$.

Similarly the number of line of force terminating on $-q_2$ at β is

$$\frac{q_2}{4\pi}\cdot 2\pi(1-\cos\beta)$$

By the property of lines of force.

$$\frac{q_1}{4\pi}\cdot 2\pi(1-\cos\alpha) = \frac{q_2}{4\pi}\cdot 2\pi(1-\cos\beta)$$

$$\Rightarrow \qquad \frac{q_1}{2}\cdot 2\sin^2\frac{\alpha}{2} = \frac{q_2}{2}\cdot 2\sin^2\frac{\beta}{2}$$

$$\Rightarrow \qquad \sin\frac{\beta}{2} = 2\sin\frac{\alpha}{2}\sqrt{\frac{q_1}{q_2}}$$

$$\Rightarrow \qquad \beta = 2\sin^{-1}\left(\sin\frac{\alpha}{2}\sqrt{\frac{q_1}{q_2}}\right)$$

Q.17 Two particles of mass m and 2m carry a charge q each. Initially the heavier particle is at rest on a smooth horizontal plane and the other is projected along the plane directly towards the first from a distance d with speed u. Find the closest distance of approach.

Sol. As the mass 2m is not fixed, it will also move away from m due to repulsion. The distance between the particles is minimum when their relative velocity is zero i.e., when they have equal velocities.

Hence at closest approach, $\quad v_1 = v_2$

By conservation of momentum

$$mu = mv_1 + 2mv_2$$

$$v_2 = v_1 = u/3$$

By conservation of energy

Loss in KE = gain in PE

$$\frac{1}{2}mu^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}2mv_2^2\right) = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{x} - \frac{1}{d}\right)$$

$$\frac{1}{2}mu^2 - \frac{1}{2}m\frac{u^2}{9}(1+2) = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{x} - \frac{1}{d}\right)$$

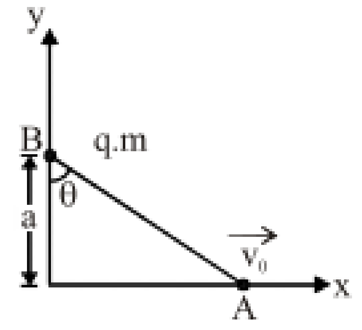
$$\frac{1}{3}mu^2 = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{x} - \frac{1}{d}\right)$$

$$\frac{1}{x} = \frac{1}{d} + \frac{4\pi\varepsilon_0 mu^2}{3q^2}$$

$$x = \frac{3q^2 d}{3q^2 + 4\pi\varepsilon_0 mu^2 d}.$$

Q.18 A charged particle having a charge q moves along the x-axis with a constant velocity υ_0. Another particle B with charge q and mass m is lying on thy y-axis at y = a. The particle B is constrained to move along the y-axis. While the particle A moves along the x-axis.

Assuming that the velocity υ_0 is very large, find the impulse imparted to B along, the y-axis as the particle A moves from $-\infty$ to ∞, assuming that the motion of particle B is negligible.



Sol. First we will determine the angular speed a A relative to B at angular position θ shown in Fig. For particle A,

$$\frac{dx}{dt} = \frac{d}{dt}(a t \tan\theta) = \upsilon_0 \qquad ...(i)$$

or $$a \sin^2\theta \frac{d\theta}{dt} = \upsilon_0$$

or $$\left(\frac{d\theta}{dt}\right) = \frac{\upsilon_0}{a}\cos^2\theta \qquad ...(ii)$$

As particle A is moving very fast, we can assume that when it crosses along the x-axis, the y-component of the force on B due to A is

$$F_y = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2\sec^2\theta} \times \cos\theta \qquad ...(iii)$$

Impulse on $$B = \int F_y\, dt = \int \frac{F_y\, d\theta}{(d\theta/dt)}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\frac{a}{\upsilon_0}\int \frac{\cos^3}{\cos^2} d\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a\upsilon_0}\int_{\theta=-/2}^{\pi/2} \cos\theta\, d\theta$$

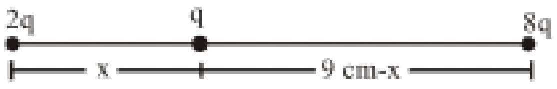
The impulse delivered to B,

$$= \Delta P_y = \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a\upsilon_0} = \frac{q^2}{2\pi\varepsilon_0(a\upsilon_0)} \qquad ...(iv)$$

Q.19 Three point charges q, 2q and 8q are to be placed on a 9 cm long straight line. Find the positions where the charges should be placed such that the potential energy of this system is minimum. In this situation, what is the electric field at the charge q due to the other two charges ?

Sol. The maximum contribution may come from the charge 8q forming pairs with others. To reduce its effect, it should be placed at a corner and the smallest charge q in the middle. This arrangement shown in figure ensures that the charges in the strongest pair 2q, 8q are at the largest separation.

The potential energy is



$$U = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{2}{x} + \frac{16}{9cm} + \frac{8}{9cm - x}\right].$$

This will be minimum if

$$A = \frac{2}{x} + \frac{8}{9cm - x} \text{ is minimum.}$$

For this, $$\frac{dA}{dx} = -\frac{2}{x^2} + \frac{8}{(9cm - x)^2} = 0 \qquad .... (i)$$

or, $9\text{ cm} - x = 2x$ or, $x = 3\text{ cm}$

The electric field at the position of charge q is

$$\frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2} - \frac{8}{(9cm - x)^2}\right)$$

$= 0.$ From (i).

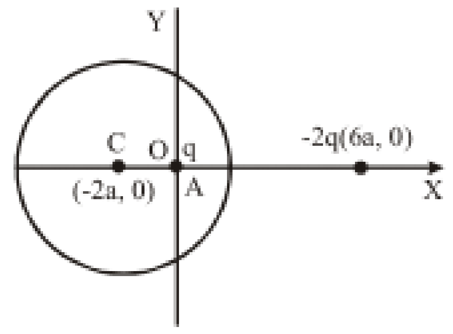
Q.20 Two points charge q and –2q are placed at a distance 6a apart. Find the locus of the point in the plane of charges where the field potential is zero.

Sol. Let us take the charge on X-axis;

q at A (0, 0) and –2q at B(6a, 0)

Potential at a point P(x, y) is

$$V = \frac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2}} + \frac{-2q}{4\pi\varepsilon_0\sqrt{(x-6a)^2+y^2}}$$



$$V = 0$$

$$\Rightarrow \quad \frac{q^2}{x^2+y^2} = \frac{4q^2}{(x-6a)^2+y^2}$$

$\Rightarrow$ the locus is $(x-6a)^2 = 4x^2 + 3y^2$.

$$3x^2 + 3y^2 + 2(6a)x = 36a^2$$

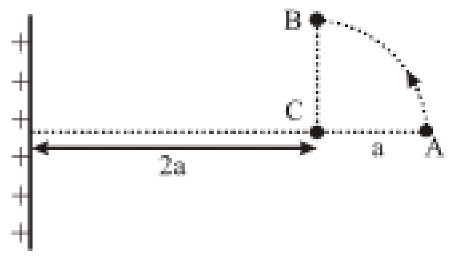
$$\Rightarrow \quad x^2 + y^2 + 4ax = 12a^2$$

$$(x+2a)^2 + y^2 = 16a^2$$

A circle with centre (-2a, 0) and radius 4a.

Q.21 The arc AB with the centre C and the infinitely long wire having linear charge density λ are lying in the same plane. Find the minimum amount of work to be expended to move a point charge q_0 from point A to B through a circular path AB of radius a is equal to :

Sol. $E = \dfrac{\lambda}{2\pi\epsilon_0 x}$

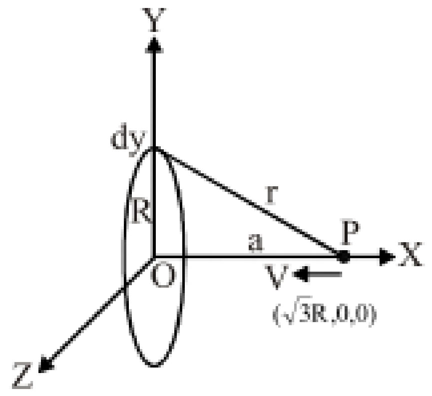


$$\int_{V_A}^{V_B} dV = -E\,dx = -\frac{\lambda}{2\pi\varepsilon_0}\int_{3a}^{2a}\frac{dx}{x}$$

$$\Rightarrow \quad V_B - V_A = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{3}{2}\right)$$

work done by agent $= \dfrac{q_0\lambda}{2\pi\varepsilon_0}\ln\dfrac{3}{2}$.

Q.22 A circular ring of radius R with uniform positive charge density λ per unit length is located in the Y-Z plane with its centre at the origin O. A particle of mass m and positive charge q is projected from the point P (R$\sqrt{3}$, 0, 0) on the positive x-axis directly towards O, whit initial velocity υ. Find the smallest (non-zero) value of the speed υ such that the particle does not returen to P.



Sol. The situation is shown in figure. Total potential at the centre of a ring is given by

$$V = \frac{\lambda R}{2\varepsilon_0\sqrt{(a^2+R^2)}}$$

$$= \frac{\lambda R}{2\varepsilon_0\sqrt{[(\sqrt{3}R)^2+R^2)}} = \frac{\lambda}{4\varepsilon_0}$$

Potential energy at P $= \lambda q/4\varepsilon_0$

Energy of particle at P $= \frac{1}{2}m\upsilon^2$

$\therefore$ Total energy at P $= \frac{\lambda q}{4\varepsilon_0} + \frac{1}{2}m\upsilon^2$

The potential energy at centre $= (\lambda q/2\varepsilon_0)$

The particle will not return to P, when

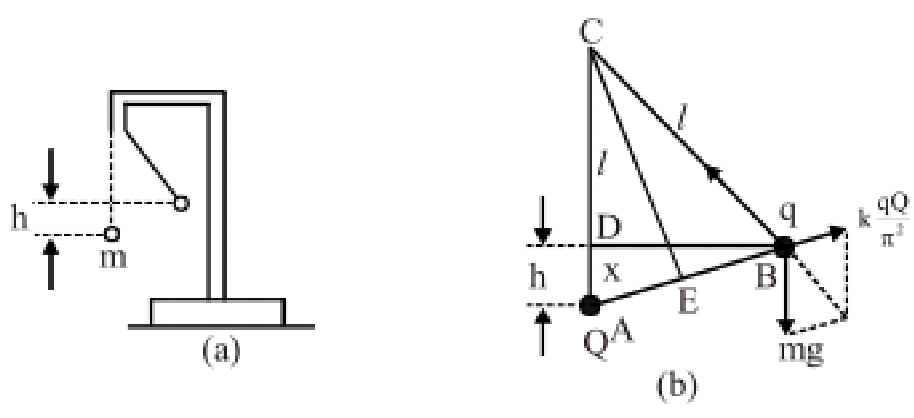
$$\frac{\lambda q}{4\varepsilon_0} + \frac{1}{2}m\upsilon^2 = \frac{\lambda q}{2\varepsilon_0}$$

$$\frac{1}{2}m\upsilon^2 = \frac{\lambda q}{2\varepsilon_0} - \frac{\lambda q}{4\varepsilon_0} = \frac{\lambda q}{4\varepsilon_0}$$

$$\upsilon^2 = \frac{\lambda q}{2\varepsilon_0 m}$$

or $$\upsilon = \sqrt{\left(\frac{\lambda q}{2\varepsilon_0 m}\right)}$$

Q.23 A small positively charge ball of mass m is suspended by an insulating thread of negligible mass. Another positively charged small ball is moved very slowly from a large distance until it is in the original position of the first ball. As a result, the first ball rises by h. How much work has been done?



Sol. Using the notation of figure the equilibrium condition for the first ball is

$$s\ \frac{mg}{F} = \frac{l}{x}$$

Where $F = kqQ/x^2$ is the Coulomb force acting on the first ball and x is the distance between the balls carrying charges q and Q.

It is clear the triangles ABD and CAE are similar and that consequently

$$\frac{x}{2} : l = h : x$$

From the three equations above we can calculate the separation of the charge and the electrostatics energy of the system:

$$x = k\frac{qQ}{2mgh}$$

and $$E_{electro} = k\frac{qQ}{x}$$

$$= 2mgh.$$

The work done is the sum of the changes in electrical and gravitational potential energy.

$$W = 2mgh + mgh$$
$$= 3mgh$$

Note that the work done does not depend on either the magnitudes of the charges or the length of the thread.

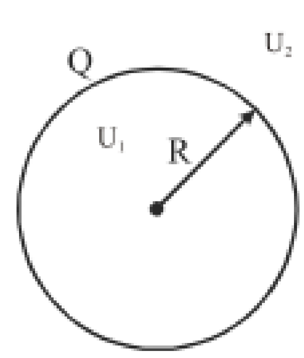
Q.24 A charge Q is uniformly distributed over a spherical volume of radius R. Obtain an expression for the energy of the system.

Sol. In this case, the electric field exists from centre of the sphere to infinity. Potential energy is stored in electric field with energy density.

$$u = \frac{1}{2}\varepsilon_0 E^2 \text{ (energy / volume)}$$

(i) Energy stored within the sphere (V_1)

Energy field at a distance r is ($r \le R$)



$$E = \frac{1}{4\pi\varepsilon_0}.\frac{Q}{R^3}.r$$

$\therefore$ $$u = \frac{1}{2}\varepsilon_0 E^2$$

$$u = \frac{\varepsilon_0}{2}\left\{\frac{1}{4\pi\varepsilon_0}.\frac{Q}{R^3}r\right\}^2$$

Volume of element, $dV = (4\pi r^2)dr$

$\therefore$ Energy stored in this volume, $dU = u(dV)$

$$dU = \left(4\pi r^2 dr\right)\frac{\varepsilon_0}{2}\left\{\frac{1}{4\pi\varepsilon_0}.\frac{Q}{R^3}r\right\}^2$$

$$dU = \frac{1}{8\pi\varepsilon_0}.\frac{Q^2}{R^6}.r^4 dr$$

$$U_1 = \int_0^R dU = \frac{1}{8\pi\varepsilon_0}\frac{Q^2}{R^6}\int_0^R f^4 dr$$

$$= \frac{Q^2}{40\pi\varepsilon_0 R^6}\left[r^5\right]_0^R$$

$$U_1 = \frac{1}{40\pi\varepsilon_0}.\frac{Q^2}{R} \qquad \text{.... (1)}$$

(ii) Energy stored outside the sphere (U_2)

Electric field at a distance r is ($r \ge R$)

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}$$

$$u = \frac{1}{2}\varepsilon_0 E^2$$

$$u = \frac{\varepsilon_0}{2}\left\{\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}\right\}^2$$

$$dV = (4\pi r^2 dr)$$

$$dU = udV = (4\pi r^2 dr)\left[\frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}\right)^2\right]$$

$$dU = \frac{Q^2}{8\pi\varepsilon_0} \cdot \frac{dr}{r^2}$$

$$U_2 = \int_R^\infty dV = \frac{Q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$U_2 = \frac{Q^2}{8\pi\varepsilon_0 R} \qquad \ldots (2)$$

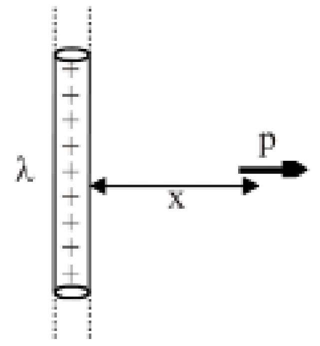
Therefore, total energy of the system is

$$U = U_1 + U_2 = \frac{Q^2}{40\pi\varepsilon_0 R} + \frac{Q^2}{8\pi\varepsilon_0 R}$$

$$U = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 R}.$$

Q.25 A dipole of mass m is placed in front of a line charge at a separation x (as shown in figure). Find (i) interaction energy between point charge and dipole (ii) force acting on the dipole due to line charge(iii) what will be its velocity if it reaches at separation x_0

Sol.



(i) $U = -\vec{p}\cdot\vec{E} = -pE\cos 0^\circ = -p\dfrac{\lambda}{2\pi\epsilon_0 x}(1) = -\dfrac{\lambda p}{2\pi\epsilon_0 x}$

(ii) $F = -\dfrac{dU}{dx} = -\dfrac{\lambda p}{2\pi\epsilon_0 x^2} \Rightarrow \vec{F}\ \dfrac{\lambda p}{2\pi\epsilon_0 x^2}$ directed along -ve x axis

(iii) using conservation of energy

$$K_f + U_f = K_i + U_i$$

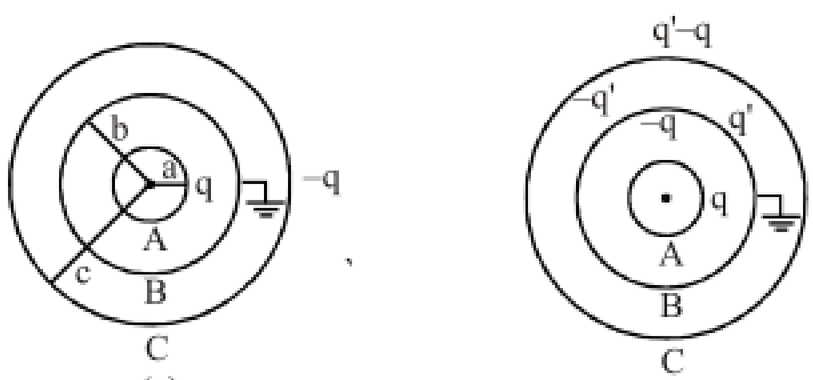
$$\frac{1}{2}mv^2 - \frac{\lambda p}{2\pi\epsilon_0 x_0} = 0 - \frac{\lambda p}{2\pi\epsilon_0 x}$$

$$\frac{1}{2}mv^2 = \frac{\lambda p}{2\pi\epsilon_0}\left(\frac{1}{x_0} - \frac{1}{x}\right)$$

$$v = \sqrt{\frac{2\lambda p}{2\pi\epsilon_0 m}\left(\frac{1}{x_0} - \frac{1}{x}\right)}$$

Q.26

Figure shows three concentric thin spherical shells A, B and C of radii a, b and c repectively. The shells A and C are given charges q and –q respectively and the shell B is earthed. Find the charges appearing on the surface of B and C.



(a) (b)

Sol. As shown in the previous worked out exmple. The inner surface of B must have a charge –q from the Gauss's law. Suppose, the outer surface of B has a charge q'. The inner surface of C must have a charge –q' from the Gauss's law. As the net charge on C must be –q, its outer surface should have a charge q' – q. The charge distribution is shown in figure.

The potential at B due to the charge q on A

$$= \frac{q}{4\pi\varepsilon_0 b},$$

due to the charge – q on the inner surface of B

$$= \frac{-q}{4\pi\varepsilon_0 b}$$

due to the charge q' on the outer surface of B

$$= \frac{q'}{4\pi\varepsilon_0 b}$$

due to the charge – q' on the outer surface of C

$$= \frac{-q'}{4\pi\varepsilon_0 c}$$

and due to the charge q' – q on the outer surface of C

$$= \frac{q'-q}{4\pi\varepsilon_0 c}$$

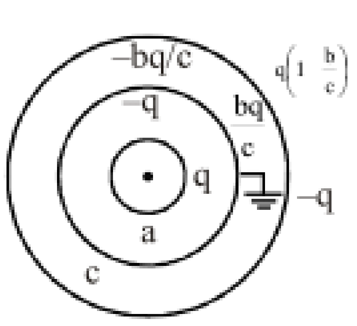
The net potential is

$$V_B = \frac{q'}{4\pi\varepsilon_0 b} - \frac{q}{4\pi\varepsilon_0 c}$$

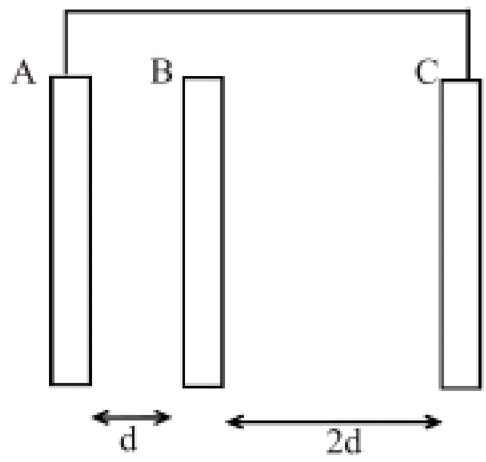
This should be zero as the shell B is eartdhed. Thus,

$$q' = \frac{b}{c} q.$$

The charges on various surface are as shown in figure



Q27 Figure shows three plates A, B and C each of area S in which A and C are joined by a conducting wire. Plate A and B are given charge Q and 3Q respectively. Find the final charge distribution of each side of each conductor.



Sol. Middle plate is isolated. Hence sum of charge on left and right side will be 3Q. But left+right plate forms isolated system and using conservation of charge for this system.

$y+(-x)+\{-(3Q-x)\}+z=0$ ....(i)

but y=z ....(ii) [$\because$ E=0 inside conductor]

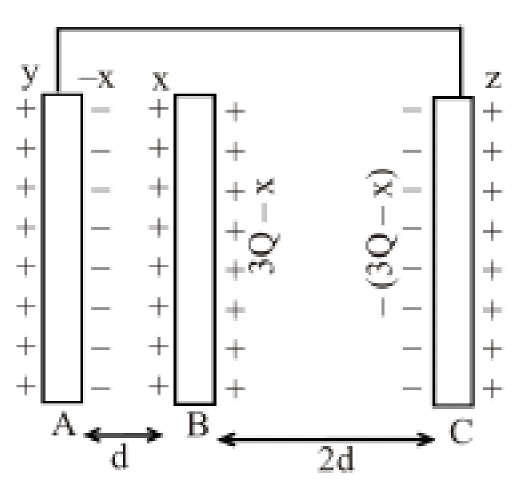
Also left and right plate have same potential. Hence potential difference between A and C will be zero. i.e,

$$\int_A^C \vec{E}\cdot d\vec{r}=0$$

$$\Rightarrow \int_A^B \vec{E}\cdot d\vec{r}+\int_B^C \vec{E}\cdot d\vec{r}=0$$

$$\Rightarrow -E_1 d+E_2(2d)=0$$

$$\Rightarrow -\left\{\frac{x}{2\in_0 S}\times 2\right\}d+\left\{\frac{3Q-x}{2\in_0 S}\times 2\right\}(2d)=0 \quad ....(iii)$$



After solving (i), (ii) and (iii) we get

x=2Q y=z=2Q

GRAVITATION

Introduction

So far we have discussed various forces : pushes and pulls, elastic forces, friction, and other forces that act when one body is in contact with another. In this chapter we study the properties of one particularly important noncontact force, gravitation, which is one of the fundamental and universal forces of nature.

Origin of the Law of Gravitation

From at least the time of the ancient Greeks, two problems were puzzling : (1) the falling of objects released near the Earth's surface, and (2) the motions of the planets. Although there was no reason at that time to connect these two problems, today we recognize that they result from the effect of the same force – gravitation. In fact, this force also determines the motion of the Sun in our Milky Way galaxy, as well as the motion of the galaxy in our Local Cluster of galaxies, the motion of the galaxy in our Local Cluster of galaxies, the motion of the Local Cluster in the Local Supercluster, and so on through the universe. In short, the gravitational force, and the law that describes that force, controls the structure, the developement, and the eventual fate of the universe.

The earliest serious attempt to explain the motions of the planets was due to Claudius Ptolemy (A.D. 2nd century), who developed a model of the solar system in which the planets, including the Sun and Moon, revolved about the Earth. Unfortunately, to explain the complicated orbits of the planets in this geocentric frame of reference, Ptolemy was forced to introduce epicycles, in which a planet moves around a small circle whose center moves around another larger circle centered on the Earth. Of course, today we would reject such a model because it violates the law that every accelerated motion must be accounted for by a force due to a body in its environment - there is no boy at the center of the small circles that would supply the force necessary for the centripetal acceleration.

Famous Indian astronomer and mathematician, Aryabhat, studied motion of earth in great detail, most likely in the 5th century A.D., and wrote his conclusions in his book Aryabhatiy. He established that the earth revolves about its own axis and moves in a circular orbit about the sun, and that the moon moves in a circular orbit about the earth. But these ideas could not be communicated to the world.

It was not until the 16th century that Nicolaus Copernicus (1473-1543) proposed a heliocentric (Sun-centered scheme, in which the Earth and the other planets move about the Sun. Like Ptolemy's model, Copernicus' solar system was still based only on geometry because the notion of a force had not yet between introduced.

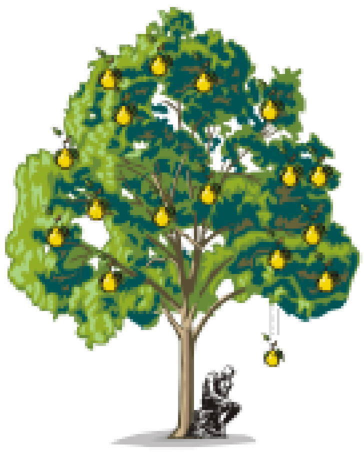
Based on careful analysis of observational data of histeacher tyco brahe (1546-1601) on planetary motions, Johannes Kepler (1571-1630) proposed three laws that describe those motions. However, Kepler's laws were only empirical-they simply described the motions of the planets without any basis in terms of forces. It was a great triumph for the newly developed field of mechanics later in the 17th century when Isaac Newton was able to derive Kepler's laws from his laws of mechanics and his proposed law of gravitation. With this tunning development, Newton was able to use the same concept to account for the motion of the planets and of bodies falling near the Earth's surface.

The year 1665 was very fruitful for Isaac Newton aged 23. He was forced to take rest at his home in Lincolnshire after his college at Cambridge was closed for an indefinite period due to plague. In this year, he performed brilliant theoretical and experimental tasks mainly in the field of mechanics and optics. In this same year he focussed his attention on the moiton of the moon about the earth.

The moon makes a revolution about the earth in T = 27.3 days. The distance of the moon from the earth T = 27.3 days. The distance of the moon from the earth is R = 3.85×10^5 km. The acceleration of the moon is, therefore,

$$a = \omega^2 R = \frac{4\pi^2 \times (3.85 \times 10^5 \,\text{km})}{(27.3\ \text{days})^2} = 0.0027\ \text{m s}^{-2}.$$

The first question before Newton was that what is the force that produces this acceleration. The acceleration is towards the center of the orbit, that is towards the centre of the earth. Hence the force must act towards the centre of the earth. A natural guess was that the earth is attracting the moon. The saying goes that Newton was sitting under an apple tree when an apple fell down from the tree on the earth. This sparked the idea that the earth attracts all bodies towards its centre. The next question was what is the law governing this force.



Newton had to make several daring assumptions which proved to be turning points in science and philosophy. He declared that the law of nature are the same for earthly and celestial bodies. The force operating between the earth and an apple and that operating between the earth and the moon, must be governed by the same laws. This statement may look vary obvious today but in the era before Newton, there was a general belief in the western countries that the earthly bodies are governed by certain rules and the heavenly bodies are governed by different rules. In particular, this heavenly structure was supposed to be so perfect that there could not be any change in the sky. This distincition was so sharp that when Tycho Brahe saw a new star in the sky, he did not believe his eyes as there could be no change in the sky. So the Newton's declaration was indeed revolutionary.

The acceleration of a body falling near the earth's surface is about 9.8 ms^{-2}. Thus,

$$\frac{a_{\text{apple}}}{a_{\text{moon}}} = \frac{9.8\,\text{ms}^{-2}}{0.0027\ \text{ms}^{-2}} = 3600.$$

Also,

$$\frac{\text{distance of the moon from the earth}}{\text{distance of the apple from the earth}}$$

$$= \frac{d_{\text{moon}}}{d_{\text{apple}}} = \frac{3.85 \times 10^5\ \text{km}}{6400\,\text{km}} = 60$$

Thus, $$\frac{a_{\text{apple}}}{a_{\text{moon}}} = \left(\frac{d_{\text{moon}}}{d_{\text{apple}}}\right)^2.$$

Newton guessed that the acceleration of a body towards the earth is inversely proportional to the square of the distance of the body from the centre of the earth.

Thus, $a \propto \frac{1}{r^2}$

Also, the force is mass times acceleration and so it is proportional to the mass of the body. Hence,

$$F \propto \frac{m}{r^2}$$

By the third law of motion, the force on a body due to the earth must be equal to the force on the earth due to the body. Therefore, this force should also be proportional to the mass of the earth. Thus, the force between the earth and a body is

$$F \propto \frac{Mm}{r^2} \text{ or } F = \frac{GMm}{r^2}$$

Newton further generalised the law by saying that not only the earth but all material bodies in the universe attract each other.

Here G, called the gravitational constant, has the experimentally determined value

$G = 6.67 \times 10^{-11}$ N.m²/kg²

G is a universal constant, with the same value for any pair of particles at any location in the universe.

Note :

1. Gravitation, the force that acts between bodies due only to their masses, is one of the four basic forces of physics. It acts throughout the universe : between bodies on Earth, where it is weak and difficult to measure ; between the Earth and bodies in its vicinity, where it is the controlling feature of our lives ; and among the stars and galaxies, where it controls their evolution and structure.

2. Normally, however, it is only when the mass of at least one of the interacting bodies is large (planet-sized) that the effects of the gravitational force become significant.

3. In this argument, the distance of the apple from the earth is taken to be equal to the radiusof the earth. This means we have assumed that earth can be treated as a single particle placed at its centre. This is of course not obvious. Newton had spent several years to prove that indeed this can be done. A spherically symmetric body can be replaced by a point particle of equal mass placed at its centre for the purpose of calculating gravitational force.

Characteristics of The Gravitational force :

(a) Gravitational force is always attractive and directed along the line joining the particles.

(b) It is independent of the nature of the medium surrounding the particles.

(c) It holds good for long distances like inter-planetary distances and also short distances like inter-atomic distances.

(d) Interaction means that, both the particles experience force of equal magnitude in opposite directions. If $\vec{F}_1$, $\vec{F}_2$ are the forces acting on particle 1 by particle 2 and particle 2 by particle 1 respectively, then $\vec{F}_1 = -\vec{F}_2$. Since the forces $\vec{F}_1$ and $\vec{F}_2$ are exerted on different bodies, they are known as action-rection pair.

(e) It is a conservative force. Therefore the work done by the gravitational force on a particle is independent of the path described by the particle. It depends upon the initial and final position of the particle. Therefore no work is done by the gravity if a particle moves in a closed path.

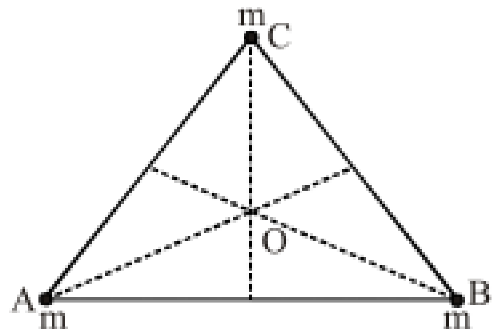
(f) If a particle is acted by n particles, say, the net force $\vec{F}$ exerted on it must be equal the vector sum of the forces due to surrounding particles.

$$\Rightarrow \quad \vec{F} = \sum_{i=1}^{i=n} \vec{F}_i$$

where $\vec{F}_i$ = force acted on the particle , by the i^{th} particle.

Illustration :

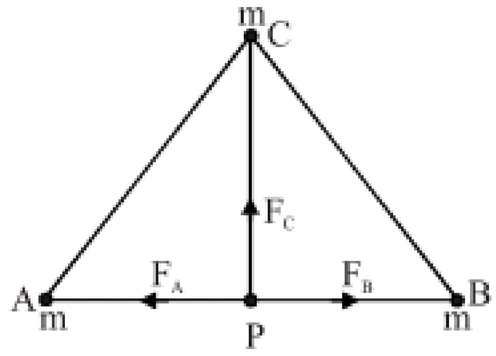
Three identical particles each of mass m are placed at the vertices of an equilateral triangle of side a. Find the force exerted by this system on a particle P of mass m placed at the



(a) the mid point of a side *(b) centre of the triangle*

Sol. *Using the superposition principle, the net gravitational force on P is* $\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C$

(a) *As shown in the figure, when P is at the mid point of a side,* $\vec{F}_A$ *and* $\vec{F}_B$ *will be equal in magnitude but opposite in direction. Hence they will cancel each other. So the net force on the particle P will be the force due to the particle placed at C only.*



$$\Rightarrow \quad F = F_C = G\frac{m\,.\,m}{(CP)^2} = G\frac{m^2}{(a\sin 60)^2} = \frac{4Gm^2}{3a^2} \text{ along PC.}$$

(b) *At the centre of triangle O, the forces* $\vec{F}_A$, $\vec{F}_B$ *and* $\vec{F}_C$ *will be equal in magnitude and will subtend* $120°$ *with each other. Hence the resultant force on P at O is* $\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C = 0$.

Illustration :

Two balls of mass m each are hung side by side by side by two long threads of equal length l. If the distance between upper ends is r, show that the distance r' between the centres of the ball is given by $g\,r^2\,(r - r') = 2\,l\,G\,m$

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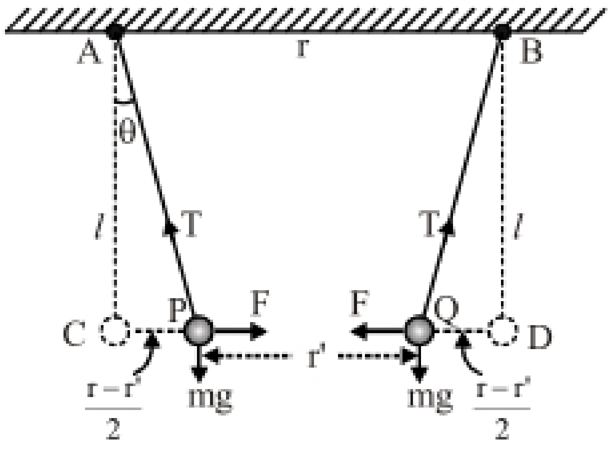
Sol. *The situation is shown in figure*



Following force act on each ball

(i) Weight of the ball m g in downward direction

(ii) Tension in thread T along string

(iii) Force of gravitation attraction towards each other

$$F = G\frac{m\,m}{r'^2}$$

Here for equilibrium of balls we have

$$T\sin\theta = \frac{Gm^2}{r'^2} \qquad ...(i)$$

$$T\cos\theta = mg \qquad ...(ii)$$

Dividing equation (i) and (ii), we get

or $$\tan\theta = \frac{Gm^2}{mgr'^2} \qquad ...(iii)$$

In ΔACP $$\tan\theta = \frac{r-r'}{2l} \qquad ...(iv)$$

From equation (iii) and (iv)

$$\frac{r-r'}{2l} = \frac{Gm^2}{mgr^2}$$

$$g\,r^2\,(r-r') = 2\,l\,G\,m$$

Practice Exercise

Q.1 Four particles of equal masses M move along a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle.

Q.2 In a double star, two stars (one of mass m and the other of 2 m) distance d apart rotate about their common centre of mass. Deduce an expression for the period of revolution. Show that the ratio of their angular momenta about the centre of mass in the same as the ratio of their kinetic energies.

Answers

Q.1 $\sqrt{\frac{GM}{R}\left(\frac{2\sqrt{2}+1}{4}\right)}$

Gravitational Field or gravitational field strength:

All the bodies on or above earth's surface experience gravitational force known as the weight of the bodies. Therefore the space surrounding the earth, where the gravitational force (weight) is experienced is known as the gravitational field of the earth. Similarly the space surrounding each and every material particle is known as gravitational field of that particle.

Gravitational field strength at any point is defined as gravitational force exerted on a unit point mass. It is equal to acceleration due to gravity.

If we ask yourself, what is your strength ? Definitely you will think of your muscular power. A boxer is stronger than an ordinary man. That means he can exert a larger force. This reveals that strength is related to force.

Now if we want to measure the strength of the gravitational field at any point we will have to calculate the force acting on a point mass placed at that point. We see that , different masses experience different forces. The larger the mass, the larger the force it will experience. When we take the ratio of the gravitational force $\vec{F}_g$ and the point mass m we obtain a constant value for that point. This constant is known as the strength of the gravitational field. If the field exerts a large force on the point mass, we say that the strength of the gravitational field is stronger at that point and vice-versa.

$\Rightarrow$ The strength of the gravitational field $\vec{g} = (\vec{F}_g / m)$

$\Rightarrow$ Gravitational field strength is defined as gravitational force per unit mass.

In earth's gravitational field $\vec{g} = \frac{\text{weight of the particle}}{\text{mass of the particle}} = \frac{\vec{W}}{m}$

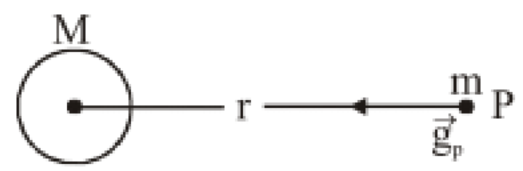
The above expression is equal to the acceleration due to gravity '$\vec{g}$'

Gravitational field unit is N/kg and dimensions LT^{-2}.

Gravitational field due to point mass at a distance r :

We want to find g due to M at point P as per the procedure, place a point mass m at P. Measure the force imparted by M on the test mass m. That is equal to $F_g = \frac{GMm}{r^2}$

$$\Rightarrow g_p = \frac{F_g}{m} = \frac{1}{m}\left(\frac{GMm}{r^2}\right)$$



$$\Rightarrow g_p = \frac{GM}{r^2}$$ and it is directed towards the mass M.

Hence, $\vec{g}_p = \frac{GM}{r^2}\hat{a}_r$, where $\hat{a}_r = (\vec{r}/r)$

Acceleration due to gravity on surface on earth

$$g = \frac{GM}{r^2}$$

Where $G = 6.67 \times 10^{-11}\ \frac{N-m^2}{Kg^2}$ (universal gravitational constant)

$M = 5.983 \times 10^{24}$ Kg (mass of earth)
$R = 6.378 \times 10^{6}$ m (equation radius of earth)
r = distance between the particle and centre of earth

If the particle is very close to the earth's surface) then $= R + h \simeq R$.
Putting all the values we obtain

$$g = 9.8\ m/sec^2$$

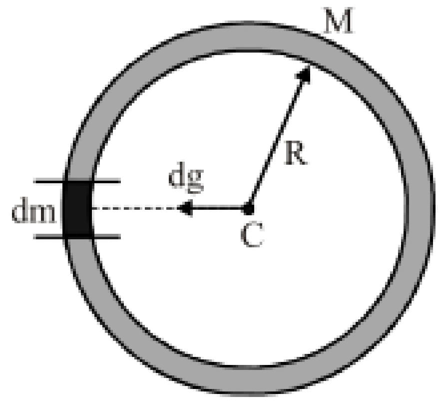
Note : (i) All objects on or above the earth's surface (at low altitudes) experience an acceleration of 9.8 m/sec^2 (approximately). The motion of particles under gravity is known as free fall. For example (a) releasing any object in earth's gravity (b) falling of fruits from the trees (c) falling of meteorites (d) motion of the satellites and (e) projectile motion.

(ii) Newton's second law of motion for any particle falling freely under gravity can be written as $\vec{F} = m\vec{a} = m\vec{g}$. The acceleration due to gravity is independent of mass of the particle. That means all the particles move with same acceleration $\vec{g}$ at a particular point.

Gravitational Field Strength due to a Ring

Case-I: At the centre of ring

To find gravitational field strength at the centre of a ring of mass M and radius R, we consider an elemental mass dm on it as shown in figure.

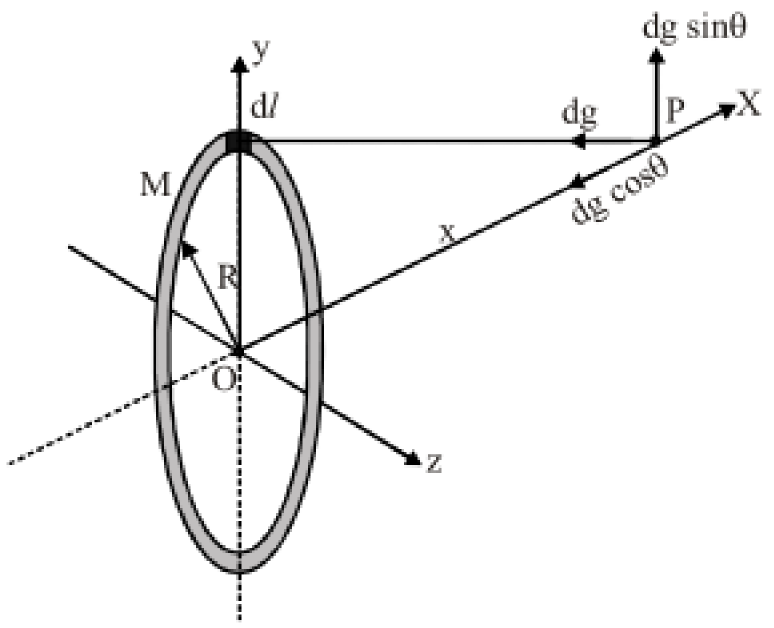


Here we can simply state that another element of same exactly opposite to dm on other half of ring will produce an equal gravitational field at C in opposite direction. Thus due to all the elements on ring, the net gravitational field at centre C will be vectorially nullified and hence net gravitational field strength at C will be 0.

Case-II: At a point on the axis of ring

To find this we consider an element dl on ring as shown figure. The mass dm of this element can be given as

$$dm = \frac{M}{2\pi R} dl$$



Let the gravitational field strength at point P due to the element dm is dg then it is given as

$$dg = \frac{Gdm}{(x^2 + R^2)}$$

Thus here net gravitational field strength at P is given as

$$g = \int dg \cos\theta \;=\; \int_0^{2\pi R} \frac{GM\, dl}{2\pi R(x^2 + R^2)} \times \frac{x}{\sqrt{x^2 + R^2}}$$

$$= \frac{GM\,x}{2\pi R(x^2+R^2)^{3/2}}[2\pi R]$$

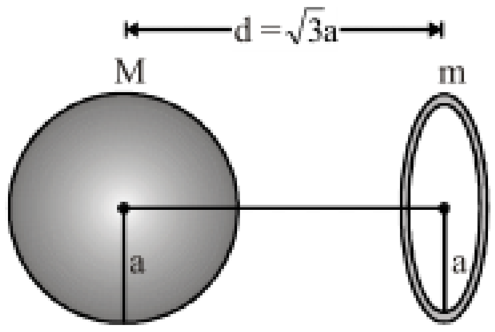
$$= \frac{GMx}{(x^2+R^2)^{3/2}}$$

Illustration :

A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The centre of the ring is at a distance $\sqrt{3}$ a from the centre of the sphere. Find the gravitational force exerted by the sphere on the ring.

Sol. *The gravitational field at any point on the ring due to the sphere is equal to the field due to single particle of mass M Placed at the centre of the sphere. Thus, the force on the ring due to the sphere is also equal to the force on it by particle of mass M placed at this point. By Newton's third law it is equal to the force on the particle by the ring. Now the gravitational field due to the ring at a distance $d = \sqrt{3}\,a$ on its axis is given as*

$$g = \frac{Gmd}{(a^2+d^2)^{3/2}} = \frac{\sqrt{3}Gm}{8a^2}$$



The force on sphere of mass M placed here is

$$F = Mg$$

$$= \frac{\sqrt{3}GMm}{8a^2}$$

Illustration :

If the radius of the earth were to shrink by one percent, its mass remaining the same. What would happen to the acceleration due to gravity on the earth's surface.

Sol. *Consider the case of a body of mass m placed on the earth's surface (mass of the earth M and radius R).*

$$g_s = \frac{GM_e}{R_e^2} \qquad ...(i)$$

Now, when the radius reduced by 1%, i.e. become 0.99 R, let acceleration due to gravity be g', then

$$g' = \frac{GM}{(0.99)^2} \qquad ...(ii)$$

From equation (i) and (ii), we get

$$\frac{g'}{g} = \frac{R^2}{(0.99)R^2} = \frac{1}{(0.99)^2}$$

or $$g' = g \times \left(\frac{1}{(0.99)}\right)^2$$

or $$g' = 1.02\ g$$

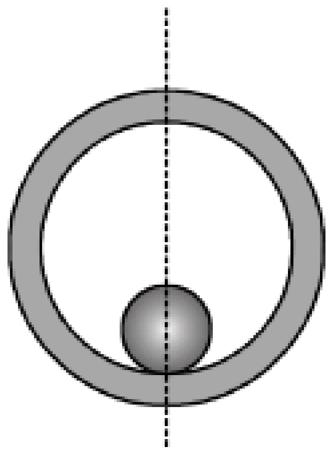
Thus the value of g is increased by 2%

Practice Exercise

Q.1 Two concentric spherical shells have masses M_1 , M_2 and radii R_1, R_2 ($R_1 < R_2$). What is the force exerted by this system on a particle of mass m_1 if it is placed at a distance $(R_1 + R_2)/2$ from the centre ?

Q.2 A particle would take a time t_1 to move down a straight tunnel from the surface of earth to its centre. If g is assumed to be constant, time would be t_2. Find t_1 / t_2 .

Q.3 A solid sphere of mass m and radius r is placed inside a hollow thin spherical shell of mass M and radius R as shown in figure. A particle of mass m' is placed on the line joining the two centre at a distance x from the point of contact of the sphere and the shell. Find the magnitude of the resultant gravitational force on this particle due to the sphere and the shell if (a) $r < x < 2r$, (b) $x < 2R$ and (c) $x > 2R$.



Answers

Q.1 $\dfrac{2GM_1m}{(R_1+R_2)}$ Q.2 $\dfrac{\pi}{2\sqrt{2}}$ Q.3 (a) $\dfrac{Gmm'(x-r)}{r^3}$ (b) $\dfrac{Gmm'}{(x-r)^2}$ (c) $\dfrac{GMm'}{(x-R)^2} + \dfrac{Gmm'}{(x-r)^2}$

Variation in the value of g

The acceleration due to gravity is given by

$$g = \frac{F}{m} = \frac{GM}{R^2}$$

where F is the force exerted by the earth on an object of mass m. This force is affected by a number of factors and hence g also depends on these factors.

(a) height from the surface of the Earth

If the object is placed at a distance h above the surface of the earth, the force of gravitation on it due to the earth is

$$F = \frac{GMm}{(R+h)^2}$$

where M is the mass of the earth and R is its radius.

Thus, $g = \frac{F}{m} = \frac{GM}{(R+h)^2}$

We see that the value of g decreases as one goes up. We can write,

$$g = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2} = \frac{g_0}{\left(1+\frac{h}{R}\right)^2}$$

where $g_0 = \frac{GM}{R^2}$ is the value of g at the surface of the earth. If h < < R,

$$g_0 = g_0\left(1+\frac{h}{R}\right)^{-2} \approx \left(1-\frac{2h}{R}\right).$$

If one goes a distance h inside the earth such as in mines, the value of g again decreases. The force by the earth is, by equation

$$F = \frac{GMm}{R^3}(R-h)$$

or, $g = \frac{F}{m} = \frac{GM}{R^2}\left(\frac{R-h}{R}\right)$

$$g = g_0\left(1-\frac{h}{g}\right)$$

The value of g is maximum at the surface of the earth and decreases with the increase in height as well as with depth.

Illustration :

Calculate the value of acceleration due to gravity at a point (a) 5.0 km above the earth's surface and (b) 5.0 km below the earth's surface. Radius of earth = 6400 km and the valeu of g at the surface of the earth is 9.80 m s^{-2}.

Sol. *(a) The value of g at a height h is (for h < < R)*

$$g_0 = g_0\left(1-\frac{2h}{R}\right)$$

$$= (9.80\ m\ s^{-2})\left(1-\frac{2\times 5.0\ km}{6400\ km}\right)$$

$$= 9.78\ ms^{-2}$$

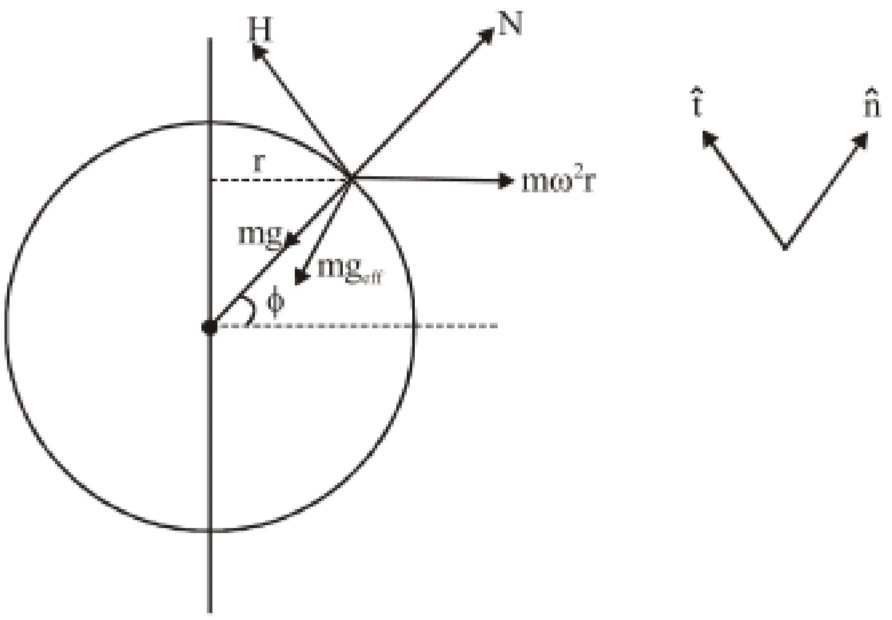
(b) The value at a depth h is

$$g = g_0\left(1 - \frac{h}{R}\right)$$

$$= (9.8\ m\ s^{-2})\ \left(1 - \frac{5.0\ km}{6400\ km}\right)$$

$$= 9.79\ m\ s^{-2}$$

Variation with latitude (Due to rotation of earth)



In figure m is a mass placed on a weighing machine situated at a latitude of ϕ
the real forces acting on it are :
(i) the gravitational force mg towards the center of earth.
(ii) the normal reaction N of the weighing machine directed away from the center of earth,
(iii) The horizontal reaction H of the weighing machine directed along the tangent as shown.
In the reference frame fixed to the earth's surface the body would also be acted upon by the pseudo force (centrifugal force) mw^2r directed as shown.
where r is the distance of P from earth's axis, $r = R_e\cos\phi$, where R_e is radius of earth.
Now in this reference frame m is at rest. Resolving forces along the radial and normal direction we get,

$$mg = m\omega^2 r\cos\phi + N$$

$$= m\omega^2 R_e \cos^2\phi + N \Rightarrow N = mg - m\omega^2 R_e \cos^2\phi$$

and $\quad H = m\omega^2 R_e \cos\phi\sin\phi$

Now the effective weight of the body is the net force experienced by the weighing machine which is equal and opposite to the force exerted by the weighing machine on the body.

$$\therefore\ m\vec{g}_{eff} = -\left(\vec{N} + \vec{H}\right) = -\left(N\hat{n} + H\hat{t}\right)$$

$$\Rightarrow \vec{g}_{eff} = -g\left[\left(\frac{1 - \omega^2 R_e \cos^2\phi}{g}\right)\hat{n} - \frac{\omega^2 R_e \cos\phi \sin\phi\, \hat{t}}{g}\right]$$

Now $\dfrac{\omega^2 R_e}{g} \simeq 0.0337$, Hence its square may be neglected and

thus $\boxed{g_{eff} \simeq g - \omega^2 R\cos^2\phi}$

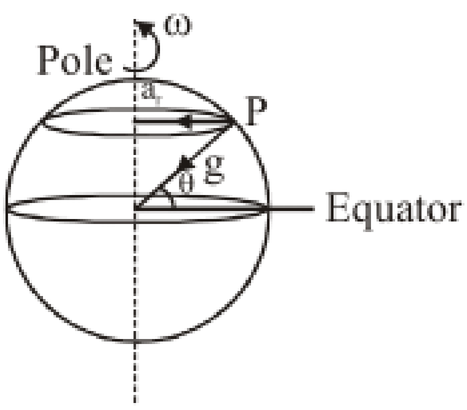
g_{eff} = apparant according due to gravity at a latitude of ϕ

* At poles, $\phi = \pi/2$ and hence $g_{eff} = g$
* At the equator, $\phi = 0$ and hence $g_{eff} = g - \omega^2 R$

Illustration :

If earth stops spinning about its own axis, what will be the change in acceleration due to gravity on its equation ? The radius of earth is 6.4×10^6 m and its angular speed is 7.27×10^{-5} rad/s.

Sol. *Effective acceleration due to gravity is given by*



$g' = g - R\omega^2 \cos^2\theta.$

Hence change in acceleration due to gravity is given as

$\Delta g = g - g' = R\omega^2 \cos^2\theta,$

at equator $\theta = 0$

$= 6.4 \times 10^6 \times (7.27 \times 10^{-5})^2$

$= 0.0338 \text{ m/s}^2$

(c) Nonsphericity of the Earth

All formulae and equations have been derived by assuming that the earth is a uniform solid sphere. the shape of the earth slightly deviates from the perfect sphere. The radius in the equatorial plane is about 21 km larger than the radius along the poles. Due to this the force of gravity is more at the poles and less at the equator. The value of g is accordingly larger at the poles and less at the equator. Note that due to rotation of earth also, the value of g is smaller at the equator than that at the poles.

(d) Nonuniformity of the Earth

The earth is not a uniformly dense object. There are a veriety of minerals, metals, water, oil, etc., inside the earth. Then at the surface there are mountains, seas, etc. Due to these nonuniformities in the mass distribution, the value of g is locally affected.

"Weighing" the Earth

The exerted force by the earth on a body is called the weight of the body. In this sence "weight of the earth" is a meaningless concept. However, the mass of the earth can be determined by noting the acceleration due to gravity near the surface of the earth. We have,

$$g = \frac{GM}{R^2}$$

or, $M = gR^2/G$

Putting $g = 9.8\ m\ s^{-2}$, $R = 6400$ km

and $G = 6.67 \times 10^{-11}\ \frac{N-m^2}{kg^2}$

the mass of the earth comes out to be 5.98×10^{-24} kg.

Illustration :

At what rate should the earth rotate so that the apparent g at the equator becomes zero ? What will be the length of the day in this situation ?

Sol. *At earth's equator effective value of gravity is*

$$g_{eq} = g_s - \omega^2 R_e$$

If g_{eff} at equator to be zero, we have

$$g_s - \omega^2 R_e = 0$$

or $$\omega = \sqrt{\frac{g_e}{R_e}}$$

Thus length of the day will be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_e}{g_s}}$$

$$= 2 \times 3.14\sqrt{\frac{6.4\times10^6}{9.8}} = 5074.77s$$

$$\simeq 84.57\ min.$$

Practice Exercise

Q.1 Earth's mass is 80 times that of the moon and their diameters are 12800 and 3200 kms respectively. What is the value of g at the moon ? g on earth = 980 cm/s^2.

Q.2 The diameter of a planet is four times that of the earth. Find the time period of pendulum on the planet, if it is a second pendulum on the earth. Take the mean density of the planet equal to that of the earth,

Q.3 A tunnel is dug along a chord of the earth at perpendicular distance R/2 from the earth's centre. The wall of the tunnel may be assumed to be frictionless. Find the force exerted by the wall on particle of mass m when it is at a distance x from the centre of the tunnel.

Q.4 Find the height over earth's surface at which the weight of a body becomes half of its value at the surface.

Answers

Q.1 196 cm/s^2 Q.2 1 s Q.3 $\frac{GM_e m}{2R^2}$ Q.4 $(\sqrt{2}-1)$

Gravitational potential energy

In analyzing the motion of planets and satellites, it is often easier and more informative to use energy rather than force. In this section we shall evaluate the potential energy of a system consisting of two bodies that interact through the gravitational force. we obtained the potential energy change due to gravity for a body that moves through a height y near the Earth's surface : $\Delta U = mgy$. However, this applies only near the Earth's surface, where (for changes in height that are small compared with the distance from the center of the Earth) we can regard the gravitational force as approximately constant. Our goal here is to find a general expression that applies at all location, such as at the altitude of an orbiting satellite.

The potential energy difference can be found from equation. $\Delta U = U_b - U_a = -W_{ab}$, where W_{ab} is the work done to configuration b. However, this equation applies only if the force is conservative.

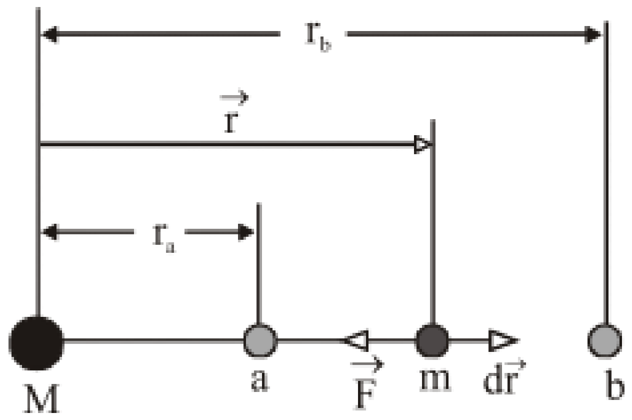
Calculating the Potential Energy

The gravitational force is conservative so we can calculate the potential energy. Figure shows a particle of mass m moving from a to b along a radial path. A particle of mass M, which we assume to be at rest at the origin, exerts a gravitational force on m. The vector $\vec{r}$ locates the position of m relative to M by the gravitational force is

$$W_{ab} = -\int_a^b F dr$$

$$= -\int_{r_a}^{r_b} \frac{GMm}{r^2} dr = -GMm\int_{r_a}^{r_b} \frac{dr}{r^2}$$

$$= -GMm\left(-\frac{1}{r}\right)\Bigg|_{r_a}^{r_b} = GMm\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$



The negative sign in the first line of this equation arises because the (attractive) force $\vec{F}$ and the infinitesimal radial vector $d\vec{r}$ point in opposite directions. Equation shows that, when $r_b > r_a$ (as in figure), the work W_{ab} is negative, as we expect.

Applying equation ($\Delta U = -W_{ab}$), we can find the change in the potential energy of the system as m oves between points a and b

$$\Delta U = U_b = -W_{ab} = GMm\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

A particle of mass M exerts a gravitational force $\vec{F}$ on a particle of mass m that moves from a to b. If m moves outward from a to b, the change in potential energy is positive ($U_b > U_a$). That is, if the particle passes through point a with a certain kinetic energy K_a, as it travels to b its gravitational potential energy increases as its kinetic energy decreases ($K_b < K_a$). Conversely, if the particle is moving inward, its potential energy decreases as its kinetic energy increases.

Instead of differences in potential energy, we can consider the value of the potential energy at a single point if wedefine a reference point. We choose our reference configuration to be an infinite separation of the particles, and the we define the potential energy to be zero in that configuration. Let us evaluate equation for $r_b = \infty$ and $U_b = 0$. If a represents any arbitrary point, where the separation between the particles is r, then equation becomes

$$U(\infty) - U(r) = GMm\left(\frac{1}{r} - 0\right)$$

or $$U(r) = -\frac{GMm}{r}$$

Note :

1. We can reverse the previous calculation and derive the gravitational force from the potential energy. For spherically symmetric potential energy functions, the relation $F = -dU/dr$ gives the radial component of the force; see equation. With the potential energy of equation, we obtain

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}$$

The minus sign here shows that the force is attractive, directed inward along a radius.

2. We can show that the potential energy defined according to equation leads to the familiar mgy for a small difference in elevation y near the surface of the Earth. Let us evluate equation for the difference in potential energy between the location at a height y above the surface (that is, $r_b = R_E + y$, where R_E is the radius of the Earth) and the surface ($r_a = R_E$)

$$\Delta U = U(R_E + y) - U(R_E) = GM_E m\left(\frac{1}{R_E} - \frac{1}{R_E + y}\right)$$

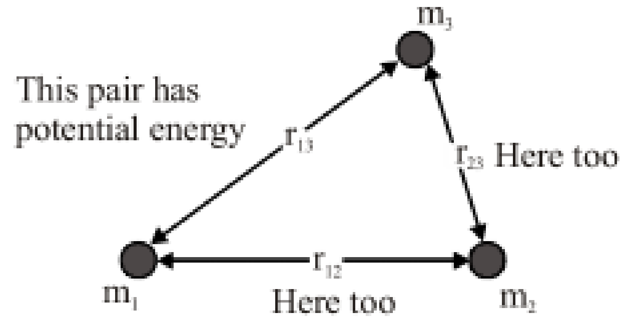
$$= \frac{GM_E m}{R_E}\left(1 - \frac{1}{1 + y/R_E}\right)$$

When $y << R_E$, which would be the case for small displacements of bodies near the Earth's surface, we can use the binomial expansion to approximate the last term as $(1 + x)^{-1} = 1 - x + ... \approx 1 - x$, which gives

$$\Delta U \approx \frac{GM_E m}{R_E}\left[1 - \left(1 - \frac{y}{R_E}\right)\right] = \frac{GM_E my}{R_E^2} = mgy$$

using equation to replace GM_E/R_E^2 with g.

3. If our system contains more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair with Equation as if the other particles were not there, and then algebraically sum the results. Each of the three pairs of Figure, for example, gives the potential energy of the system as



If our system contains more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair with Eq. below as if the other particles were not there, and then algebraically sum the results. the potential energy of the system as

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

Gravitational Potential

The gravitational potential at a point in gravitational field is the gravitational potential energy per unit mass placed at the point in gravitational field. Thus at a certain point in gravitational field, a mass m_0 has a potential energy U the gravitational potential at that point is given as

$$V = \frac{U}{m_0}$$

or if at a point in gravitational field gravitational potential V is known then the interaction potential energy of a point mass m_0 at the point in the field is given as

$$U = m_0 V$$

Interaction energy of a point mass m_0 in a field is defined as work done in bringing that mass from infinity to the point. In the same fashion we can define gravitational potential at a point in field, alternatively as "Work done in bringing a unit mass from infinity to that point against gravitational force."

When a unit mass is brought to a point in a gravitational field, force on the unit mass is $\vec{g}$ at a point in the field. Thus the work done in bringing this unit mass from infinity to a point P in gravitational field or gravitational potential at point P is given as

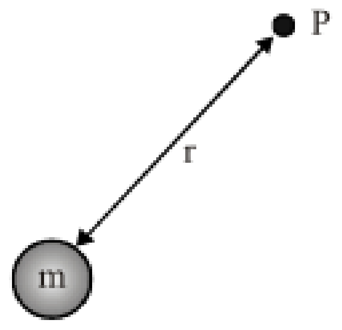
$$V_P = -\int_{\infty}^{P} \vec{g} \cdot dx$$

Here negative sing shown that V_P is the negative of work done by gravitation field or it is the external required work for the purpose against gravitational force.

Gravitational Potential due to a Point Mass

We place a test mass m_0 at P and we find the interaction energy of m_0 with the field of m, which is given as

$$U = -\frac{Gmm_0}{x}$$



Now the gravitational potential at P due to m can be written as

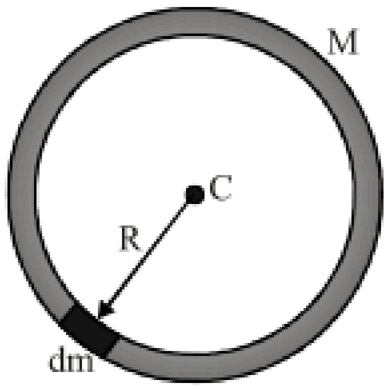
$$V = \frac{U}{m_0} = -\frac{Gm}{r}$$

Gravitational potential due to a ring

At the centre of ring

In gravitational potential the situation is not like this as it is a scaler quantity and here the distane of centre from each element dm on ring circumference is equal to R, thus every element dm produces an equal gravitational potential at C, given as

$$dV = -\frac{Gdm}{R}$$



Now due to the whole ring the gravitational potential at its centre C is given as

$$V_C = \int dV = -\int \frac{Gdm}{R} = -\frac{Gm}{R}$$

Note : Now we have seen that most of the expressions of $\vec{g}$ and V are same as that we have calculated in the topic electrostatics. So we will not prove them again rather we can replace some sysbols as :

Electrostatics	Gravitation
Q	M
K	G
$\in_0$	$\frac{1}{4\pi G}$

For Formula conversion :

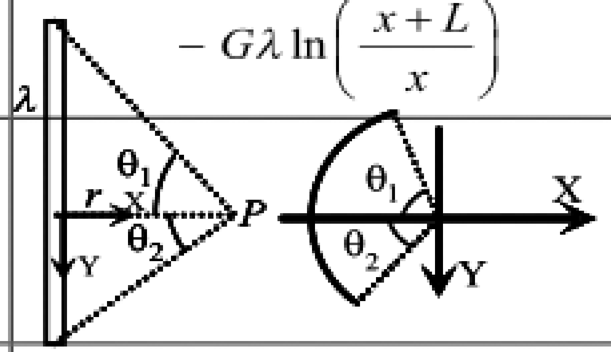
Body	Gravitational Field	Gravitational potential
Point mass	$-\frac{Gm}{r^2}$	$-\frac{Gm}{r}$
Ring	$\frac{Gmx}{\left(R^2+x^2\right)^{3/2}}$; $E_{\max}\big\vert_{x=R/\sqrt{2}}$	$-\frac{Gm}{\sqrt{x^2+R^2}}$
Disc	$2\pi G\sigma\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$	$-2\pi G\sigma\left[\sqrt{x^2+R^2}-x\right]$
infinite wire	$E = 2G\lambda / r$	$\Delta V = 2G\lambda \ln\frac{r_1}{r_2}$
wire	$\frac{G\lambda L}{x(x+L)}$	$-G\lambda \ln\left(\frac{x+L}{x}\right)$
circular arc & rod	$-\frac{G\lambda}{r}\begin{bmatrix}(\sin\theta_1+\sin\theta_2)\hat{i} \\ -(\cos\theta_1 - \cos\theta_2)\hat{j}\end{bmatrix}$	
Apex of cone	∞	$-2\pi G\sigma R$
infinite plate	$E = 2\pi G\sigma$	$\Delta V = 2\pi G\sigma d$
Hollow Sphere	$\vec{E} = \begin{cases} 0 & 0 \le r < R \\ -\frac{GM}{r^2}\hat{r} & r \ge R \end{cases}$	$V = \begin{cases} -\frac{GM}{R} & 0 \le r < R \\ -\frac{GM}{r} & r \ge R \end{cases}$
Solid sphere	$\vec{E} = \begin{cases} -\frac{GM\vec{r}}{R^3} = -\frac{4\pi G\rho}{3}\vec{r} & 0 \le r < R \\ -\frac{Kq}{r^2}\hat{r} & r \ge R \end{cases}$	$V = \begin{cases} -\frac{GM}{2R}\left(3-\frac{r^2}{R^2}\right) & 0 \le r < R \\ -\frac{GM}{r} & r \ge R \end{cases}$

Illustration :

The magnitude of gravitational field intensities at distance r_1 and r_2 from the centre of a uniform solid sphere of radius R and mass M are I_1 and I_2 respectively. Find the ratio of I_1/I_2 if (a) $r_1 > R$ and $r_2 > R$ and (b) $r_1 < R$ and $r_2 < R$ (c) $r_1 > R$ and $r_2 < R$.

Sol. *Gravitational field intensity for a uniform spherical distribution of mass is given by:*

$$I = \frac{GM}{r^2} \qquad \text{for } r > R \text{ and}$$

$$= \frac{GM}{R^3} \qquad \textit{for } r < R.$$

(a) $r_1 > R$ and $r_2 > R$

$$\frac{I_1}{I_2} = \frac{(GM/r_1^2)}{(GM/r_2^2)} = \frac{r_2^2}{r_1^2}$$

(b) $r_1 < R$ and $r_2 < R$

$$\frac{I_1}{I_2} = \frac{(GM/R^3)\, r_1}{(GM/R^3)\, r_2} = \frac{r_1}{r_2}$$

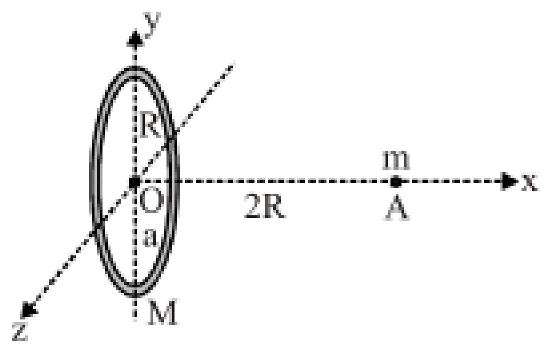
(c) $r_1 > R$ and $r_2 < R$

$$\frac{I_1}{I_2} = \frac{(GM/r_1^2)}{(GM/R^3)\, r_2} = \frac{R^3}{r_1^2 r_2}$$

Illustration :

A circular ring of mass M and radius R is placed in YZ plane with centre at origin. A particle of mass m is released from rest at a point x = 2R. Find the speed with which it will pass the centre of ring

Sol.



As shown in figure, first we find potential at A due to the ring, it is given as

$$V_A = -\frac{GM}{\sqrt{R^2 + (2R)^2}} = -\frac{GM}{\sqrt{5}R}$$

Now potential at origin O due to ring is

$$V_0 = -\frac{GM}{R}$$

When m moves from A to O, work done on it due to gravitational force is

$$W = m\,(V_A - V_0) = m\left(-\frac{GM}{\sqrt{5}R} + \frac{GM}{R}\right)$$

$$= \frac{GMm}{R}\left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)$$

This work done by gravitational force on m must be equal to the increase in kinetic energy of the mass m, thus we have

$$\frac{1}{2}mv^2 = \left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)\frac{GMm}{R}$$

or $$v = \left[\frac{2(\sqrt{5}-1)GM}{\sqrt{5}R}\right]^{1/2}$$

This problem can also be solved simply by using energy conservation. These initially when m was at rest at point A. The total energy of system is only gravitational potential energy given as

$$E_i = m \,.\, V_A = -\frac{GMm}{\sqrt{5}R}$$

Finally when m passes through O, the total energy of system is

$$E_f = \frac{1}{2}mv^2 + mV_0$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{R}$$

As no external work is done on the system in this case, the total energy of system must be conserved, thus according to energy conservation we have

$$E_i - E_f$$

$$-\frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

$$v = \left[\frac{2(\sqrt{5}-1)GM}{\sqrt{5}R}\right]^{1/2}$$

Illustration :

A small mass m is transferred from the centre of a hollow sphere of mass M to infinity. Find work done in the process. Compare this with the situation if instead of a hollow sphere, a solid sphere of same mass were there.

Sol. *We know at infinity, gravitational potential is taken zero. Thus if V_C be the gravitational potential at centre of hollow sphere then external work required in the process is*

$$W = m\,(0 - V_C)$$

or $$= m\left(0 - \left(\frac{GM}{R}\right)\right) = \frac{GMm}{R}$$

Here $V_C = -\frac{GM}{R}$*, the potential at the centre of a hollow sphere of mass M and radius R. If a solid sphere we here, we have at its centre*

$$V_C = -\frac{3}{2}\frac{GM}{R}$$

Thus work required will be

$$W = m\left[0 - \left(-\frac{3}{2}\frac{GM}{R}\right)\right] = \frac{3}{2}\frac{GMm}{R}$$

We can see in second case more work is required for the process.

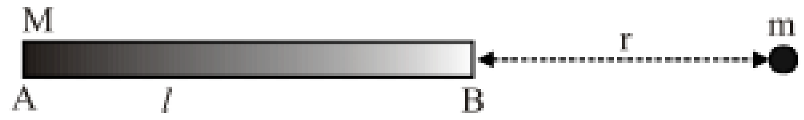
Practice Exercise

Q.1 Find the kinetic energy needed to project a body of mass m from the centre of a ring of mass M and radius R so that it will never come back.

Q.2 How much work is done in circulating a small object of mass m around a sphere of mass m in a circle of radius R.

Q.3 Find the gravitational potential due to a hemispherical cup of mass M and radius R, at its centre of curvature.

Q.4 Find the gravitational potential energy of system consisting of uniform rod AB of mass M, length l and a point mass m as shown in figure.



Answers

Q.1 $\dfrac{GMm}{R}$ Q.2 0 Q.3 $-\dfrac{GM}{R}$ Q.4 $-\dfrac{GMm}{l}\ln\left(1+\dfrac{l}{r}\right)$

The motion of planets and satellites

(a) Planets

Planets move round the sun due to the gravitational attraction of the sun. The path of these planets are elliptical with the sun at a focus.

(b) Satellite

Satellites are launched from the earth so as to move round it. A number of rockets are fired from the satellite at proper time to establish the satellite in the desired orbit. Once the satellite is placed in the desired orbit with the correct speed for that orbit, it will continue to move in that orbit under gravitational attraction of the earth.

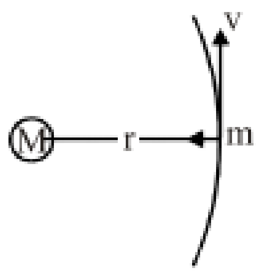
We make two assumptions that simplify the analysis:

(1) We consider the gravitational force only between the orbiting body (the Earthy, for instance and the central body (the Sun), ignoring the perturbing effect of the gravitational force of other bodies (such as other planets)

(2) We assume that the central body is so much more massive than the orbiting body that we can ignore its motion under their mutual interaction.

Let the speed of an artificial earth satellite in its orbits of radius r be v_0. The satellite is accelerating towards the centre of earth by earth's gravitational pull $\dfrac{GMm}{r^2}$

$$\Rightarrow \quad F_{CP} = \frac{GMm}{r^2}$$



$$\frac{mv_0^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow \quad v_0 = \sqrt{\frac{GM}{r}} \qquad \text{...(i)}$$

Putting $\frac{GM}{r^2} = g$ (acceleration due to gravity at the orbit), we obtain,

$$\Rightarrow \quad v_0 = \sqrt{gr} \qquad \text{...(ii)}$$

When it orbits at an altitude h, putting $r = (R+h)$

$$\Rightarrow \quad v_0 = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{GM}{R(1+h/R)}} = \sqrt{\frac{gR}{(1+h/R)}}$$

Angular speed

The angluar speed

$$\omega = \frac{v_0}{r}$$

Putting $v_0 = \sqrt{\frac{GM}{r}}$, we obtain

$$\omega = \sqrt{\frac{GM}{r^3}} = \sqrt{\frac{GM}{(R+h)^3}}$$

Angular momentum

The angular momentum of an earth satellite or a planet is given as

$L = mvr$

$$= m\sqrt{\frac{GM}{r}} \times r$$

$$= m\sqrt{GMr}$$

Time period of Revolution

The period of revolution

$$T = \frac{2\pi}{\omega}$$

Puting $\omega = \sqrt{\frac{GM}{r^3}}$

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

Energy consideration in planetary and satellite motion

$$U(r) = -\frac{GMm}{r}$$

Where r is the radius of the circular orbit.
The kinetic energy of the system is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 r^2$$

the Sun being at rest

$$\omega^2 r^2 = \frac{GM}{r}$$

so that (with $v = \omega r$)

$$K = \frac{GMm}{2r}$$

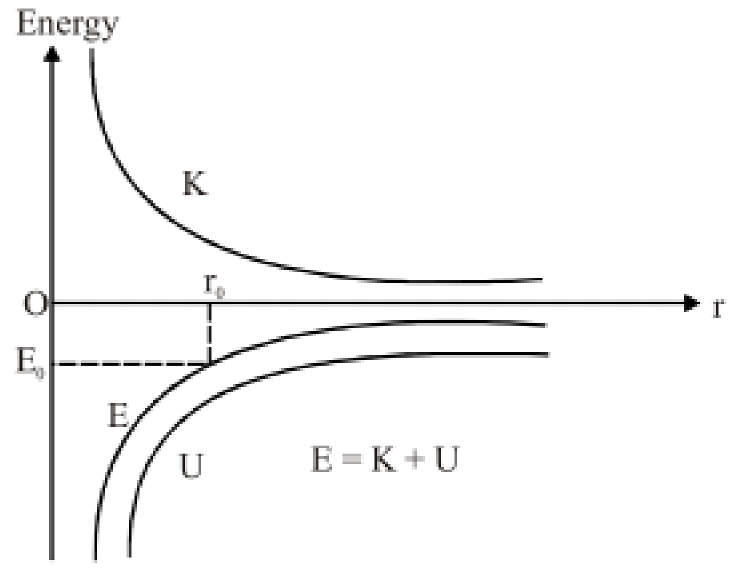
The total mechanical energy is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = \frac{GMm}{2r}$$

This energy is constant and negative. The kinetic energy can never be negative, but from equation we see that it

Note :

(1) Graph of kinetic energy K, potential energy U, and total energy E = K + U of a body in circular planetary motion.



A planet with total energy $E_0 < 0$ will remain in a orbit. The greater the distance from the Sun, the greater (that is, less negative) its total energy E.

(2) Gravitational binding energy

We have seen that if a particle of mass m placed on the earth is given an energy $\frac{1}{2}mu^2 = \frac{GMm}{R}$ or more, it finally escapes from earth. The minimum energy needed to take the particle infinitely away from the earth is called the binding energy of the earth-particle system. Thus, the binding energy of the earth-particle system is $\frac{GMm}{R}$

Geostationary satellite :

A satellite which appears to be stationary when seen from earth is called a Geostationary satellite.
For a satellite to be geostationary.

(i) Its orbit must be circular

(ii) It must rotate about the same axis as earth, i.e. it must move in the equatorial* plane.

(iii) It must revolve form west to east.

(iv) Its time period must be 24 hours.

$$\Rightarrow \quad m\omega^2(R+h) = \frac{GMm}{(R+h)^2}$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{GM}{(R+h)^3}$$

$$(R+h) = \left\{GM\left(\frac{T}{2\pi}\right)^2\right\}^{1/3} \quad \text{where } T = 24 \times 3600 \text{ sec.}$$

Solving $h = 36000$ km (approx)

* Any satellite must rotate about the center of earth.

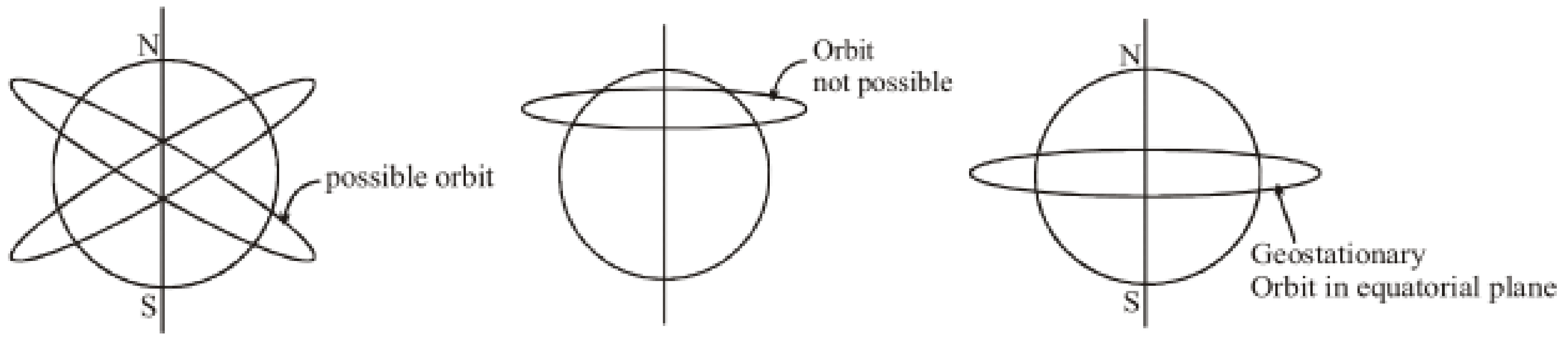


Illustration :

Estimate the mass of the sun, assuming the orbit of the earth round the sun to be a circle. The distance between the sun and earth is 1.49×10^{11} m and $G = 6.66 \times 10^{-11}$ Nm²/kg².

Sol. *Here the revolving speed of earth can be given as*

$$v = \sqrt{\frac{GM}{r}} \qquad \text{[Orbital speed]}$$

Where M is the mass of sun and r is the orbit radius of earth.

We know time period of earth around sun is T = 365 days, thus we have

$$T = \frac{2\pi r}{v}$$

or
$$T = 2\pi r\sqrt{\frac{r^3}{GM}}$$

or
$$M = \frac{4\pi^2 r^3}{GT^2}$$

$$= \frac{4\times(3.14)^2\times(1.49\times1011)^3}{(365\times24\times3600)^2\times(6.66\times10^{-11})} = 1.972\times10^{22}\text{ kg}$$

Illustration :

If the earth be one-half of its present distance from the sun, how many days will be in one year ?

Sol. *If orbit of earth's radius is R, in previous example we've discussed that time period is given as*

$$T = 2\pi r\sqrt{\frac{r}{GM}} = \frac{2\pi}{\sqrt{GM}}r^{3/2} \qquad ...(i)$$

If radius changes to $r' = \frac{r}{2}$, *new time period become*

$$T' = \frac{2\pi}{\sqrt{GM}}r^{3/2} \qquad ...(ii)$$

From equation (i) and (ii) we have

$$\frac{T}{T'} = \left(\frac{r}{r'}\right)^{3/2}$$

or

$$T' = T\left(\frac{r'}{r}\right)^{3/2}$$

$$= 365\left(\frac{1}{2}\right)^{3/2} = \left(\frac{365}{2\sqrt{2}}\right) \text{ days}$$

Illustration :

An artificial satellite is describing an equatorial orbit at 1600 km above the surface of the earth. Calculate its orbital speed and the period of revolution. If the satellite is travelling in the same direction as the rotation of the earth (i.e., from west to east), calculate the interval between two successive times at which it will appear vertically overhead to an observer at a fixed point on the equator. Radius of earth = 6400 km.

Sol. *We know that the period of the satellite is*

$$T = \frac{2\pi}{\sqrt{GM}}r^{3/2} = \frac{2\pi}{\sqrt{gR^2}}r^{3/2}$$

Where $r = 6400 + 1600 = 8000 \times 10^3$ m,

$g = 9.8$ m/sec^2 and $R = 6400 \times 10^3$ m

Substituting values we get

$$T = 2 \times 3.14\left[\frac{(8000\times 10^3)^3}{9.8\times(6400\times 10^3)^2}\right]^{1/2}$$

$$= 7096\ x$$

Futher, orbital speed,

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{gR^2}{r}}$$

or

$$v = \sqrt{\left(\frac{9.8}{8000\times 10^3}\right)}\times(6400\times 10^3)$$

$$= 7083.5 \text{ m/s}$$

Let t be the time interval between two successive moments at which the satellite is overhead to an observer at fixed position on the equator. As both satellite and earth are moving in same direction with angular speed ω_S and ω_E respectively, we can write the time of separation as

$$t = \frac{2\pi}{\omega_S - \omega_E}$$

Here $\omega_S = \frac{2\pi}{7096}$ *and* $\omega_E = \frac{2\pi}{86400}$

Thus we have $t = \frac{86400 \times 7096}{86400 - 7096}$

$= 7731\ s$

Escape Velocity

Definition: If a particle of mass m, kept in an attractive gravitational field is given sufficient kinetic energy, it may escape the gravitational pull due to the field. The particle will escape to infinity depending on whether its path allows it to do so. For example, a particle of mass m kept on the surface of the earth requires a minimum velocity v_{esc} so that it moves to infinity.

$$\frac{1}{2}mv_{esc}^2 - \frac{GMm}{R} = 0$$

Where M is the mass of the earth and R is its radius.

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Note :

The escape speed does not depend on the direction in which the projectile is fired. The Earth's rotation which we have not considered in this calculation does play a role, however. Firing eastward has an advantage in that the Earth's tangential surface speed, which is 0.46 km/s at Cape Canaveral, provides part of the kinetic energy needed for escape, and thus less thrust from the rocket engines would be required to escape the Earth's gravity.

Illustration :

The minimum velocity of projection of a body to send it to infinity from the surface of a planet is $\frac{1}{\sqrt{6}}$ times that is required from the surface of the earth. The radius of the planet is $\frac{1}{36}$ times the radius of the earth. The planet is surrounded by an atmosphere which contains monoatomic innert gas ($\gamma = 5/3$) of constant density up to a height h(h << radius of the planet). Find the velocity of sound on the surface of the planet.

Sol. *Escape velocity from the surface of the planet*

$$v_p = \sqrt{2g_pR_p}$$

Given $v_p = \frac{v_e}{\sqrt{6}} = \sqrt{\frac{2g_eR_e}{6}}$

$$\sqrt{\frac{g_e R_e}{3}} = \sqrt{2g_p R_e/36} \Rightarrow \qquad g_p = 6g_e$$

Pressure exerted by the atmospheric column of height h on the surface of the planet $P = \rho g_p h$

Using equation of state $P = \dfrac{\rho RT}{M}$

Hence speed of the sound $v = \sqrt{\dfrac{\gamma RT}{M}} = \sqrt{\gamma g_p h} = \sqrt{6\gamma\, g_e h} = \sqrt{10\, g_e h}$

Practice Exercise

Q.1 Two satellites A and B of the same mass are orbiting the earth at altitudes R and 3R respectively, where R is the radius of the earth. Taking their orbits to be circular, obtain the ratios of their kinetic and potential energies.

Q.2 A satellite is to revolve round the earth in a circle of radius 8000 km. With what speed should this satellite be projected into orbit ? What will be the time period ? Take g at the surface = 9.8 m./s^2 and radius of the earth = 6400 km.

Q.3 A satellite of mass 2×10^3 kg has to be shifted from an orbit of radius 2R to another of radius 3R, where R is the radius of the earth. Calculate the minimum energy required.

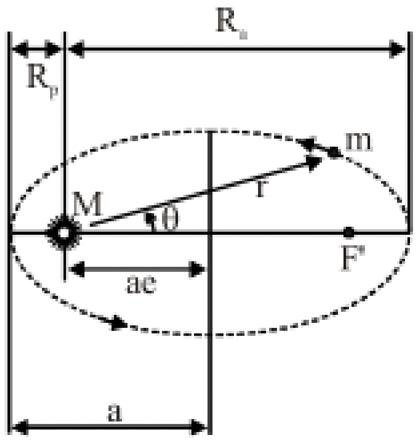
Answers

Q.1 2 : 1 Q.2 7.08 km/s. 118 minutes Q.3 1.04×10^{10} J

Kepler Laws :

The empirical basis for understanding the motions of the planets is there laws deduced by Kepler (1571 - 1630, well before Newton) from studies of the motion of the planet Mars.

1. **The law of orbits :** *All planets move in elliptical orbits having the Sun at one focus.* Newton was the first realize that there is a direct mathematical relationship between inverse-square ($1/r^2$) force and elliptical orbits. Figure shows a typical elliptical orbit.



A planet of mass m moving in a elliptical orbit around the Sun. The Sun, of mass M, is at one focus of the elipse. F' marks the other or "empty" focus. The semimajor axis a of the ellipse, the perihelion distance R_p, and the aphelion distance R_e are also shown. The distance *ae* locates the focal points, e being the eccentricity of the orbit.
other planets in the solar system, the eccentricities are small and the orbits are nearly circular.

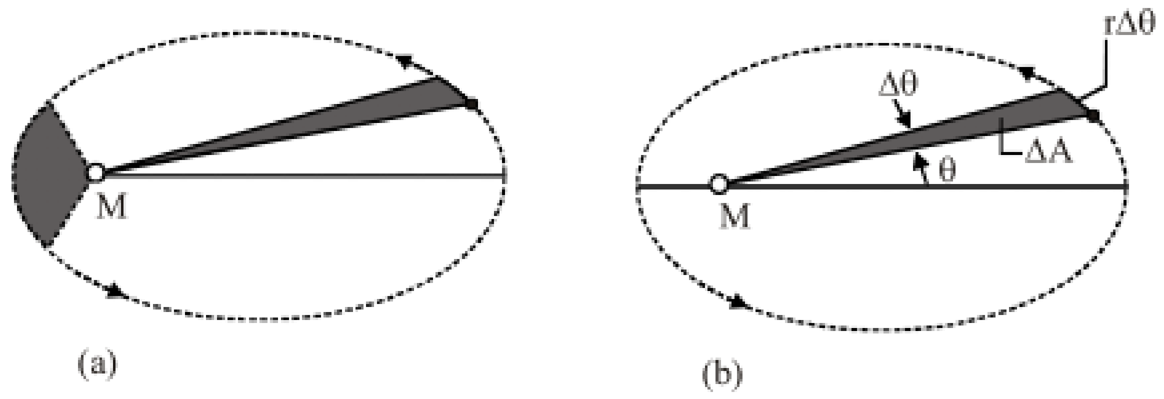
The maximum distance R_a of the orbiting body from the central body is indicated by the prefix apo - (or sometimes ap-), as in aphelion (the maximum distance from the Sun) or apogee (the maximum distance from Earth). Similarly, the cloest distance R_p is indicated by the prefix peri-, as in perihelion or perigee. As you can see from figure $R_a = a(1 + e)$ and $R_p = a(1 - e)$. For circular orbits. $R_a = R_p = a$

2. **The Law of Areas :** *A line joining any planet to the Sun sweeps out equal area in equal times.* Figure illustrates this law : in effect it says that the orbiting body moves more rapidly when it is close to the central body than it does when it is far aways. We now show that the law of areas is identical with the law of conservation of angular momentum.

Consider the small area increament ΔA covered in a time interval Δt, as shown in figure. The area of this approximately triangular wedge in one-half its base, $r\,\Delta\theta$, times its height r. The rate at which this area is swept out is $\Delta A / \Delta t = \frac{1}{2}(r\Delta\theta)(r)/\Delta t$. In the instantaneous limit this becomes

$$\frac{dA}{dt} = \lim_{\Delta\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta\to 0}\frac{1}{2}r^2\frac{\Delta\theta}{\Delta t} = \frac{1}{2}r^2\omega$$

Assuming we can regard the more massive body M as at rest, the angular momentum of the orbiting body m relative



(a) The equal shaded areas are covered in equal times by a line connecting the planet to the Sun, demonstrating the law of areas.

(b) The area ΔA is covered in a time Δt, during which the line sweeps through an angle $\Delta\theta$.

to the origin at the central body is , according to equation $L_z = I\omega = mr^2\omega$ (choosing the z axis perpendicular to the plane of the orbit). Thus

$$\frac{dA}{dt} = \frac{L_z}{2m}$$

If the system of M and m is isolated, meaing that there is no net external torque on the system, the L_z is a constant; therefore dA/dt is also constant. That is in every interval dt in the orbit, the line connecting m and M sweeps out equal areas dA, which verifies kepler's second law. The speeding up of a comet as it passes close to the Sun is an example of this effect and is thus a direct consequence of the law of conservation of angular momentum.

3. **The law of Periods :** *The square of the period of any planet about the Sun is proportional to the cube of seminajor axis of the* . Let us prove this result for circular orbits. The gravitational force provides the necessary centripetal acceleration for circular motion.

If 'T' is the period of revolution and 'a' be the semi-major axis of the path of planet then according to kepler's III Law, we have

$$T^2 \propto a^3$$

For circular orbits, it is a special case of ellipse when its major and minor axis are equal. If a planet is in a circular orbit of radius R around the sun then its revolution speed must be given as

$$v = \sqrt{\frac{GM_s}{r}}$$

Where M_S is the mass of sun. There you can recall that this speed is independent from the mass of planet. Here the time period of revolution can be given as

$$T = \frac{2\pi r}{v}$$

or $$T = \frac{2\pi r}{\sqrt{\frac{GM_s}{r}}} \qquad ...(A)$$

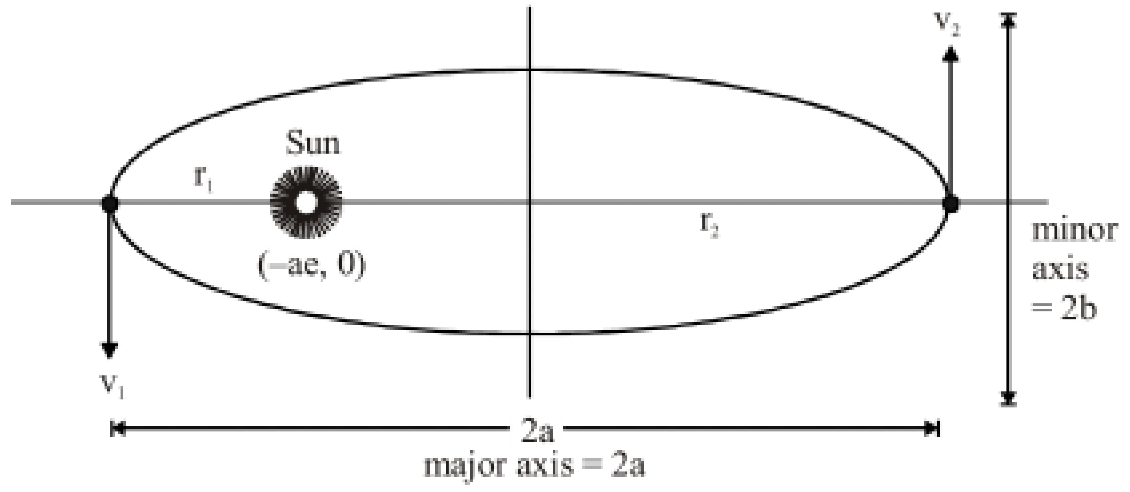
Squaring equation no. (A) we get

$$T^2 = \frac{4\pi^2}{GM_S} r^3 \qquad ...(B)$$

Equation (B) verifies the statement of Kepler's third law for circular orbits. Similarly we can also verify it from elliptical orbits. For this we start from the relation we've derived earlier for rate sweeping area by the position vector of planet with respect to sun which is given as

$$\frac{dA}{dt} = \frac{L}{2m}$$

Where L is the total angular momentum of planet during its motion consider the path of planet shown in figure is an elliptical path with sun on focus (–ae, 0).



Here r_1 and r_2 are the shortest and farthest distance of planet from sun during its motion, which are given as

$$r_1 = a\,(1 - e)$$

and $$r_2 = a\,(1 + e)$$

Where e is the centricity. From geometry we know that the relation in semi major axis and semiminor axis be is given as

$$b = a\sqrt{1-e^2}$$

If v_1 and v_2 are the planet speeds at perihelion and aphelion points then from conservation of momentum we have

$$L = mv_1 r_1 = mv_2 r_2$$

From energy conservation we have

$$\frac{1}{2}m_1 v_1^2 - \frac{GM_S m}{r_1} = \frac{1}{2}mv_2^2 - \frac{GM_s m}{r_2}$$

or $$v_1^2 - v_2^2 = 2GM_S\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$v_1^2\left[1 - \frac{r_1^2}{r_2^2}\right] = 2GM_S\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

or $$v_1 = \sqrt{\frac{2GM_S r_2}{(r_1 + r_2)r_1}}$$

From equation (vi) and (vii) we have

$$v_1 = \sqrt{\frac{GM_S}{a}\left(\frac{1+e}{1-e}\right)}$$

Now from equation (v) we have the total area of ellipse traced by the planet is given as

$$A = \frac{L}{2m}T$$

or $$T = \frac{2m}{L}A = \frac{2m\pi ab}{L} = \frac{2m\pi ab}{mv_1 r_1}$$

or $$T = \frac{2m\pi a\left[a\sqrt{1-e^2}\right]}{m\left[\sqrt{\frac{GM_S}{a}\left(\frac{1+e}{1-e}\right)}[a(1-e)]\right]}$$

or $$T^2 = \frac{4\pi^2}{GM_s}a^3$$

Illustration :

A satellite is launched tangentially from a height h above earth's surface as shown.

I. *Find v_{min} so that it just touches the earth's surface*

II. *If h = R and satellite is launched tangentially with speed = $\sqrt{\frac{3GM}{5R}}$ find the mximum distance of satellite from earth's center*

Sol. *(I) Satellite just Grazes from surface of earth*

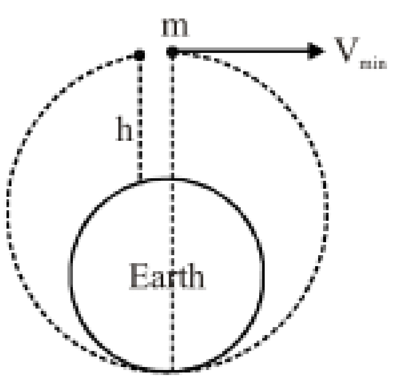
when $2a = 2R + h$

$$a = \left(R + \frac{h}{2}\right)$$

$$\text{Total energy } (E) = -\frac{GMm}{2a} = -\frac{GMm}{(2R+h)}$$

$$\frac{1}{2}mV_{0\ min}^{2} - \frac{GMm}{R+h} = -\frac{GMm}{2R+h}$$

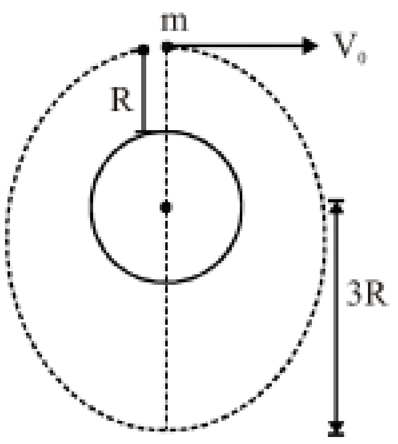
$$V_{0\ min}^{2} = 2GMm\left[\frac{1}{R+h} - \frac{1}{2R+h}\right]$$



$$= \frac{2GMR}{(R+h)(2R+h)} = \frac{2GMR}{r(R+r)}$$

$$V_{0\ min} = \sqrt{\frac{2GMR}{r(R+r)}} \qquad \text{where } r = h + R$$

(II) To find maximum distance from center.



$$E = -\frac{GMm}{2a} = \frac{1}{2}mv_0^2 - \frac{GMm}{2R}$$

$$= \frac{1}{2}m \cdot \frac{3GM}{5R} - \frac{GMm}{2R}$$

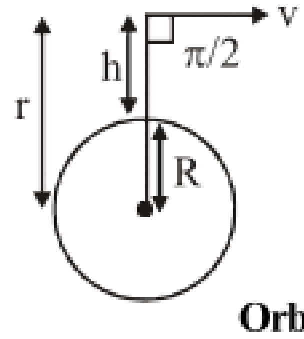
$$-\frac{1}{2a} = \frac{3}{10R} - \frac{1}{2R} = -\frac{2}{10R}$$

$2a = 5R = \text{Major axis}$

$= 3R$

Conditions for different trajectroy :

For a body being projected tangentially from above earth's surface, say at a distance r from earth's center, the trajectory would depend on the velocity of projection v.



	Velocity	**Orbit**
1.	velocity, $v < \sqrt{\frac{GM}{r}\left(\frac{2R}{r+R}\right)}$	Body returns to earth following ellipitical Path.
2.	$\sqrt{\frac{GM}{r}} > v > \sqrt{\frac{GM}{r}\left(\frac{2R}{r+R}\right)}$	Body acquires an elliptical orbit with earth as the far-focus w.r.t. the point of projection.
3.	Velocity is equal to the critical velocity of the orbit, $v = \sqrt{\frac{GM_e}{r}}$	Circular orbit with radius r
4.	Velocity is between the critical and escape velocity of the orbit $\sqrt{\frac{2GM_e}{r}} > v > \sqrt{\frac{GM_e}{r}}$	Body acquires an elliptical orbit with earth as the near focus w.r.t. the point of projection.
5.	$v = v_{esc} = \sqrt{\frac{2GM_e}{r}}$	Body just escapes earth's gravity, along a parabolic path.
6.	$v > v_{esc} = \sqrt{\frac{2GM_e}{r}}$	Body escape earth's gravity along a hyperbolic path.

Practice Exercise

Q.1 A planet of mass M moves around the sun along an ellipse so that its minimum distance from the sun is equal to r and the maximum distance to R. Making use of Kepler's law, find the its period of revolution around the sun.

Q.2 What should be the orbit radius of a communication satellite so that it can cover 75% of the surface area of earth during its revolution.

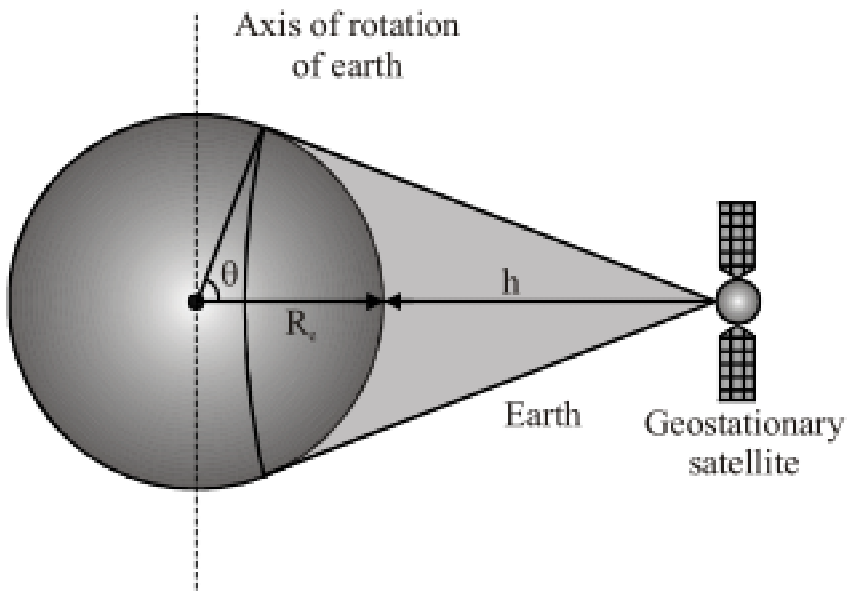
Practice Exercise

Q.1 $\sqrt{\frac{(r+R)^3}{2GM_s}}$ Q.2 $1.15\,R_e$

Broadcasting Region of a Satellite :

Now as we know the height of a geostationary satellite we can easily find the area of earth exposed on the satellite or area of the region in which the communication can be mode using this satelline.
Figure shows earth and its exposed area to a geostationary satellite. Here the angle θ can be given as

$$\theta = \cos^{-1}\left(\frac{R_e}{R_e + h}\right)$$



Now we can find the solid angle Ω which the exposed area subtend on earth's centre as

$$\Omega = 2\pi(1 - \cos\theta)$$

$$= 2\pi\left(1 - \frac{R_e}{R_e + h}\right) = \frac{2\pi h}{R_e + h}$$

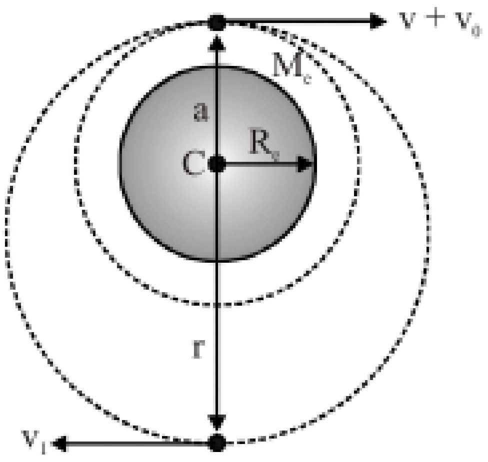
Thus the area of earth's surface to geostationary satellite is

$$S = \Omega R_e^{\,2} = \frac{2\pi h R_e^2}{R_e + h}$$

Solved Example

Q.1 A satellite is revoling round the earth in a circular orbit a rasius a with velocity v_0. A particle is projected from the satellite in forward direction with relative velocity $v = \left(\sqrt{5/4} - 1\right)v_0$. Calculate, during subsequent motion of the particle its minimum and maximum distance from earth's centre.

Sol. The corresponding situation is shown in figure



Initial velocity of statellite $\quad v_0 = \sqrt{\left(\frac{GM}{a}\right)}$

When particle is thrown with the velocity v relative to satellite, the resultant velocity of particle will become

$$v_R = v_0 + v$$

$$= \sqrt{\left(\frac{5}{4}\right)}\, v_0 = \sqrt{\left(\frac{5}{4}\frac{GM}{a}\right)}$$

As the particle velocity is greater than the velocity requried for ciruclar orbit, hence the particle path deviates from circular path to elliptical path. At positions of minimum and maximum distance velocity vector are perpendicular to instantaneous radius vector. In this elliptical path the minimum distance of particle from earth's centre is a and maximum speed in the path is v_R and let the maximum distance and minimum speed in the path is r and v_1 respectively.

Now as angular momentum and total energy remain conserved. Applying the law of conservation of angular momentum, we have

$$mv_1 r = m(v_0 + v)a \qquad [m = \text{mass of particle}]$$

or

$$v_1 = \frac{(v_0 + v)a}{r}$$

$$= \frac{a}{r}\left[\sqrt{\left(\frac{5}{4}\frac{GM}{a}\right)}\right]$$

$$= \frac{1}{r}\left[\sqrt{\left(\frac{5}{4} \times GMa\right)}\right]$$

Applying the law of conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r} = \frac{1}{2}m(v_0 + v)^2 - \frac{GMm}{a}$$

or $$\frac{1}{2}m\left(\frac{5}{4}\frac{GMa}{r^2}\right)-\frac{GMm}{r}=\frac{1}{2}m\left(\frac{5}{4}\frac{GM}{a}\right)-\frac{GMm}{a}$$

$$\frac{5}{8}\times\frac{a}{r^2}-\frac{1}{r}=\frac{5}{8}\times\frac{1}{a}-\frac{1}{a}=\frac{3}{8a}$$

or $$3r^2-8a\,r+5\,a^2=0$$

or $$r=a \text{ or } \frac{5a}{3}$$

Thus minimum disatnce of the particle = a

And maximum distance of the particle = $\frac{5a}{3}$

Q.2 A satellite is revolving around a planet of mass M in an elliptic orbit of semimajor axis a. Show that the orbital speed the satellite when it is at a distance r from the focus will be given by :

$$v^2=GM\left[\frac{2}{r}-\frac{1}{a}\right]$$

Sol. As in case of elliptic orbit with semi major axis a, of a satellite total mechanical energy remains constant, at any position of satellite in the orbit, given as

$$E=-\frac{GMm}{2a}$$

or $$KE+PE=-\frac{GMm}{2a} \qquad ...(i)$$

Now, if at postion r, v is the orbital speed of satellite, we have

$$KE=\frac{1}{2}mv^2 \text{ and } PE=-\frac{GMm}{r} \qquad ...(ii)$$

so from equation (i) and (ii), we have

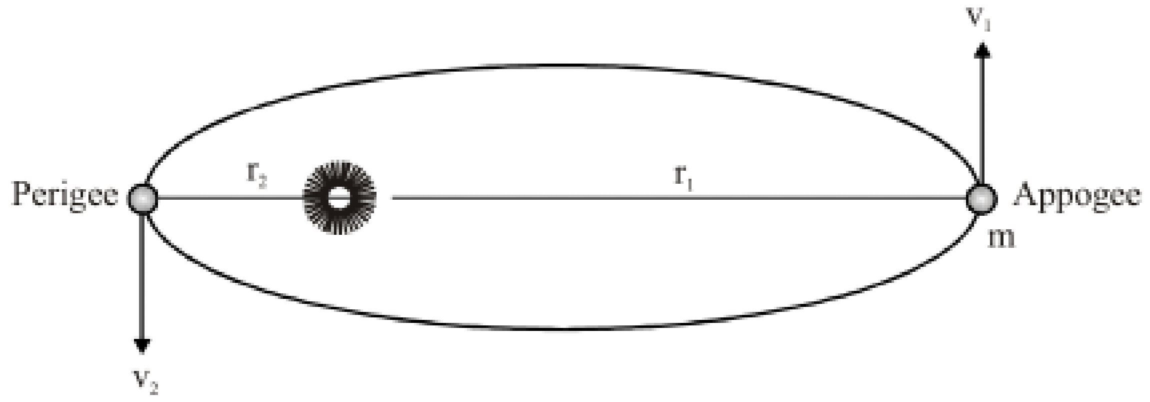
$$\frac{1}{2}mv^2-\frac{GMm}{r}=-\frac{GMm}{2a}, \text{ i.e. } \qquad v^2=GM\left[\frac{2}{r}-\frac{1}{a}\right]$$

Q.3 A planet of mass m moves along an ellipse around the sun so that its maximum and minimum distance from the sun are equal to r_1 and r_2 respectively. Find the angular momentum of this plane relative to the centre of the sun.

Sol. If v_1 and v_2 are the velocity or planet at its apogee and perigee respectively then according to conservation of angular momentum, we have

$$m\,v_1\,r_1=mv_2\,r_2$$

or $$v_1\,r_1=v_2\,r_2$$



As the total energy of the planet is also constant, we have

$$-\frac{GMm}{r_1}+\frac{1}{2}mv_1^2=-\frac{GMm}{r_2}+\frac{1}{2}mv_2^2$$

Where M is the mass of the sun.

or $$GM\left[\frac{1}{r_1}-\frac{1}{r_2}\right]=\frac{v_2^2}{2}-\frac{v_1^2}{2}$$

or $$GM\left(\frac{r_1-r_2}{r_1r_2}\right)=\frac{v_1^2r_1}{2r_2^2}-\frac{v_1^2}{2}$$

or $$GM\left(\frac{r_1-r_2}{r_1r_2}\right)=\frac{v_1^2}{2}\left(\frac{r_1^2}{r_2^2}-1\right)$$

$$=\frac{v_1^2}{2}\left(\frac{v_1^2-v_2^2}{v_2^2}\right)$$

or $$=v_1^2=\frac{2GM(r_1-r_2)r_2^2}{r_1r_2(r_1^2-r_2^2)}=\frac{2Gr_2}{r_1(r_1+r_2)}$$

or $$v_1=\sqrt{\left[\frac{2GMr_2}{r_1(r_1+r_2)}\right]}$$

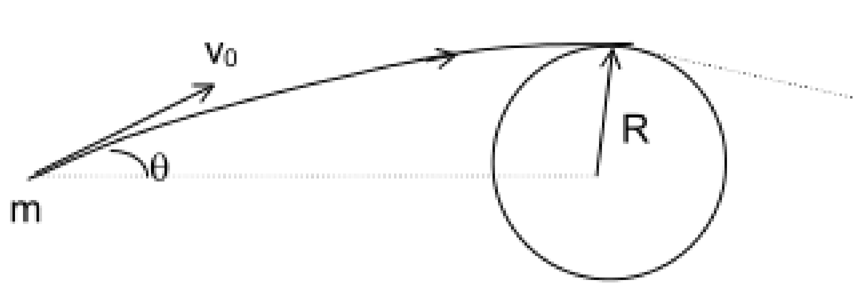
Now Angular momentum of planet is given as

$$L=mv_1r_1$$

$$=m\sqrt{\left[\frac{2GMr_1r_2}{(r_1+r_2)}\right]}$$

Q.4 A spaceship is sent to investigate a planet of mass M and radius R. While hanging motionless in space at a distance 5R from the center of the planet, the spaceship fires an instrument package with speed v_0 as shown in the figure. The packages has mass m, which is much smaller than the mass of the spaceship. For what angle θ will the package just graze the surface of the planet?

Sol. Let the speed of the instrument package is v when it grazes the surface of the planet.



Conserving angular momentum of the package about the centre of the planet $mv_0\times 5R\sin(\pi-\theta)=mvR\sin 90^\circ$

Conserving mechanical energy

$$-\frac{GMm}{5R}+\frac{1}{2}mv_0^2=-\frac{GMm}{R}+\frac{1}{2}mv^2$$

$$\Rightarrow \frac{1}{2}m(v^2-v_0^2)=\frac{4GMm}{5R}$$

$$v^2-v_0^2=\frac{8GM}{5R}$$

substituting the value of v from equation (I) in equation (ii)

$$25\ v_0^2\sin^2\theta-v_0^2=\frac{8GM}{5R}\Rightarrow \sin\theta\ \frac{1}{5}\sqrt{1+\frac{8GM}{5v_0^2R}}$$

$$\text{or } \theta=\sin^{-1}\left[\frac{1}{5}\sqrt{\left(1+\frac{8GM}{5v_0^2R}\right)}\right]$$

Q.5 A missile is launched at an angle of 60° to the vertical with a velocity $\sqrt{0.75gR}$ from the surface of the earth (R is the radius of the earth). Find its maximum height from the surface of earth. (Neglect air resistance and rotation of earth.)

From conservation of mechanical energy

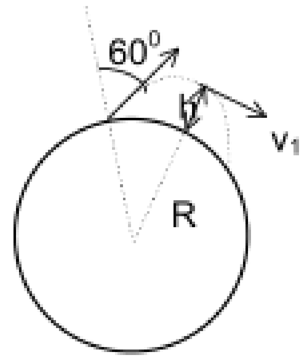
$$\frac{1}{2}mv_0^2-\frac{GMm}{R}=\frac{1}{2}mv_1^2-\frac{GMm}{R+h} \qquad ...(i)$$

Also from conservation of angular momentum

$$mv_0R\sin 60^\circ=mv_1(R+h) \qquad ...(ii)$$

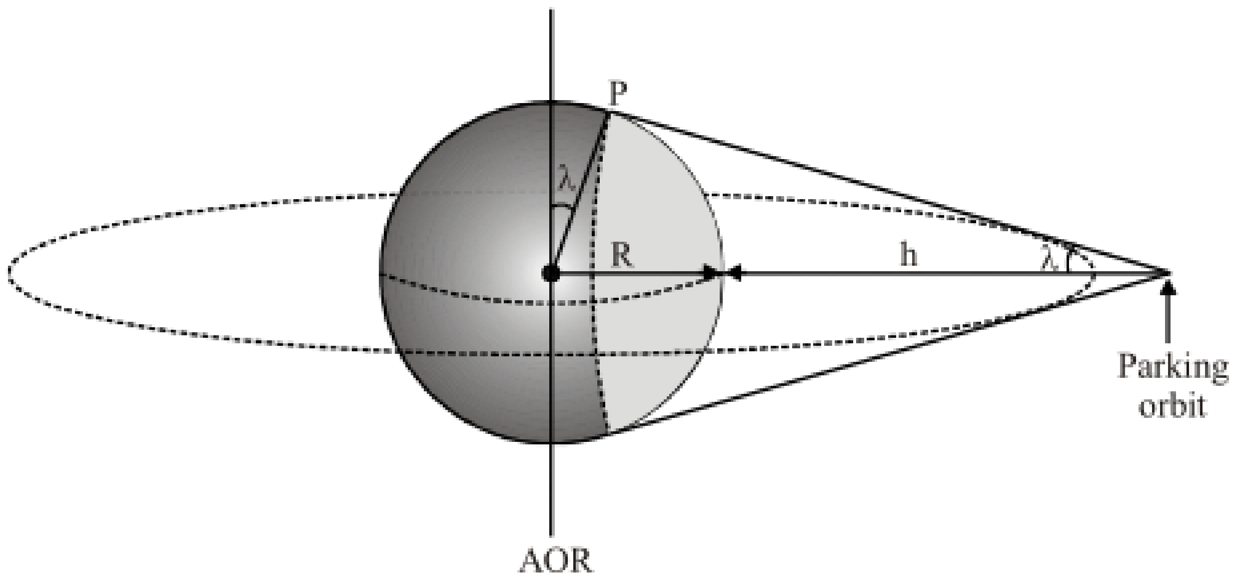
Solving (i) and (ii) and putting $v_0=\sqrt{\frac{3GM}{4R}}$, we get

$h \approx 0.25$ R.



Q.6 Find the minimum colatitude which can directly receive a signed from a geostationary satellite.

Sol. The farthest point on earth, which can receive signals from the parking orbit is the point where a length is drawn on earth surface from satellite as shown in figure



The colatitude λ of point P can be obtained from figure as

$$\sin\lambda = \frac{R_e}{R_e+h} \simeq \frac{1}{7}$$

We know for a partking orbit $h \simeq 6\,R_e$

Thus we have $$l = \sin^{-1}\left(\frac{1}{7}\right)$$

Q.7 A satellite of mass m is orbiting the earth in a circular orbit of radius r. It starts losing energy slowly at a constant rate C due to friction. If M_e and R_e denote the mass and radius of the earth respectively, show that the satellite falls on the earth in a limit time t given by

$$t = \frac{G\,m\,M_e}{2C}\left(\frac{1}{R_e} - \frac{1}{r}\right)$$

Sol. Let velocity of satellite in its orbit of radius r be v then we have

$$v = \sqrt{\frac{G\,M_e}{r}}$$

When satellite approaches earth's surface, if its velocity becomes v' then it is given as

$$v' = \sqrt{\frac{G\,M_e}{R_e}}$$

The total initial energy of satellite at a distance r is

$$E_{T_i} = K_i + U_i$$

$$= \frac{1}{2}mv^2 - \frac{GM_e m}{r}$$

$$= -\frac{1}{2}\frac{GM_e m}{r}$$

The total final energy of satellite at a distance R_e is

$$EE_{T_f} = K_f + U_f$$

$$= \frac{1}{2}mv'^2 - \frac{GM_e m}{R_e}$$

$$= \frac{1}{2}\frac{GM_e m}{R_e}$$

As satellite is loosing energy at a rate C, if it take a time t in reaching earth, we have

$$Ct = E_{T_f} - E_{T_f}$$

$$= \frac{1}{2} GM_e m \left[\frac{1}{R_e} - \frac{1}{r}\right]$$

or $$t = \frac{GM_e m}{2C}\left[\frac{1}{R_e} - \frac{1}{r}\right]$$

Q.8 Two Earth's satellites move in a common plane along circular orbits. The orbital radius of one satellite r = 7000 km while that of the other satellite is $\Delta r = 70$ km less. What time interval separates the periodic approaches of the satellites to each other over the minimum distance ?

Sol. Now for first satellite which is revolving about the earth (mass M and radius r) the orbital speed is

$$v = \sqrt{\frac{GM}{r}}$$

Let T_1 and T_2 be the time period for first and second satellites respectively. Then we know that

$$T_1 = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{GM}} r^{3/2}$$

and $$T_2 = \frac{2\pi}{\sqrt{GM}} (r - \Delta r)^{3/2}$$

As second satellite is revolving in a radius $(r - \Delta r)$. Know the period interval $(T_1 - T_2)$ is given by

$$T_1 - T_2 = \frac{2\pi}{\sqrt{GM}}\left[r^{3/2} - r^{3/2}\left(1 - \frac{\Delta r}{r}\right)^{3/2}\right]$$

$$= \frac{2\pi}{\sqrt{GM}} r^{3/2}\left[1 - \left(1 - \frac{3}{2}\frac{\Delta r}{r}\right)\right]$$

$$= \frac{2\pi}{\sqrt{GM}} r^{3/2}\left(\frac{3}{2}\frac{\Delta r}{r}\right).$$

Current Electricity

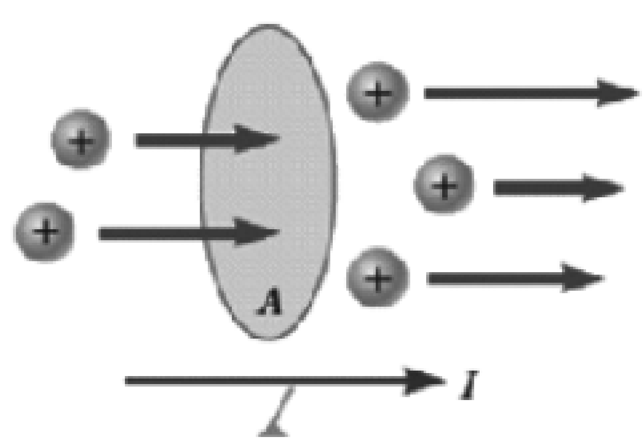
Introduction

In the last chapter we discussed electrostatics-the physics of stationary charges. In this chapter, we discuss the physics of electric currents-that is, charges in motion.

Examples of electric currents abound and involve many professions. Meteorologists are concerned with lightning and with the less dramatic slow flow of charge through the atmosphere. Biologists, physiologists, and engineers working in medical technology are concerned with the nerve currents that control muscles and especially with how those currents can be reestablished after spinal cord injuries. Electrical engineers are concerned with countless electrical systems, such as power systems, lightning protection systems, information storage systems, and music systems. Space engineers monitor and study the flow of charged particles from our Sun because that flow can wipe out telecommunication systems in orbit and even power transmission systems on the ground.

In this chapter we discuss the basic physics of electric currents and why they can be established in some materials but not in others. We begin with the meaning of electric current.

Electric Current



The time rate of flow of charge through any cross-section is called current.

$$i = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

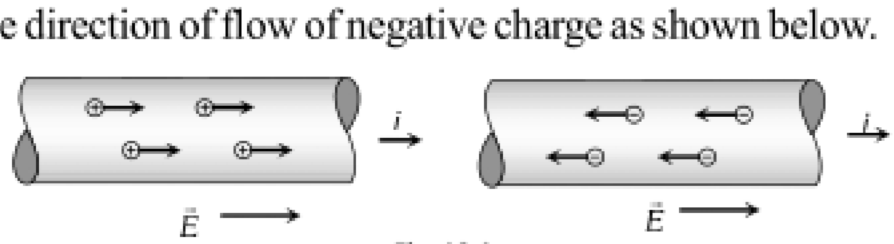
If flow is uniform then $i = \frac{Q}{t}$.

Current is a scalar quantity. It's S.I. unit is ampere (A) and C.G.S. unit is emu and is called biot (Bi), or ab ampere. 1A = (1/10) Bi (ab amp.)

Note :

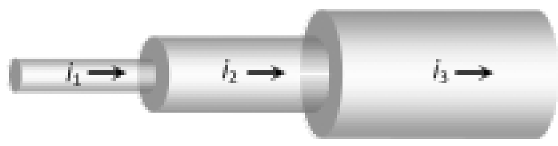
(1) Ampere of current means the flow of 6.25×10^{18} electrons/sec through any cross-section of the conductor.

(2) The conventional direction of current is taken to be the direction of flow of positive charge, i.e. field and is opposite to the direction of flow of negative charge as shown below.

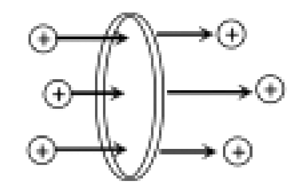


(3) The net charge in a current carrying conductor is zero.

(4) For a given conductor current does not change with change in cross-sectional area. In the following figure $i_1 = i_2 = i_3$



(5) Current due to translatory motion of charge : If n particle each having a charge q, pass through a given area in time t then $i = \frac{nq}{t}$

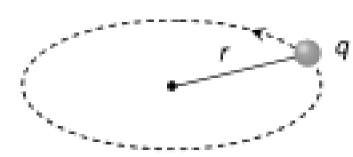


If n particles each having a charge q pass per second per unit area, the current associated with cross-sectional area A is $\mathbf{i = nqA}$

If there are n particle per unit volume each having a charge q and moving with velocity v, the current thorough, cross section A is $\mathbf{i = nq\upsilon A}$

(6) Current due to rotatory motion of charge : If a point charge q is moving in a circle of radius r with speed ν (frequency ν, angular speed ω and time period T) then corresponding current

$$i = q\nu = \frac{q}{T} = \frac{q\upsilon}{2\pi r} = \frac{q\omega}{2\pi}$$



(7) **Current carriers :** The charged particles whose flow in a definite direction constitutes the electric current are called current carriers. In different situation current carriers are different.

(i) **Solids :** In solid conductors like metals current carriers are free electrons.

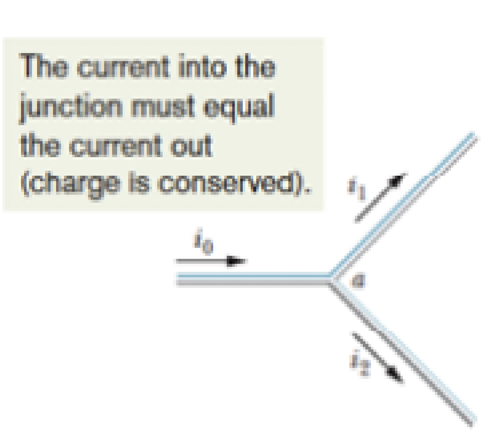
(ii) **Liquids :** In liquids current carriers are positive and negative ions.

(iii) **Gases :** In gases current carriers are positive ions and free electrons.

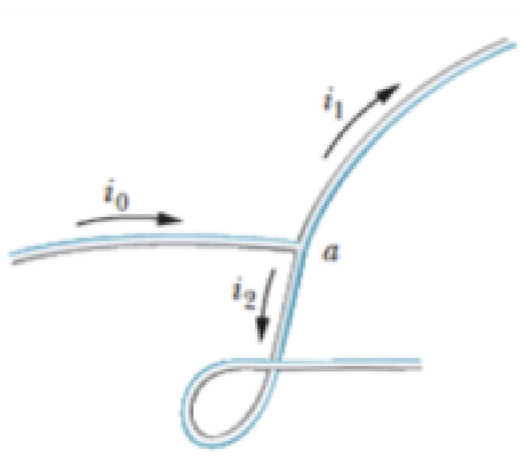
(iv) **Semi conductor :** In semi conductors current carriers are holes and free electrons.

(8) Current, as defined by above Equation, is a scalar because both charge and time in that equation are scalars. Yet, as in Figure (a), we often represent a current with an arrow to indicate that charge is moving. Such arrows are not vectors, however, and they do not require vector addition. Figure shows a conductor with current i_0 splitting at a junction into two branches. Because charge is conserved, the magnitudes of the currents in the branches must add to yield the magnitude of the current in the original conductor, so that

$$\mathbf{i_0 = i_1 + i_2}$$



(a)



(b)

As Figure (b) suggests, bending or reorienting the wires in space does not change the validity of above Equation. Current arrows show only a direction (or sense) of flow along a conductor, not a direction in space.

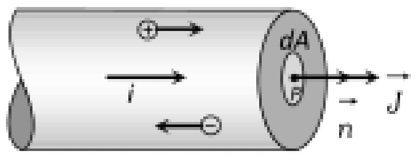
The relation $i_0 = i_1 + i_2$
is true at junction a no matter what the orientation in space of the three wires. Currents are scalars, not vectors.

9. Total charge flown through a cross section of conductor whoes current (i) is given will be $q = \int i\ dt$, we integrate with in prescribed limits to time

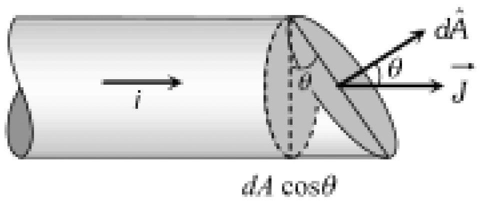
Current Density ($\vec{J}$)

Current density at any point inside a conductor is defined as a vector having magnitude equal to current per unit area surrounding that point.

Current density at point P is given by $\vec{J} = \frac{di}{dA}\vec{n}$



If the cross-sectional area is not normal to the current, but makes an angle to θ with the direction of current then



$$J = \frac{di}{dA\cos\theta} \Rightarrow di = JdA\cos\theta = \vec{J}.\vec{dA} = i = \int \vec{J}\cdot\vec{dA}$$

Note :

(1) Direction of $\vec{J}$ coincides with the direction of current flow at that point. So it is a vector quntity whose direction is defined with the electric field at that point.

(2) If current density $\vec{J}$ is uniform for a normal cross-section $\vec{A}$ then $J = \frac{i}{A}$

(3) Current density $\vec{J}$ is a vector quantity. It's direction is same as that of $\vec{E}$. It's S.I. unit is amp/m^2 and dimension [$L^{-2}A$].

(4) In case of uniform flow of charge through a cross-section normal to it as $i = nqvA$

so $J = \frac{i}{A} = nqv$

(5) Current density relates with electric field as $\vec{J} = \sigma\vec{E} = \frac{\vec{E}}{\rho}$; where σ = conductivity and ρ = resistivity or specific resistance of substance.

Illustration :

A copper wire of diameter 1.02 mm carries a current of 1.7 amp. Find the drift velocity (v_d) of electrons in the wire. Given n, number density of electrons in copper = 8.5×10^{28} /m³.

Sol. $I = 1.7$ A

J = current density

$$= \frac{1}{\pi r^2} = \frac{1.7}{\pi \times (0.51 \times 10^{-3})^2}$$

$= nev_d$

$= 8.5 \times 10^{28} \times (1.6 \times 10^{-19}) \times v_d$

$$\therefore \quad v_d = \frac{1.7}{\pi \times (0.51 \times 10^{-3})^2 \times 8.5 \times 10^{28} \times 1.6 \times 10^{-19}}$$

$= 1.5 \times 10^{-3}$ m/sec. = 1.5 mm/sec.

Illustration :

A solution of NaCl discharges 6.5×10^{16} Na^+ ions and 4.2×10^{16} Cl^- ions in 1 sec. Find the total current passing through the solution.

Sol. *The total current through a solution (conductor) is due to all the charge carriers (moving in opposite directions if they are oppositely charged.*

$$I_{tot} = \frac{6.5 \times 10^{16} + 4.2 \times 10^{16}}{1\,\text{sec}} \times e$$

$= 10.7 \times 10^{16}$ 1.6×10^{-19} coulomb/sec.

$= 1.7 \times 10^{-4}$ A

Illustration :

The magnitude J of the current density in a certain lab wire with a circular cross section of radius R = 2.00 mm is given by J = $(3.00 \times 10^8)r^2$, with j in amperes per square meter and radial distance r in meters. What is the current through the outer section bounded by r = 0.900R and r = R ?

Sol. *Assuming J is directed along the wire (with no radial flow) we integrate, starting*

$$i = \int |\vec{J}|\,dA = \int_{9R/10}^{R} (kr^2) 2\pi r dr = \frac{1}{2} k\pi (R^4 - 0.656R^4)$$

Where k = 3.0×10^8 and SI units are understood. Therefore if R = 0.00200m. We obtain i = 2.59×10^{-3} A.

Illustration :

What is the current in a wire of radius R = 3.40 mm if the magnitude of the current density is given by (a) $J_a = J_0/R$ and (b) $J_b = J_0 (1 - r/R)$, in which r is the radial distance and $J_0 = 5.50 \times 10^4$ A/m²? (c) Which function maximizes the current density near the wire's surface ?

Sol. *(a) The current resulting from this nonuniform current density is*

$$i = \int_{\text{cylinder}} J_a\, dA = \frac{J_0}{R}\int_0^R r\cdot 2\pi r dr = \frac{2}{3}\pi R^2 J_0 = \frac{2}{3}\pi(3.40\times10^{-3}\,\text{m})(5.50\times10^4\,\text{A/m}^2)$$

$= 1.33\ A.$

(b) *In this case*

$$i = \int_{\text{cylinder}} J_a dA = \frac{J_0}{R}\int_0^R r\cdot 2\pi r dr = \frac{2}{3}\pi R^2 J_0 = \frac{1}{3}\pi R^2 J_0 = \frac{1}{2}\pi(3.40\times10^{-3}\,\text{m})^2(5.50\times10^4\,\text{A/m}^2)$$

$= 0.666\ A$

(c) *The result is different from that in part (a) because J_b is higher near the center of the cylinder (where the area is smaller for the same radial interval) and lower outward, resulting in lower average current density over the cross section and consequently a lower current than that in part (a). So J_a has its maximum value near the surface of the wire.*

Practice Exercise

Q.1 A steady current passes through a cylindrical conductor. Is there an electric field inside the conductor?

Q.2 If 0.6 mol of electrons flow through a wire in 45 min what are (a) the total charge that passes through the wire, and (b) the magnitude of the current.

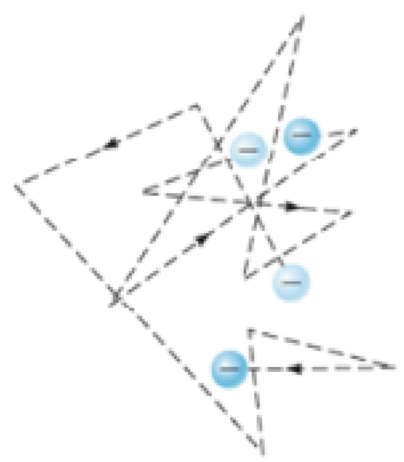
Answers

Q.1 Yes Q.2 (a) 5.7×10^4 C (b) 21.41 Amp

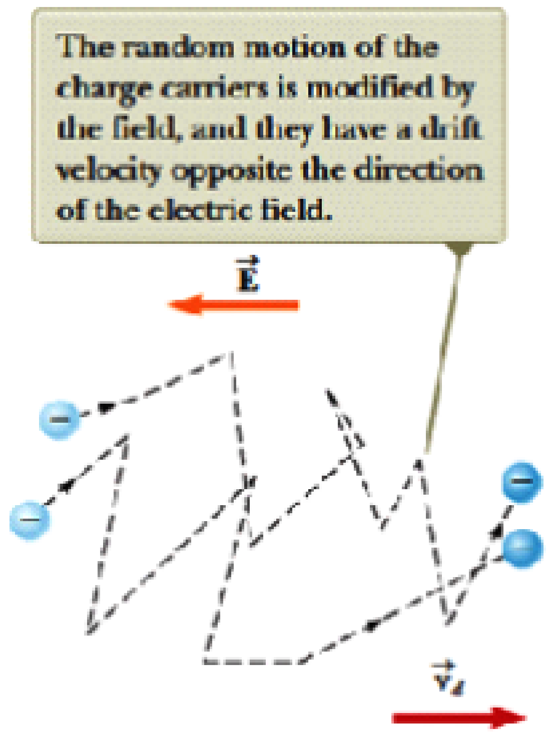
Model for Electric Conduction

We describe a classical model of electrical conduction in metals that was first proposed by Paul Drude (1863-1906) in 1900.

Consider a conductor as a regular array of atoms plus a collection of free electrons, which are sometimes called conduction electrons. The conduction electrons, although bound to their respective atoms when the atoms are not part of a solid, become free when the atoms condense into a solid. In the absence of an electric field, the conduction electrons move in random directions through the conductor Fig below. The situation is similar to the motion of gas molecules confined in a vessel. In fact, some scientists refer to conduction electrons in a metal as an electron gas.



When an electric field is applied, the free electrons drift slowly in a direction opposite that of the electric field (Figure Below), with an average drift speed v_d that is much smaller (typically 10^{-4}m/s) than their average speed between collisions (typically 10^6m/s).



In our model, we make the following assumptions:

1. The electron's motion after a collision is independent of its motion before the collision.
2. The excess energy acquired by the electrons in the electric field is transferred to the atoms of the conductor when the electrons and atoms collide.

We are now in a position to derive an expression for the drift velocity. When a free electron of mass m_e and charge q (= – e) is subjected to an electric field $\vec{E}$, it experiences a force $\vec{F} = q\vec{E}$. The electron is a particle under a net force, and its acceleration can be found from Newton's second law, $\sum\vec{F} = m\vec{a}$:

$$\vec{a} = \frac{\sum\vec{F}}{m} = \frac{q\vec{E}}{m_e}$$

Because the electric field is uniform, the electron's acceleration is constant, so the electron can be modeled as a particle under constant acceleration. If $\vec{v}_i$ is the electron's initial velocity the instant after a collision (which occurs at a time defined as t = 0), the velocity of the electron at a very short time t later (immediately before the next collision occurs) is, from equation

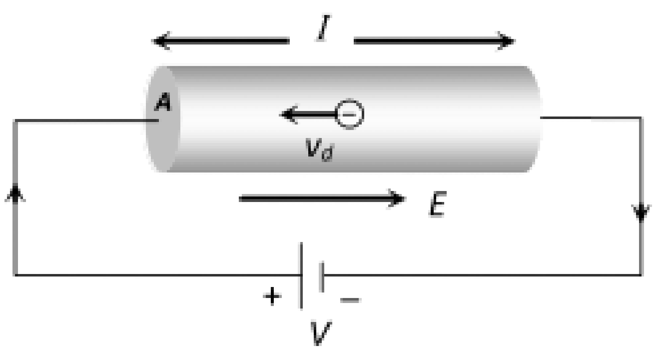
$$\vec{v}_f = \vec{v}_i + \vec{a}t = \vec{v}_i + \frac{q\vec{E}}{m_e}t$$

Let's now take the average value of $\vec{v}_f$ for all the electrons in the wire over all psoible collision times t and all possible value of $\vec{v}_i$. Assuming the initial velocities are randomly distributed over all possible directions, the average value of $\vec{v}_i$ is zero. The average value of the second terms of equation is $(q\vec{E}/m_\ell)\,\tau$, where τ is the average time interal between successive collisions. Because the average value of $\vec{v}_f$ is equal to the drift velocity.

$$\vec{v}_{f,avg} = \vec{v}_d = \frac{q\vec{E}}{m_e}\tau$$

Drift Velocity

Drift velocity is the average uniform velocity acquired by free electrons inside a metal by the application of an electric field which is responsible for current through it. Drift velocity is very small it is of the order of 10^{-4} m/s as compared to thermal speed ($\simeq 10^5$ m/s) of electrons at room temperature.



If suppose for a conductor
n = Number of electron per unit volume of the conductor
A = Area of cross-section
V = potential difference across the conductor
E = electric field inside the conductor

i = current, J = current density, ρ = specific resitance, σ = conductivity $\left(\sigma = \frac{1}{\rho}\right)$

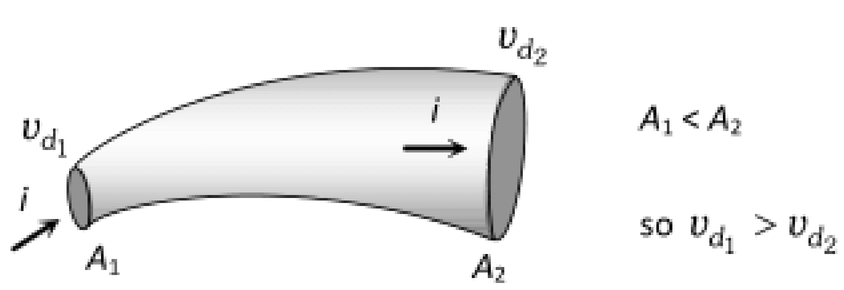
then current relates with drift velocity as $i = neAv_d$ ***we can also write***

$$v_d = \frac{i}{neA} = \frac{J}{ne} = \frac{\sigma E}{ne} = \frac{E}{\rho ne} = \frac{V}{\rho l ne}$$

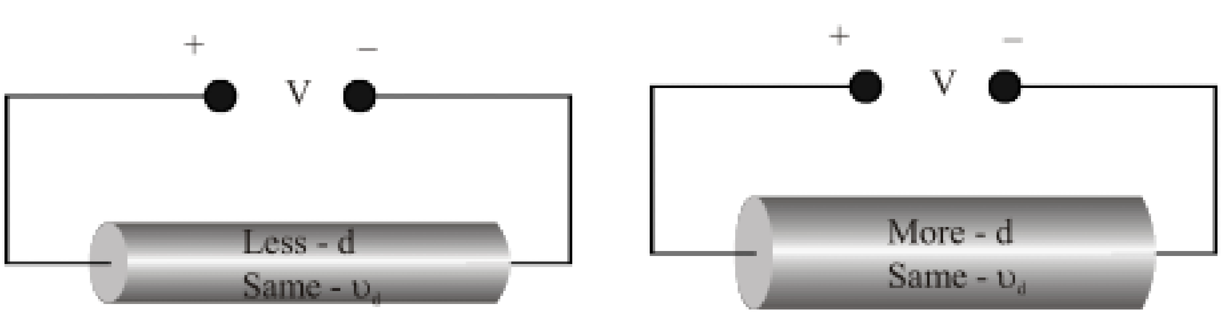
(1) The direction of drift velocity for electron in a metal is opposite to that of applied electric field (i.e. current density $\vec{j}$).

$v_d \propto E$ i.e. greater the electric field, larger will be the drift velocity

(2) When a steady current flows through a conductor of non-uniform cross-section drift velocity varies inversely with area of cross-section $\left(v_d \propto \frac{1}{A}\right)$



(3) If diameter (d) of a conductor is doubled, then drift velocity of electrons inside it will not change.



(4) **Relaxation time (τ) :** The time interval between two successive collisions of electrons with the positive ions in the metallic lattice is defined as relaxation time $\tau = \dfrac{\text{mean free path}}{\text{r.m.s. velocity of electrons}} = \dfrac{\lambda}{v_{rms}}$. With rise in temperature υ_{rms} increases consequently τ decreases.

(5) **Mobility :** Drift velocity per unit electric field is called mobility of electron i.e. $\mu = \dfrac{\upsilon_d}{E}$. It's unit is $\dfrac{m^2}{\text{volt} - \text{sec}}$

Illustration :

Find the electric current in a conductor (copper) of cross-section $A = 1nm^2$, conduction electron density $n = 8.69 \times 10^{28}/m^2$ and drift speed $v_d = 1$ cm /s.

Sol. $i = nev_d A$

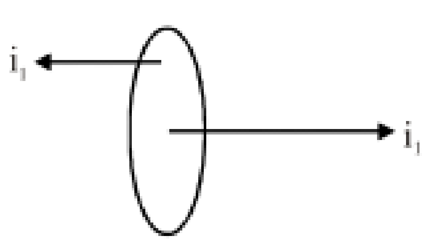
$= 8.69 \times 10^{28} \times 1.6 \times 10^{-19} \times 10^{-2} \times 1 \times 10^{-4}$

$= 8.69 \times 1.6 \times 10^{5}$ amp

Illustration :

n_1 electron/s passes through a given cross-section towards right with velocity v_1 and n_2 proton/s passes through the same cross-section with velocity v_2 in the same direction. Find the current through a given cross-sectional. Put $n_1 = 1.5 \times 10^{10}$ and $n_2 = 10^{10}$.

Sol. $i_1 = \dfrac{\Delta q}{\Delta r} = \dfrac{\Delta N_1 q_1}{\Delta r} = \dfrac{dN_1}{dt} q_1$



$i_2 = \dfrac{dN_2}{dt} q_2$

$i = i_1 + i_2$

$= \left(\dfrac{dN_1}{dt}\right)(-e) + \left(\dfrac{dN_2}{dt}\right)e$

$i = (n_2 - n_1)\, e$

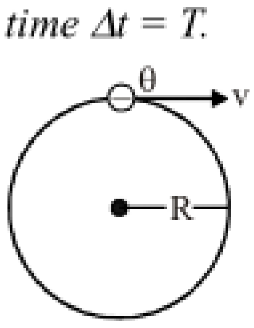
$= (1.5 \times 10^{10} - 1 \times 10^{10})\ 1.6 \times 10^{-19} = 0.5 \times 10^{-9}$ amp

Illustration :

Find the current associated with an electron revolving with a speed $v = 10^6$ m/s in an orbit of radius $R = 1Å$.

Sol. *The charge Δq (= –e) flows (passes) through a fixed point during a time $\Delta t = T$.*

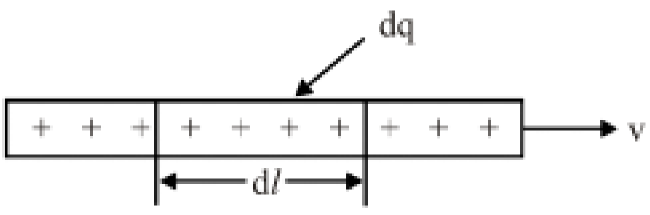
Then, $i = \dfrac{\Delta q}{\Delta t} = \dfrac{e}{T}$, *where* $T = \dfrac{2\pi R}{v}$



or, $i = \dfrac{ev}{2\pi R} = \dfrac{(1.6 \times 10^{-19})(10^6)}{2 \times \dfrac{22}{7} \times (10^{-10})} \approx 0.26 \times 10^{-3}$ A

Illustration :

Find the current associated with a moving straight wire of linear charge density $\lambda = 2\ \mu C/m$ and of cross-section $A = 2\ mm^2$, when the wire is pulled with a speed $v = 2\ m/s$.



Sol. *Let dq (= λ dl) passes through a given vertical plane in time dt.*

Then, $i = \frac{dq}{dt}$

$= \frac{\lambda dl}{dt}$

$= lv \qquad \left(\because v = \frac{dl}{dt}\right)$

$= 2 \times 10^{-3} \times 2 = 4\ mA$

Illustration :

A homogeneous beam of proton accelerated through a potential difference $V = 500\ KV$ has a circular cross-section of radius $R = 4\ mm$. Assuming beam current $i = 32 \times 10^{-3}\ A$. Find the
(i) number of protons passing through a cross-section per second.
(ii) electric field at the surface of the beam.
(iii) potential difference between the surface and axis of the beam.

Sol. *(i) The numer of protons/second*

$$= \frac{i}{e} = \frac{32\times10^{-3}}{1.6\times10^{-19}} = 2\times10^{16}$$

(ii) $E = \frac{\lambda}{2\pi\varepsilon_0 r}$, *where* $l = \frac{i}{v}$

or, $$E = \frac{i}{2\pi\varepsilon_0 rv} \qquad ...(i)$$

since $\frac{1}{2}mv^2 = eV$, *substituting* $v = \sqrt{\frac{2eV}{m}}$ *in eq. (i).*

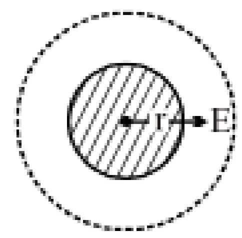
$$E = \frac{1}{2\pi\varepsilon_0 R}\sqrt{\frac{m}{2eV}}$$

$$= \frac{2\times9\times10^9\times32\times10^{-3}}{4\times10^{-3}}\sqrt{\frac{1.6\times10^{-27}}{2\times1.6\times10^{-19}\times500\times10^3}} \qquad \left(\because \frac{1}{4\pi\varepsilon_0} = 9\times10^9\right)$$

$= 144 \times 10^9 \times 10^{-7}\ V/m$

$= 14.4\ KV/m$

(iii) Applying Gauss Law.



$$E.\ 2\pi rl = \left(\frac{Q_0}{Q\pi R^2 l}\right)\left(\frac{Q\pi r^2 l}{\varepsilon_0}\right)$$

or, $$E = \frac{Qr}{2\pi\varepsilon_0 R^2 l} = \frac{\lambda r}{2\pi\varepsilon_0 R^2}$$

Then, $$\Delta V = \int_0^R E\ dr = \frac{\pi}{2\pi\varepsilon_0 R^2}\int_0^R r\ dr = \frac{\lambda}{4\pi\varepsilon_0}, \qquad \text{where } \lambda = i\sqrt{\frac{m}{2eV}}$$

or, $$\Delta V = \frac{i}{4\pi\varepsilon_0}\sqrt{\frac{m}{2eV}} = \frac{ER}{2} = \frac{14.4\times10^3\times4\times10^{-3}}{2} = 28.8\ V$$

Practice Exercise

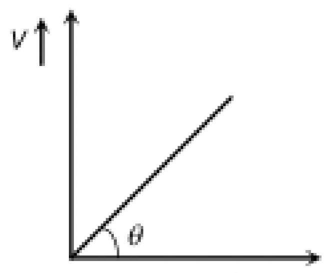
Q.1 A beam of fast moving electrons having cross-sectional area $A = 1\ cm^2$ falls normally on a flat surface. The electrons are absorbed by the surface and the average pressure exerted by the electrons on this surface is found to be $P = 9.1$ Pa. If the electrons are moving with a speed $v = 8\times10^7$ m/s, then find the effective current (in A) through any cross-section of the electron beam.
(mass of electron $= 9.1\times10^{-31}$ kg)

Answers

Q.1 0002 A

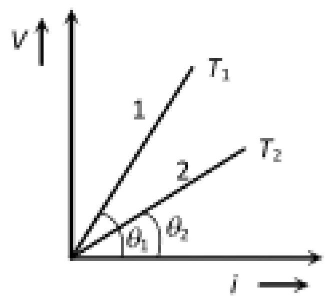
Ohm's Law

If the physical conditions of the conductor (length, temperature, mechanical strain etc.) remains same, then the current flowing through the conductor is directly proportional to the potential difference across it's two ends i.e. $i \propto V \Rightarrow V = iR$ where R is a proportionality constant, known as electric resistance.
(1) Ohm's law is not a universal law, the substances, which obey ohm's law are known as ohmic substance.
(2) Graph between V and i for a metallic conductor is a straight line as shown. At different temperatures V-i curves are different.



(A) Slope of the line

$= \tan\theta = \frac{V}{i} = R$



(B) Here $\tan\theta_1 > \tan\theta_2$

So $R_1 > R_2$

i.e. $T_1 > T_2$

Resistance

(1) The property of substance by virtue of which it opposes the flow of current through it, is known as the resistance.

(2) Formula of resistance : For a conductor if l = length of a conductor A = Area of cross-section of conductor, n = No. of free electrons per unit volume in conductor, τ = relaxation time then resistance of conductor $R = \rho \frac{l}{A} = \frac{m}{ne^2\tau} \cdot \frac{l}{A}$; where ρ = resistiveity of the material of conductor

(3) **Unit and dimension :** It's S.I. unit is Volt/Amp. or Ohm (Ω). Also 1 ohm = $= \frac{1\,\text{volt}}{1\,\text{Amp}} = \frac{10^8\,\text{emu of potential}}{10^{-1}\,\text{emu of current}} = 10^9$ emu of resistance. It's dimension is $[ML^2T^{-3}A^{-2}]$

(4) **Dependence of resistance :** Resistance of a conductor depends upon the following factors.

(i) Length of the conductor : Resistance of a conductor is directly proportional to it's length i.e. $R \propto l$ and and inversely proportional to it's area of cross-section i.e. $R \propto \frac{1}{A}$

(ii) Temperature : For a conductor
Resistance $\propto$ temperature.

If R_0 = resistance of conductor at 0°C
R_t = resistance of conductor at t°C

and α, β = temperature co-efficient of resistance then $R_t = R_0\,(1+\alpha t+\beta t^2)$ for $t > 300$°C and $R_t = R_0\,(1+\alpha t)$ for $t \le 300$°C or $\alpha = \frac{R_t - R_0}{R_0 \times t}$

If R_1 and R_2 are the resistance at t_1°C and t_2°C respectively then $\frac{R_1}{R_2} = \frac{1+\alpha t_1}{t+\alpha t_2}$

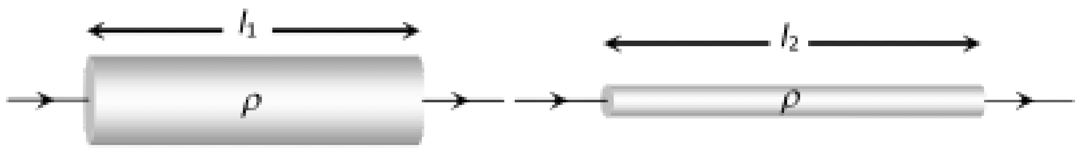
The value of α is different at different temperature rage t_1°C to t_2°C is given by $\alpha = \frac{R_2 - R_1}{R_1(t_2 - t_1)}$ which given $R_2 = R_1\,[1+\alpha(t_2 - t_1)]$. This formula gives an approximate value.

(5) **Stretching of wire**

If a conducting wire stretches, it's length increases, area of cross-section decreases so resistance increases but volume remain constant.

Suppose for a conducting wire before stretching it's length = l_1, area of cross-section = A_1 radius = r_1 diameter = d_1, and resistance $R_1 = \rho \frac{l_1}{A_1}$

Before stretching After streching



Volume remains constant *i.e.* $A_1 l_1 =$

After streching length = l_2 area of cross-section = A_2 radius = r_2 diameter = d_2 and resistance = $R_2 = \rho \frac{l_2}{A}$

Ratio of resistance before and after streching

$$\frac{R_1}{R_2}=\frac{l_1}{l_2}\times\frac{A_2}{A_1}=\left(\frac{l_1}{l_2}\right)^2=\left(\frac{A_2}{A_1}\right)^2=\left(\frac{r_2}{r_1}\right)^4=\left(\frac{d_2}{d_1}\right)^4$$

(i) If length is given then $R\propto l^2 \Rightarrow \frac{R_1}{R_2}=\left(\frac{l_1}{l_2}\right)^2$

(ii) If radius is given then $R\propto\frac{1}{r^4}\Rightarrow\frac{R_1}{R_2}=\left(\frac{r_2}{r_1}\right)^4$

Resistivity (ρ), Conductivity (σ) and Conductance (C)

(1) **Resistivity :** From $R=\rho\frac{l}{A}$; If l = 1m, A = 1 m² then R = ρ i.e. resistivity is numerically equal to the resistance of a substance having unit area of cross-section and unit length.

(i) Unit and dimension : It's S.I. unit ohm × m and dimension is $[ML^3T^{-3}A^{-2}]$

(ii) It's formula : $\rho=\frac{m}{ne^2\tau}$

(iii) Resistivity is the intrinsic property of the substance. It is independent of shape and size of the body (i.e. l and A).

(v) Resistivity depends on the temperature. For metals $\rho_t=\rho_0\,(1+\alpha\Delta t)$ i.e. resitivity increases with temperature.

(vi) Resistivity increases with impurity and mechanical stress.

(vii) Magnetic field increases the resistivity of all metals except iron, cobalt and nickel.

(viii) Resistivity of certain substances like selenium, cadmium, sulphides is inversely proportional to intensity of light falling upon them.

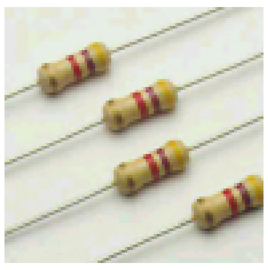
(2) **Conductivity :** Reciprocal of resistivity is called conductivity i.e. $s=\frac{1}{\rho}$ with unit mho/m and dimensions $[M^{-1}L^3T^{-3}A^2]$

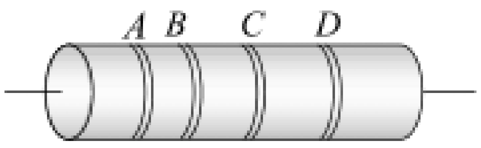
(3) **Conductance:** Reciprocal of resistance is known as conductance. $C=\frac{1}{R}$. It's unit is $\frac{1}{\Omega}$ or Ω^{-1} or "siemen".

Colour Coding of Resistance

To know the value of resistance colour code is used. These code are printed in form of set of rings or strips. By reading the values of colour bands, we can estimate the value of resistance.

The carbon resistance has normally four coloured rings or bands say A, B, C and D as shown in following figure.





Colour band A and B : Indicate the first two significant figures of resistance in ohm.

Band C : Indicates the decimal multiplier i.e. the number of zeros that follows the two significant figures A and B.

Band D : Indicates the tolerance in percent about the indicated value or in other words it represents the percentage accuracy of the indicated value.

The tolerance in the case of gold is ± 5% and in silver is ± 10%. If only three bands are marked on carbon resistance, then it indicate a tolerance of 20%.

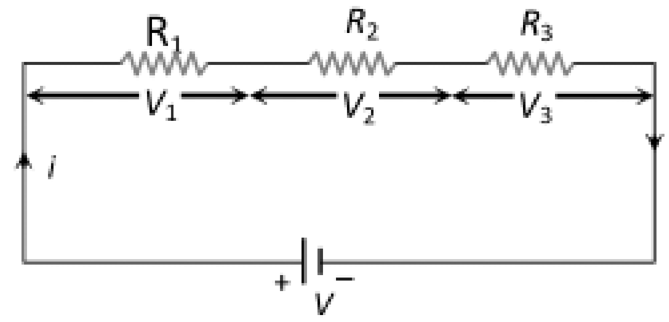
Table : Colour code for carbon resistance

Letters as an aid to memory	Colour	Figure (*A*, *B*)	Multiplier (C)
B	Black	0	10^0
B	Brown	1	10^1
R	Red	2	10^2
O	Orange	3	10^3
Y	Yellow	4	10^4
G	Green	5	10^5
B	Blue	6	10^6
V	Violet	7	10^7
G	Grey	8	10^8
W	White	9	10^9

Grouping of Resistance

(1) Series grouping

(i) Same current flows through each resistance but potential difference distributes in the ratio of resistance i.e. $V \propto R$

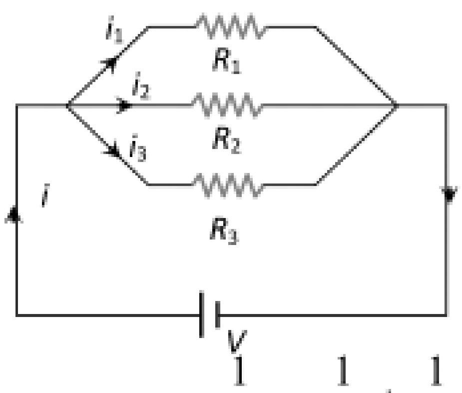


(ii) $R_{eq} = R_1 + R_2 + R_3$ equivalent resistance is greater than the maximum value of resistance in the combination.

(iii) If n identical resistance are connected in series $R_{eq} = nR$ and potential different across and resistance $V' = \dfrac{V}{n}$

(2) Parallel grouping

(i) Same potential difference appeared across each resistance but current distrbutes in the reverse ratio of their resistance i.e. $i \propto \dfrac{1}{R}$



(ii) Equivalent resistance is given by $\dfrac{1}{R_{eq}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$ or $R_{eq} = (R_1^{-1} + R_2^{-1} + R_3^{-1})^{-1}$ or

$$R_{eq} = \frac{R_1R_2R_3}{R_1R_2 + R_2R_3 + R_2R_1}$$

Equivalent resistance is smaller than the minimum value of resistance in the combination.

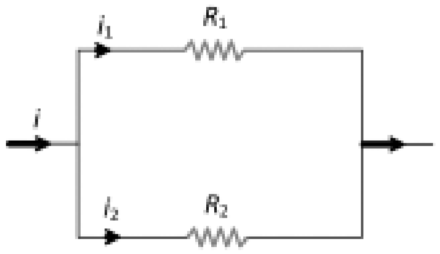
(iv) If two resistance in parallel $R_{eq} = \dfrac{R_1R_2}{R_1 + R_2}$

(v) Current through any resistance $i' = i \times \left[\dfrac{\text{Resistance of opposite branch}}{\text{Total resistance}}\right]$

Where i' = required current (branch current), i = main current

$$i_1 = i\left(\frac{R_2}{R_1 + R_2}\right)$$

and $$i_2 = i\left(\frac{R_1}{R_1 + R_2}\right)$$



(vi) In n identical resistance are connected in parallel $R_{eq} = \frac{R}{n}$ and current through each resistance $i' = \frac{i}{n}$

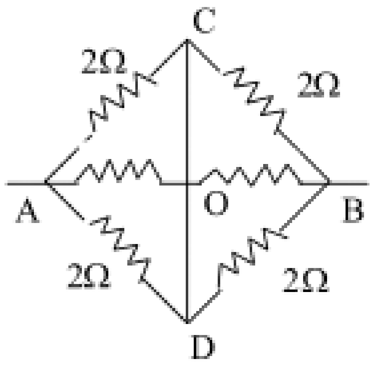
Note :

Rules for finding Req complicated resistance circuit:

- We can join any number of points in a circuit that are connected by a simple conducting wire as they will be at same potential.
- We can join any number of points in a circuit that are lying on plane of symmetry.
- We can break a single point in multiple points if after breaking new points formed are lying on plane of symmetry.

Illustration :

Find the equivalent resistance between A and B in the circuit shown here. Every resistance shown here is of 2Ω.



Sol. *Points C, O & D are at the same potential. Therefore, resistances AO, AC and AD are in parallel . Similarly BC, BO and BD are in parallel.*

$$\therefore \quad R_{AB} = \frac{1}{3} \times (2\Omega) + \frac{1}{3} \times (2\Omega)$$

Plane of symmetrg passes through c,o and D

$$= \frac{4}{3}\Omega = 1.33\Omega$$

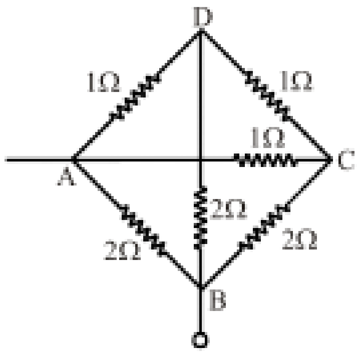
Illustration :

It is desired to make a 20 Ω coil of wire which has a zero thermal coefficient of resistance. To do this, a carbon resistor of resistance R_1, is placed in series with an iron resistor of resistance R_2. The proportions of iron and carbon are so chosen that $R_1 + R_2 = 20\ \Omega$ for all temperatures near 20°C, how large are R_1 and R_2 ? $(\alpha_c = -0.5 \times 10^{-3}, \alpha_{Fe} = 5 \times 10^{-3})$

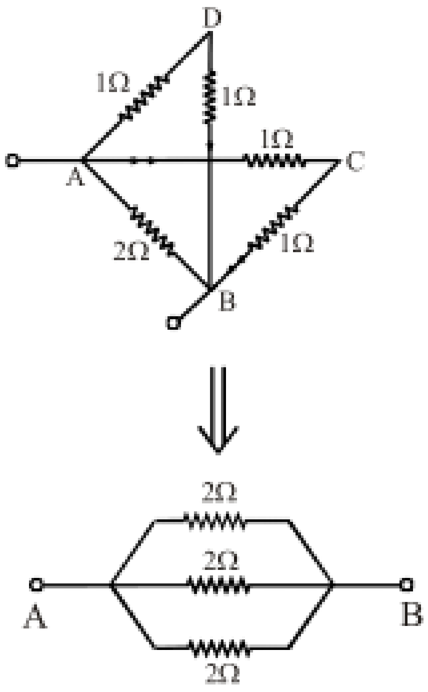
Sol. *We need $R_1 (1 + \alpha_1 \Delta t) + R_2 (1 + \alpha_2 \Delta T) = 20$ because $R_1 + R_2 = 20$ where $\Delta t = 0$,*
We must have $R_1\alpha_1 = - R_1\alpha_2$ with $\alpha_1 = - 0.5 \times 10^{-3}$ solving the two equation $R_1+R_2 = 20$ and $R_1 = 18.18\ \Omega$ and $R_2 = 1.82\ \Omega$.

Illustration :

Six resistors form a pyramid. Find the effective resistance between A and B.



Sol. *The branches ADB and ACB are symmetrical relative tot the terminals A and B. Hence, the points D and C are equipotential. Since, $R_{DC} \neq 0$ $i_{DC} = 0$. Then remove the branch DC and then the circuit is reduced to a simpler one as shown in the figure.*

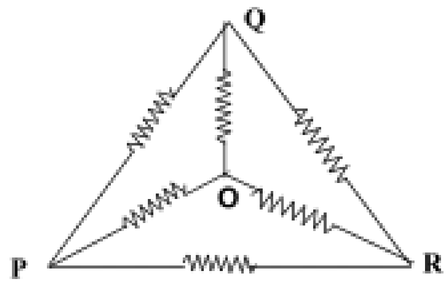


Then $$\frac{1}{R_{AB}} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

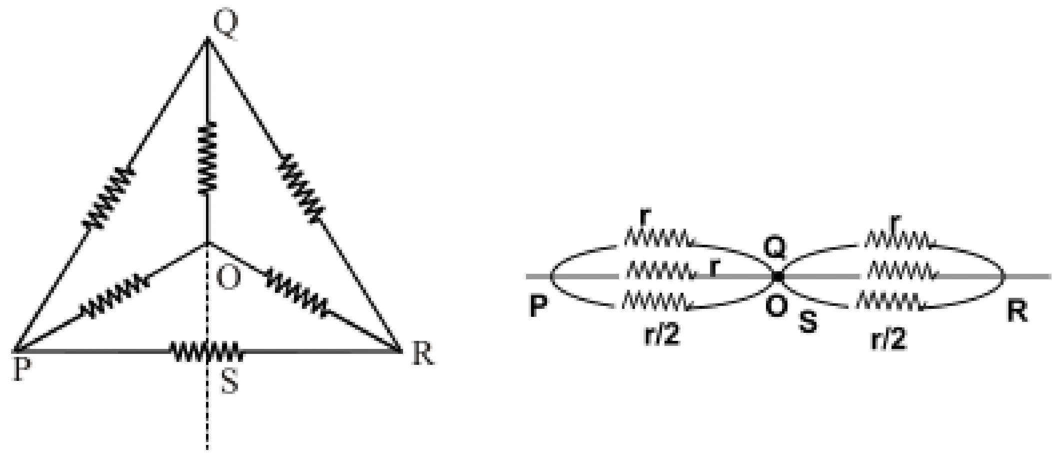
or, $$R_{AB} = \frac{2}{3}\Omega$$

Illustration :

Six equal resistances each of resistance 4Ω are connected to form the following figure. What is the resistance between any two corners.



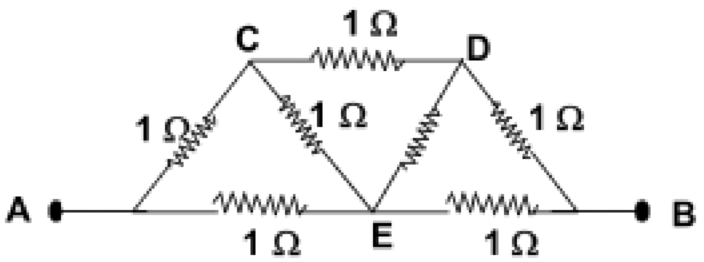
Sol. *There is symmetry about the line passing through QO and mid point of PR.*



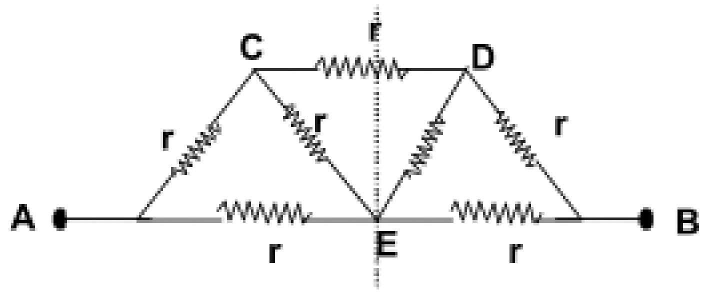
$$= \frac{r}{2} = 2\Omega$$

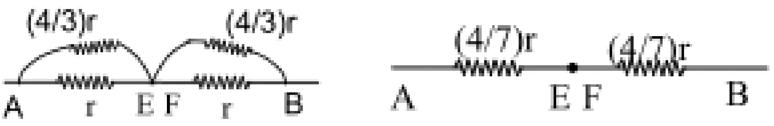
Illustration :

In the network shown in figure, each resistance is 1 Ω. What is the effective resistance between A and B



Sol. *There is a symmetry about line passing through E and mid point of CD.*

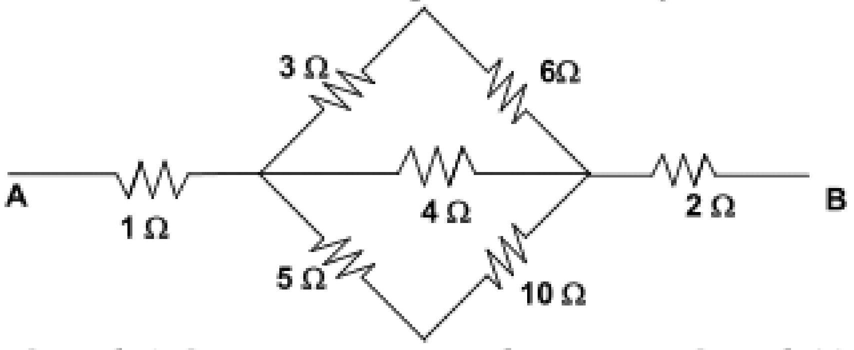




$$= \frac{8}{7}r = \frac{8}{7}\Omega$$

Illustration :

Find the equivalent resistance between points A & B of the network shown in the given diagram

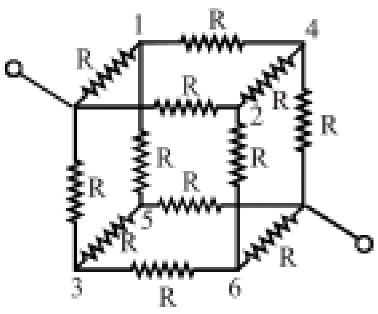


Sol. *The resistors 3 Ω and 6 Ω are in series and so are 5 Ω and 10 Ω resistors. These two series equivalents are in parallel to each other and also to the 4 Ω resistors. Hence the network reduces to the one given below :*

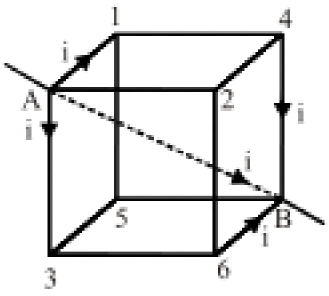
$$R_{eq} = 5.34\ \Omega$$

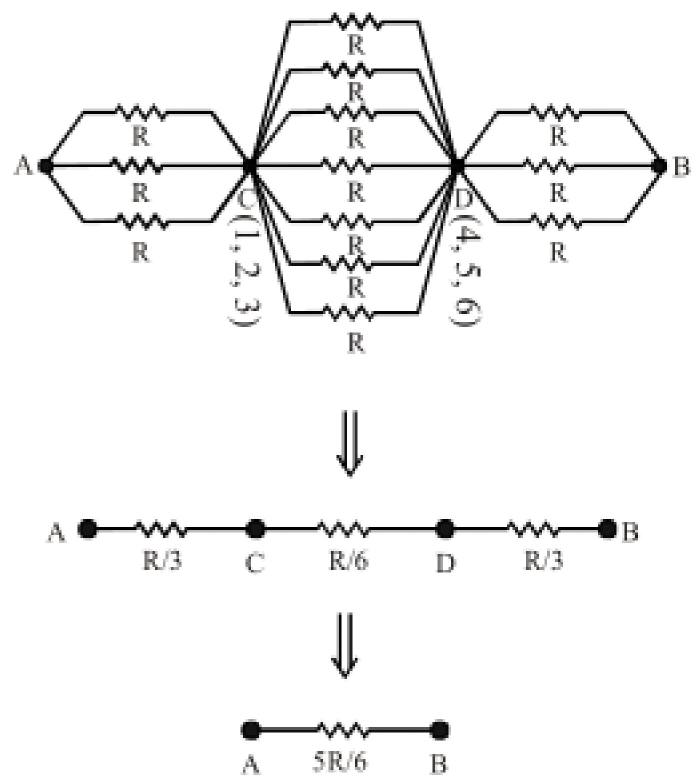
Illustration :

Find R_{AB} in the cubic network of twelve resistors each of resistance R.



Sol. *The network is symmetrical about the body diagonal AB. Since equal currents flow in the branches between A and (1, 2 and 3), the points 1, 2 and 3 are equipotential,. Similarly, the point 4,5,6 are equipotential. Let us now superimpose the points 1,2 and 3 at C and 4, 5, and 6 at D. You can now see that there are 3 resistors between A and C, six resistors between C and D and three resistors between D and B.*





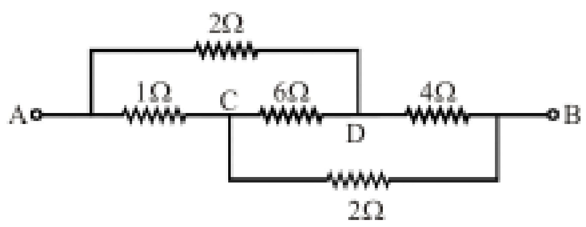
Then, $R_{AB} = R_{AC} + R_{CD} + R_{DB}$

$$= \frac{R}{3} + \frac{R}{6} + \frac{R}{3}$$

$$= \frac{5R}{6}$$

Illustration :

Find R_{AB} in the network



Sol. *The given network is a Wheatstone bridge as shown in the figure.*

Since, $\frac{R_{AC}}{R_{AD}} = \frac{R_{CB}}{R_{DB}} = \frac{1}{2}$

The remove the branch CD to obtain a simple circuit.

Hence $R_{AB} = \frac{3 \times 6}{3+6} = 2\Omega$

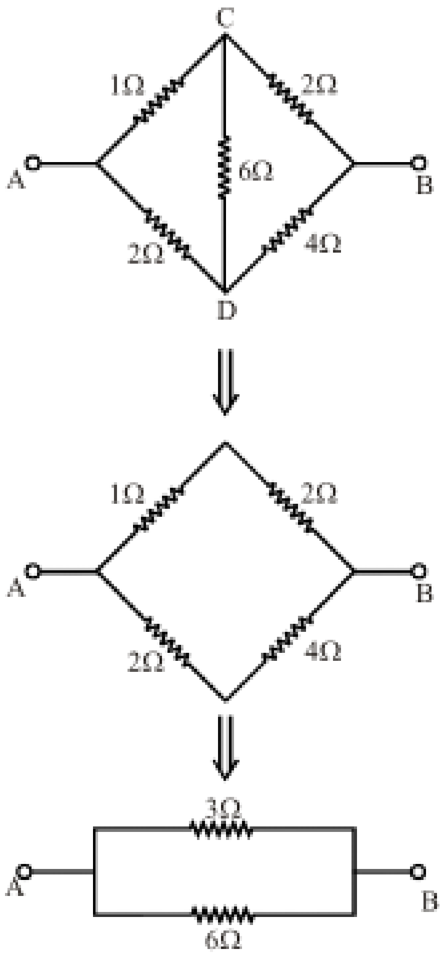
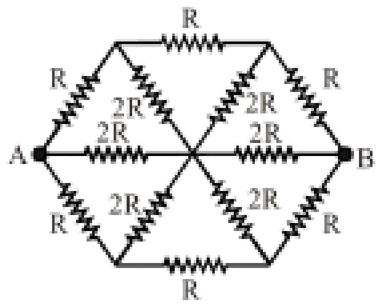
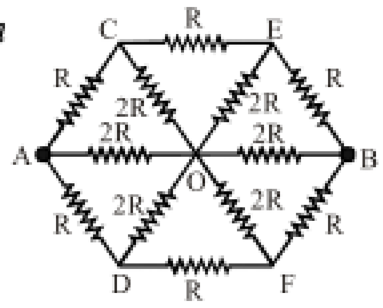


Illustration :

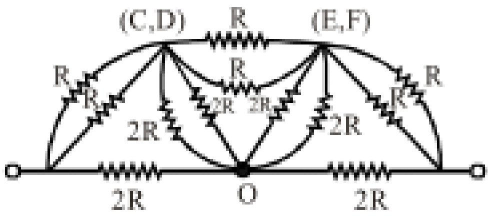
Find R_{AB}



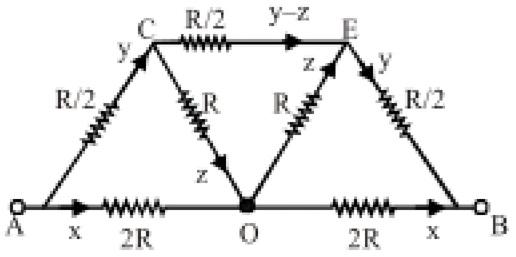
Sol. *By inspection we can say that lower half and upper half of the given circuit is symmetrical about AB. Then, C and D are equipotential; E and F are equipotential. Superimposing D with C and F with E we have the following circuite. You can see that (AC and AD), (CE and DF), (EB and FB), (CO and DO) and (EO and FO) are superimposed.*



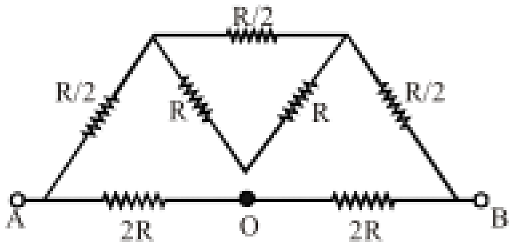
By current distribution following KCL, we understand that equal current passes through the branches CO and OE. Then, you can separate the branch COE, from AOB as shown in the figure and solve it by the processes of series and parallel combination.



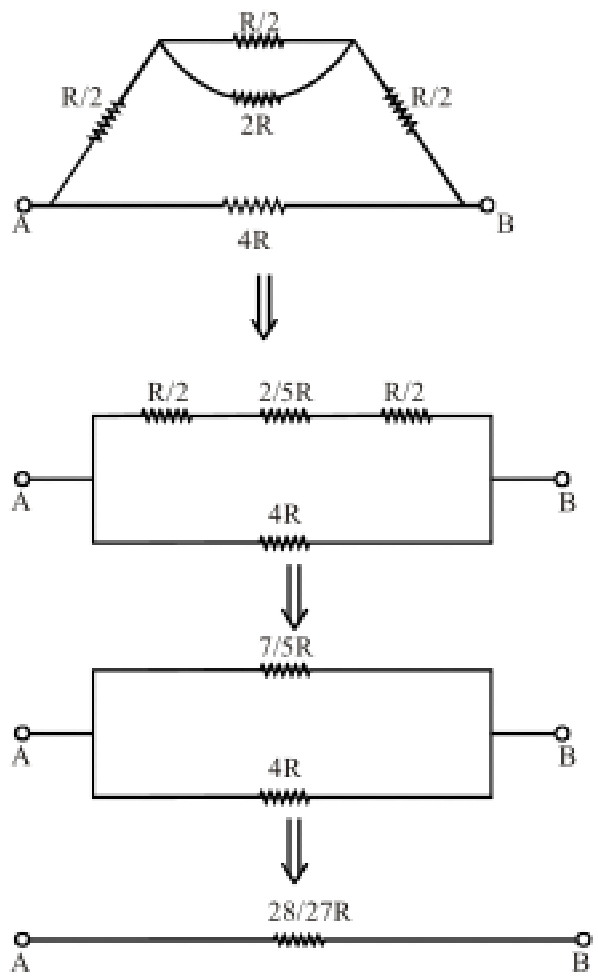
$\Downarrow$



$\Downarrow$



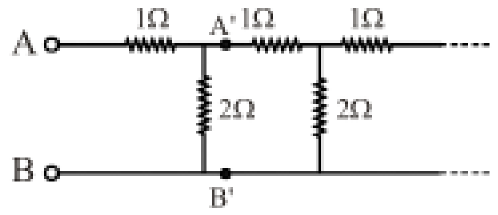
$\Downarrow$



$$R_{AB} = \frac{\left(\frac{7}{5}R\right)(4R)}{\frac{7}{5}R + 4R} = \frac{28}{27}R$$

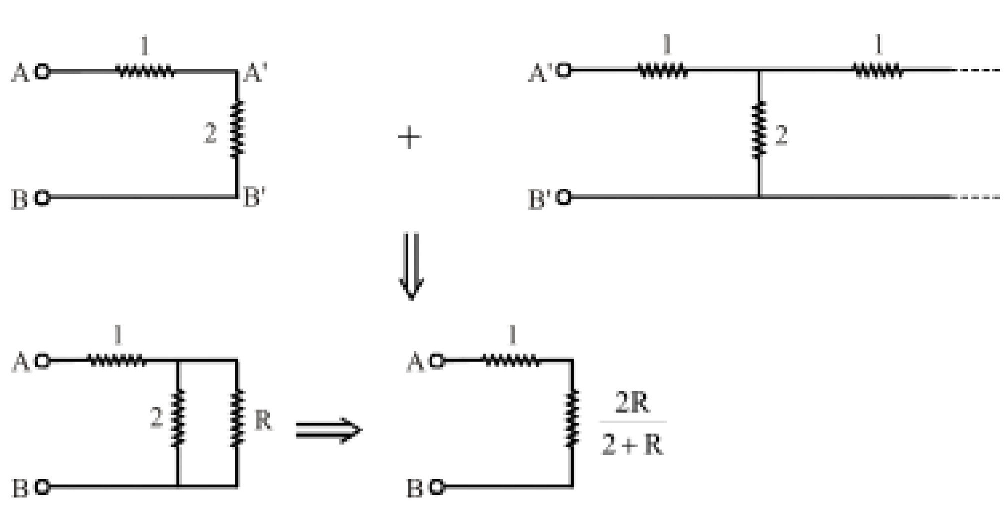
Illustration :

Find R_{AB}.



Sol. *Let $R_{AB} = R$ since infinite minus something is infinite, if you cut one well $R_{A'B'} = R_{AB} = R$.*

Hence $$\frac{(R_{A'B'})(2)}{2 + R_{A'B'}} + 1$$



Then, $$R_{AB} = 1 + \frac{2R}{2+R}$$

$$= \frac{2+3R}{2+R}$$

Putting $R_{AB} = R$, *we have*

$$R = \frac{2+3R}{2+R}$$

or, $$R^2 - R - 2 = 0$$

or, $$R = \frac{1 \pm \sqrt{1+8}}{2}$$

It gives $$R = 2\Omega$$

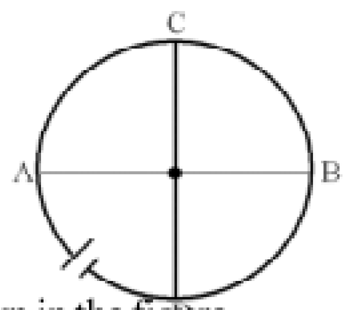
Practice Exercise

Q.1 A square pyramid is formed by joining 8 equal resistances R across the edges. The square base of the pyramid has the corner at A, B, C, D. The vertex is at M. Calculate the

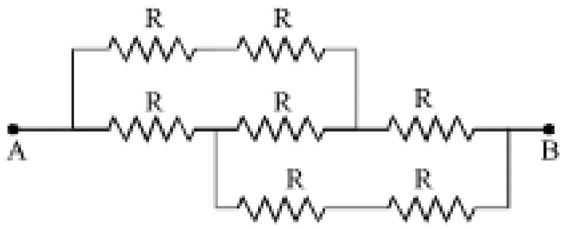
(a) current in the edge MC if an ideal cell of emf E is connected across the adjacent corners A and B.

(b) current in the edge MA if an ideal cell of emf E is connected across the opposite corners A and C.

Q.2 Calculate the equivalent resistance between the terminals of the cell shown in figure. The resistance of each quadrant is 1 ohm and the intersecting diameters have resistance 2 ohm each.



Q.3 Find the equivalent resistance of the configuration of equal valued resistors shown in the figure.



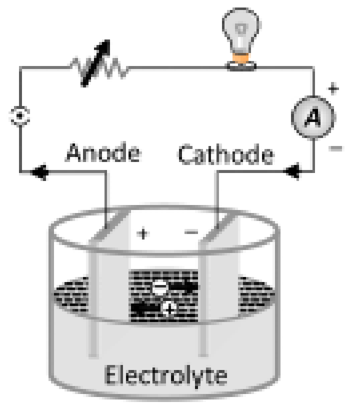
Q.4 Two conducting plates each of area A are separated by a distance d and they are parallel to each other. A conducting medium of varying conductivity fills the space between them. The conductivity varies linearly from σ and 2σ as you move from one plate to the other plate. Find the resistance of the medium between the conducting plates.

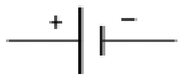
Answers

Q.1 (a) E/8R, (b) E/2R Q.2 $\frac{15}{7}\Omega$ Q.3 7/5 R Q.4 $\left[\frac{d}{\sigma A}\right]\ln(2)$

Electrice cell or Battery :

The device which converts chemical energy into electrical energy is known as electric cell. Cell is a source of constant emf but not constant current.





Symbol of cell

(1) **Emf of cell (E) :** The potential difference across the terminals of a cell when it is not supplying any current is called it's emf.

(2) **Potential difference (V) :** The voltage across the terminals of a cell when it is supplying current to external resistance is called potential difference or terminal voltage.

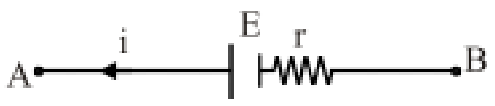
(3) Internal resistance (r) : In case of a cell the opposition of electrolyte to the flow of current through it is called internal resistance of the cell. The internal resistance of a cell depends on the distance between electrodes ($r \propto d$). area of electrodes [$r \propto (1/A)$] and nature, concentration ($r \propto C$) and temperture of elctrolyte [$r \propto (1/temp)$]

A cell is said to be ideal, if it has zero internal resistance.

Note : (i) During charging

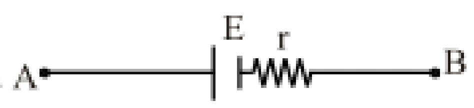
$$V_A - V_B = E + ir$$

During discharging



$$V_A - V_B = E - ir$$

If no current is drawn



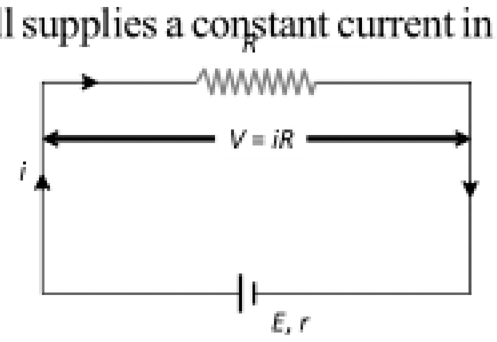
$$V_A - V_B = E$$

(ii) Inside a battery during discharging, charge is taken from -ve terminal (lower Potential) to +ve terminal (higher potential) by battery mechanism.

(iii) Work done by a battery during discharging = charge flown from +ve to -ve in outer circuit × emf of battery.

Cell in various Positions

(1) **Closed circuit :** Cell supplies a constant current in the circuit.



(i) Current given by the cell $i = \frac{E}{R+r}$

(ii) Potential difference across the resistance $V = iR$

(iii) Potential drop inside the cell $= ir$

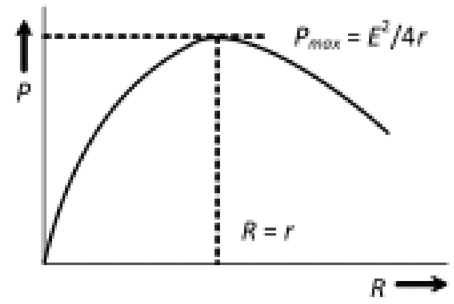
(iv) Equation of cell $E = V + ir$ $(E > V)$

(v) Internal resistance of the cell $r = \left(\frac{E}{V} - 1\right) \cdot R$

(vi) Power dissipated in external resistance (load) $P = Vi = i^2R = \frac{V^2}{R} = \left(\frac{E}{R+r}\right)^2 .R$

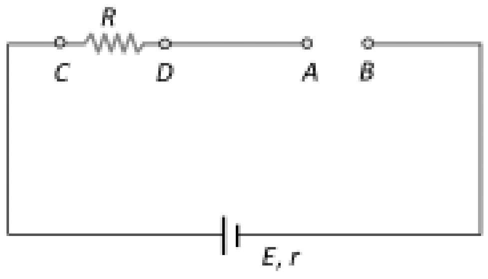
Power delivered will be maximum when $R = r$ so $P_{max} = \frac{E^2}{4r}$

This statement in generalised from is called "maximum power transfer theorem".



(vii) When the cell is being charged i.e. current is given to the cell then $E = V - ir$ and $E < V$.

(2) Open circuit : When no current is taken from the cell it is said to be in open circuit

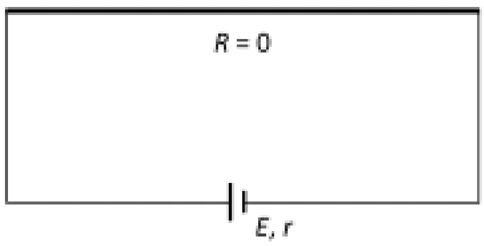


(i) Current through the circuit $i = 0$

(ii) Potential difference between A and B, $V_{AB} = E$

(iii) Potential difference between C and D, $V_{CD} = 0$

(3) **Short circuit :** If two terminals of cell are join together by a thick conducting wire

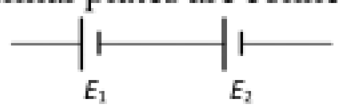


(i) Maximum current (called short circuit current) flows momentarily $i_{sc} = \frac{E}{r}$

(ii) Potential difference $V = 0$

Grouping of cells

In series grouping of cell's their emf's are additive or subtractive while their internal resistances are always additive. If dissimilar plates of cells are connected together their emf's are added to each other while if their similar plates are connected together their emf's are subtractive.



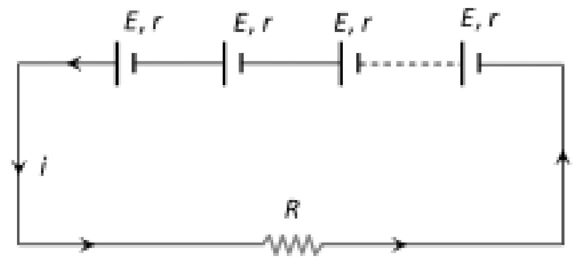
$E_{eq} = E_1 + E_2$

$r_{eq} = r_1 + r_2$

$E_{eq} = E_1 - E_2\ (E_1 > E_2)$

$r_{eq} = r_1 + r_2$

(1) **Series grouping :** In series grouping anode of one cell is connected to cathode of other cell and so on. If n identical cells are connected in series



(i) Equivalent emf of the combination $E_{eq} = nE$

(ii) Equivalent internal resistance $r_{eq} = nr$

(iii) Main current = Current from each cell $= i = \dfrac{nE}{R + nr}$

(iv) Potential difference across external resistance $V = iR$

(v) Potential difference across each cell $V' = \dfrac{V}{n}$

(vi) Power dissipated in the external circuit $= \left(\dfrac{nE}{R + nr}\right)^2 .R$

(vii) Condition for maximum power $R = nr$ and $P_{max} = n\left(\dfrac{E^2}{4r}\right)$

(viii) This type of combination is used when $nr << R$.

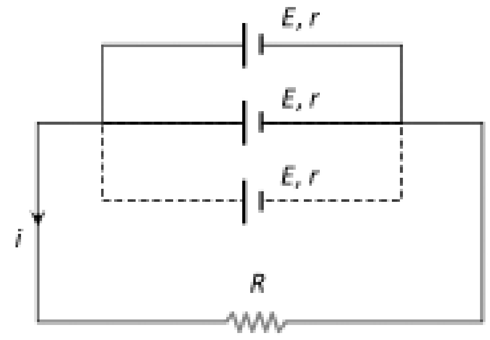
Note :

If Batteries are different

$E_{eq} = E_1 + E_2 + ...E_n$ if they are connected in same sense

$r_{eq} = r_1 + r_2 + ... + r_n$

(2) **Parallel grouping :** In parallel grouping all anodes are connected at one point and all cathode are connected together at other point. If n identical cells are connected in parallel



(i) Equivalent emf $E_{ef} = E$

(ii) Equivalent internal resistance $R_{eq} = r/n$

(iii) Main current $i = \dfrac{E}{R + r/n}$

(iv) potential difference across external reistance = p.d. across each cell $V = iR$

(v) Current from each cell $i' = \dfrac{i}{n}$

(vi) Power dissipated in the circuit $P = \left(\dfrac{E}{R + r/n}\right)^2 . R$

(vii) Condition for max. power is $R = r/n$ and $P_{max} = n\left(\dfrac{E^2}{4r}\right)$

(viii) This type of combination is used when $nr >> R$

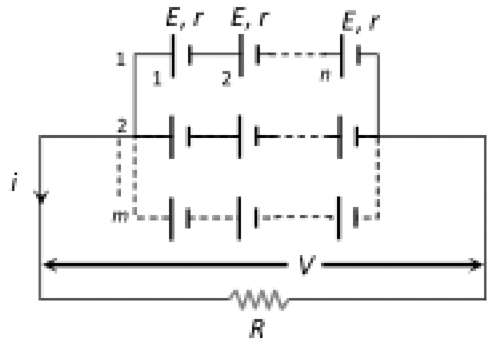
Note :

If Batteries are different

$$E_{eq} = \frac{\dfrac{E_1}{r_1} + \dfrac{E_2}{r_2} + ...}{\dfrac{1}{r_1} + \dfrac{1}{r_2} + ...}$$ If they are connected in same sense

$$\frac{1}{r_{eq}} = \frac{1}{r_1} + \frac{1}{r_2} + ...\frac{1}{r_n}$$

(3) **Mixed Grouping :** If n identical cell's are connected in a row and such m row's are connected in parallel as shown.



(i) Equivalent emf of the combination $E_{eq} = nE$

(ii) Equivalent internal resistance of the combination $r_{eq} = \dfrac{nr}{m}$

(iii) Main current flowing through the load $i = \dfrac{nE}{R + \dfrac{nr}{m}} = \dfrac{mnE}{mR + nr}$

(iv) Potential difference across load $V = iR$

(v) Potential difference across each cell $V' = \dfrac{V}{n}$

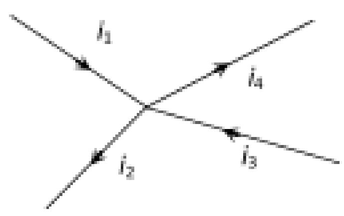
(vi) Current from each cell $i' = \dfrac{i}{n}$

(vii) Condition for maximum power $R = \frac{nr}{m}$ and $P_{max} = (mn)\frac{E^2}{4r}$

(viii) Total number of cell = mn

Kirchoff's Laws

Kirchoff's first law : This law is also known as junction rule or current law (KCL). According to it the algebraic sum of currents meeting at a junction is zero i.e.$\Sigma i = 0$



In a circuit, at any junction the sum of the currents entering the junction must equal the sum of the currents leaving the junction. $i_1 + i_3 = i_2 + i_4$

This law is simply a statement of ***"conservation of charge".***

Kirchoff's second law : This law is also known as loop rule or voltage law (KVL) and according to it "the algebraic sum of the changes in potential in complete traversal of a mesh (closed loop) is zero", i.e. $\Sigma V = 0$

This law represents ***"conservation of energy".***

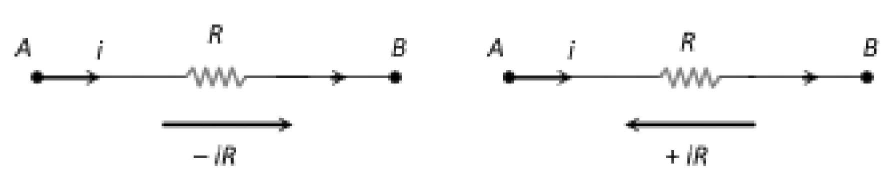
If there are n meshes in a circuit, the number of independent equations in accordance with loop rule will be (n - 1).

Note :

Sign convention for the application of Kirchoff's law :

For the application of Kirchoff's laws following sign convention are to be considered

(i) The change in potential in traversing a resistance in the direction of current is - iR while in the opposite direction +iR



(ii) The change in potential in traversing an emf source from negative to positive terminal is +E while in the opposite direction - E irrespective of the direction of current in the circuit.

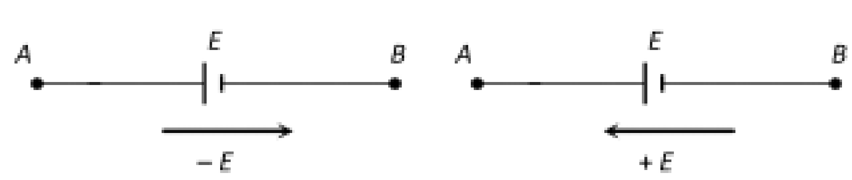


Illustration :

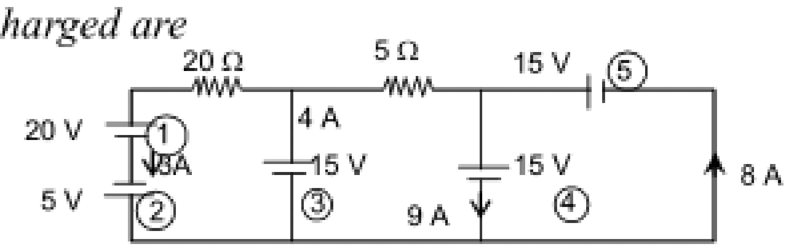
In the given network, the batteries getting charged are

(A) 1 and 3

(B) 1, 3 and 5

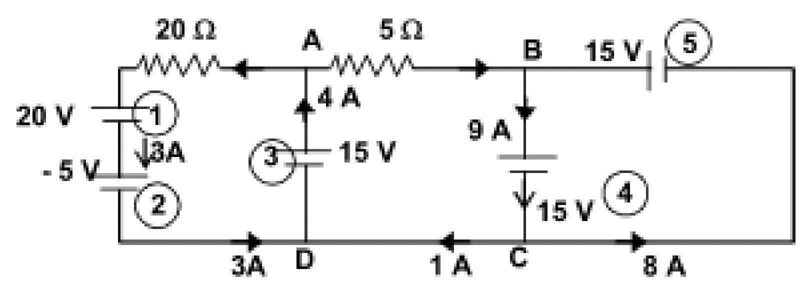
(C) 1 and 4

(D) 1, 2 and 5



Sol. *Applying Kirchhoff law at A, C and D, the direction of the currents in each branch will be as shown in the figure. It is clear from the figure that the batteries 1 and 4 are being charged.*

$\therefore$ *Hence (C) is correct*



Circuit solving Techniques

Case - (I)

Circuits having single Battery :

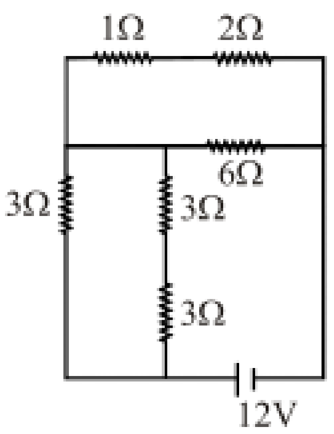
Step 1 - Remove Battery and find R_{eq} across the terminals of Battery

Step 2 -Total current through Battery $I_{total} = \dfrac{\text{Emf of battery}}{R_{eq}}$

Step 3 - Now divide the current as series- parallel combination.

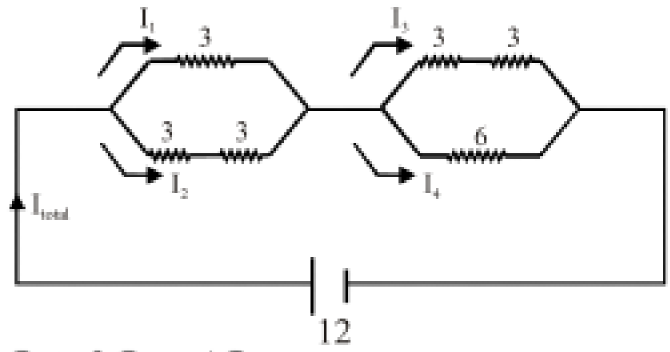
i.e. In series branches current remains same and in parallel current divides in inverse proportion of resistance.

Illustration :



Find current through each ressistance.

Sol.



$$R_{eq} = 2\Omega + 2\Omega = 4\Omega$$

$$I_{total} = \frac{12}{4} = 3 \;\; Amp$$

$$I_1 + I_2 = 3\ Amp \text{ and } \frac{I_1}{I_2} = \frac{2}{1} \quad hence \quad I_1 = 2A$$

$$I_2 = 1A$$

Similarly $I_3 = 2A$
$I_4 = 1A$

Case - (II)

Circuits having many Batteries (can be reduced to single battery using Battery combination)

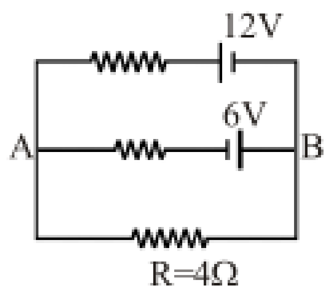
Step - 1

Apply Battery combination formula to reduced multiple batteries in single battery.

Step - 2

Solve as pervious case (I).

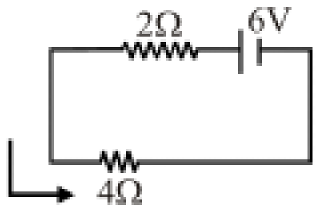
Illustration :



Find current through
$R = 4\ \Omega$
Also find $V_A - V_B$

Sol. *Applying parallel combination of Batteries,*

$$E_{eq} = \frac{\frac{+12}{3} - \frac{6}{6}}{\frac{1}{3} + \frac{1}{6}} = \frac{4-1}{1/2} = +6\text{V}$$

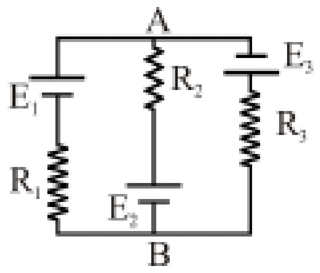


$$V_{eq} = 2\Omega$$

$$i = \frac{6}{2+4} = 1Amp$$

$$V_A - V_B = iR = 2\ volt$$

Illustration :

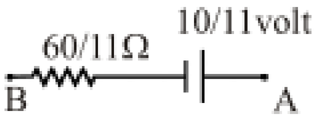


Find Potential difference ($V_A - V_B$) in the circuit
Shown $E_1 = 1.5\ V, E_2 = 2.0\ V$
$E_3 = 2V, R_1 = 10\ \Omega,$
$R_2 = 20\ \Omega, R_3 = 30\ \Omega$

Sol. *We can reduced the wholde circuit into one Battery and on resistance.*

$$E_{eq} = \frac{\frac{1.5}{10} + \frac{2.0}{20} - \frac{2.5}{30}}{\frac{1}{10} + \frac{1}{20} + \frac{1}{30}} = \frac{\frac{1}{6}}{\frac{11}{60}} = \frac{10}{11}\text{volt}$$

$$R_{eq} = \frac{60}{11}\Omega$$



$$V_A - V_B = \frac{10}{11} volt$$

Case - (III)

Circuits having many Batteries.

(Using loop rule)

Step - I

Assume current in each Independent loop.

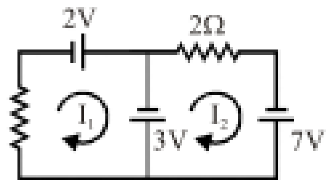
Step - II

Apply kirchoff's voltage law in each independent mesh (loop).

Illustration :

Find the current through each resistance

Sol. Let us assume currents I_1 and I_2 in the directions shown.



Using KVL, $-2-3+10\,I_1=0$ ...(i)

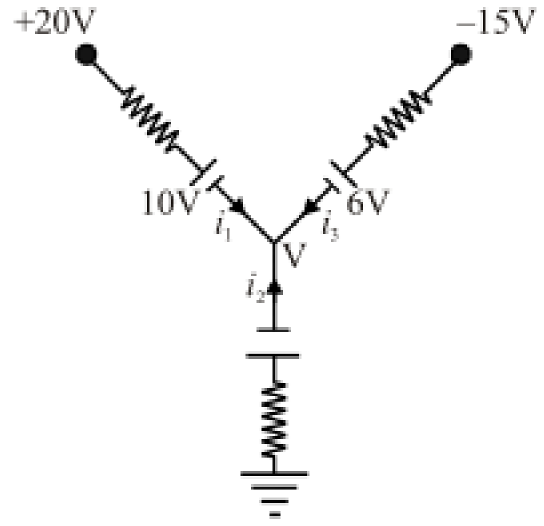
and $+3+2I_2-7=0$ ...(ii)

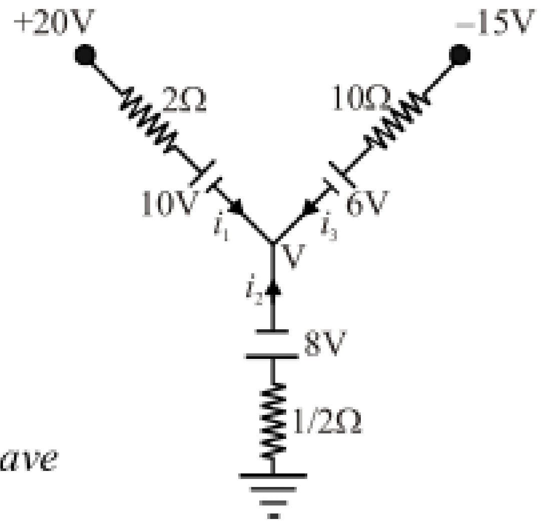
from (i) & (ii) $I_1=0.5\text{ A}$

$I_2=2\text{A}$

Illustration :

In the network of three cells, find the potential V of theri function.





Sol. *Applying KCL for the individual branches,*

$$20-i_1(2)+10=V \quad ...(i)$$

$$0-i_2\left(\frac{1}{2}\right)-8=V \quad ...(ii)$$

$$-15-i_3(1)-6=V \quad ...(iii)$$

$$i_1+i_2+i_3=0 \quad ...(iv)$$

Putting i_1, i_2 and i_3 from eqs. (i), (ii) and (iii) in eq. (iv) we have

$$\frac{30-V}{2}+\frac{V+8}{-1/2}+\frac{V+21}{-1}=0$$

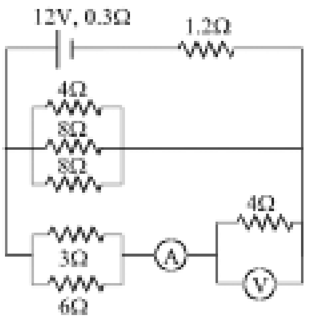
or, $V=-\dfrac{44}{7}$ *volt*

Practice Exercise

Q.1 For the circuit shown in the figure, find

(i) the equivalent external resistance of the circuit

(ii) the reading in ammeter (A) and voltmeter (V)

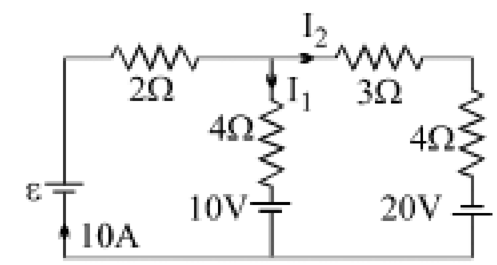


Q.2 For the circuit shown in the figure,

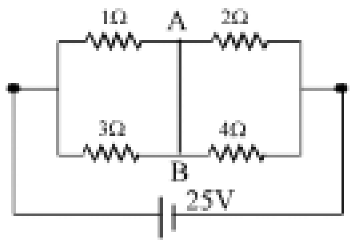
(i) find the currents I_1 and I_2, and the emf ε of the battery.

(ii) which batteries are supplying energy and at what rate to the circuit? Which batteries are absorbing energy and at what rate?

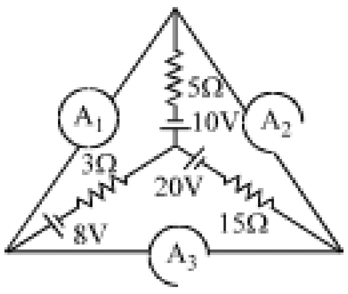
(iii) is total energy conserved? Justify.



Q.3 Find the current flowing through the segment AB of the circuit shown in figure.



Q.4 In the given circuit the ammeter A_1 and A_2 are ideal and the ammeter A_3 has a resistance of $1.9 \times 10^{-3}\Omega$. Find the readings of all the three meters.



Answers

Q.1 (i) 2.7 Ω, (ii) 1 A, 4 Volts

Q.2 (i) $\frac{40}{11}$ A, $\frac{70}{11}$ A (ii) $\frac{400}{11}$ W, $\frac{1400}{11}$ W, $\frac{4900}{11}$ W (iii) ∴ energy is conserved

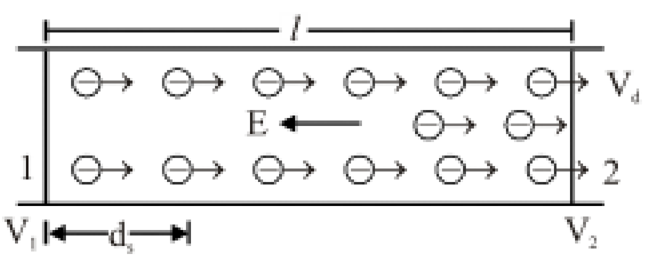
Q.3 1A from A to B   Q.4 $\frac{82}{27}$ A, $\frac{34}{27}$ A, 0

Energy conversion and electrical power

Input electrical energy

Let us consider a length l of the straight conductor of uniform cross-section A and conduction electron density n. Then the total number of conduction electrons in the considered segment is

$$N = nAl$$



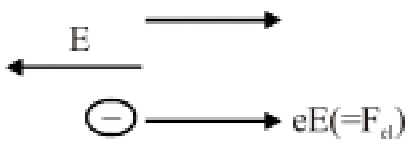
The electric field does a positive work in pushing the conduction electron in opposite direction of the field $\vec{E}$

Since, the uniform electric field E pushes each electron with a constant drift speed v_d against the resistance (offered by the fixed atoms in the lattice), the total work done by the field during a time dt in shifting the electrons by a distance ds is

$$\begin{aligned} dW &= (\text{work done each electron}) \times (\text{no. of electrons present in the segment}) \\ &= (F_{el} \,.\, ds)\,(N) \\ &= (eEds)\,(nAl) \qquad (\because\ F_{el} = eE) \end{aligned}$$

or, $dW = (eEv_d\,dt)\,(nAl) \qquad (\because\ ds = v_d\,dt)$

$= (nev_d A)\,(El)\,dt$



Each electron experiences a force $\vec{F}_{el} = -e\,\vec{E}$, opposite to the applied field $\vec{E}$

By putting $nev_d A = i$ and $El = V_1 - V_2\ (=V)$, we have

Then, the total work done by the electric field on the assumed portion of theconductor during a time t is

$$W = \int_0^t iV\,dt$$

Where V = potential fifferece between the terminals 1 and 2 of the given portion ofthe conductor.

Input Electron Power

The electrical power of a voltage V while sending a current i can be give as rate of electrical work done.

or, $$P_{el} = \frac{dW}{dt}$$

or, $$P_{el} = iV$$

Heat Disslpated

As the electrons travel from lower potential V_1 to higher potenial V_2 they must lose their electrostatic potential energy or excess kinetic energy while accelerating in the applied electric field. This appears in the form of heat, light and sound etc., due to the resistance offered by the coductor. Hence, the amount of eat liberated in the considered portion of the conductor is

$$Q = \int_0^t iV\,dt$$

$$= \int_0^t i^2 R\,dt \qquad (\because V = iR)$$

$$= \int_0^t \frac{V^2}{R}\,dt \qquad \left(\because i = \frac{V}{R}\right)$$

Thermal Power

The rate of heat is liberated, that is power loss in the resistaor is called Ohmic heating, or Joule heating or Copper-loss or thermal power or $i^2 R$ loss which can be given as

$$\frac{dQ}{dt}(= P_R) = iV$$

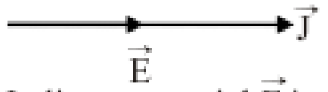
$$= i^2 R$$

$$= \frac{V^2}{R}$$

We can use thermal energy in room heater, toaster, electric iron etc. and in other electric circuits (power distribution and transmission) power lost cannot be used.

Joule - Lenz Law

The above expression is called macroscopic form of Joule-Lenz law.



In linear material $\vec{E}$ is parallel to $\vec{J}$ at any point

Substing $i = JA$, $R = \rho \dfrac{l}{A}$ in the formula $\dfrac{dQ}{dt} = i^2 R$,

we have $\dfrac{dQ}{dt} = (JA)^2 \left(\rho \dfrac{l}{A}\right)$

$= \rho J^2 (Al)$, where $Al = V$ (volume of the segment)

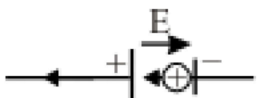
Then, the power loss (rate of heat generated) per unit volume is

$$\frac{dQ}{dt} \Big/ V = Q_V = \rho J^2 = J.E = E^2/\rho \qquad (\because J = \rho E)$$

This expression is valid for any point of the conductor. Hence, we call i "point (or differenial) from" of Joule-Lenz law.

Micro-interportation of Heat Dissipation

The emf (battery) sets on electric field which pushes the electrons in the conductor. As a result, the electrons gain kinetic energy or loses electrostatic potential energy. The gain in K.E. is lost due to their repeated collision with the site atoms of the lattice. the exchange in kinetic energy and momenta of the electrons cause the lattics atoms to vibrate with more amplitudes. The vibrating metallic kernels of the lattice radiate electromagnetic energy in the form of heat, light etc., obeying the principle of electromagnetic radiation.



A +ve charge is brought from –ve to +ve electrode of the battery by its emf against the electric field $\vec{E}$ inside the battery

The excess K.E. of the elctrons received from the electric field (ultimately from the battery) is spent in exciting the atoms of the lattice which in turn radiate electromagnetic energy in the form of heat and light.

Power of an EMF

A pattery is ultimately respeonsible for setting electric field inside and outside of the conducting wires. Hence, the battery does work in circulating the charges. The rate of work done by a seat of emf (battery) to establish a current is defined as electrical power of a battery.

$$P_{el} = \frac{dW_b}{dt}$$

As discussed earlier, the work is done by a battery to push the conventinal +ve charge dq from its – ve terminal to +ve termial against the electrostatic force can be given as

$$dW_b = \varepsilon\, dq$$

Then, the power delivered by the batter in setting a current i is

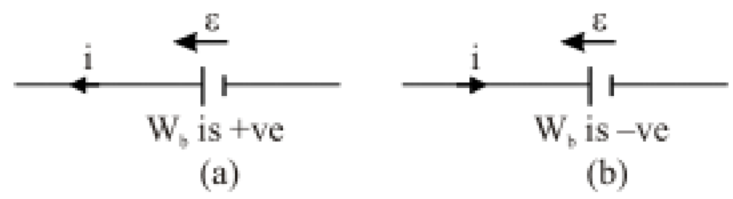
$$P_{el} = \frac{dW_b}{dt}$$

$$= \varepsilon \frac{dq}{dt}$$

$$= \varepsilon\, i$$

or, $\quad P_{el} = \varepsilon\, i$

If current (or dq) flows in the direction of the emf, work done and power deliverd by the battery is +ve and vice-versa.



Illustraion :

Two bulb's of powers P_1 and P_2 are connected in series. If the supply voltage is equal to the rated voltage, find the power of the combination.

Sol. *Let their resistance be R_1 and R_2 respectively.*

For a rated voltage V, the power of the combination is

$$P = \frac{V^2}{R_1 + R_2} \quad (\because \text{ the resistance are connected in series})$$

Putting $\quad R_1 = \frac{V^2}{P_1}$ *and* $R_2 = \frac{V^2}{P_2}$ *we obtain*

$$P = \frac{P_1 P_2}{P_1 + P_2}$$

Illustration :

A 1000 watt heater coil can be cut into two parts and when each part is used in the rated supply voltage, it gives more power as $P \propto \frac{1}{R}$, but we do not recommed this , explain.

Sol. *Since, the power dissipate in the coil is*

$$P = \frac{V^2}{R}$$

and R decreases by two fold if we cut it into two equal halves (say), power dissipation will be doubled. The heat liberation will be doubled which in turn, damages the coil by heating it or reduces its life.

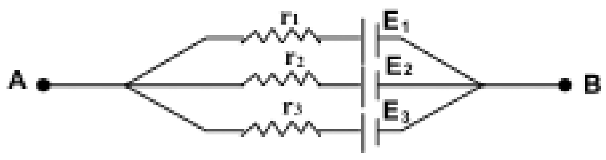
Illustration :

In the circuit shown in figure,

$E_1 = 3$ V, $E_2 = 2$ V, $E_3 = 1$ V and $r_1 = r_2 = r_3 = 1$ ohm.

(a) Find the potential difference between the points A and B and the currents through each branch.

(b) If r_2 is short circuited and the point A is connected to point B through a resistance R, find the currents through E_1, E_2, E_3 and the resistor R.



Sol. *(a) Applying Kirchoff's loop law to mesh PLMQP and PLMQONP in the figure shown below, we have*

Shown below, we have

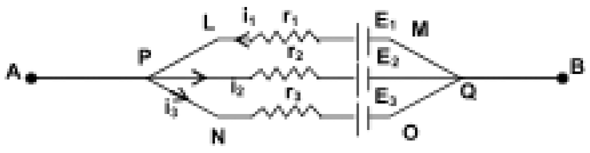
$$i_1 r_1 + i_2 r_2 = E_1 - E_2 \quad \text{or} \quad i_1 + i_2 = 1 \qquad \ldots \text{(i)}$$

$$i_1 r_1 + i_3 r_3 = E_1 - E_3 \quad \text{or} \quad i_1 + i_3 = 2 \qquad \ldots \text{(ii)}$$

At P, $i_2 + i_3 = i_1$ ... (iii)

On solving (i), (ii) and (iii)

$$i_1 = 1 \text{ amp}, \; i_2 = 0 \text{ amp}, \; i_3 = 1 \text{ amp}.$$



Since no current is drown along the brance AP

$\therefore \quad V_{AB} = V_{PQ}$

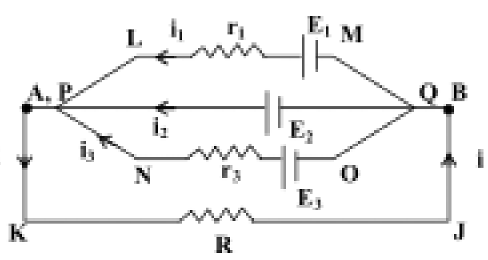
Potential difference across PQ,

$$V_{PQ} = E_1 - r_1 r_1 = 2 \text{ volt}$$

(b) *The figure shows the circuit when point A is connected to point B and r_2 is short-circuited.*

Applying Kirchoff's junction rule at P, we get

$$i = i_1 + i_2 + i_3 \qquad \ldots \text{(iv)}$$



Applying Kirchoff's law to mesh ABMLA

$$i_1 r_1 = E_1 - E_2 \quad \text{or} \quad i_1 = 1 \text{ amp}.$$

Applying Kirchoff's law to mesh ANOQML

$$i_1 r_1 - i_3 r_3 = E_1 - E_3 \quad \text{or} \quad i_1 - i_3 = 2 \qquad \ldots \text{(v)}$$

From above equations

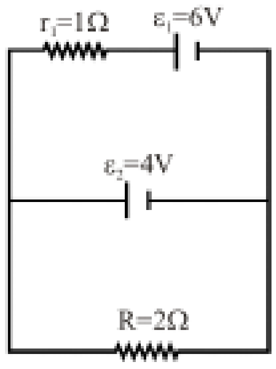
$$i_1 = 1 \text{ amp}, \; i_2 = 2 \text{ amp}, \; i_3 = 1 \text{ amp}.$$

(direction of current is opposite)

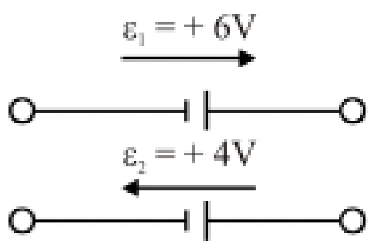
So, current through resistor R will be $I = I_1 + I_2 + I_3 = 2$ amp.

Illustration :

Two cells are connected to an external load of resistance R = 2 W. Find the current in the resistor.



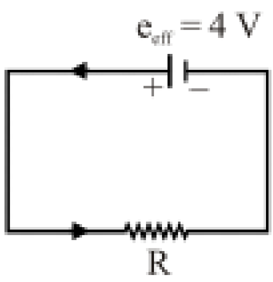
Sol. $\varepsilon_{eff} = \dfrac{\dfrac{\varepsilon_1}{r_1}+\dfrac{\varepsilon_2}{r_2}}{\dfrac{1}{r_1}+\dfrac{1}{r_2}}$



$= \dfrac{\varepsilon_1 r_2 + \varepsilon_2 r_1}{r_1 + r_2}$

$= \dfrac{6(0)+(-4)(1)}{1}$

$= -4\ V$



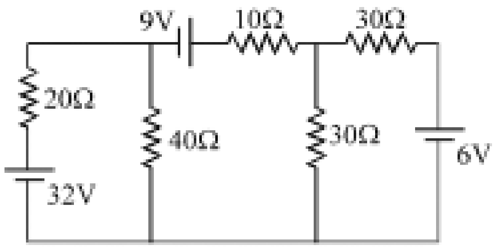
$r_{eff} = \dfrac{r_1 r_2}{r_1 + r_2} = \dfrac{(0)(1)}{0+1} = 0$

$i = \dfrac{e_{eff}}{R} = \dfrac{4}{2} = 2A$

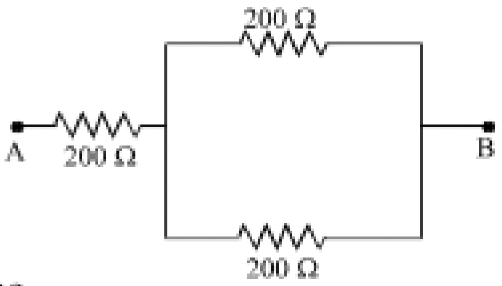
Practice Exercise

Q.1 In a house there are 3 lamps of 40W each, 8 lamps of 60W each, a radio of 40W and a TV of 160W. The lamps are in operation, on an average, for 2hrs a day, the radio for 4hrs a say and the TV for an hr a day. On Sundays an electric iron of 750W is used for an hour and the TV for an extra 3 hrs. Calculate the electricity bill for the month of February of a leap year at the rate of 45 paise per unit. The first Sunday falls on 3rd February.

Q.2 Obtain the power imparted to the 10 Ω resistor in the shown network.



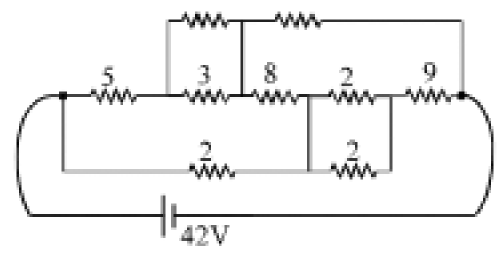
Q.3 Three 200 Ω resistors are connected as shown in figure. The maximum power that can be dissipated in any one of the resistor is 50 W. Find:



(i) the maximum voltage that can be applied to the terminals A and B.

(ii) the total power dissipated in the circuit for maximum voltage across the terminals A and B.

Q.4 Find the power dissipated in 5Ω and 8Ω resistors.



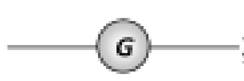
Answers

Q.1	22.05	Q.2	5.1 W	Q.3	(a) 150 V, (b) 75 W	Q.4	5W in 5Ω, 0 in 8Ω

Different Measuring Instruments

(1) **Galvanometer :**

It is an instrument used to detect small current passing through it by showing deflection. Galvanometers are of different types e.g. moving coil galvanometer, moving magnet galvanometer, hot wire galvanometer. In dc circuit usually moving coil galvanometer are used.

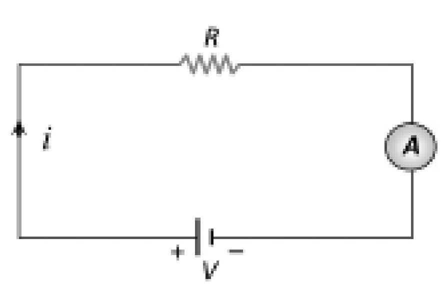
(i) It's symbol : 

; where G is the total internal resistance of the galvanometer.

(ii) Full scale deflection current : The current required for full scale deflection in a galvanometer is called full scale deflection current and is represented by i_g.

(iii) Shunt : The small resistance connected in parallel to galvanometer coil, in order to control current flowing through the galvanometer is known as shunt.

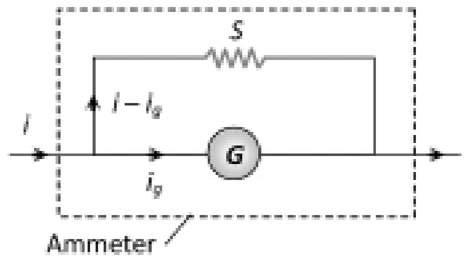
(2) **Ammeter :**



(i) The reading of an ammeter is always lesser than actual current in the circuit.

(ii) Smaller the resistance of an ammeter more accurate will be its reading. An ammeter is said to be ideal if its resistance r is zero.

(iii) **Conversion of galvanometer into ammeter :** A galvanometer may be converted into an ammeter by connecting a low resistance (called shunt S) in parallel to the galvanometer G as shown in figure.



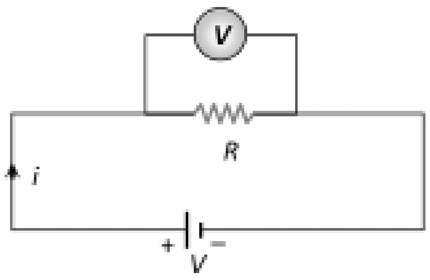
(a) Equivalent resistance of the combination $= \frac{GS}{G+S}$

(b) G and S are parallel to each other hence both will have equal potential difference i.e. $i_g G = (i - i_g)S$;

which gives Required shunt $S = \frac{i_g}{(i - i_g)} G$

(c) To pass nth part of main current (i.e. $i_g = \frac{i}{n}$) through the galvanometer, required shunt $S = \frac{G}{(n-1)}$

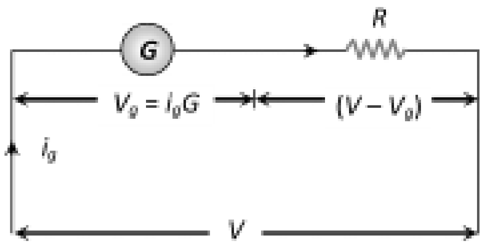
(3) **Voltmeter :** It is a device used to measure potential difference and is always put in parallel with the 'circuit element' across which potential difference is to be measured.



(i) The reading of a voltmeter is always lesser than true value.

(ii) Greater the resistance of voltmeter, more accurate will be its reading. A voltmeter is said to be ideal if its resistance is infinite, i.e., it draws no current from the circuit element for its operation.

(iii) **Conversion of galvanometer into voltmeter :** A galvanometer may be converted into a voltmeter by connecting a large resistance R in series with the galvanometer as shown in the figure.



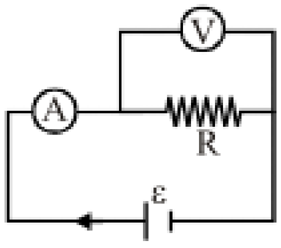
(a) Equivalent resistance of the combination $= G + R$

(b) According to ohm's law $v = i_g (G+R)$; which gives required series resistance $R = \frac{v}{i_g} - G = \left(\frac{v}{v_g} - 1\right) G$

(c) If n^{th} part of applied voltage appeared across galvanometer (i.e. $v_g = \frac{v}{n}$) then required series resistance $R = (n-1)G$.

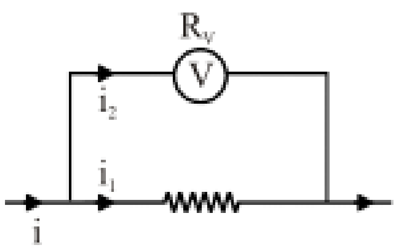
Illustration :

To measure the value of the resistance R, we have connected the voltmeter and ammeter as shown in the figure. Can the ratio of voltmeter and ammeter reading $\frac{V}{i}$ given the correct value of R ? Discuss.



Sol. Let $\frac{\text{Voltmeter reading}}{\text{Ammeter reading}} = R_m$

Where, R_m = meter reading of resistance



or, $R_m = \frac{V}{i}$ ...(i)

Since, R_V and R are parallel,

$i_2 R_V = i_1 R$ ...(ii)

According to KCL (1st law),

$i = i_1 + i_2$ ...(iii)

Using these three equation, we have

$$\frac{1}{R} = \frac{1}{R_m} - \frac{1}{R_V}$$

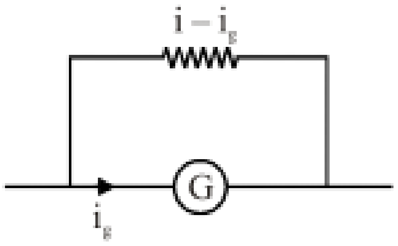
If $R_V \to \infty$, $R \to R_m$

Hence, the ration of voltmeter and ammeter reading cannot give the exact volue of the resistance R.

Illustration :

The deflection of a moving coil galvanometer falls from 60 divisions to 12 divisions when a shunt of 12 Ω is connected. What is the resistance of the galvanometer ?

Sol. The current i in the galvanometer is directly proportional to the angle of deflection ($i \propto \theta$)



Then, $\frac{i_g}{i} = \frac{12}{60} = \frac{1}{5}$

or, $i_g = \frac{i}{5}$ ...(i)

For shunted galvanometer,

$(i - i_g)\, S = i_g\, G$

$G = (i - i_g)\, \frac{S}{i_g}$ ...(ii)

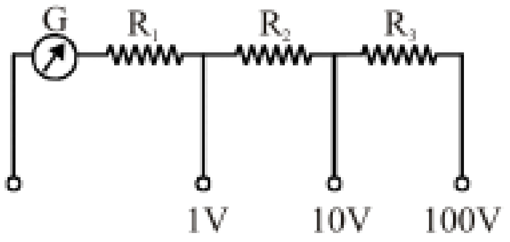
Putting i_g from eq. (i) in eq. (ii) and $S = 12$ ohm.

$G = 48$ ohm

Illustration :

The galvanometer G has internal resistance $G = 50\ \Omega$ and full scale deflection occurs at $i = 1$ mA. Find the series resistors R_1, R_2 and R_3 needed to use the arrangement as a voltmeter with different rages as shown in the figure.

Sol. For the range of $V_1 = 1$ volt,



$i_g = \frac{V_1}{G + R_1}$

or, $10^{-3} = \frac{1}{50 + R_1}$

or, $R_1 = 950$ ohm

For the range of $V_2 = 10$ volt

$$i_g = \frac{V_2}{G + R_1 + R_2}$$

or, $$10^{-3} = \frac{10}{50 + 950 + R_2}$$

or, $R_2 = 9 \times 10^3$ *ohm*

For the range of $V_3 = 100$ *volt*

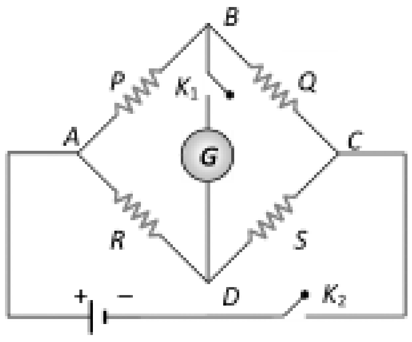
$$i_g = \frac{V_3}{G + R_1 + R_2 + R_3}$$

or, $$10^{-3} = \frac{100}{50 + 950 + 9000 + R_3}$$

or, $R_3 = 90 \times 10^3$ *ohm*

Wheatstone bridge :

Wheatstone bridge is an arrangement of four resistance which can be used to measure one of them in terms of rest. Here arms AB and BC are called ratio arm and arms AC and BD are called conjugate arms



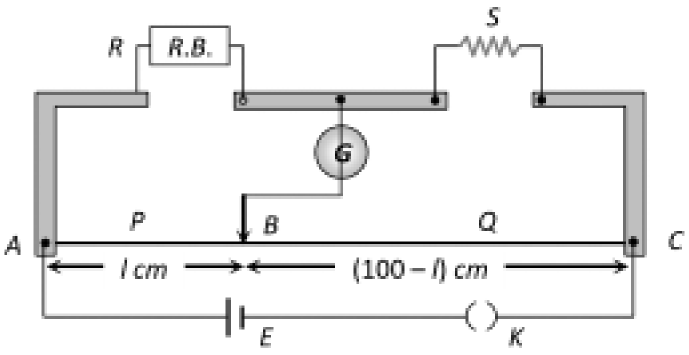
(i) Balanced bridge : The bridge is said to be balanced when deflection in galvanometer is zero i.e. no current flows through the galvanometer or in other words $V_B = V_D$. In the balanced condition $\frac{P}{Q} = \frac{R}{S}$, on mutually changing the position of cell and galvanometer this condition will not change.

(ii) Unbalanced bridge : If the bridge is not balanced current will flow from D to B if VD > VB i.e. $(V_A - V_D) < (V_A - V_B)$ which gives PS > RQ.

Applications of wheatstone bridge :

Meter bridge, post office box and Carey Foster bridge are instruments based on the principle of wheatstone bridge and are used to measure unknown resistance.

(4) Meter bridge : In case of meter bridge, the resistance wire AC is 100 cm long. Varying the position of tapping point B, bridge is balanced.



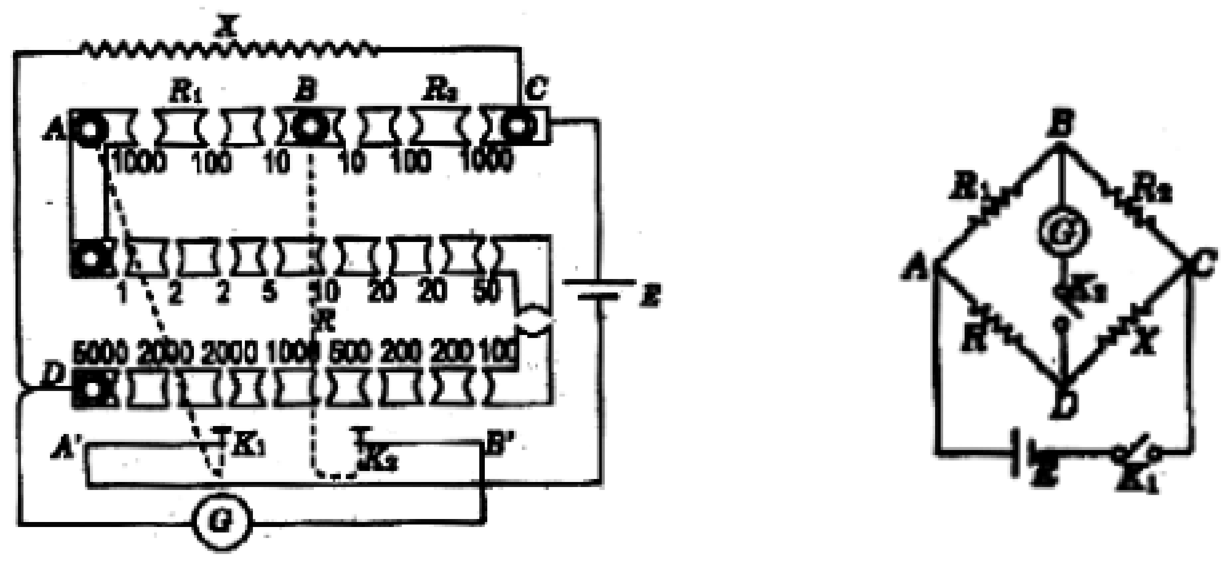
If in balanced position of bridge AB = ℓ, BC (100 - l)

so that $\frac{Q}{P} = \frac{(100-\ell)}{\ell}$ Also $\frac{P}{Q} = \frac{R}{S} \Rightarrow S = \frac{(100-\ell)}{\ell}R$

Note that :

* The balance-point is obtained by trial and error–not by scraping the jockey along the wire.
* The value of R in the resistance box should be chosen so that the balance point comes near to the center of the wire, i.e. from 40 cm to 60 cm from the end A.
* If the length either ℓ_1 or ℓ_2 is small, then the resistance of its end connections AA' and BB' will not be negligible in comparison with R_{AB} or R_{CB}. Then, the equation will not valid.
* The end resistance error can be minimized by interchanging R and X, and balancing again. The average values of ℓ_1 and ℓ_2 are taken to calculate the value of X.
* Since galvanometer is a sensitive instrument, therefore, a high resistance is sometimes connected in series with it until a near balance point is obtained. Then the high resistor is shunted or removed and the final balance point is obtained.
* The lowest resistance that can be measured with this bridge is about 1Ω.

(5) The post office Box



(a) The post office Bax (b)

It is a compact form of the Wheatstone bridge. It consists of compact resistance so arranged that different desired values of resistances may be selected in the three arms of Wheatstone bridge, as shown in figure.

Each of the arms AB and BC contains three resistances of 10, 10^2 and 10^3 Ω, respectively. These are called the ratio arms. Using these resistances the ratio $\frac{R_2}{R_1}$ can b made to have any of the following values : 100 : 1, 10 : 1, 1 : 1, 1 : 10 or 1 : 100.

The arm AD is a complete resistance box containing resistances from 1 to 5000 Ω. The tap keys K_2 and K_2 are are also provided in the post office box. The key K_1 is internally connected to the point A and the key K_2 to the point B (as shown by dotted line in the figure). The unknown resistance X is connected

between C and D, the battery between C and the key K_1 and the galvanometer between D and the key K_2. The circuit shown in figure (A) is exactly the same as that of the Wheatstone bridge shown in figure. Hence, the value of the unknown resistance is given by

$$X = R\left(\frac{R_2}{R_1}\right)$$

Note that :

* The accuracy of the post office box depends on the choice of ratio arm $\frac{R_2}{R_1}$.
* If $R_2 : R_1$ is 1 : 1, then the value of the unknown resistance is obtained within $\pm 1\Omega$.
* If the ratio $R_2 : R_1$ is selected as 1 : 10, then the unknown resistance $X = R\left(\frac{1}{10}\right)$ is accurately measured upto $\pm 0.1\Omega$.
* If the ratio $R_2 : R_1$ is adjusted to 1 : 100, then the value of unknown resistance $X = R\left(\frac{1}{100}\right)$ is obtained to an accuracy of $\pm 0.01\ \Omega$.

Illustration :

The value of an unknown resistance is obtained by using a post office box. Two consecutive readings of R are observed at which the galvanometer deflects in the opposite directions for three different value of R_1. These two values are recorded under the column-I and II in the following observation table.

S.No.	R_1 (Ω)	R_2 (Ω)	R lies in-between		X = R (R_2/R_1)	
			I (Ω)	II (Ω)	I (Ω)	II (Ω)
1	10	10	16	17		
2	100	10	163	164		
3	1000	10	1638	1639		

Determine the value of the unknown resistance.

Sol. *The observation table may be complete as follows :*

S.No.	R_1 (Ω)	R_2 (Ω)	R lies in-between		X = R (R_2/R_1)	
			I (Ω)	II (Ω)	I (Ω)	II (Ω)
1	10	10	16	17	16.0	17.0
2	100	10	163	164	16.3	16.4
3	1000	10	1638	1639	16.38	16.39

The value of the unknown resistance lies in-between 16.38 Ω and 16.39 Ω.
The unknown value may be the average of the two

i.e. $X = \dfrac{16.38 + 16.39}{2}$

or $X = 16.385\ \Omega$

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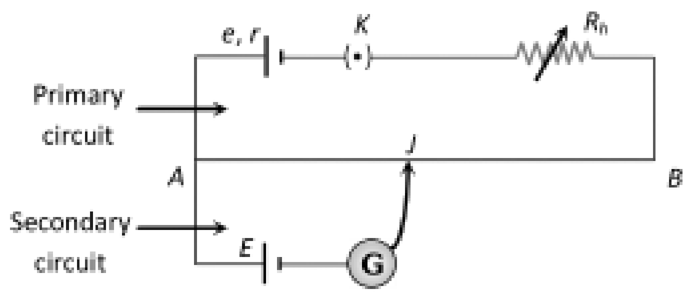
(6) Potentiometer

Potentiometer is a device mainly used to measure emf of a given cell and to compare emf's of cells. It is also used to measure internal resistance of a given cell.

Circuit diagram :

Potentiometer consists of a long resistive wire AB of length L (about 6m to 10 m long) made up of mangnine or constantan and a battery of known voltage e and internal resistance r called supplier battery or driver cell. Connection of these two forms primary circuit.

One terminal of another cell (whose emf E is to be measured) is connected at one end of the main circuit and the other terminal at any point on the resistive wire through a galvanometer G. This forms the secondary circuit. Other details are as follows



J = Jockey

K = Key

R = Resistance of potentiometer wire,

ρ = Specific resistance of potentiometer wire.

R_h = Variable resistance which controls the current through the wire AB

(i) The specific resistance (ρ) of potentiometer wire must be high but its temperature coefficient of resistance (α) must be low.

(ii) All higher potential points (terminals) of primary and secondary circuits must be connected together at point A and all lower potential points must be connected to point B or jockey.

(iii) The value of known potential difference must be greater than the value of unknown potential difference to be measured.

(iv) The potential gradient must remain constant. For this the current in the primary circuit must remain constant and the jockey must not be slided in contact with the wire.

(v) The diameter of potentiometer wire must be uniform everywhere.

(vi) Potential gradient (x) : Potential difference (or fall in potential) per unit length of wire is called potential gradient i.e. $x = \frac{V}{L}\frac{\text{volt}}{\text{m}}$ where $V = iR = \left(\frac{e}{R + R_h + r}\right).R$.

So $x = \frac{V}{L} = \frac{iR}{L} = \frac{i\rho}{A} = \frac{e}{(R + R_h + r)}.\frac{R}{L}$

Potential gradient directly depends upon

(a) The resistance per unit length (R/L) of potentiometer wire.

(b) The radius of potentiometer wire (i.e. Area of cross-section)

(c) The specific resistance of the material of potentiometer wire (i.e.ρ)

(d) The current flowing through potentiometer wire (i)

(ii) potential gradient indirectly depends upon

(a) The emf of battery in the primary circuit (i.e. e)

(b) The resistance of rheostat in the primary circuit (i.e. R_h)

Working :

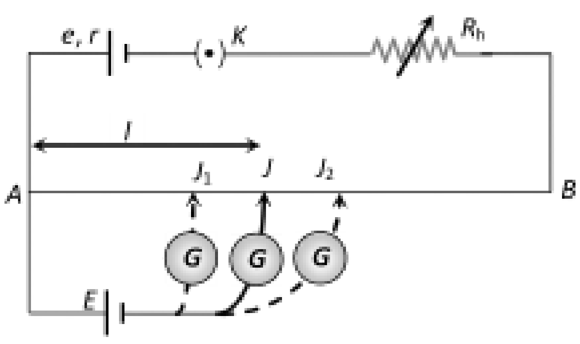
Suppose "jockey" is made to touch a point J on wire then potential difference between A and J will be

$V = xl$

At this length (l) two potential difference are obtained

(i) V due to battery e and

(ii) E due to unknown cell



If V > E then current will flow in galvanometer circuit in one direction

If V < E then current will flow in galvanometer circuit in opposite direction

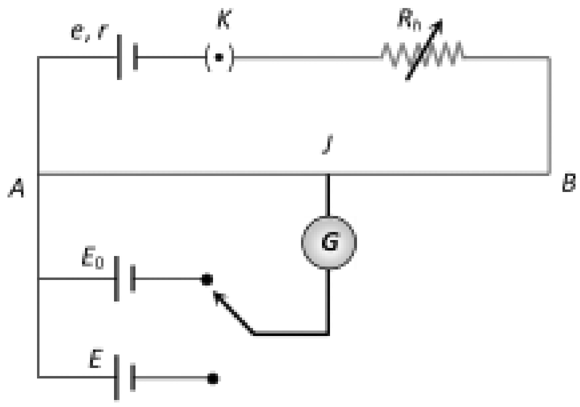
If V = E then no current will flow in galvanometer circuit this condition to known as null deflection position, length l is known as balancing length.

In balanced condition $E = xl$

or $$E = xl = \frac{V}{L}l = \frac{iR}{L}l = \left(\frac{e}{R + R_h + r}\right).\frac{R}{L} \times l$$

If V is constant then $L \propto l \Rightarrow \frac{x_1}{x_2} = \frac{L_1}{L_2} = \frac{l_1}{l_2}$

(vii) **Standardization of potentiometer :** The process of determining potential gradient experimentally is known as standardization of potentiometer.



Let the balancing length for the standard emf E_0 is l_0 then by the principle of potentiometer

$E_0 = xl_0 \Rightarrow x = \frac{E_0}{l_0}$

(viii) **Sensitivity of potentiometer :** A potentiometer is said to be more sensitive, if it measures a small potential difference more accurately.

(a) The sensitivity of potentiometer is assessed by its potential gradient. The sensitivity is inversely proportional to the potential gradient.

(b) In order to increase the sensitivity of potentiometer

(c) The resistance in primary circuit will have to be decreased.

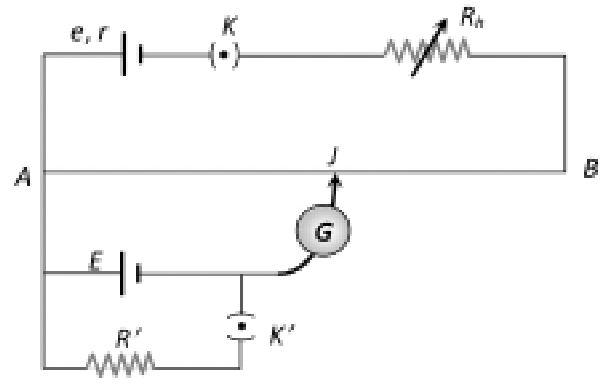
(d) The length of potentiometer wire will have to be increased so that the length may be measured more accuracy.

Difference between voltmeter and potentiometer

Voltmeter	Potentiometer
It's resistance is high but finite	Its resistance is infinite
It draws some current from source of emf	It does not draw any current from the source of unknown emf
The potential difference measured by it is lesser than the actual potential difference	The potential difference measured by it is equal to actual potential difference
Its sensitivity is low	Its sensitivity is high
It is a versatile instrument	It measures only emf or potential difference
It is based on deflection method	It is based on zero deflection method

Application of Potentiometer

(1) **To determine the internal resistance of a primary cell**



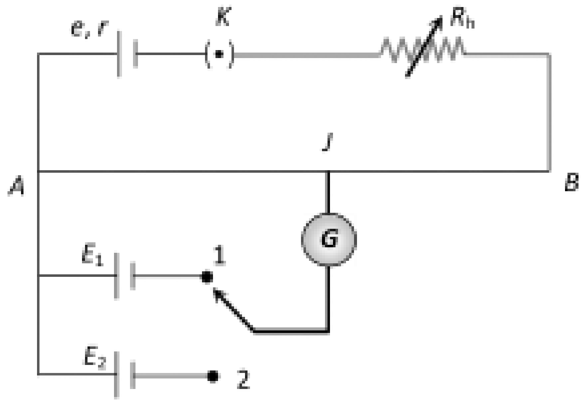
(a) Initially in secondary circuit key K' remains open and balancing length (l_1) is obtained. Since cell E is in open circuit so it's emf balances on length l_1 i.e. $E = xl_1$ .... (i)

(b) Now key K is closed so cell E comes in closed circuit. If the process of balancing repeated again then potential difference V balances on length l_2 i.e. $V = xl_2$ .... (ii)

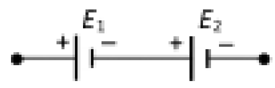
(c) By using formula internal resistance $r = \left(\frac{E}{V} - 1\right).R'$

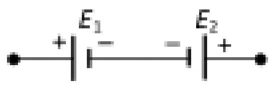
$$r = \left(\frac{l_1 - l_2}{l_2}\right).R'$$

(2) **Comparison of emf's of two cell :** Let l_1 and l_2 be the balancing lengths with the cells E_1 and E_2 respectively then $E_1 = xl_1$ and $E_2 = xl_2 \Rightarrow \frac{E_1}{E_2} = \frac{l_1}{l_2}$



Let $E_1 > E_2$ and both are connected in series. If balancing length is l_1 when cell assist each other and it is l_2 when they oppose each other as shown then :





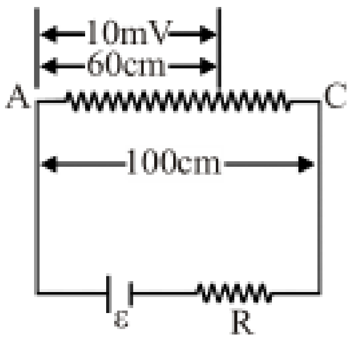
$(E_1 + E_2) = xl_1$ $\qquad$ $(E_1 - E_2) = xl_2$

$$\Rightarrow \quad \frac{E_1 + E_2}{E_1 - E_2} = \frac{l_1}{l_2} \qquad \frac{E_1}{E_2} = \frac{l_1 + l_2}{l_1 - l_2}$$

Illustration :

A potentiometer wire of length 1 m has a resistance of 10 ohm. It is connected in series with a resistance R and a cell of emf 3 V and negligible internal resistance. A source of emf 10 mV is balanced against a length of 60 m of the potentiometer wire. Find the volue of R.

Sol. *Following the theory of potentiometer,*



$$V_{AB} = i\,R_{AB}$$

$$= \left(\frac{\varepsilon}{R + R_{AB}}\right) R_{AB}$$

$$\varepsilon = 3\ V,\ R_{AB} = 10\ \Omega,\ V_{AB} = 10 \times 10^{-3}\ V$$

and $\qquad R_{AC} = \frac{AB}{AC} R_{AB} = \frac{60}{100} \times 10 = 6\ \Omega,$

We have $\qquad 10 \times 10^{-3} = \left(\frac{3}{R + 10}\right) \times 6$

or, $\qquad R = 1790\ ohm$

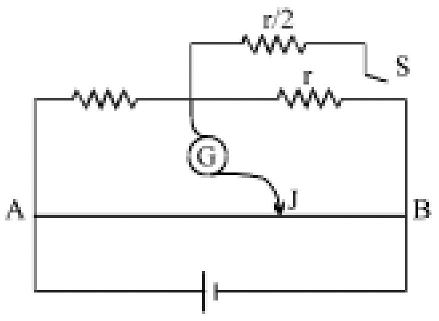
Practice Exercise

Q.1 A moving coil galvanometer of resistance 20Ω gives a full scale deflection when a current of 1mA is passed through it. It is to be converted into an ammeter reading 20A on full scale. But the shunt of 0.005Ω only is available. What resistance should be connected in series with the galvanometer coil ?

Q.2 In a potentiometer experiment it is found that no current passes through the galvanometer when the terminals of the cell are connected across 0.52 m of the potentiometer wire. If the cell is shunted by a resistance of 5Ω, a balance is obtained when the cell is connected across 0.4m of the wire. Find the internal resistance of the cell.

Q.3 There is a milliammeter each division of which reads 1mA. It has a resistance of 15Ω. How would you convert it into a voltmeter so that each division of its graduation would read 1 volt.

Q.4 The diagram shows a meter bridge with the wire AB having uniform resistance per unit length. When the switch S is open, AJ is the balance length and when the switch is closed, AJ' is the balance length. If AB = L and AJ = L/2 then what is the value of AJ'?



Q.5 How can the sensitivity of a potentiometer be increased?

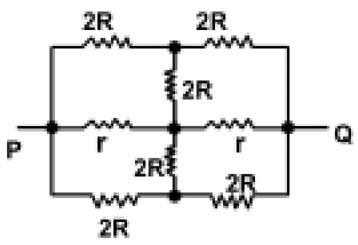
Q.6 An ammeter and a voltmeter are connected in series to a cell of e.m.f. 12 volts. When a certain resistance is connected in parallel with voltmeter the reading of voltmeter is reduced 3 times whereas the reading of ammeter increases 3 times. Find the voltmeter reading after the connection of resistance.

Answers

Q.1	79.995 Ω	Q.2	1.5Ω	Q.3	985 Ω in series	Q.4	$\frac{3l}{4}$
Q.5	Increasing rheostat in primary circuit it ↓ potential drop per unit length of wire					Q.6	3 volts

Solved Examples

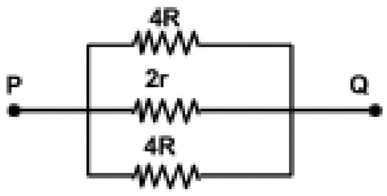
Q.1 The effective resistance between points P and Q of the electrical circuit shown in the figure is



(A) $\frac{2Rr}{R+r}$ (B) $\frac{8(R+r)}{3R+r}$

(C) $2r+4R$ (D) $\frac{5R}{2}+2r$

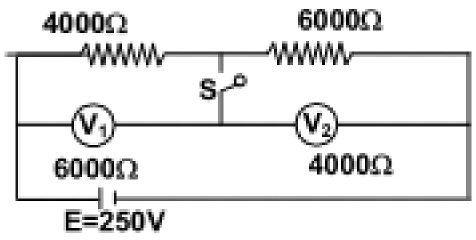
Sol. Sol. The circuit can be reduced to the one given alongside



$$R_e = \frac{2rR}{r+R}$$

Hence, (A) is correct

Q.2 In the circuit shown in the figure, V_1 and V_2 are two voltmeters having resistances 6000Ω and 4000Ω respectively emf of the battery is 250 volts, having negligible internal resistance. Two resistances R_1 and R_2 are 4000Ω and 6000Ω, respectively. Find the reading of the voltmeters V_1 and V_2 when



(i) switch S is open
(ii) switch S is closed

Sol.

(a) When switch S is open

R_1 and R_2 are in series. Let their equivalent resistance be R'

$R' = 4000 + 6000 = 10000$

The voltmeter are also in series. Let their resistance be R'', then

$R'' = 6000 + 4000 = 10000$

The resistance R' and R'' are connected in parallel. Their equivalent resistance is given by

$$R_{eq} = \frac{R' \times R''}{R' + R''} = \frac{10000 \times 10000}{20000} = 5000\ \Omega$$

$$\text{Current from battery} = \frac{E}{R_{eq}} = \frac{250}{5000} = \frac{1}{20}\text{A}$$

$$\text{Current } i_1 \text{ in the voltmeter branch} = \frac{1}{2} \times \frac{1}{20} = \frac{1}{40} \text{ amp}$$

$$\text{Potential difference across } V_1 = \frac{1}{40} \times 6000 = 150 \text{ volt}$$

$$\text{Potential difference across } V_2 = \frac{1}{40} \times 4000 = 100 \text{ volt}$$

(b) When switch S is closed. The circuit redrawn in this case is shown in figure. In this case V_1 and R_1 are in parallel. Similarly V_2 and R_2 are in parallel.

Equivalent resistance of V_1 and R_1

$$R' = \frac{6000 \times 4000}{6000 + 400} = 2400\ \Omega$$

Similarly for R_2 and V_2

$$R'' = \frac{6000 \times 4000}{6000 + 400} = 2400\ \Omega$$

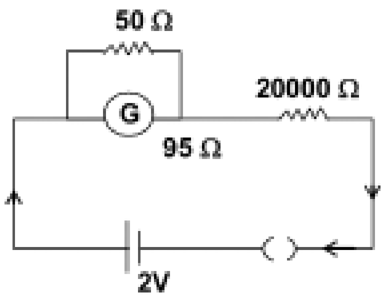
So, the two equal resistances are connected in series.

Hence reading of $V_1 = 125$ volt

And reading of $V_2 = 125$ volt

Q.3 A galvanometer of resistance 95Ω , shunted by a resistance of 50 Ω gives a deflection of 50 divisions when joined in series with a resistance of 20 kΩ and a 2 volt battery, what is the current sensitivity of galvanometer (in div/μA)?

Sol. Current in the circuit

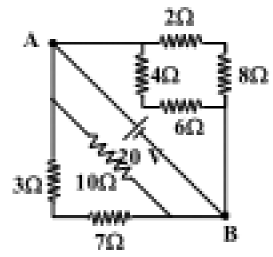


$$I = \frac{2}{20 \times 10^3} = 100\ \mu A$$

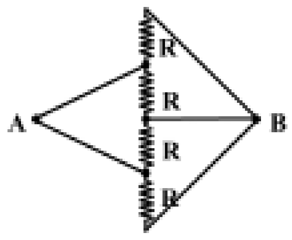
This current produces deflection of 50 div in the galvanometer

$$CS = \frac{\theta}{I} = \frac{50\,\text{Div}}{100\mu A} = \frac{1}{2}\frac{\text{Div}}{\mu A}$$

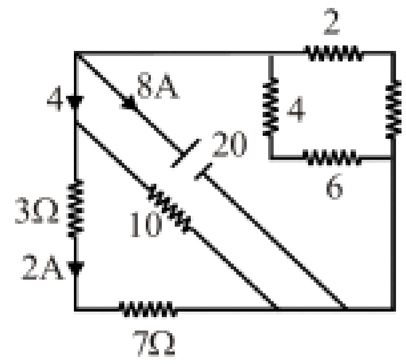
Q.5 (a) The potential difference across 7Ω resistor is equal to _______ and the current flowing through the battery is equal to _______.



(b) The equivalent resistance across A and B is equal to _____________.



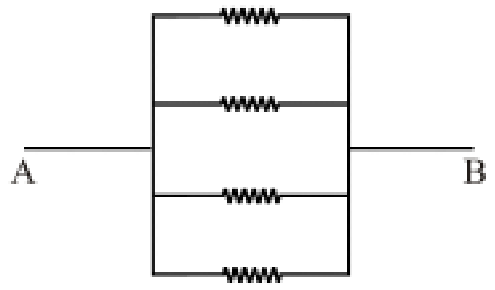
Sol. (a) $R_{AB} = \frac{5}{2}\Omega$, $i_{total} = \frac{20}{5/2} = 8A$



$\Delta V_{7\Omega} = 7 \times 2 = 14$ V

14 volt, 8A

(b) The circuit can redrawn as,



$R_{AB} = \frac{R}{4}$

Q.6 Two resistors, 400 ohm and 800 ohm, are connected in series with a 6 V battery. It is desired to measure the current in the circuit. An ammeter of 10 ohm resistance is used for this purpose. What will be the reading in the ammeter? Similarly, if a voltmeter of 10,000 ohm resistance is used to measure the potential difference across 400 ohm, what will be the reading of the voltmeter?

Sol. Ammeter has law resistance and voltmeter has high resistance as compared with resistance of circuit hence

$$i = \frac{6}{400+800} = \frac{6}{1200} = 5 \text{ mA}, V = 400 \times 5\text{mA} = 2 \text{ volt}$$

Q.7 Two cells, having emfs of 10 V and 8 V, respectively, are connected in series with a resistance of 24 Ω in the external circuit. If the internal resistances of each of these cells in ohm are 200% of the value of their emf's, respectively, find the terminal potential difference across 8 V battery.

Sol. We determine the internal resistance of each these cells :

$r_1 = 2\Omega/V \times 10V = 20,$

$r_2 = 2\Omega/V \times 8V = 16\Omega$

$\therefore$ Total resistance in circuit $= (24+16+20) = 60\Omega$

$$\therefore \text{ Current } = \frac{18V}{60\Omega} = 0.3 \text{ A.}$$

Thus terminal potential difference $V = E \quad ir = 8 - 0.3(16) = 3.2$ V

Q.8 A galvanometer having 50 divisions provided with a variable shunt S is used to measure the current when connected in series with a resistance of 90Ω and a battery of internal resistance 10Ω. It is observed that when the shunt resistances are 10Ω and 50Ω, the deflections are, respectively, 9 and 30 divisions. What is the resistance of the galvanometer?

Sol. $$I = \frac{\varepsilon}{\left(90+10+\frac{SG}{S+G}\right)} = \frac{\varepsilon}{\left(100+\frac{SG}{S+G}\right)} \quad \ldots(i)$$

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Applying kirchhoff's of law

we get, $$i_g = \frac{IS}{S+G}$$

$$\Rightarrow \quad i_g = \frac{S}{S+G} \times \frac{\varepsilon}{\left(100 + \frac{SG}{S+G}\right)} \qquad \text{...(ii)}$$

Let $i_g = i_1$ of $S = 10\Omega$ and $i_g = i_2$ for $S = 50\Omega$

$$\frac{i_1}{i_2} = \frac{\left(\frac{10}{10+G}\right) \times \left(\frac{\varepsilon}{100 + \frac{100G}{10+G}}\right)}{\left(\frac{50}{50+G}\right) \times \left(\frac{\varepsilon}{100 + \frac{50G}{50+G}}\right)}$$

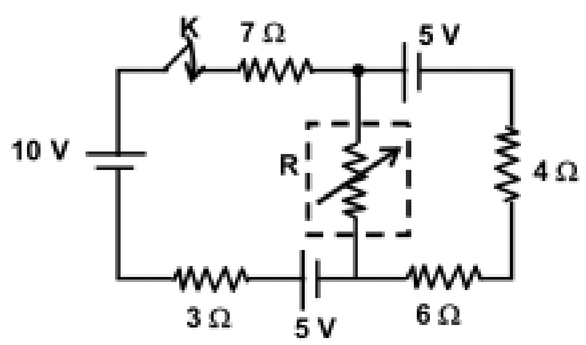
$$\frac{i_1}{i_2} = \frac{100+3G}{100+11G}$$

$\therefore$ Defletion is proportional to the current

$$\Rightarrow \quad \frac{9}{30} = \frac{100+3G}{100+11G}$$

Solving we get, $G = 233.3\ \Omega$

Q.9. In the circuit shown the resistance R is kept in a chamber whose temperature is 20°C which remains constant. The initial temperature and resistance of R is 50°C and 15Ω respectively. The rate of change of resistance R with temperature is $\frac{1}{2}\ \Omega/°C$ and the rate of decrease of temperature of R is $ln\left(\frac{3}{100}\right)$ times the temperature difference from the surrounding (Assume the resistance R loses heat only in accordance with Newton's law of cooling). If K is closed at $t = 0$, then find the



(a) value of R for which power dissipation in it is maximum.
(b) temperature of R when power dissipation is maximum.
(c) time after which the power dissipation will be maximum.

Sol.

(a) Let i_1 and i_2 be the current in two loops respectively

$\therefore$ $(10 - 10)\, i_1 - R(i - i_2) + 5 = 0$ (for loop 1)

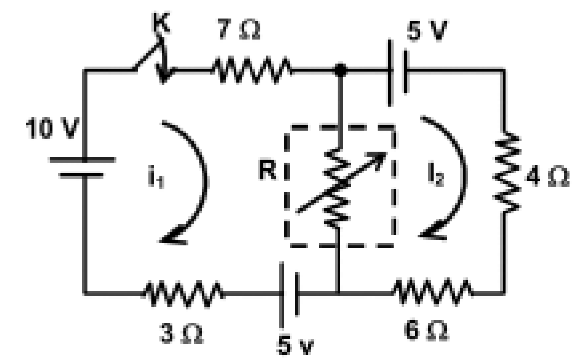
$(10 + R)\, i_2 - Ri_1 = -5$ (for loop 2)

Power dissipated in R,

$$P = (i_1 - i_2)^2 R = \frac{25}{(5+R)^2} \times R$$

$\Rightarrow$ For maximum power dissipation $\frac{dP}{dR} = 0$

$\Rightarrow$ $R = 5\Omega$



(b) $R = R_0 - \left(\frac{dR}{d\theta}\right)\Delta\theta$

$$5 = 15 - \frac{1}{2}\Delta\theta$$

$\Rightarrow \Delta\theta = 20^{o}C \Rightarrow$ temperature at that instant $= 30^{o}C$

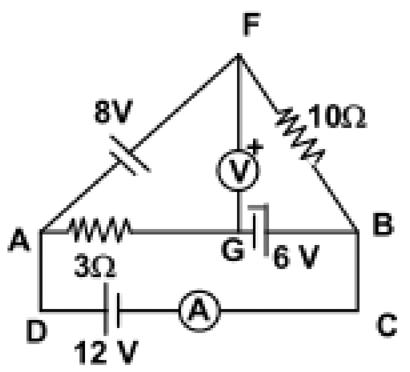
(c) According to Newton's law :

$$\frac{d\theta}{dt} = -k(\theta - 20^{o})$$

$$\int_{50}^{20} \frac{d\theta}{\theta - 20} = -kt, \quad = \frac{-\ln 3}{100} t, = -\ln 3$$

$\therefore$ $t = 100$ sec.

Q.10 Find the reading of ammeter A and voltmeter V shown in the figure assuming the instruments to be ideal.



Sol. Distributing the currents in the circuit according to Kirchhoff's I law is shown in the figure. In ideal voltmeter current = 0. Applying Kirchhoff's law in mesh ABCDA

$$-3I_1 + 6 + 12 = 0$$

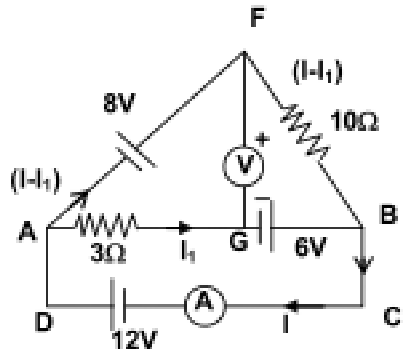
i.e. $I_1 = 6A$

Now apply Kirchoff's law in AFBA

$$8 - (I - I_1) \times 10 - 6 + 3I_1 = 0$$

i.e $10I - 13I_1 = 2$

or $I = \frac{2}{10} + \frac{13}{10} \times 6 = 8A$



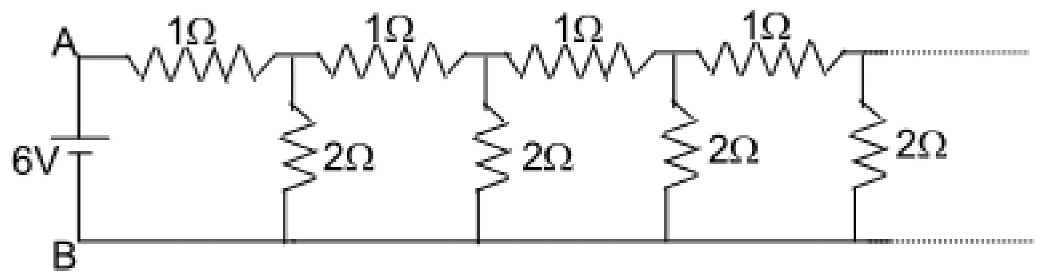
Hence reading of ammeter = 8A

Reading of the voltmeter $V = V_F - V_G$
applying Kirchhoff's law in mesh AFGA

$$8 - V + 6 \quad 3 = 0 \qquad \text{i.e. } V = 26\text{ V}$$

Hence reading of voltmeter $= 26$ V.

Q.11 An infinite ladder network of resistance is constructed with 1 and 2 resistance, as shown in fig. The 6V battery between A and B has negligible internal resistance.
(i) Show that the effective resistance between A and B is 2 .
(ii) What is the current that passes through 2 resistance nearest to the battery?



Sol.
(i) Since the network is an infinite ladder, we can assume that resistance across AB is equal to that of A' B'

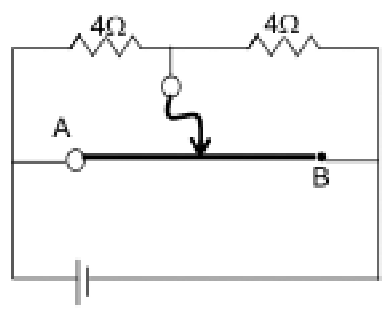
$$R = 1 + \frac{2R}{2+R}$$

$\Rightarrow$ $2R + R^2 = 2 + R + 2R$ or $R = 2$ ohm.

(ii) $i = \frac{6}{2} = 3$ amp.

$i' = \frac{3}{2} = 1.5$ amp

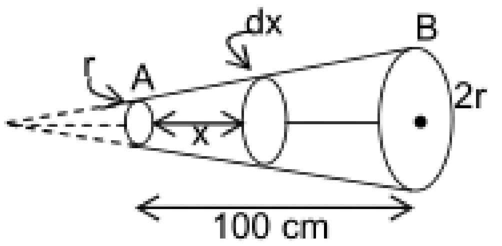
Q.12 The wire AB of a meter bridge continuously changes from radius r to $2r$ from left end to right end. Where should the free end of galvanometer be connected on AB so that the deflection in the galvanometer is zero?



Sol. Let the galvanometer be connected at a point $x = x_1$ from end A where $x = 0$.
Let R_1 = resistance of left part i.e. AX_1 and
R_2 = resistance of right part i.e. X_1B
Length = 100 cm = 1 m.
Consider an element of thickness dx at a distance x from end A and of radius r_x.

Thus, $r_x = \left(r + \frac{r}{1}x\right) = r(1 + x)$

Resistance of this element will be, $dRx = \frac{\rho dx}{\pi r_x^2}$



$$R_1 = \int_0^{x_1} \frac{\rho\, dx}{\pi(1+x)^2 r^2} = \frac{\rho}{\pi r^2}\left[1 - \frac{1}{x_1}\right]$$

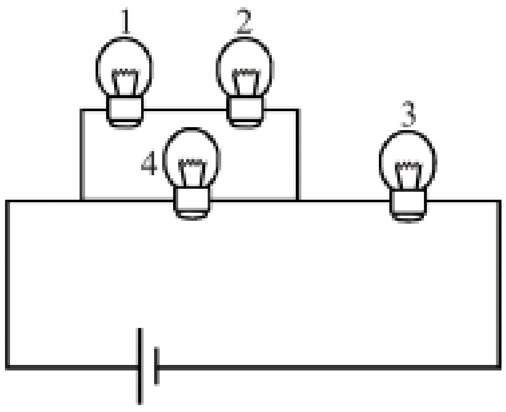
$$R_2 = \int_{x_1}^{4} \frac{\rho\, dx}{(1+x)^2 r^2} = \frac{\rho}{\pi r^2}\left[\frac{1}{1+x_1} - \frac{1}{1+1}\right]$$

For null point or zero deflection,

$$\frac{R_1}{R_2} = \frac{4}{4} \Rightarrow 1 - \frac{1}{1+x_1} = \frac{1}{1+x_1} - \frac{1}{1+1}$$

$$\Rightarrow \quad x_1 = \frac{1}{3}\text{ m} = 33.33 \text{ cm}$$

Q.13 Four identical bulbs, each of same rating (100 W, 220 V) are connected across an ideal battery of emf 550 volts. Which of the 4 bulbs will have a voltage across it, which is greater than voltage rating. (i.e. which of them will fuse)



Sol. By voltage division
$v_3 = 330$ volts
$v_4 = 220$ volts
$v_1 = v_2 = 100$ volts
Ans. only bulb (3)

Q.14 What amount of heat will be generated in a coil of resistance R due to a charge q passing through it if the current in the coil
(a) decreases down to zero uniformly during a time interval Δt
(b) decreases down to zero halving its value every Δt seconds?

Sol. (a) As current i is linear function of time, and at $t = 0$ and Δt, it equals i_0 and zero respectively, it may be represented as,

$$i = i_0\left(1 - \frac{t}{\Delta t}\right)$$

Thus $\quad q = \int_0^{\Delta t} i\,dt = \int_0^{\Delta t} i_0\left(1 - \frac{t}{\Delta t}\right)dt = \frac{i_0 \Delta t}{2}$

So, $\quad i_0 = \frac{2q}{\Delta t} \qquad$ Hence $\quad i = \frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)$

The heat generated.

$$H = \int_0^{\Delta t} i^2 R dt = \int_0^{\Delta t} \left[\frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)\right]^2 R dt = \frac{4q^2 R}{3\Delta t}$$

(b) Obviously the current through the coil is given by

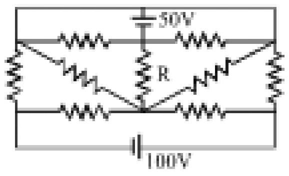
$$i = i_0 \left(\frac{1}{2}\right)^{t/\Delta t}$$

Then charge $q = \int_0^{\infty} i dt = \int_0^{\infty} i_0 2^{-t/\Delta t} dt = \frac{i_0 \Delta t}{\ln 2}$ So, $i_0 = \frac{q \ln 2}{\Delta t}$

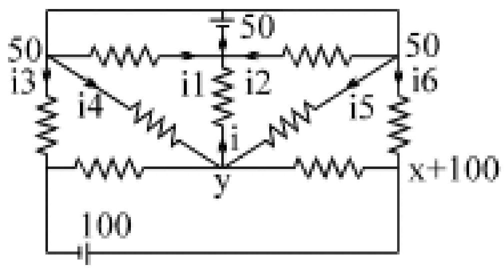
And hence, heat generated in the circuit in the time interval $t[0, \infty]$,

$$H = \int_0^{\infty} i^2 R dt = \int_0^{\infty} \left[\frac{q \ln 2}{\Delta t} 2^{-t/\Delta t}\right]^2 R dt = -\frac{q^2 \ln 2}{2\Delta t} R$$

Q.15 Find the current in the resistance R. Each resistance is of 2Ω.



Sol.



Nodal analysis $\frac{y-0}{2} + 2 . \frac{y-50}{2} + \frac{y-x}{2} + \frac{y-x-100}{2} = 0$

$\Rightarrow$ $y + 2y - 100 + (y - x) - 100 = 0$

$5y - 2x = 200$ ... (1)

$i = i3 + i4 + i5 + i6$

$$\frac{y-0}{2} = \frac{50-x}{2} + \frac{50-y}{2} + \frac{50-y}{2} + \frac{50-x-100}{2}$$

$y = 150 - x - y - 50 - x$

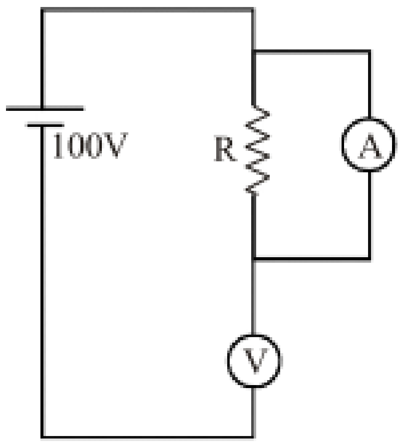
$2x + 3y = 100$ ... (2)

$-2x + 5y = 200$ ... (1)

$8y = 300$

$\frac{y}{2} = \frac{300}{16}$ $= 18.75$ A

Q.16 A voltmeter of resistance 995 Ω and an ammeter of resistance 10 Ω is connected as shown to calculate the unknown resistance R which is connected to the ideal battery. Voltmeter reading is 99.5 volts. The value of resistance R is calculated as $\frac{\text{Voltmeter reading}}{\text{Ammeter reading}}$ by student A.



(i) Find his answer.

(ii) Also find the actual value of resistance.

Sol. (i) Voltage across ammeter = 0.5 volts

Resistance = 10 Ω

Ammeter reading = 0.05 A

$$R = \frac{\text{Voltmeter reading}}{\text{Ammeter reading}} = \frac{99.5}{0.05} = 1990\ \Omega$$

(ii) Current across voltmeter = $\frac{99.5}{995} = 0.1$ **A**

and current through ammeter = 0.05 A

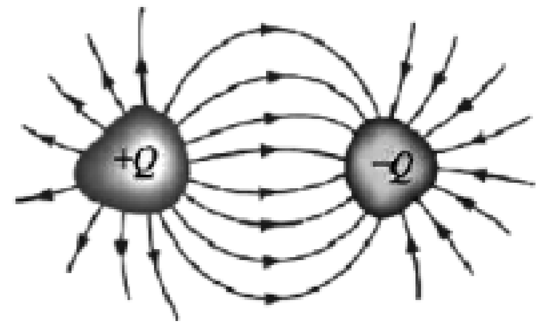
∴ Current through R = 0.05 A and voltage across R = 0.5 V

∴ $R = \frac{0.5}{0.05} = 10\ \Omega$

Capacitance

Introduction

Capacitor is an arrangement of two conductors generally carrying charges of equal magnitudes and opposite sign and separated by an insulating medium . A capacitor is a device which stores electric charge. Capacitors vary in shape and size, but the basic configuration is two conductors carrying equal but opposite charges (Figure). Capacitors have many important applications in electronics. Some examples include storing electric potential energy, delaying voltage changes when coupled with resistors, filtering out unwanted frequency signals, forming resonant circuits and making frequency-dependent and independent voltage dividers when combined with resistors. Some of these applications will be discussed in latter chapters

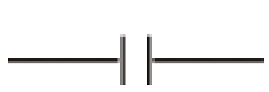


When charges are pulled apart, energy is associated with the pulling apart of charges, just like energy is involved in stretching a spring. Thus, some energy is stored in capacitors.

In the *uncharged* state, the charge on either one of the conductors in the capacitor is zero. During the charging process, a charge Q is moved from one conductor to the other one, giving one conductor a charge $+Q$, and the other one a charge $-Q$. A potential difference ΔV is created, with the positively charged conductor at a higher potential than the negatively charged conductor. Note that whether charged or uncharged, the net charge on the capacitor as a whole is zero.

Note :

1. The net charge on the capacitor as a whole is zero. When we say that a capacitor has a charge Q, we mean that the positively charged conductor has charge +Q and negatively charged conductor has a charge –Q.

2. In a circuit, a capacitor is represented by the symbol : 

Limitations on charging a conductor

How much electric charge can be placed on a conductor?

As more air is pumped into the tank, the pressure opposing the flow of additional air becomes greater so it becomes further difficult to pump more air. . Similarly, as more charge Q is transferred to the conductor, the potential V of the conductor becomes higher, making it increasingly difficult to transfer more charge. Suppose we try to place an indefinite quantity of charge Q on a spherical conductor of radius r. The air surrounding the conductor is an insulator, sometimes called a dielectric, which contains few charges free to move. The electric field intensity E and the potential V at the surface of the sphere are given by

$$E = \frac{kQ}{r^2} \quad \text{and} \quad V = \frac{kQ}{r}$$

Since the radius r is constant, both the field intensity and the potential at the surface of the sphere increase in direct proportion to the charge Q. There is a limit, however, to the field intensity that can exist on a conductor without ionizing the surrounding air. When this occurs, the air essentially becomes a conductor, and any additional charge placed on the sphere will "leak off" to the air. This limiting value of electric field intensity for which a material loses its insulation properties is called the dielectric strength of that material.

The dielectric strength for a given material is that electric field intensity for which the material ceases to be an insulator and becomes a conductor.

The dielectric strength for dry air at 1 atm pressure is around 3MN/C. Since the dielectric strength of a material varies considerably with environmental conditions, such as pressure and humidity, it is difficult to compute accurate values.

Note that the amount of charge that can be placed on a spherical conductor decreases with the radius of the sphere. Thus, smaller conductors can usually hold less charge. But the shape of a conductor also influences its ability to retain charge. Consider the charged conductors. If these conductor are tested with an electroscope, it will be discovered that the charge on the surface of a conductor is concentrated at points of greatest curvature. Because of the greater charge density in these regions, the electric field intensity is also greater in regions of higher curvature. If the surface is reshaped to a sharp point, the field intensity may become great enough to ionize the surrounding air. A show leakage of charge sometimes occurs at these locations, producing a corona discharge, which is often observed as a faint violet glow in the vicinity of the sharply pointed conductor. It is important to remove all sharp edges from electrical equipment to minimize this leakage of charge.

Practice Exercise

Q.1 What is the maximum charge that may be placed on a spherical conductor 1m in diameter? Assume it is surrounded by air. Assume the dielectric strength for dry air at 1 atm pressure is around 3MN/C.

Answers

Q.1 $\frac{1}{12}\times 10^{-3}\,C$

Capacitance

We can say that the increase in potential V is directly proportional to the charge Q placed on the conductor. Symbolically: $V \propto Q$

Therefore, the ratio of the quantity of charge Q to the potential V produced will be a constant for a given conductor. This ratio reflects the anility of a conductor to store charge and is called its capacitance C.

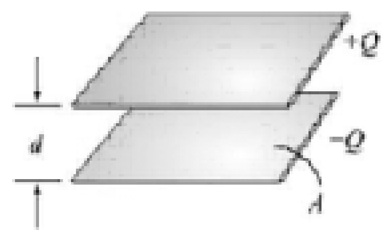
$$C = \frac{Q}{V}$$

The unit of capacitance is the coulomb per volt, which is redefined as a farad (F). Thus, if a conductor has a capacitance of 1 farad, a transfer of 1 coulomb of charge to the conductor will raise its potential by 1 volt.

The value of C for a given conductor is not a function of either the charge placed on a conductor or the potential produced. In principle, the ratio Q/V will remain constant as charge is added indefinitely, but the capacitance depends on the size and shape of a conductor as well as on the nature of the surrounding medium.

The capacitor

The simplest example of a capacitor consists of two conducting plates of area A, which are parallel to each other, and separated by a distance d, as shown in Figure.



A parallel-plate capacitor

Experiments show that the amount of charge Q stored in a capacitor is linearly proportional to $|\Delta V|$, the electric potential difference between the plates. Thus, we may write

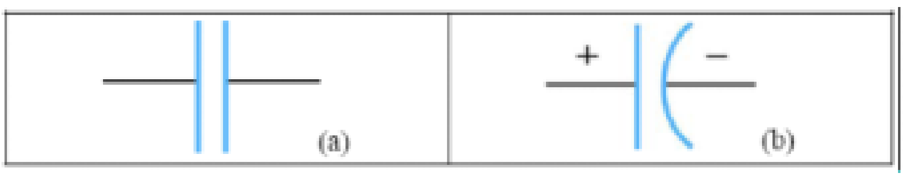
$$Q = C\,|\Delta V|$$

where C is a positive proportionality constant called *capacitance*. Physically, capacitance is a measure of the capacity of storing electric charge for a given potential difference ΔV. The SI unit of capacitance is the *farad* (F) :

$$1\text{ F} = 1\text{ farad} = 1\text{ coulomb / volt} = 1\text{C/V}$$

A typical capacitance is in the picofarad ($1\text{pF} = 10^{-12}$ F) to millifarad range, ($1\text{ mF} = 10^{-3}\text{ F} = 1000\ \mu\text{F}$; $1\mu\text{F} = 10^{-6}$ F).

Figure (a) shows the symbol which is used to represent capacitors in circuits. For a polarized fixed capacitor which has a definite polarity, Figure (b) is sometimes used.



Capacitor symbols.

Practice Exercise

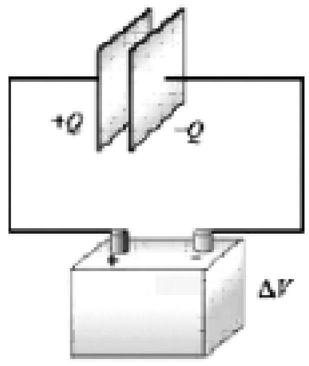
Q.1 A capacitor having a capacitance of 4μF is connected to a 60V battery. What is the charge on the capacitor?

Answers

Q.1 240 μC

Capacitors in Electric Circuits

A capacitor can be charged by connecting the plates to the terminals of a battery, which are maintained at a potential difference ΔV called the ***terminal voltage.***

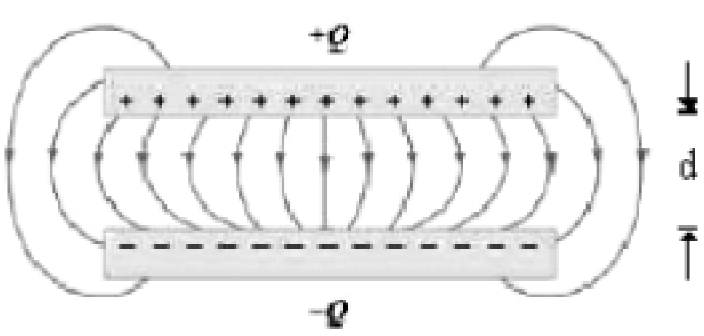


Charging a capacitor.

The connection results in sharing the charges between the terminals and the plates. For example, the plate that is connected to the (positive) negative terminal will acquire some (positive) negative charge. The sharing causes a momentary reduction of charges on the terminals, and a decrease in the terminal voltage. Chemical reactions are then triggered to transfer more charge from one terminal to the other to compensate for the loss of charge to the capacitor plates, and maintain the terminal voltage at its initial level. The battery could thus be thought of as a charge pump that brings a charge Q from one plate to the other.

Parallel-Plate Capacitor

Consider two metallic plates of equal area A separated by a distance d, as shown in Figure below. The top plate carries a charge $+Q$ while the bottom plate carries a charge $-Q$. The charging of the plates can be accomplished by means of a battery which produces a potential difference. Find the capacitance of the system.



The electric field between the plates of a parallel-plate capacitor

To find the capacitance C, we first need to know the electric field between the plates. A real capacitor is finite in size. Thus, the electric field lines at the edge of the plates are not straight lines, and the field is not contained entirely between the plates. This is known as *edge effects*, and the non-uniform fields near the edge are called the *fringing fields*. In Figure the field lines are drawn by taking into consideration edge effects. However, in what follows, we shall ignore such effects and assume an idealized situation, where field lines between the plates are straight lines.

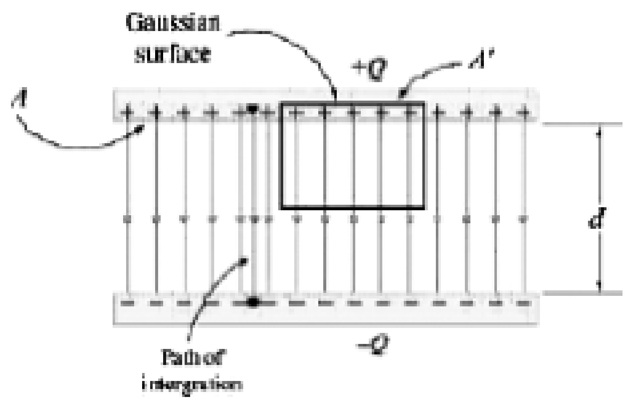
In the limit where the plates are infinitely large, the system has planar symmetry and we can calculate the electric field everywhere using Gauss's law given in Eq. :

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

By choosing a Gaussian "pillbox" with cap area A' to enclose the charge on the positive plate (see Figure), the electric field in the region between the plates is

$$EA' = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma A'}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{\varepsilon_0}$$

The same result has also been obtained using superposition principle.



Gaussian surface for calculating the electric field between the plates.

The potential difference between the plates is

$$\Delta V = V_- - V_+ = -\int_{+}^{-} \vec{E} \cdot d\vec{s} = -Ed$$

where we have taken the path of integration to be a straight line from the positive plate to the negative plate following the field lines. Since the electric field lines are always directed from higher potential to lower potential, $V_- < V_+$. However, in computing the capacitance C, the relevant quantity is the magnitude of the potential difference:

$$|\Delta V| = Ed$$

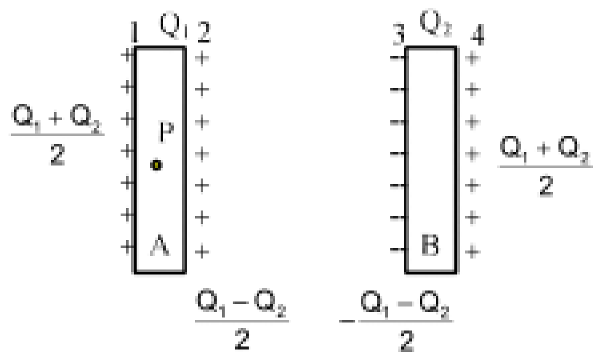
and its sign is immaterial. From the definition of capacitance, we have

$$C = \frac{Q}{|\Delta V|} = \frac{\varepsilon_0 A}{d} \text{ (parallel plate)}$$

Note that C depends only on the geometric factors A and d. The capacitance C increases linearly with the area A since for a given potential difference ΔV, a bigger plate can hold more charge. On the other hand, C is inversely proportional to d, the distance of separation because the smaller the value of d, the smaller the potential difference $|\Delta V|$ for a fixed Q.

Plates of a Parallel Plate Capacitor carrying Different Charges

Two identical plates of parallel plate capacitor are given unequal charges Q_1 and Q_2. The charges appearing on the inner surface be $+\frac{Q_1 - Q_2}{2}$ and $-\frac{Q_1 - Q_2}{2}$ and the charges appearing on outer surfaces are $\frac{Q_1 + Q_2}{2}$ (as shown in the figure). Here the potential difference between the plates is



$$V = \frac{\left(\frac{Q_1 - Q_2}{2}\right)}{\left(\frac{\varepsilon_0 A}{d}\right)}$$

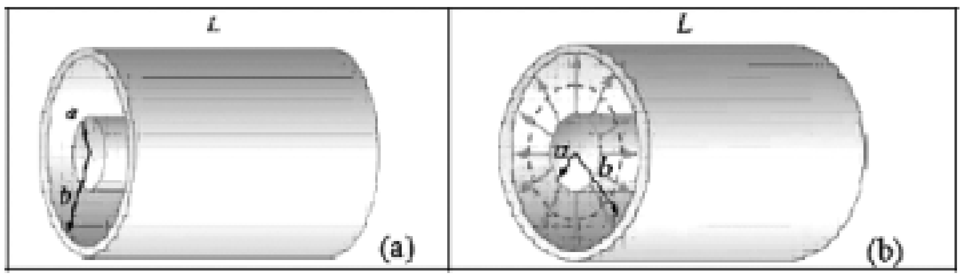
This means

charge of the capacitor is $q = \frac{Q_1 - Q_2}{2}$

capacitance is still $C = \frac{\varepsilon_0 A}{d}$

Cylindrical Capacitor

Consider next a solid cylindrical conductor of radius a surrounded by a coaxial cylindrical shell of inner radius b, as shown in Figure. The length of both cylinders is L and we take this length to be much larger than $b- a$, the separation of the cylinders, so that edge effects can be neglected. The capacitor is charged so that the inner cylinder has charge $+Q$ while the outer shell has a charge $-Q$. What is the capacitance?



(a) A cylindrical capacitor. (b) End view of the capacitor. The electric field is non-vanishing only in the region $a < r < b$.

The potential difference is given by

$$\Delta V = V_b - V_a = -\int_a^b E_r dr = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

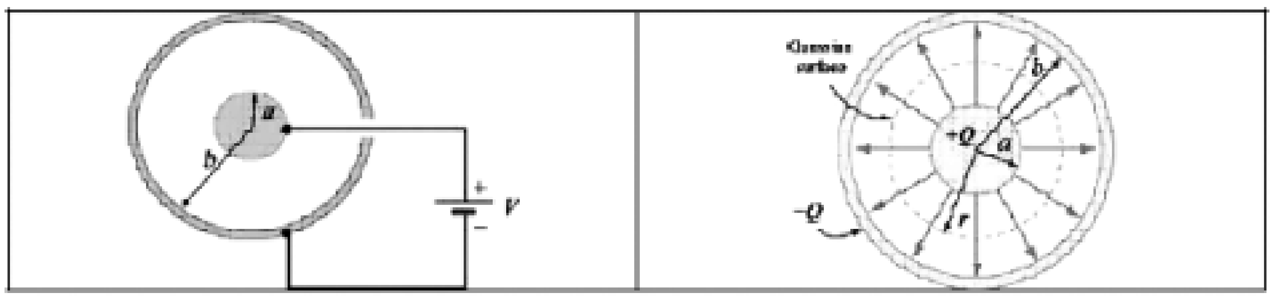
where we have chosen the integration path to be along the direction of the electric field lines. As expected, the outer conductor with negative charge has a lower potential. This gives

$$C = \frac{Q}{|\Delta V|} = \frac{\lambda L}{\lambda \ln(b/a)/r\pi\varepsilon_0} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

Once again, we see that the capacitance C depends only on the geometrical factors, L, a and b.

Spherical Capacitor

As a third example, let's consider a spherical capacitor which consists of two concentric spherical shells of radii a and b, as shown in Figure. The inner shell has a charge $+Q$ uniformly distributed over its surface, and the outer shell an equal but opposite charge $-Q$. What is the capacitance of this configuration?



(a) spherical capacitor with two concentric spherical shells of radii a and b.
(b) Gaussian surface for calculating the electric field.

The potential difference between the two conducting shells is :

$$\Delta V = V_b - V_a = -\int_a^b E_r dr = -\frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

which yields

$$C = \frac{Q}{|\Delta V|} = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$

Again, the capacitance C depends only on the physical dimensions, a and b.

An "isolated" conductor (with the second conductor placed at infinity) also has a capacitance. In the limit where $b \to \infty$, the above equation becomes.

$$\lim_{b\to\infty} C = \lim_{b\to\infty} 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)\lim_{b\to\infty} 4\pi\varepsilon_0 \frac{a}{\left(1-\frac{a}{b}\right)} = 4\pi\varepsilon_0 a$$

Thus, for a single isolated spherical conductor of radius R, the capacitance is

$$C = 4\pi\varepsilon_0 R$$

The above expression can also be obtained by noting that a conducting sphere of radius R with a charge Q uniformly distributed over its surface has $V = Q/4\pi\varepsilon_0 R$, using infinity as the reference point having zero potential, $V(\infty) = 0$. This gives

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{Q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R$$

As expected, the capacitance of an isolated charged sphere only depends on its geometry, namely, the radius R.

Illustration:

A parallel plate air capacitor is made using two plates 0.2 m square, spaced 1 cm apart. It is connected to a 50 V battery.

(a) *what is the capacitance ?*

(b) *what is the charge on each plate ?*

(c) *what is the electric field between the plates ?*

(d) *if the battery is disconnected and then the plates are pulled apart to a separation of 2 cm, what are the Answers to the above parts ?*

Sol. *(a)* $C_0 = \frac{\varepsilon_0 A}{d_0} = \frac{8.85\times10^{-12}\times0.2\times0.2}{0.01}$

$$C_0 = 3.54\times10^{-5}\ \mu F$$

(b) $Q_0 = C_0V_0 = (3.54\times10^{-5}\times50)\mu C \ = 1.77\times10^{-3}\mu C$

(c) $E_0 = \frac{V_0}{d_0} = \frac{50}{0.01} = 5000\,V/m.$

(d) *If the battery is disconnected, the charge on the capacitor plates remains constant while the potential difference between plates can change.*

$$C = \frac{A\,\epsilon_0}{2d} = 1.77\times10^{-5}\ \mu F$$

$$Q = Q_0 = 1.77\times10^{-3}\ \mu C$$

$$V = \frac{Q}{C} = \frac{Q_0}{C_0/2} = 2V_0 = 100\text{ volts.}$$

$$E = \frac{V}{c}\frac{2V_0}{2d_0} = E_0 = 5000\,V/m.$$

Practice Exercise

Q.1 A capacitor having plate area A, separation between plates d is connected to a battery having potential difference across it as V. Find what happens to its provided the battery remains connected

(a) Capacitance (c) P.d, across capacitor plates

(b) Charge (d) Field between the plates

when its area is doubled

Q.2 A capacitor having plate area A, separation between plates d is connected to a battery having potential difference across it as V. Find what happens to its provided the battery is disconnected

(a) Capacitance (c) P.d, across capacitor plates

(b) Charge (d) Field between the plates

when its area is doubled

Q.3 Two identical metal plates are given positive charges Q_1 and Q_2 ($<Q_1$) respectively. If they are now brought close together to form a parallel plate capacitor with capacitance C, the potential difference between them is :

Answers

Q.1 (a) doubled (b) doubled (c) remain same (d) remain same

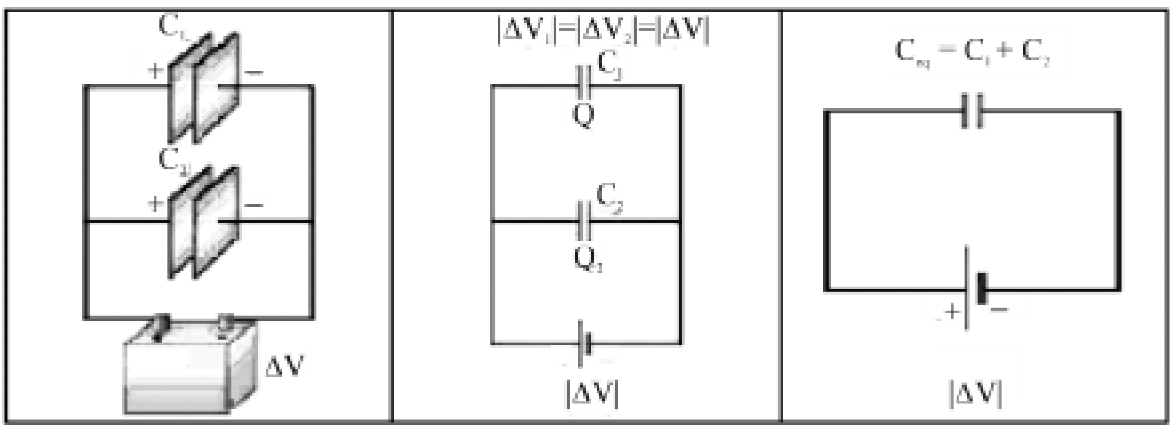
Q.2 (a) doubled (b) remain same (c) halved (d) halved

Q.3 $\frac{Q_1 - Q_2}{2C}$

Grouping of capacitors

Parallel Connection :

Suppose we have two capacitors C_1 and C_2 that are connected in parallel, as shown in Figure.



Capacitors in parallel and an equivalent capacitor.

The left plates of both capacitors C_1 and C_2 are connected to the positive terminal of the battery and have the same electric potential as the positive terminal. Similarly, both right plates are negatively charged and have the same potential as the negative terminal. Thus, the potential difference $|\Delta V|$ is the same across each capacitor. This gives

$$C_1 = \frac{Q_1}{|\Delta V|}, \; C_2 = \frac{Q_2}{|\Delta V|}$$

These two capacitors can be replaced by a single equivalent capacitor with a total charge Q supplied by the battery. However, since Q is shared by the two capacitors, we must have

$$Q = Q_1 + Q_2 = C_1 \,|\Delta V| + C_2 \,|\Delta V| = (C_1 + C_2)\,|\Delta V|$$

The equivalent capacitance is then seen to be given by

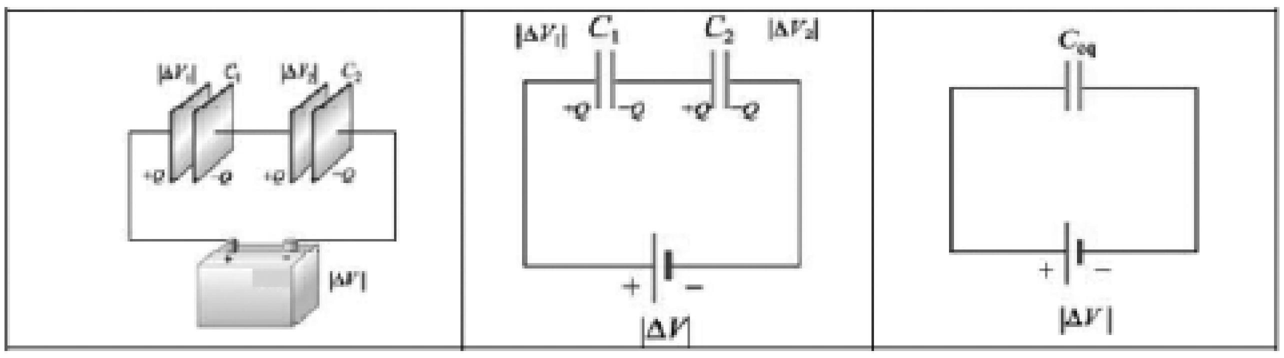
$$C_{eq} = \frac{Q}{|\Delta V|} = C_1 + C_2$$

Thus, capacitors that are connected in parallel add. The generalization to any number of capacitors is

$$C_{eq} = C_1 + C_2 + C_3 + ... + C_N = \sum_{i=1}^{N} C_i \quad \text{(parallel)}$$

Series Connection :

Suppose two initially uncharged capacitors C_1 and C_2 are connected in series, as shown in Figure. A potential difference $|\Delta V|$ is then applied across both capacitors. The left plate of capacitor 1 is connected to the positive terminal of the battery and becomes positively charged with a charge $+Q$, while the right plate of capacitor 2 is connected to the negative terminal and becomes negatively charged with charge $-Q$ as electrons flow in. What about the inner plates? They were initially uncharged; now the outside plates each attract an equal and opposite charge. So the right plate of capacitor 1 will acquire a charge $-Q$ and the left plate of capacitor 2 will acquire a charge of $+Q$.



Capacitors in series and an equivalent capacitor

The potential differences across capacitors C_1 and C_2 are

$$|\Delta V_1| = \frac{Q}{C_1}, \; |\Delta V_2| = \frac{Q}{C_2}$$

respectively. From Figure, we see that the total potential difference is simply the sum of the two individual potential differences:

$$|\Delta V| = |\Delta V_1| + |\Delta V_2|$$

In fact, the total potential difference across any number of capacitors in series connection is equal to the sum of potential differences across the individual capacitors. These two capacitors can be replaced by a single equivalent capacitor $C_{eq.} = \dfrac{Q}{|\Delta V|}$. Using the fact that the potentials add in series,

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

and so the equivalent capacitance for two capacitors in series becomes

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Also

$$|\Delta V_1| = \frac{Q}{C_1} = \frac{C_2 V}{C_1 + C_2} \qquad |\Delta V_2| = \frac{Q}{C_2} = \frac{C_1}{C_1 + C_2} V$$

The generalization to any number of capacitors connected in series is

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots.. + \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i} \text{ (series)}$$

Illustration:

Two capacitors of capacitance $C_1 = 6\,\mu F$ and $C_2 = 3\,\mu F$ are connected in series across a cell of emf 18 V.

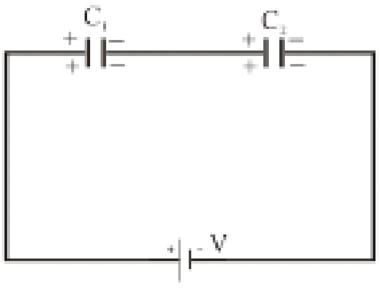
Calculate :

(a) the equivalent capacitance

(b) the potential difference across each capacitor

(c) the charge on each capacitor.

Sol. (a) $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$



$\Rightarrow$ $C = \frac{C_1 C_2}{C_1 + C_2} = \frac{6 \times 3}{6 + 3} = 2\mu F.$

(b) $V_1 = \frac{C_2}{C_1 + C_2} V = \frac{3}{6+3} \times 18 = 6 \text{ volts}$

$V_2 = \frac{C_1}{C_1 + C_2} V = \frac{6}{6+3} \times 18 = 12 \text{ volts}$

Note that the smaller capacitor C_2 has a larger potential difference across it.

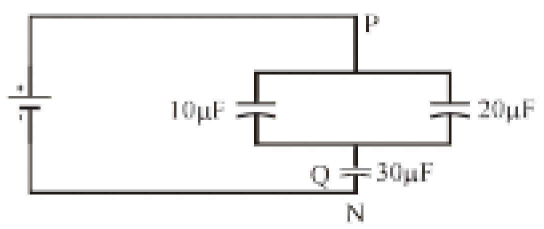
(c) $Q_1 = Q_2 = C_1 V_1 = C_2 V_2 = CV$

charge on each capacitor $= C_{eq} V$

$= 2\mu F \times 18 \text{ volts} = 36\mu C$

Illustration:

Find the equivalent capacitance of the combination shown in figure between the points P and N.

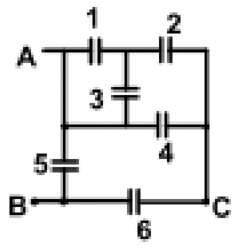


Sol. *The* $10\ \mu F$ *and* $20\ \mu F$ *capacitors are connected in parallel. Their equivalent capacitance is* $10\,\mu F + 20\,\mu F = 30\ \mu F$. *We can replace the* $10\ \mu F$ *and the* $20\ \mu F$ *capacitors by a single capacitor of capacitance* $30\,\mu F$ *between P and Q. This is connected in series with the given* $30\ \mu F$ *capacitor. The equivalent capacitance C of this combination is given by*

$$\frac{1}{C} = \frac{1}{30\mu F} + \frac{1}{30\mu F} \quad or, \quad C = 15\ \mu F.$$

Illustration:

Find the equivalent capacitance between points A and B capacitance of each capacitor is 2 μF*.*



Sol. *The circuit can be redrawn as*

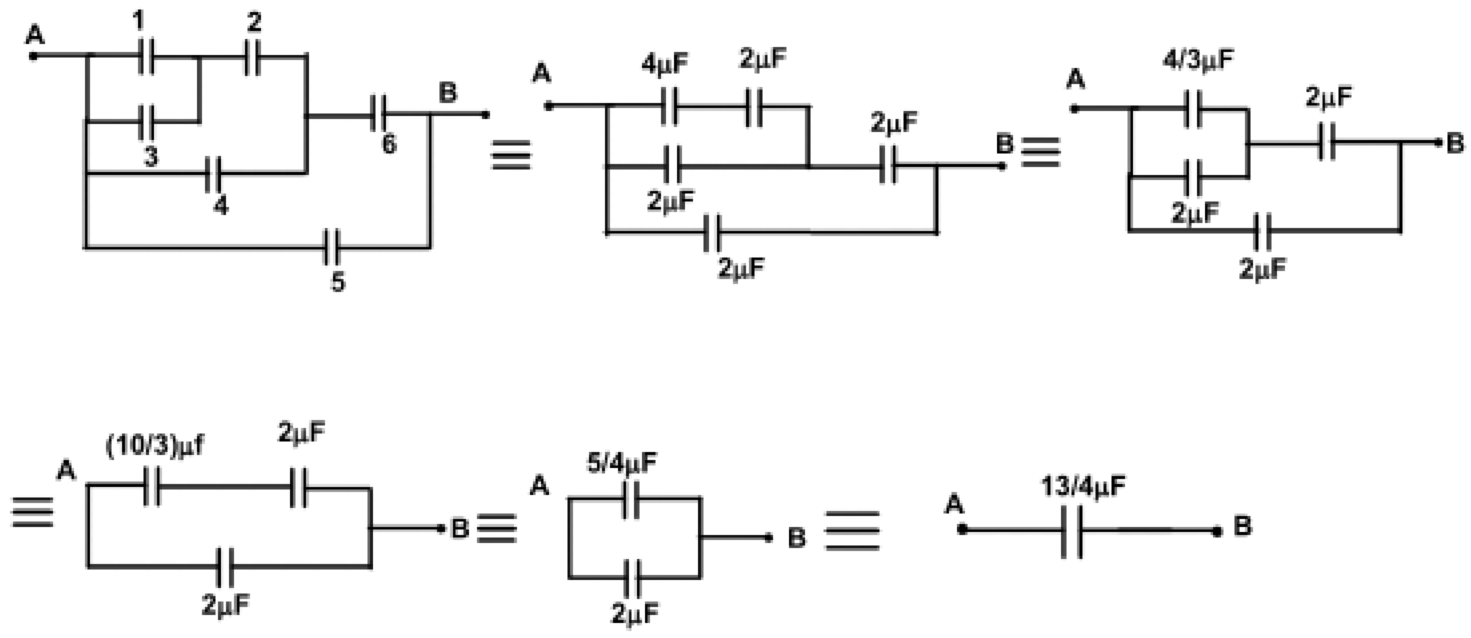
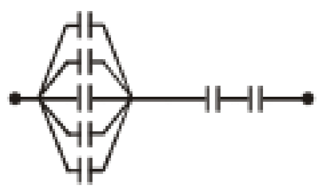
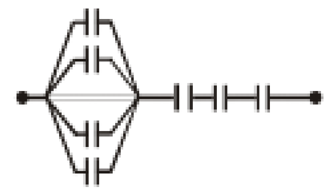
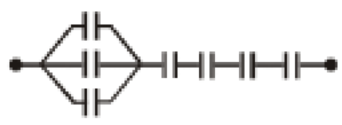


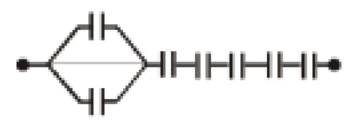
Illustration :

Seven capacitors each of capacitance $2\mu F$ *are to be connected in a configuration to obtain an effective capacitance of* $(10/11)\mu F$. *Which of the combination(s), shown in figure below, will achieve the desired result?*

(a) 

(b) 

(c) 

(d) 

Sol.

(a) $\frac{1}{C} = \frac{1}{5 \times 2} + \frac{2}{2} = \frac{11}{10}$ or $C = \frac{10}{11}\mu F$

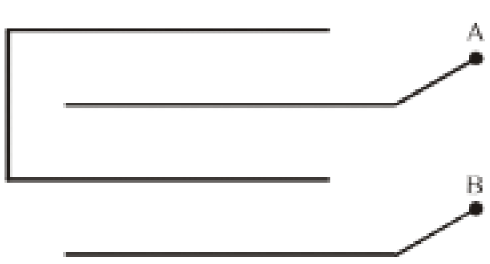
(b) $\frac{1}{C} = \frac{1}{4 \times 2} + \frac{3}{2} = \frac{13}{8}$ or $C = \frac{8}{13}\mu F$

(c) $\frac{1}{C} = \frac{1}{3 \times 2} + \frac{4}{2} = \frac{13}{6}$ or $C = \frac{6}{13}\mu F$

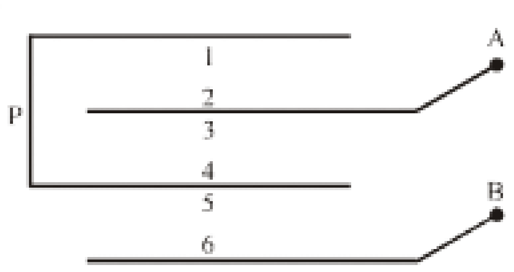
(d) $\frac{1}{C} = \frac{1}{2 \times 2} + \frac{5}{2} = \frac{11}{4}$ or $C = \frac{4}{11}\mu F$.

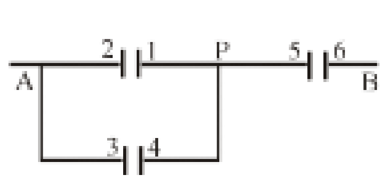
Illustration :

Four identical metal plates are located in air at equal separations d as shown. The area of each plate is A. Calculate the effective capacitance of the arrangement across A and B.



Sol. *Let us call the isolated plate as P. A capacitor is formed by a pair of parallel plates facing each other. Hence we have three capacitor formed by the pairs (1, 2), (3, 4) and (5, 6). The surface 2 and 3 are at same potential as that of A. The arrangement can be redrawn as a network of three capacitors.*

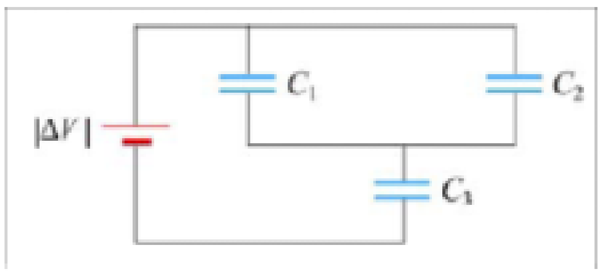




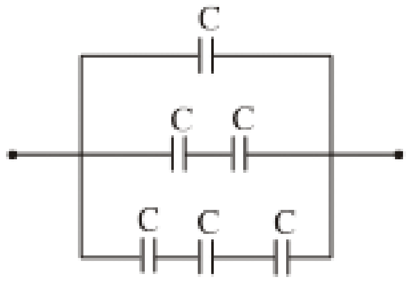
$$C_{AB} = \frac{2C.C}{2C + C} = \frac{2C}{3} \qquad = \frac{2}{3}\frac{\varepsilon_0 A}{d}$$

Practice Exercise

Q.1 Find the equivalent capacitance for the combination of capacitors shown in Figure.



Q.2 Find the equivalent capacitance, assuming that all the capacitors have the same capacitance C.



Q.3 Four identical metal plates are located in air at equal distances d from one another. The area of each plate is equal to A. Find the capacitance of the system between points A and B if the plates are interconnected as shown (a) in Fig. (a) (b) in Fig. (b)

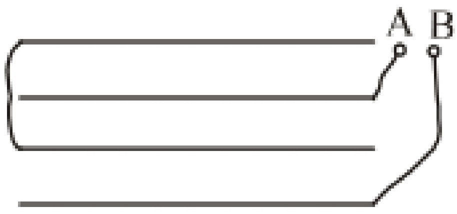


Fig. : (a)

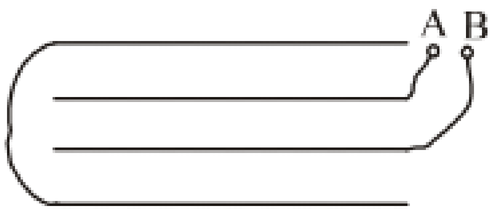
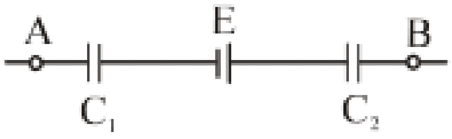


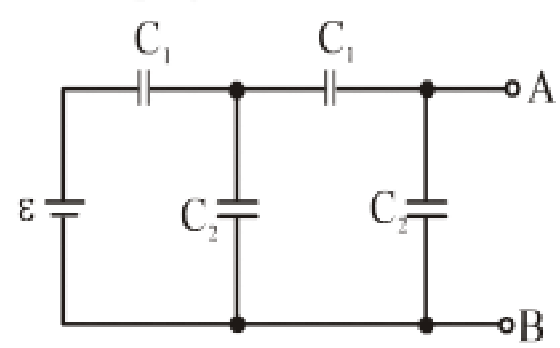
Fig. (b)

Q.4 A capacitor of capacitance $C_1 = 1.0\ \mu F$ with stands the maximum voltage $V_1 = 6.0$ kV while a capacitor of capacitance $C_2 = 2.0\ \mu F$, the maximum voltage $V_2 = 4.0$ kV. What voltage will the system of these two capacitors withstand if they are connected in series?

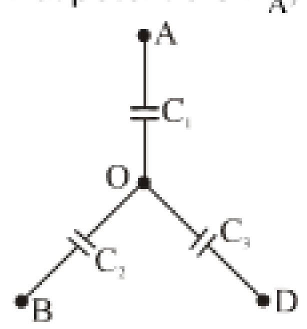
Q.5 A circuit has a section AB shown in Fig. The emf of the source equals E = 10V, the capacitor capacitances are equal to $C_1 = 1.0\ \mu F$ and $C_2 = 2.0\ \mu F$, and the potential difference $V_A - V_B = 5.0$ V. Find the voltage across each capacitor.



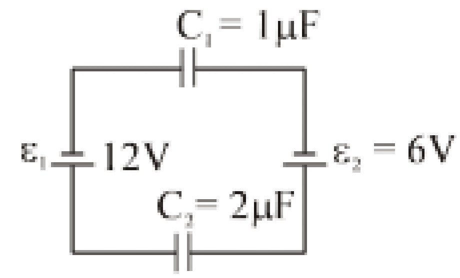
Q.6 Find the potential difference between points A and B of the system shown in Fig. If the emf is equal to $\xi = 110$ V and the capacitance ratio $C_2/C_1 = 2.0$.



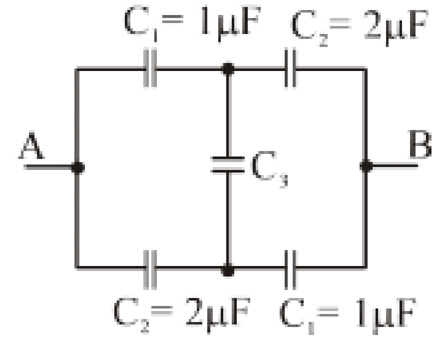
Q.7 Three uncharged capacitors of capacitance C_1, C_2 and C_3 are connected as shown in figure to one another and to points A, B and D at potentials V_A, V_B and V_D. Determine the potential V_0 at point O.



Q.8 In a circuit shown in Fig. Find the potential difference between the left and right plates of each capacitor.

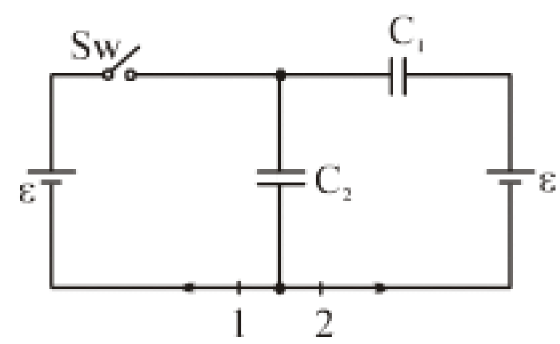


Q.9 Find the capacitance of the circuit shown in Fig. between points A and B.



Q.10 A capacitor of capacitance $C_1 = 1.0\ \mu F$ charged up to a voltage $V = 110$ V is connected in parallel to the terminals of a circuit consisting of two uncharged capacitors connected in series and possessing the capacitances $C_2 = 2.0\ \mu F$ and $C_3 = 3.0\ \mu F$. What charge will flow through the connecting wires?

Q.11 What charges will flow after the shorting of the switch Sw in the circuit illustrated in Fig. through sections 1 and 2 in the directions indicated by the arrows?



Practice Exercise

Q.1 $\dfrac{(C_1+C_2)C_3}{C_1+C_2+C_3}$

Q.2 $\dfrac{11}{6}$ C

Q.3 (a) $C = 2\varepsilon_0 A / 3d$; (b) $C = 3\varepsilon_0 A/2d$

Q.4 $V \le V_1 (1 + C_2 / C_2) = 9$ kV

Q.5 $V_1 = q / C_1 = 10V$, $V_2 = q / C_2 = 5V$, where $q = (V_A - V_B + E)\, C_1C_2 / (C_1 + C_2)$

Q.6 V= 10V

Q.7 $\dfrac{V_AC_1+V_BC_2+V_DC_2}{C_1+C_2+C_3}$

Q.8 $V_1 = -4V$, $V_2 = 2V$

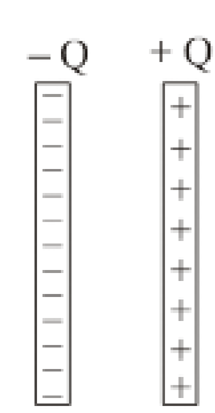
Q.9 $\dfrac{7}{5}$ μF

Q.10 0.06 mC

Q.11 $q_1 = \xi C_2$, $q_2 = -\xi C_1C_2 / (C_1 + C_2)$

Force on the Plates of a Capacitor

The plates of a parallel-plate capacitor have area A and carry total charge $\pm Q$ (see Figure). Electric field due to negative plate at the location of positive plate



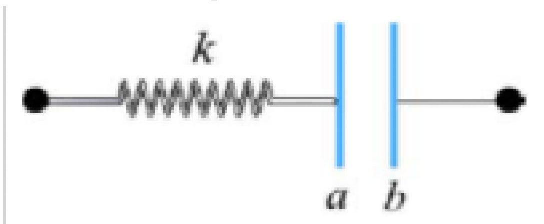
$$E = \frac{\sigma}{2\epsilon_0} = \frac{Q}{2\epsilon_0 A}$$

force on the positive plate

$$F = q\,E_{ext} = (Q)\left(\frac{Q}{2\epsilon_0 A}\right) = \frac{Q^2}{2\epsilon_0 A} \text{ (attracting)}$$

Illustration:

Consider an air-filled parallel-plate capacitor with one plate connected to a spring having a force constant k, and another plate held fixed. The system rests on a table top as shown in Figure



Sol. For equilibrium

$$\frac{Q^2}{2A\epsilon_0} = kx \quad \Rightarrow \quad x = \frac{Q^2}{2kA\varepsilon_0}$$

Practice Exercise

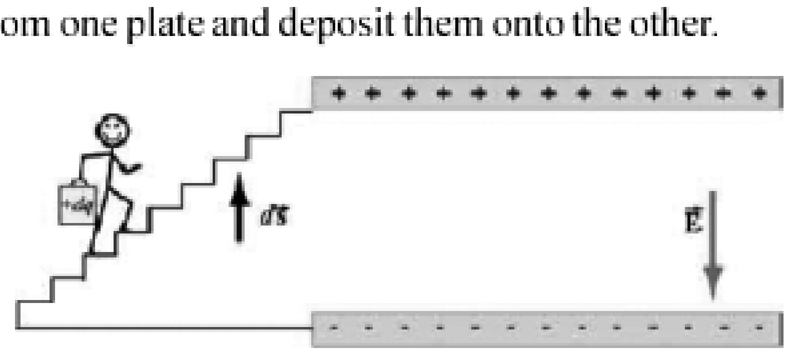
Q.1 Plates of a parallel plate of area A and separation between the plates d. Is charged to a potential difference of V. Find the attraction force between plates.

Answers

Q.1 $\dfrac{\epsilon_0 AV^2}{2d^2}$

Storing Energy in a Capacitor :

As discussed in the introduction, capacitors can be used to stored electrical energy. The amount of energy stored is equal to the work done to charge it. During the charging process, the battery does work to remove charges from one plate and deposit them onto the other.



Work is done by an external agent in bringing $+dq$ from the negative plate and depositing the charge on the positive plate.

Let the capacitor be initially uncharged. In each plate of the capacitor, there are many negative and positive charges, but the number of negative charges balances the number of positive charges, so that there is no net charge, and therefore no electric field between the plates. We have a magic bucket and a set of stairs from the bottom plate to the top plate (Fig.).

We start out at the bottom plate, fill our magic bucket with a charge +dq, carry the bucket up the stairs and dump the contents of the bucket on the top plate, charging it up positive to charge +dq. However, in doing so, the bottom plate is now charged to $-dq$. Having emptied the bucket of charge, we now

descend the stairs, get another bucketful of charge $+dq$, go back up the stairs and dump that charge on the top plate. We then repeat this process over and over. In this way we build up charge on the capacitor, and create electric field where there was none initially.

Suppose the amount of charge on the top plate at some instant is $+q$, and the potential difference between the two plates is $|\Delta V| = q/C$. To dump another bucket of charge $+dq$ on the top plate, the amount of work done to overcome electrical repulsion is $dW = |\Delta V|\, dq$. If at the end of the charging process, the charge on the top plate is $+Q$, then the total amount of work done in this process is

$$W = \int_0^Q dq\,|\Delta V| = \int_0^Q dq\,\frac{q}{C} = \frac{1}{2}\frac{Q^2}{C}$$

This is equal to the electrical potential energy U_E of the system:

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q\,|\Delta V| = \frac{1}{2}C\,|\Delta V|^2$$

Energy Density of the Electric Field :

One can think of the energy stored in the capacitor as being stored in the electric field itself. In the case of a parallel-plate capacitor, with $C = \varepsilon_0 A/d$ and $|\Delta V| = Ed$, we have

$$U_E = \frac{1}{2}C|\Delta V|^2 = \frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2 = \frac{1}{2}e_0 E^2\,(Ad)$$

Since the quantity Ad represents the volume between the plates, we can define the electric energy density as

$$u_E = \frac{U_E}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2$$

Note that u_E is proportional to the square of the electric field. Alternatively, one may obtain the energy stored in the capacitor from the point of view of external work. Since the plates are oppositely charged, force must be applied to maintain a constant separation between them. From Eq., we see that a small patch of charge $\Delta q = \sigma(\Delta A)$ experiences an attractive force $\Delta F = \sigma^2(\Delta A)/2\varepsilon_0$. . If the total area of the plate is A, then an external agent must exert a force $F_{ext} = \sigma^2 A/2\varepsilon_0$ to pull the two plates apart. Since the electric field strength in the region between the plates is given by $E = \sigma/\varepsilon_0$, the external force can be rewritten as

$$F_{ext} = \frac{\varepsilon_0}{2}E^2 A$$

Note that F_{ext} is independent of d. The total amount of work done externally to separate the plates by a distance d is then

$$W_{ext} = \int \vec{F}_{ext}.d\vec{s} = F_{ext}\,d = \left(\frac{\varepsilon_0 E^2 A}{2}\right)d$$

consistent with Eq. Since the potential energy of the system is equal to the work done by the external

agent, we have $u_E = W_{ext} / Ad = \varepsilon_0 E^2 / 2$. The electric energy density u_E can also be interpreted as electrostatic pressure P.

First law of thermodynamics in Capacitors:

If heat liberated by system = $Q_{liberated}$

$\Rightarrow \quad \Delta Q = -Q_{liberated}$

Work done by the system

$$\Delta W = -\Delta W_{battery}$$

Now using

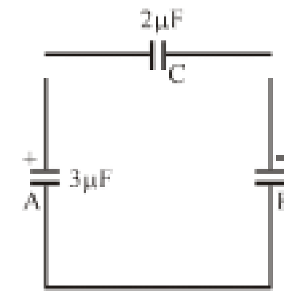
$$\Delta Q = \Delta U + \Delta W$$

$\Rightarrow \quad -Q_{liberated} = \Delta U_{capacitor} - \Delta W_{battery}$

$\therefore \quad \Delta W_{battery} = \Delta U_{capacitor} + Q_{liberated}$

Therefore, as mentioned in the beginning of chapter, the work done by battery goes in storing energy in capacitor and rest goes as heat loss in resistor.

Illustration :



Two capacitors A and B with capacities 3 μF and 2μF are charged to a potential difference of 100V and 180V respectively. The plates of the capacitors are connected as shown in the figure with one wire of each capacitor free. The upper plate of A is positive and that of B is negative. An uncharged 2 μF capacitor C with lead wires falls on the free ends to complete the circuit. Calculate :

(i) the final charge on the three capacitors, and

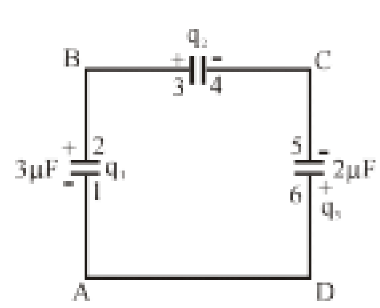
(ii) the amount of electrostatic energy stored in the system before and after the completion of the circuit.

Sol. *Charge on capacitor A, before joining with an uncharged capacitor,*

$$q_A = CV = (100) \times 3 \ \mu c = 300 \ \mu c$$

similarly charge on capacitor B,

$$q_B = 180 \times 2 \ \mu c$$
$$= 360 \ \mu c$$



Let q_1, q_2 and q_3 be the charges on the three capacitors after joining them as shown in fig.

From conservation of charge,

Net charge on plates 2 and 3 before joining

= *Net charge after joining*

$\therefore \quad 300 = q_1 + q_2 \qquad .... (1)$

Similarly, net charge on plates 4 and 5 before joining

= *Net charge after joining*

$$-360 = -q_2 - q_3$$

$$360 = q_2 + q_3 \qquad .... (2)$$

applying Kirchoff's 2nd law in loop ABCDA,

$$\frac{q_1}{3} - \frac{q_2}{2} + \frac{q_3}{2} = 0$$

$$2q_1 - 3q_2 + 3q_3 = 0 \qquad .... (3)$$

From equations (1), (2) and (3),

$q_1 = 90\ \mu c,\ q_2 = 90\ \mu c$ *and* $q_3 = 150\ \mu c$

(ii) (a) Electrostatic energy stored before completing the circuit,

$$U_i = \frac{1}{2}(3 \times 10^{-6})\ (100)^2 + \frac{1}{2}(2 \times 10^{-6})(180)^2 \qquad (U = \frac{1}{2}CV^2)$$

$$= 4.74 \times 10^{-2}\ J$$

$$= 47.4\ mJ.$$

(b) Electrostatic energy stored after completing the circuit,

$$U_f = \frac{1}{2}(90\times10^{-6})^2.\frac{1}{3\times10^{-6}} + \frac{1}{2}(90\times10^{-6})^2.\frac{1}{2\times10^{-6}} + \frac{1}{2}(150+10^{-6})^2.\frac{1}{2\times10^{-6}} \qquad \left(U = \frac{1}{2}\frac{q}{c^2}\right)$$

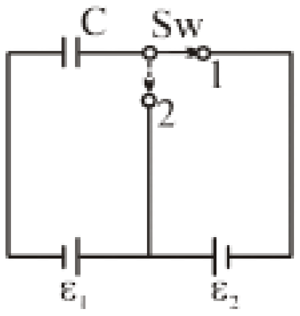
$$= 90\times10^{-4}\ J$$

$$= 9\ mJ.$$

Practice Exercise

Q.1 A capacitor of capacitance $C_1 = 1.0\ \mu F$ carrying initially a voltage $V = 300$ V is connected in parallel with an uncharged capacitor of capacitance $C_2 = 2.0\ \mu F$. Find the loss of the electric energy of this system by the moment equilibrium is reached. Explain the result obtained.

Q.2 What amount of heat will be generated in the circuit shown in Fig. after the switch Sw is shifted from position 1 to position 2?



Q.3 Each plate of a parallel-plate air capacitor has an area A. What amount of work has to be performed to slowly increase the distance between the plates from x_1 to x_2 if
(a) the charge on the capacitor, which is equal to q, or (b) the voltage across the capacitor, which is equal to V, is kept constant in the process?

Answers

Q.1 $\Delta W = -1/2\ V^2C_1C_2 / (C_1 + C_2)$ 0.03 mJ Q.2 $Q = 1/2C\,\xi_2^2$

Q.3 (a) $W = q^2 (x_2 - x_1) / 2\varepsilon_0 A$; (b) $W = \varepsilon_0 AV^2 (x_2 - x_1) / 2x_1x_2$

Dielectrics

Dielectric is any insulating substance (insulator). It can be rubber, plastic wood, oil etc. In contrast to conductors, the electrons in dielectrics are attached to specific atoms or molecules, so they are not allowed from moving randomly at will. They are in tight leash; all they can do is move a bit within the atom or a molecule.

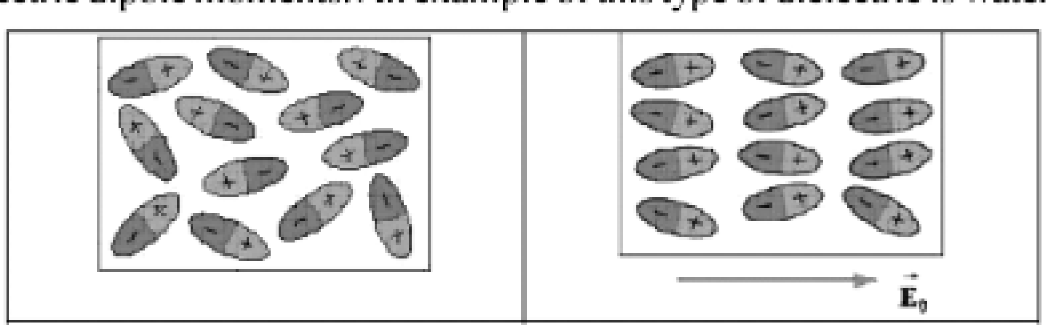
Experimentally it was found that capacitance C increases when the space between the conductors is filled with dielectrics. To see how this happens, suppose a capacitor has a capacitance when there is no material between the plates. When a dielectric material is inserted to completely fill the space between the plates, the capacitance increases to K_e times i.e.

$$C = K_e C_0$$

where K_e is called the dielectric constant or relative permittivity. In the Table below, we show some dielectric materials with their dielectric constant. Experiments indicate that all dielectric materials have $K_e > 1$.

The fact that capacitance increases in the presence of a dielectric can be explained from a molecular point of view. We shall show that K_e is a measure of the dielectric response to an external electric field.

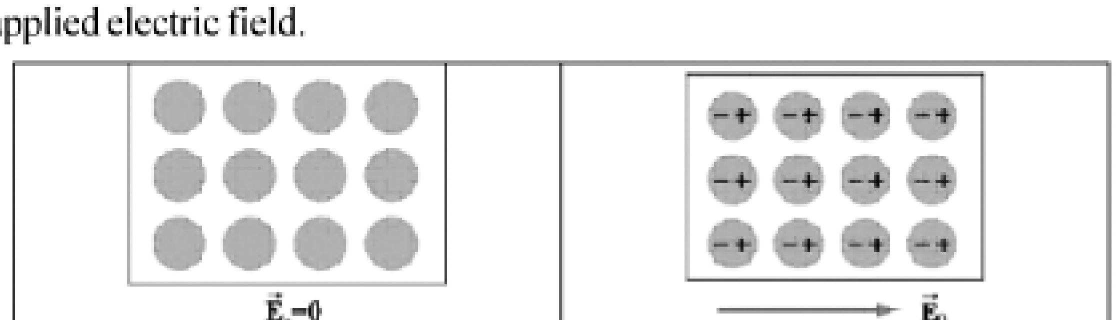
There are two types of dielectrics. The first type is polar dielectrics, which are dielectrics that have permanent electric dipole moments. An example of this type of dielectric is water.



Orientations of polar molecules when (a) $\vec{E}_0 = \vec{0}$ and (b) $\vec{E}_0 \neq 0$

As depicted in Figure, the orientation of polar molecules is random in the absence of an external field. When an external electric field $\vec{E}_0$ is present, a torque is set up and causes the molecules to align with $\vec{E}_0$. However, the alignment is not complete due to random thermal motion. The aligned molecules then generate an electric field that is opposite to the applied field but smaller in magnitude.

The second type of dielectrics is the non-polar dielectrics, which are dielectrics that do not possess permanent electric dipole moment. Electric dipole moments can be induced by placing the materials in an externally applied electric field.



Orientations of non-polar molecules when (a) $\vec{E}_0 = \vec{0}$ and (b) $\vec{E}_0 \neq \vec{0}$

Figure illustrates the orientation of non-polar molecules with and without an external field $\vec{E}_0$. The induced surface charges on the faces produces an electric field $\vec{E}_P$ in the direction opposite to $\vec{E}_0$, leading to $\vec{E} = \vec{E}_0 + \vec{E}_P$, with $|\vec{E}_P| < |\vec{E}_0|$. Below we show how the induced electric field $\vec{E}_P$ is calculated.

Let us now examine the effects of introducing dielectric material into a system. We shall first assume that the atoms or molecules comprising the dielectric material have a *permanent* electric dipole moment. If left to themselves, these permanent electric dipoles in a dielectric material never line up spontaneously, so that in the absence of any applied external electric field $\vec{p} = \vec{0}$ due to the random alignment of dipoles, and the average electric field $\vec{E}_P$ is zero as well. However, when we place the dielectric material in an external field $\vec{E}_0$, the dipoles will experience a torque $\vec{\tau} = \vec{p} \times \vec{E}_0$ that tends to align the dipole vectors $\vec{p}$ with $\vec{E}_0$. The effect is a net polarization $\vec{p}$ parallel to $\vec{E}_0$, and therefore an average electric field of the dipoles $\vec{E}_P$ *anti-parallel* to $\vec{E}_0$, i.e., that will tend to *reduce* the total electric field strength below $\vec{E}_0$.

The total electric field $\vec{E}$ is the sum of these two fields:

$$\vec{E} = \vec{E}_0 + \vec{E}_P$$

Note that every dielectric material has a characteristic dielectric strength which is the maximum value of electric field before breakdown occurs and charges begin to flow.

Material	κ_e	Dielectric strength $(10^6 \mathrm{V/m})$
Air	1.00059	3
Paper	3.7	16
Glass	4–6	9
Water	80	–

Revised energy density becomes: $u = \frac{1}{2} K \varepsilon_0 E^2$

Calculation of induced (polarised) charge on dielectric :

However, we have just seen that the effect of the dielectric is to weaken the original field E_0 by a factor K_e. Therefore,

$$E = \frac{E_0}{K_e} = \frac{Q}{K_e \varepsilon_0 A} = \frac{Q - Q_P}{\varepsilon_0 A}$$

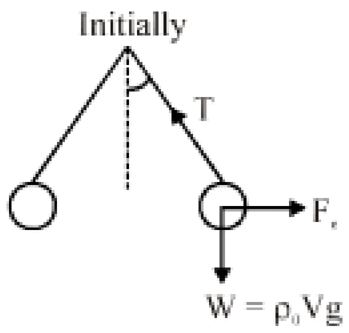
from which the induced charge Q_P can be obtained as

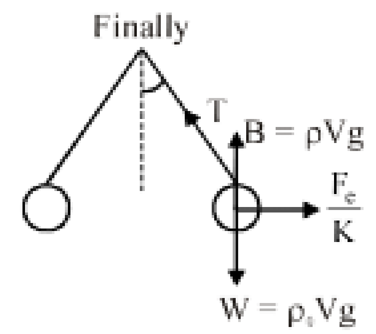
$$Q_P = Q\left(1 - \frac{1}{K_e}\right)$$

Illustration :

Two small identical balls carrying the charges of the same sign are suspended from the same point by insulating threads of equal length. When the surrounding space was filled with kerosene (of density ρ_0, dielectric constant K) the divergence angle between the threads remained constant. What is the density of the material of which the balls are made?

Sol.





When dielectric is filled, force on charge is reduced by K factor

According to question

$$\theta = \theta'$$

$$\Rightarrow \quad \tan\theta = \tan\theta' \Rightarrow \quad \frac{F}{\rho_0 Vg} = \frac{\frac{F}{K}}{(\rho - \rho_0)Vg} \quad \Rightarrow \quad \rho = \frac{K}{K-1}\rho_0$$

Practice Exercise

Q.1 The charges on the plates of a parallel-plate capacitor are of opposite sign, and they attract each other. To increase the plate separation, is the external work done positive or negative? What happens to the external work done in this process?

Q.2 How does the stored energy change if the potential difference across a capacitor is tripled?

Q.3 Does the presence of a dielectric increase or decrease the maximum operating voltage of a capacitor? Explain.

Answers

Q.1 +ve Q.2 9 times Q.3 Depends on the dielectric strength

Capacitor Containing Dielectrics without Battery :

As shown in Figure, a battery with a potential difference $|\Delta V_0|$ .across its terminals is first connected to a capacitor C_0, which holds a charge $Q_0 = C_0|\Delta V_0|$. We then disconnect the battery, leaving Q_0 = const.

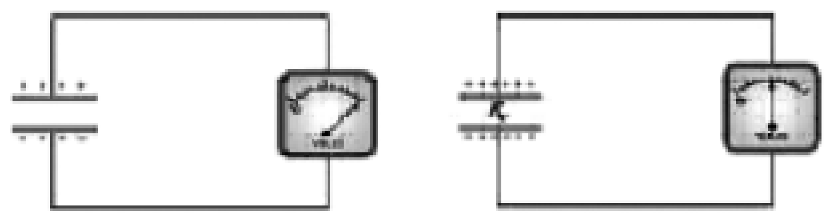


Fig.: 44 - Inserting a dielectric material between the capacitor plates while keeping the charge Q_0 constant

If we then insert a dielectric between the plates, while keeping the charge constant, experimentally it is found that the potential difference decreases by a factor of *Ke*.

$$|\Delta V| = \frac{|\Delta V_0|}{K_e}$$

This implies that the capacitance is changed to

$$C = \frac{Q}{|\Delta V|} = \frac{Q_0}{|\Delta V_0|/K_e} = K_e\frac{Q_0}{|\Delta V_0|} = K_eC_0$$

Thus, we see that the capacitance has increased by a factor of *Ke*. *The* electric field within the dielectric is now

$$E = \frac{|\Delta V|}{d} = \frac{|\Delta V_0|/K_e}{d} = \frac{1}{K_e}\left(\frac{|\Delta V_0|}{d}\right) = \frac{E_0}{K_e}$$

We see that in the presence of a dielectric, the electric field decreases by a factor of K_e.

Capacitor Containing Dielectrics with Battery :

Consider a second case where a battery supplying a potential difference remains connected as the dielectric is inserted. Experimentally, it is found (first by Faraday) that the charge on the plates is increased by a factor K_e.

$$Q = K_eQ_0$$

where Q_0 is the charge on the plates in the absence of any dielectric.

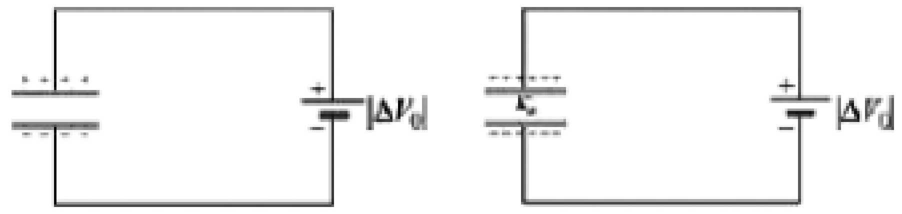


Fig. : 45 - (a) Fig. : 45 - (b)

Figure : Inserting a dielectric material between the capacitor plates while maintaining a constant potential difference $|V_0|$

The capacitance becomes

$$C = \frac{Q}{|\Delta V_0|} = \frac{K_e Q_0}{|\Delta V_0|} = K_e C_0$$

which is the same as the first case where the charge Q_0 is kept constant, but now the charge has increased.

Illustration:

A parallel plate capacitor has plates of area 4 m² separated by a distance of 0.5 mm. The capacitor is connected across a cell of emf 100 volts. Find the capacitance, charge & energy stored in the capacitor if a dielectric slab of dielectric constant k=3 and thickness 0.5 mm is inserted inside this capacitor after it has been disconnected from the cell.

Sol.

when the capacitor is without dielectric

$$C_0 = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times 4}{0.5 \times 10^{-3}}$$

$$C_0 = 7.08 \times 10^{-2}\ \mu F.$$

$$Q_0 = C_0 V_0$$

$$= (7.08 \times 10^{-2} \times 100)\mu C = 7.08\ \mu C$$

$$U_0 = \frac{1}{2} C_0 V_0^2 = 354 \times 10^{-6}\ J.$$

as the cell has been disconnected, charge on the capacitor remain constant

$$C = \frac{k\varepsilon_0 A}{d} = kC_0 = 0.2124\ \mu F$$

$$V = \frac{Q}{C} = \frac{Q_0}{kC_0} \frac{V_0}{k} = \frac{100}{3}\ \text{volts.}$$

$$U = \frac{1}{2}\frac{Q_0}{C} = \frac{1}{2}\frac{Q_0^2}{kC_0} = \frac{U_0}{k} = 118 \times 10^{-6}\ J.$$

Electric field inside the plates $= E = \frac{V}{d} = \frac{V_0}{kD} = \frac{E_0}{k}$

Note that the field becomes 1/k times by insertion of dielectric.

Illustration :

A 6 × 10^{-9} F parallel plate capacitor is connected to a 500 V battery. When air is replaced by another dielectric material. 7.5 × 10^{-6} C charge flows into the capacitor. Find the dielectric constant of the material

Sol. $Q = CV$

$Q_1 = 6 \times 10^{-9} \times 500$

$= 3 \times 10^{-6}\ C$

After insertion of dielectric

$Q'_1 = (3+7.5) \times 10^{-6}\ C$

$= 10.5 \times 10^{-6}\ C$

$Q'_1 = CVK$

$10.5 \times 10^{-6} = 6 \times 10^{-9} \times 500\, K$

$K = 3.5$

Capacitance of capacitor filled partially with dielectric :

A non-conducting slab of thickness t, area A and dielectric constant K_e is inserted into the space between the plates of a parallel-plate capacitor with spacing d, charge Q and area A, as shown in Figure(a). The slab is not necessarily halfway between the capacitor plates. What is the capacitance of the system?

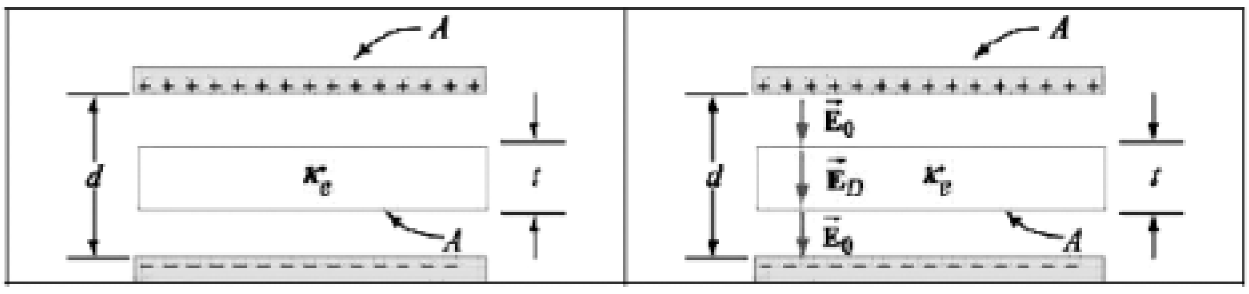


Fig. : 46 (a) Capacitor with a dielectric. (b) Electric field between the plates.

To find the capacitance C, we first calculate the potential difference ΔV. We have already seen that in the absence of a dielectric, the electric field between the plates is given by $E_0 = Q/\varepsilon_0 A$ and $E_D = E_0 / K_e$ when a dielectric of dielectric constant K_e is present, as shown in Figure (b). The potential can be found by integrating the electric field along a straight line from the top to the bottom plates:

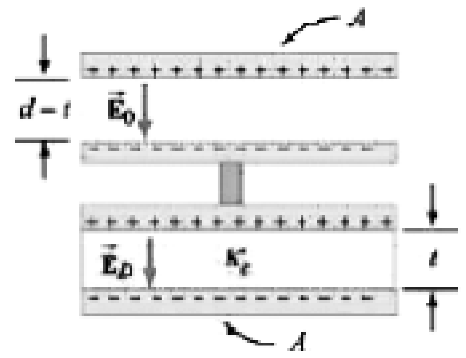
$$\Delta V = -\int_{+}^{-} E\,d\ell = -\Delta V_0 - \Delta V_D = -E_0(d-t) - E_D t = -\frac{Q}{A\varepsilon_0}(d-t) - \frac{Q}{A\varepsilon_0 K_e}t$$

$$= -\frac{Q}{A\varepsilon_0}\left[d - t\left(1 - \frac{1}{K_e}\right)\right]$$

where $\Delta V_D = E_D t$ is the potential difference between the two faces of the dielectric. This gives

$$C = \frac{Q}{|\Delta V|} = \frac{\varepsilon_0 A}{d - t\left(1 - \frac{1}{K_e}\right)}$$

We also comment that the configuration is equivalent to two capacitors connected in series, as shown in Figure.



Using Eq. for capacitors connected in series, the equivalent capacitance is

$$\frac{1}{C} = \frac{d-t}{\varepsilon_0 A} + \frac{t}{K_e \varepsilon_0 A}$$

It is useful to check the following limits :

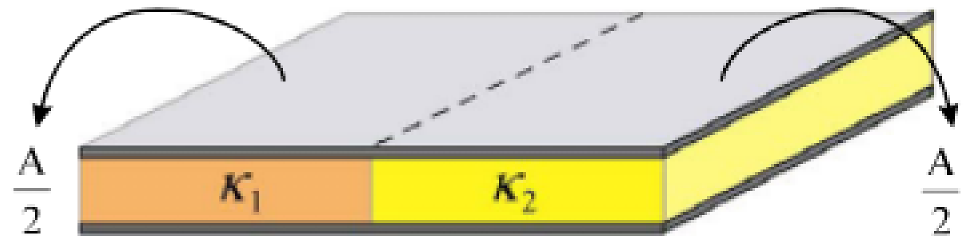
(i) *As $t \to 0$ i.e., the thickness of the dielectric approaches zero, we have $C = \varepsilon_0 A/d = C_0$, which is the expected result for no dielectric.*

(ii) *As, $K_e \to 1$, we again have $C \to \varepsilon_0 A/d = C_0$, and the situation also correspond to the case where the dielectric is absent.*

(iii) *In the limit where $t \to d$, the space is filled with dielectric, we have. $C \to K_e \varepsilon_0 A/d = K_e C_0$.*

Capacitor filled with two different dielectrics

Two dielectric with dielectric constant K_1 and K_2 each fill half the space between the plates of a parallel-plate capacitor as shown in figure



Capacitor filled with two different dielectrics

Each plate has an area A and the plates are separated by a distance d. Compute the capacitance of the system.

$$C_1 = \frac{K_1 \epsilon_0 (A/2)}{d} = \frac{K_1 \epsilon_0 A}{2d}, \qquad C_2 = \frac{K_2 \epsilon_0 A}{2d}$$

Since the two capacitors are in parallel

$$\Rightarrow \quad C = C_1 + C_2 = \frac{\varepsilon_0 A}{2d}(K_1 + K_2)$$

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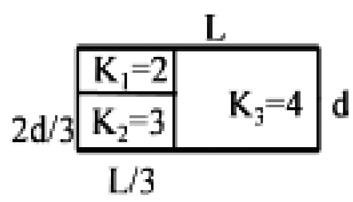
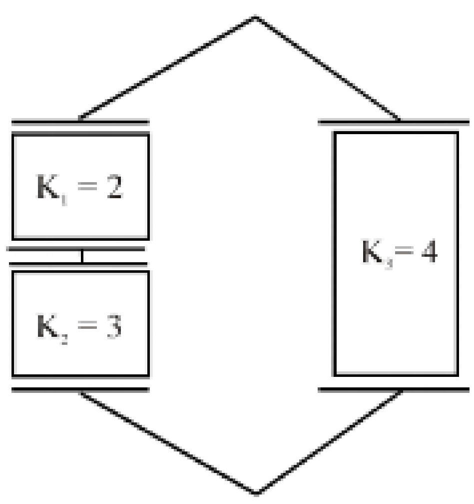
Illustration :

Find the equivalent capacitance of the system shown (assume square plates).



Sol. *The system can be represent as*



Taking $K_1 = 2$ to be series in $K_2 = 3$

$$\Rightarrow \quad \frac{1}{C_{\text{left}}} = \frac{1}{\dfrac{(2)\epsilon_0\left\{(L)\left(\dfrac{L}{3}\right)\right\}}{\left(\dfrac{d}{3}\right)}} + \frac{1}{\dfrac{(3)\epsilon_0\left\{(L)\left(\dfrac{L}{3}\right)\right\}}{\left(\dfrac{2d}{3}\right)}} \qquad \Rightarrow \quad C_{\text{left}} = \frac{6\epsilon_0 L^2}{7d}$$

Now

$$C_{\text{right}} = \frac{(4)\epsilon_0\left\{(L)\left(\dfrac{2L}{3}\right)\right\}}{d} = \frac{8\epsilon_0 L^2}{3d}$$

Now $\quad$ *C_{left} and C_{right} are in parallel*

$$\Rightarrow \quad C_{eq} = C_{left} + C_{right} = \frac{6\epsilon_0 L^2}{7d} + \frac{8\epsilon_0 L^2}{3d} = \frac{74\epsilon_0 L^2}{21d}$$

Practice Exercise

Q.1 A parallel plate capacitor has a capacitance of 112 pF, a plate area of 96.5 cm², and a mica dielectric ($k_e = 5.40$). At a 55 V potential difference, calculate

(a) The electric field strength in a the mica.

(b) The magnitude of the free charge on the plates.

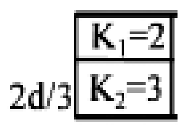
(c) The magnitude of the induced surface charge.

Q.2 Two parallel-plate air capacitors, each of capacitance C, were connected in series to a battery with emf ξ. Then one of the capacitors was filled up with slab of dielectric constant k.

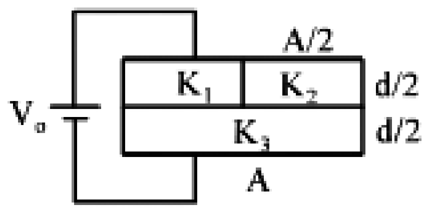
(a) What amount of charge flows through the battery?

(b) Find the factor by which electric field in each capacitor charges during the process. $\left(\text{i.e. } \frac{E_{after}}{E_{before}}\right)$

Q.3 Find the equivalent capacitance of the system shown (assume square plates each of area A).



Q.4 An ideal parallel plate capacitor of area A is filled with three dielectric slabs having dielectric constants $K_1 = 3.0$, $K_2 = 5.0$ and $K_3 = 2.0$ as shown. If a single dielectric material is to be used to have the same capacitance as this capacitor, then find its dielectric constant



Answers

Q.1 (a) 13.4 kV/m, (b) 6.16 nC, (c) 5.02 nC

Q.2 The strength decreased $1/2\,(\varepsilon + 1)$ times ; (a) $q = 1/2\, C\xi\, (\varepsilon - 1)\, /\, (\varepsilon + 1)$]

Q.3 $\dfrac{18 \in_0 A}{5d}$ Q.4 8/3

Energy Related discussion for dielectric capacitor

Illustration :

In the figure shown, a parallel plate capacitor is connected across a source of emf ε. The plates are square shaped with edge 'ℓ' and separated by a distance d. A dielectric slab of dielectric constant k and thickness d is inserted between the plates with constant speed v. Find the current in the connecting wires [ignore the resistance of connecting wires].

Sol. *Consider that length x of the dielectric is inside the capacitor. The capacitance of the system is*

$$C = \frac{\varepsilon_0 k x \ell}{d} + \frac{\varepsilon_0 \ell(\ell - x)}{d} = \frac{\varepsilon_0 \ell}{d}[(k-1)x + \ell]$$

Charge on the capacitor,

$$q = \frac{\varepsilon_0 \ell}{d}[(k-1)x + \ell]\varepsilon$$

$$\Rightarrow \quad I = \frac{dq}{dt} = \frac{\varepsilon_0 \ell}{d}(k-1)\varepsilon\frac{dx}{dt} = \frac{\varepsilon_0 \ell \varepsilon v(k-1)}{d}$$

Illustration :

A dielectric completely fills the gap between the plates of a parallel-plate capacitor whose capacitance is equal to C_0 when the dielectric is absent. Find the mechanical work which must be done against electric forces for extracting the dielectric out of the capacitor if

(i) Voltage (V) of the capacitor is maintained constant.

(ii) Charge (Q) of the capacitor is maintained constant.

Neglect the resistance of the circuit (If any)

Sol. *(i) T maintain constant voltage we have to used ideal battery*

change in capacitance

$$\Delta C = C_0 - KC_0 = -(K-1)\,C_0$$

Charge supplied by battery

$$\Delta q = V\Delta C$$

Work done by battery

$$W_b = V\Delta q = V^2\Delta C$$

Change in energy of capacitor

$$\Delta U = \frac{1}{2}V^2\Delta C$$

Now using work energy theorem

$$W_{mechanical} + W_b = \Delta K + \Delta U$$

$$\Rightarrow \quad W_{mechanical} = \Delta K + \Delta U - W_b = 0 + \frac{1}{2}V^2\Delta C - V^2\Delta C = -\frac{1}{2}V^2\Delta C = \frac{1}{2}(K-1)C_0V^2$$

(ii) To maintain constant charge capacitor should not be connected with the battery or any thing else]

$$\Delta q = 0$$

$$\Delta U = \frac{Q^2}{2C_0} - \frac{Q^2}{2KC_0} = \frac{Q^2}{2C_0}\left(1-\frac{1}{K}\right)$$

$$W_{mechanical} = \Delta K + \Delta U - W_b = 0 + \frac{Q^2}{2C_0}\left(1-\frac{1}{K}\right) - 0 = \frac{Q^2}{2C_0}\left(1-\frac{1}{K}\right)$$

Practice Exercise

Q.1 Between the plates of a parallel-plate capacitor there is a metallic plate whose thickness takes up $\eta = 0.60$ of the capacitor gap. When that plate is absent the capacitor has a capacity $C = 20$ nF. The capacitor is connected to a dc voltage source $V = 100$ V. The metallic plate is slowly extracted from the gap. Find: (a) the change in the energy of the capacitor; (b) the mechanical work performed in the process of plate extraction.

Answers

Q.1 (a) $\Delta U = -1/2\ CV^2\eta / (1-\eta) = -0.15$ mJ ; (b) $W = 1/2CV^2\eta / (1-\eta) = 0.15$ mJ

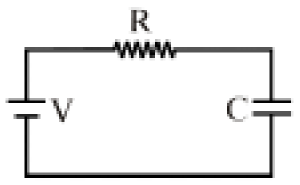
RC Circuit

Charging a capacitor :

Let the capacitor be initially uncharged. As soon as the circuit completes, the charge begins to flow. Let 'q' be the charge on the capacitor at certain instance & i be the current in the circuit. Then,

$$iR + \frac{q}{C} = V \qquad \& \qquad i = \frac{dq}{dt}$$

$$\Rightarrow \quad R\frac{dq}{dt} = \frac{CV - q}{C}$$

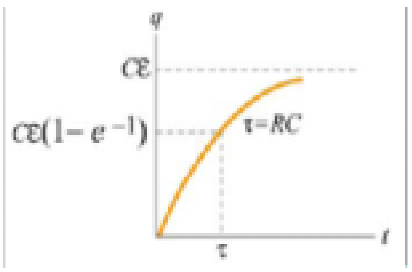


$$\Rightarrow \quad \int_0^q \frac{-dq}{CV - q} = \int_0^t \frac{-dt}{CR}$$

$$\therefore \quad \ell n\frac{CV - q}{CV} = \frac{-t}{CR}$$

$$\therefore \quad q = q_0\,(1 - e^{-t/CR})$$

where, $q_0 = CV$ = maximum amount of charge stored on the plates



Now, $\frac{dq}{dt} = i = \frac{V}{R}e^{-t/CR} = \frac{V}{R}e^{-t/\tau}$

Once we know the charge on the capacitor we also can determine the voltage across the capacitor,

$$V_C(t) = \frac{q(t)}{C} = \varepsilon(1 - e^{-t/RC})$$

The graph of voltage as a function of time has the same form as figure. From the figure, we see that after a sufficiently long time the charge on the capacitor approaches the value.

$$q(1 - \infty) = C\varepsilon = Q$$

At that time, the voltage across the capacitor is equal to the applied voltage source and the charging process effectively ends,

$$V_C = \frac{q(t = \infty)}{CQ} = \frac{Q}{C} = \varepsilon$$

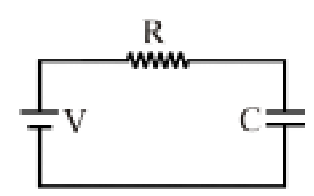
For current a **capacitor acts as :**

- Short-circuit just after closing the switch.
- Open circuit a long time after closing the switch.

Discharging a Capacitor

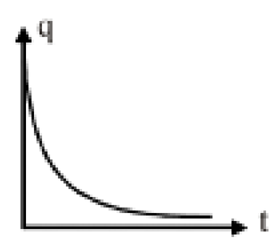
Suppose initially the capacitor has been charged to some value Q. For $t < 0$, the switch is open and the potential difference across the capacitor is given by $V_C = Q/C$. On the other hand, the potential difference across the resistor is zero because there is no current flow, that is, $I = 0$. Now suppose at $t = 0$ the switch is closed (Figure). The capacitor will begin to discharge.

$$iR - \frac{q}{C} = 0 \qquad \& \qquad i = -\frac{dq}{dt}$$



$$\Rightarrow \qquad R\frac{dq}{dt} = \frac{CV - q}{C}$$

$$\Rightarrow \qquad \int_{q_0}^{q} \frac{dq}{q} = \int_0^t \frac{-dt}{CR}$$



$$\therefore \qquad \ell n \frac{q}{q_0} = \frac{-t}{CR}$$

$$\therefore \qquad q = q_0\, e^{-t/CR} = CV_0\, e^{-t/CR}$$

$$\text{Now, } i = \frac{dq}{dt} = \frac{V_0}{R} e^{-t/CR} = \frac{V_0}{R} e^{-t/\tau}$$

Illustration :

In the circuit shown in Fig. the sources have emf's $\xi_1 = 1.0\ V$ and $\xi_2 = 2.5\ V$ and the resistances have the values $R_1 = 10\Omega$ and $R_2 = 20\ \Omega$. The internal resistances of the sources are negligible. Find the potential difference between the plates A and B of the capacitor C.

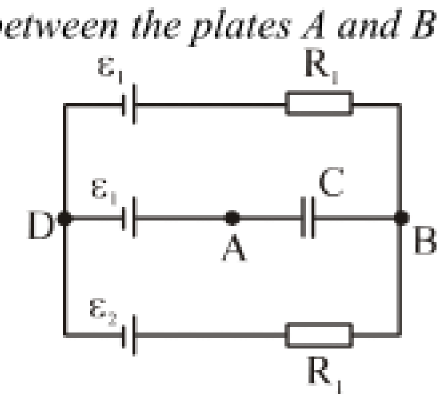


Fig. : 63

Sol. *At steady state there is no current through capacitor i.e. current exist in bigger loop anticlockwise which will be*

$$I = \frac{2.5 - 1.0}{10 + 20} = \frac{1}{20}\,\text{A}$$

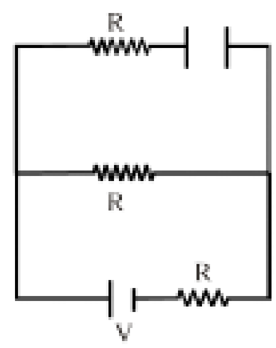
Now

$$V_{BD} = V_{BA} + V_{AD}$$

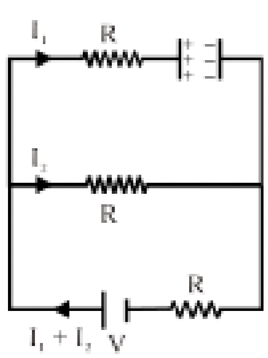
$$\Rightarrow \qquad I + \frac{1}{20} \times 10 = V_{BA} + 1 \qquad \Rightarrow \qquad V_{BA} = 0.5\ V$$

Illustration :

In the figure shown find (i) The charge of the capacitor as a function of time (ii) equivalent time constant.



Sol. $I_1 = \dfrac{dq}{dt}$ ...(i)



$$I_1 R + \frac{q}{C} + (I_1 + I_2)R = V$$

$$I_1 R + \frac{q}{C} - I_2 R = 0$$

$$2I_1 + I_2 + \frac{q}{RC} = \frac{V}{R} \quad \text{...(ii)}$$

$$I_2 = I_1 + \frac{q}{RC} \quad \text{...(iii)}$$

From (ii) and (iii)

$$2I_1 + \left(I_1 + \frac{q}{RC}\right) + \frac{q}{RC} = \frac{V}{R}$$

$$3I_1 + \frac{2q}{RC} = \frac{V}{R} \quad \text{...(iv)}$$

from (i) and (iv)

$$\frac{3dq}{dt} + \frac{2q}{RC} = \frac{V}{R}$$

$$3RC\frac{dq}{dt} + 2q = CV$$

$$3RC\frac{dq}{dt} = CV - 2q$$

$$\int_0^q \frac{dq}{CV - 2q} = \frac{1}{3RC}\int_0^t dt$$

$$\left[\frac{CV-2q}{-q}\right]_0^q = \frac{t}{3RC}$$

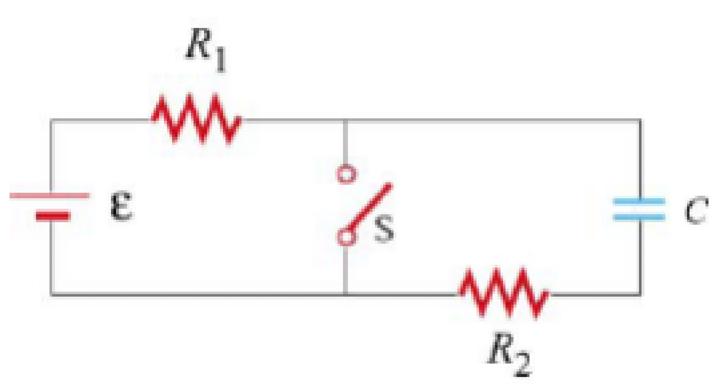
$$in\left(1-\frac{2q}{CV}\right) = \frac{-2t}{3RC}$$

$$\Rightarrow q = \frac{CV}{2}\left(1-e^{-\frac{2t}{3RC}}\right)$$

$$obviously\ \tau_{eq} = \frac{3RC}{2}$$

Practice Exercise

Q.1 In the circuit in figure, suppose the switch has been open for a very long time. At time t = 0, it is suddenly closed.

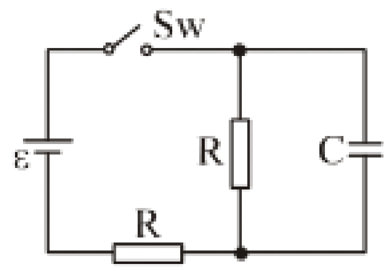


(a) What is the time constant before the switch is closed?

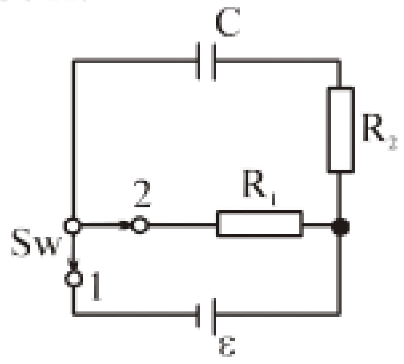
(b) What is the time constant after the switch is closed?

(c) Find the current through the switch as a function of time after the switch is closed.

Q.2 Find how the voltage across the capacitor C varies with time t (Fig.) after the shorting of the switch Sw at the moment t = 0.



Q.3 A capacitor of capacitance C = 5.00 μF is connected to a source of constant emf ξ = 200 V (Fig.). Then the switch Sw was thrown over from contact 1 to contact 2. Find the amount of heat generated in a resistance R_1 = 500 Ω if R_2 = 330 Ω.



Answers

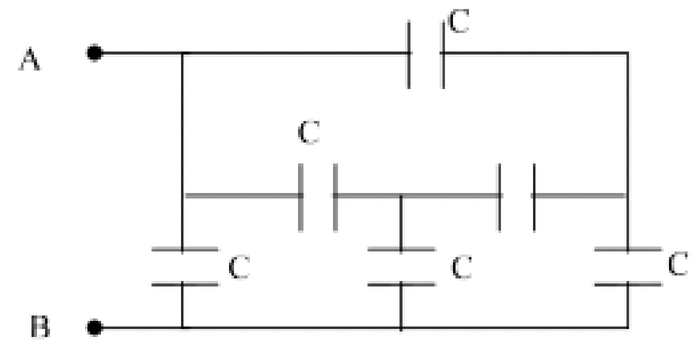
Q.1 (a) $(R_1 + R_2)C$ (b) $R_2 C$ (c) $I(t) = I_1 + I'(t) = \frac{\varepsilon}{R_1} + \left(\frac{\varepsilon}{R_2}\right) e^{-t/R_2 C}$

Q.2 $V = 1/2\,\xi\,(1 - e^{-2t/RC})$

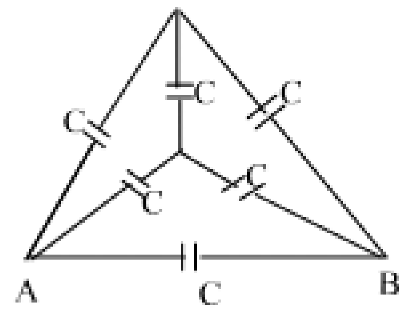
Q.3 $Q = 1/2C\,\xi^2 R_1/(R_1 + R_2) = 60$ mJ

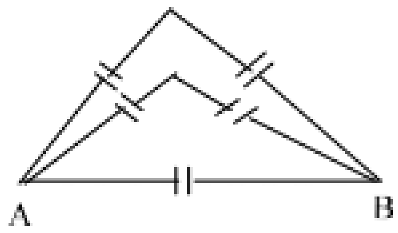
Solved Example

Q.1 In the given figure, what is the equivalent capacitance between points A and B?



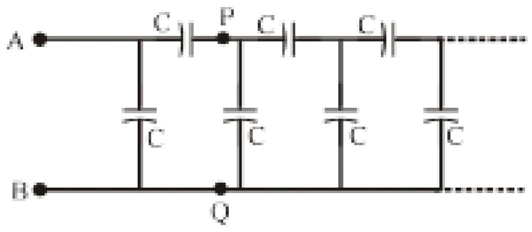
Sol. Circuit can be redrawn as.



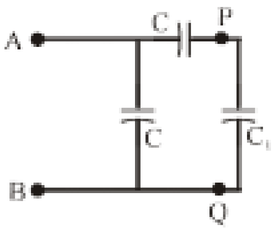


Hence, equivalent capacitance = 2C

Q.2 Find the capacitance of the infinite ladder shown in figure.



Sol. As the ladder is infinitely long, the capacitance of the ladder to the right of the points P, Q is the same as that of the ladder to the right of the points A, B. If the equivalent capacitance of the ladder is C_1, the given ladder may be replaced by the connections shown in figure.



The equivalent capacitance between A and B is easily found to be $C + \frac{CC_1}{C + C_1}$. But being equivalent to the original ladder, the equivalent capacitance is also C_1.

Thus, $C_1 = C + \frac{CC_1}{C + C_1}$

Or, $\quad C_1C + C_1^2 = C^2 + 2CC_1$

Or, $\quad C_1^2 - CC_1 - C^2 = 0$

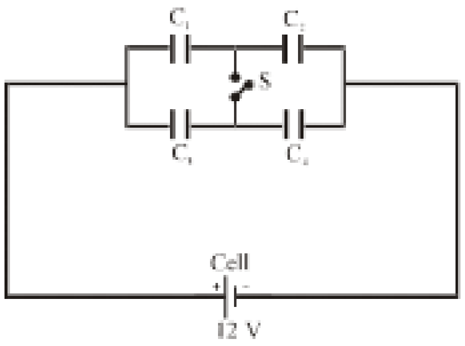
Giving $C_1 = \dfrac{C + \sqrt{C^2 + 4C^2}}{2} = \dfrac{1+\sqrt{5}}{2}C$.

Negative value of C_1 is rejected.

Q.3 The emf of the cell in the circuit is 12 volts and the capacitors are : $C_1 = 1\ \mu f, C_2 = 3\ \mu f, C_3 = 2\ \mu f, C_4 = 4\ \mu f$. Calculate the charge on each capacitor and the total charge drawn from the cell when

(a) the switch s is closed

(b) the switch s is open.



Sol. (a) Switch S is closed :

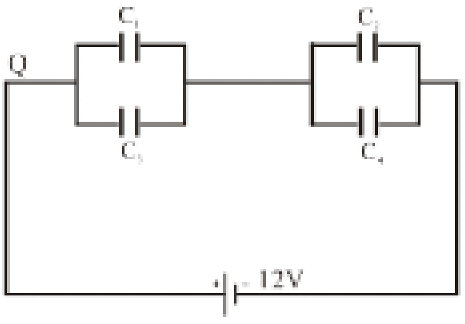
$$C = \frac{(C_1 + C_3)(C_2 + C_4)}{(C_1 + C_3) + (C_2 + C_4)}$$

$$C = \frac{3 \times 7}{3 + 7} = 2.1\mu F$$

total charge drawn from the cell is :

$$Q = C\ V = 2.1\ \mu F \times 12 \text{ volts} = 25.2\ \mu C$$

C_1, C_3 are in parallel and C_2, C_4 are in parallel.



Charge on C_1

$$Q_1 = \frac{C_1}{C_1 + C_3} Q = \frac{1}{1+2} \times 25.2\mu C = 8.4\mu C.$$

Charge on C_3

$$Q_3 = \frac{C_3}{C_1 + C_3} Q = \frac{2}{1+2} \times 25.2\mu C = 16.8\mu C.$$

Charge on C_2

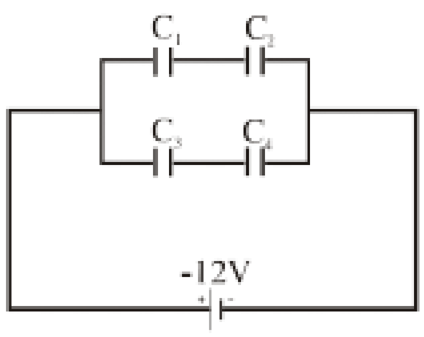
$$Q_2 = \frac{C_2}{C_2 + C_4} Q = \frac{3}{3+4} \times 25.2\mu C = 10.8\mu C.$$

Charge on C_4

$$Q_4 = \frac{C_4}{C_2 + C_4} Q = \frac{4}{3+4} \times 25.2\mu C = 14.4\mu C.$$

(b) Switch S is open :

$$C = \frac{C_1C_2}{C_1+C_2} + \frac{C_3C_4}{C_3+C_4}$$



$$C = \frac{1\times 3}{1+3} + \frac{2\times 4}{2+4} = \frac{25}{12}\,\mu F$$

total charge drawn from battery is :

$$Q = CV = \frac{25}{12}\times 12 = 25\mu C$$

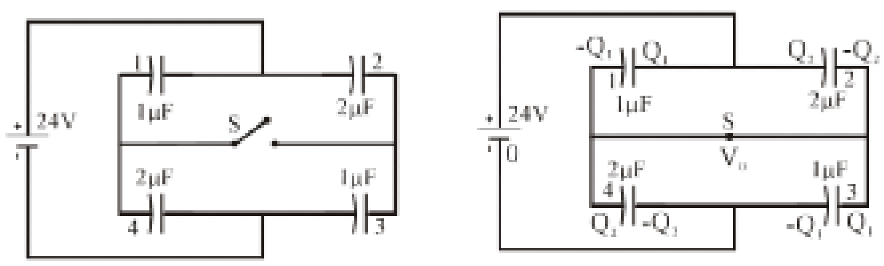
C_1 & C_2 are in series and the potential difference across combination is 12 volts.
charge on C_1 = charge on C_2

$$= \left(\frac{C_1C_2}{C_1+C_2}\right)V = \frac{3}{4}\times 12 = 9\mu C.$$

C_3 & C_4 are in series and the potential difference across combination is 12 volts.
charge on C_3 = charge on C_4

$$= \left(\frac{C_3C_4}{C_3+C_4}\right)V = \frac{8}{6}\times 12 = 16\mu C.$$

Q.4 The connections shown in figure are established with the switch S open. How much charge will flow through the switch if it is closed?



(a) (b)

Sol. When the switch is open, capacitors (2) and (3) are in series. Their equivalent capacitance is $\frac{2}{3}\mu F$. The charge appearing on each of these capacitors is, therefore, $24\,V\times\frac{2}{3}\mu F = 16\mu C$.

The equivalent capacitance of (1) and (4), which are also connected in series, is also $\frac{2}{3}\mu F$ and the charge on each of these capacitors is also $16\mu C$. The total charge on the two plates of (1) and (4) connected to the switch is, therefore, zero.

The situation when the switch S is closed is shown in figure. Let the charges be distributed as shown in the figure. Q_1 and Q_2 are arbitrarily chosen for the positive plates of (1) and (2). The same magnitude of charges will appear at the negative plates (3) and (4).

Take the potential at the negative terminal to the zero and at the switch to be V_0.

Writing equations for the capacitors (1), (2), (3) and (4).

$$Q_1 = (24V - V_0) \times 1\ \mu F \qquad ...(i)$$

$$Q_2 = (24V - V_0) \times 2\mu F \qquad ...(ii)$$

$$Q_1 = V_0 \times 1\ \mu F \qquad ...(iii)$$

$$Q_2 = V_0 \times 2\mu F \qquad ...(iv)$$

From (i) and (iii), $V_0 = 12V$.

Thus, from (iii) and (iv),

$$Q_1 = 12\mu C \text{ and } Q_2 = 24\ \mu C.$$

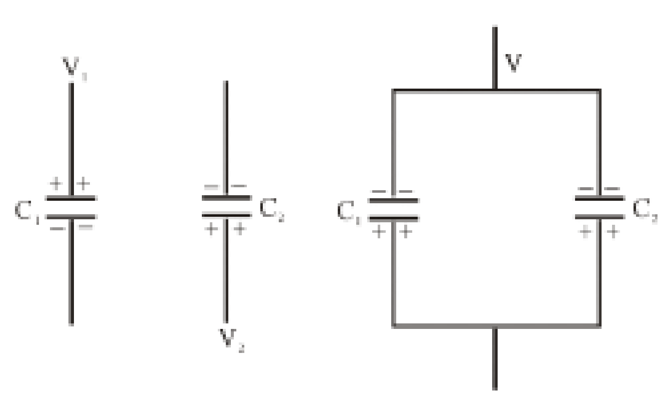
The charge on the two plates of (1) and (4) which are connected to the switch is, therefore $Q_2 - Q_1 = 12\mu C$. When the switch was open, this charge was zero. Thus, $12\mu C$ of charge has passed through the switch after it was closed.

Q.5 Two capacitors $C_1 = 1\mu F$ and $C_2 = 4\mu F$ are charged to a potential difference of 100 volts and 200 volts respectively. The charged capacitors are now connected to each other with terminals of opposite sign connected together. What is the

(a) final charge on each capacitor in steady state ?

(b) decrease in the energy of the system ?

Sol.



Initial charge on $C_1 = C_1V_1 = 100\mu C$

Initial charge on $C_2 = C_2V_2 = 800\mu C$

$$C_1V_1 < C_2V_2$$

when the terminals of opposite polarity are connected together, the magnitude of net charge finally is equal to the difference of magnitude of charges before connection.

(charge on $C_2)_i$ - (charge on $C_1)_i$

= (charge on $C_2)_f$ - (charge on $C_1)_f$

Let V be the final common potential difference across each.

The charges will be redistributed and the system attains a steady state when potential difference across each capacitor becomes same.

$$C_2V_2 - C_1V_1 = C_2V + C_1V$$

$$V = \frac{C_1V_2 - C_1V_1}{C_2 + C_1} = \frac{800-100}{5} = 140 \text{ volts}$$

Note that because $C_1V_1 < C_2V_2$, the final charge polarities are same as that of C_2 before connection.

Final charge on $C_1 = C_1 V = 140\ \mu C$

Final charge on $C_2 = C_2 V = 560\ \mu C$

Loss of energy $= U_i - U_f$

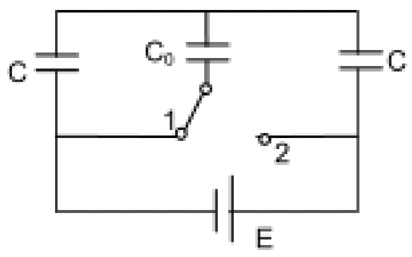
Loss of energy $= \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2 - \frac{1}{2}C_1V^2 - \frac{1}{2}C_2V^2$

$$= \frac{1}{2}1(100)^2 + \frac{1}{2}4(200)^2 - \frac{1}{2}(1+4)(140)^2$$

$$= 36000\mu J = 0.036J$$

Note: The energy is lost as heat in the connected wires due to the temporary currents that flow while the charge is being redistributed.

Q.6 Find the amount of heat generated in the circuit shown in the figure after the switch is shifted from position 1 to position 2.



Sol. When the switch is in position 1, the combination has C and C_0 in parallel and C in series for which the equivalent capacitance is

$$C_{eq} = \frac{C(C+C_0)}{2C+C_0}$$

The total charge on the combination is

$$Q = EC_{eq} = \frac{EC(C+C_0)}{2C+C_0}$$

The total charge on the three capacitors can be obtained as

$$q_3 = EC_{eq} = \frac{EC(C+C_0)}{2C+C_0}$$

$$q_2 = \frac{EC(C+C_0)C_0}{(2C+C_0)(C+C_0)} = \frac{ECC_0}{2C+C_0}$$

$$q_1 = \frac{EC(C+C_0)C}{(2C+C_0)(C+C_0)} = \frac{EC^2}{2C+C_0}$$

When the switch is in position 2, the charge distribution on the three capacitors is obtained as

$$q_3' = \frac{EC^2}{2C+C_0}, \quad q_2' = q_2 \quad \text{and} \quad q_1' = \frac{EC(C+C_0)}{2C+C_0}$$

Now, heat produced = (loss in stored electrical energy) + (extra energy drawn from the battery).

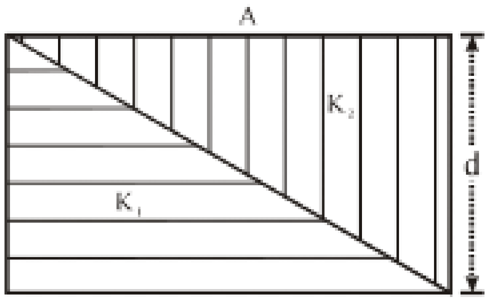
Since the equivalent capacitance C_{eq} remains unchanged in both the positions of the key, the loss in stored energy is zero. Hence,

$$\text{Heat produced} = \text{energy drawn from the battery}$$

$$= E\Delta q = E\,(q'_1 - q_1) = E(q_3 - q'_3)$$

$$= E\left[\frac{EC(C+C_0)}{2C+C_0} - \frac{EC^2}{2C+C_0}\right] = \frac{E^2CC_0}{2C+C_0}$$

Q.7 The capacitance of a parallel plate capacitor with plate area A and separation d is C. The space between the plates is filled with two wedges of dielectric constants K_1 and K_2 respectively (Fig.). Find the capacitance of the resulting capacitor.



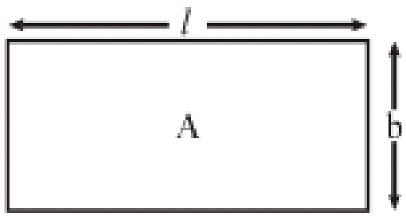
Sol. Let length and breadth of the capacitor be l and b respectively and d be the distance between the plates as shown in fig. Then consider a strip at a distance x of width dx.

Now QR = x tan θ

and PQ = d – x tan θ

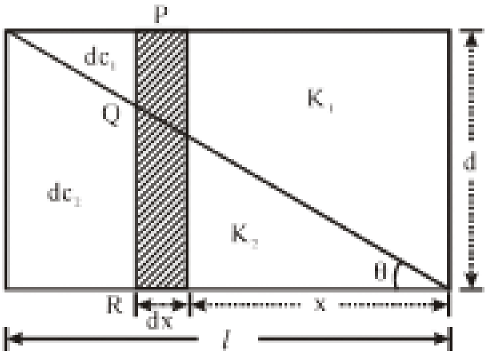
Where tan θ = d/l,

Capacitance of PQ



$$dC_1 = \frac{k_1\varepsilon_0(b\,dx)}{d - x\tan\theta} = \frac{k_1\varepsilon_0(b\,dx)}{d - \dfrac{xd}{l}}$$

$$dC_1 = \frac{k_1\varepsilon_0 bl dx}{d(l-x)} = \frac{k_1\varepsilon_0 A(dx)}{d(l-x)}$$



and dC_2 = capacitance of QR

$$dC_2 = \frac{k_2\varepsilon_0 b(dx)}{d\tan\theta}$$

$$dC_2 = \frac{k_2\varepsilon_0 A(dx)}{x\,d} \quad \ldots\ldots \quad \left\{\because \tan\theta = \frac{d}{l}\right\}$$

Now dC_1 and dC_2 are in series. Therefore, their resultant capacity dC will be given by

$$\frac{1}{dC} = \frac{1}{dC_1} + \frac{1}{dC_2}$$

then $$\frac{1}{dC} = \frac{1}{dC_1} + \frac{1}{dC_2}$$

$$= \frac{d(l-x)}{K_1\varepsilon_0 A(dx)} + \frac{x.d}{K_2\varepsilon_0 A(dx)}$$

$$\frac{1}{dC} = \frac{d}{\varepsilon_0 A(dx)}\left(\frac{l-x}{K_1}+\frac{x}{K_2}\right) \quad = \quad \frac{d[K_2(l-x)+K_1x]}{\varepsilon_0 AK_1K_2(dx)}$$

$$dC = \frac{\varepsilon_0 AK_1K_2}{d[K_2(l-x)+K_1x]}dx \quad , \quad dC = \frac{\varepsilon_0 AK_1K_2}{d[K_2 l+(K_1-K_1)x]}dx$$

All such elemental capacitor representing DC are connected in parallel.

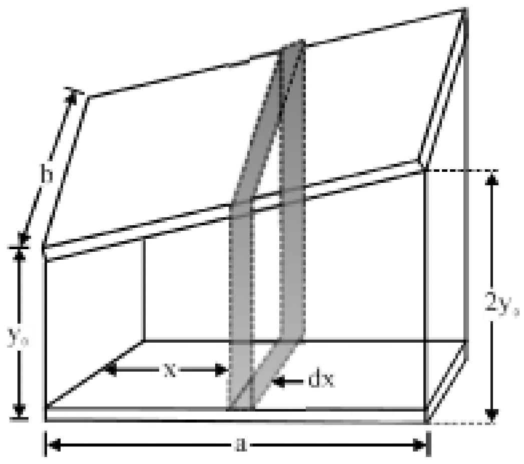
Now the capacitance of the given parallel plate capacitor is obtained by adding such infinitesimal capacitors parallel from x = 0 to x = l.

i.e. $$C = \int_{x=0}^{x=l} dC$$

$$= \int_0^l \frac{\varepsilon_0 AK_1K_2}{d[K_2 l+(K_1-K_2)x]}dx$$

$$C = \frac{K_1K_2\varepsilon_0 A}{(K_1-K_2)d}\ln\frac{K_2}{K_1}.$$

Q.8 A capacitor has rectangular plates of length a and width b. The top plate is inclined at a small angle as shown in figure The plate separation varies from $d = y_0$ at the left to $d = 2y_0$ at the right where y_0 is much less than a or b. Calculate the capacitance of the system.



Sol. We consider a differential strip of width dx and length b to approximate a differential capacitor of area b dx and separation $d = y_0 + \left(\frac{y_0}{a}\right)x$. All such differential capacitor are in parallel arrangement.

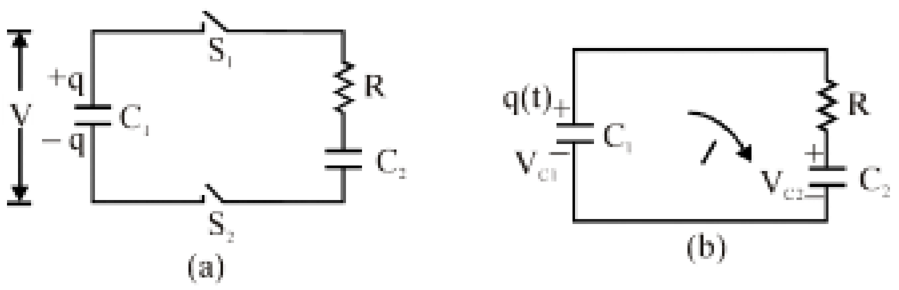
$$dC = \frac{\varepsilon_0(b\,dx)}{y_0+\left(\frac{y_0}{a}\right)x}$$

$$C = \varepsilon_0 b\int_0^a \frac{dx}{\left(y_0+\frac{y_0}{a}x\right)}$$

$$= \frac{\varepsilon_0 b}{(y_0 / a)}\left[\text{In}\left(\frac{y_0 + \frac{y_0}{a} \times a}{y_0}\right)\right]$$

$$= \frac{\varepsilon_0 ab}{y_0} \text{ in } 2$$

Q.9 The capacitor C_1 figure initially carries a charge q_0. When the switches S_1 and S_2 are shut, capacitor C_1 is connected in series to resistor R and a second capacitor C_2, which initially does not carry any charge.
(i) Find the charge deposited on the capacitor and the current through R as a function of time.
(ii) What is the heat lost in the resistor after a long time of closing the switch ?



Sol. (i) Suppose at a moment 't' the charge deposited on C_1 is q (t).

$\therefore \quad V_{C_1} = \frac{q(t)}{C_1}$

and $\quad V_{C_2} = \frac{q_0 - q(t)}{C_2}$

$V_R = IR$

and $\quad I = \frac{dq}{dt}$

Applying KVL,

$$\frac{q}{C_1} - \frac{(q_0 - q)}{C_2} = IR = -R\frac{dq}{dt}$$

$\therefore \quad R\frac{dq}{dt} + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)q = \frac{q_0}{C_2}$

Put $\quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$

$\therefore \quad q(t) = \left(1 - \frac{C}{C_2}\right)q_0\, e^{-t/RC} + \frac{C}{C_2}q_0$

or $$q(t)=\frac{C}{C_1}q_0e^{-t/RC}+\frac{C}{C_2}q_0$$

$$\therefore \quad I(t)=-\frac{dq}{dt}=\frac{q_0}{RC_1}e^{-t/RC}$$

Charge C_2,

$$q_0-q(t)=q_0\frac{C}{C_1}(1-e^{-t/RC})$$

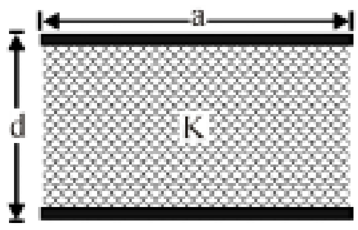
(ii) Electrostatic energy at t = 0 is

$$U(0)=\frac{q_0^2}{2C_1}$$

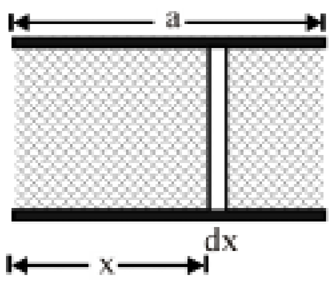
$$\text{Final energy}=U(\infty)=\frac{q_0^2}{2(C_1+C_2)}$$

$$\Delta U=U(0)-U(\infty)=\frac{q_0^2C_2}{2C_1(C_1+C_2)}$$

Q.10 Shows a parallel-plate capacitor having square plates of edge a and plate-separation d. The gap between the plates is filled with a dielectric of dielectric constant K which varies parallel to an edge as



(a)



(b)

Where K and α are constants and x is the distance from the left ends. Calculate the capacitance.

Sol. Consider a small strip of width dx at separation x from the left end. This strip forms small capacitor of plate area adx. Its capacitance is

$$dC=\frac{(K_0+ax)\varepsilon_0 adx}{d}$$

The given capacitor may be divided into such strips with x varying from 0 to a. All these strips are connected in parallel. The capacitance of the given capacitor is,

$$C=\int_0^a\frac{(K_0+ax)\varepsilon_0 adx}{d}$$

$$=\frac{\varepsilon_0a^2}{d}\left(K_0+\frac{a\alpha}{2}\right)$$

Q.11 A capacitor of capacitance C is charged by connecting it to battery of emf ε. The capacitor is now disconnected and reconnected to the battery with the polarity reversed. Calculate the heat developed in the connecting wires.

Sol. When the capacitor is connected to the battery, a charge. When the polarity is reversed, a charge - Q on the other. When the polarity is reversed, a charge – Q appears on the first plate and + Q on the second. A charge 2Q, therefore passes through the battery from the negative to the positive terminal. The battery does a work

$$W = (2Q)\varepsilon = 2C\varepsilon^2$$

in the process. The energy stored in the capacitor is the same in the two cases. Thus, the work done by the battery appears as heat in the connecting wires. The heat produced is, therefore, $2C\varepsilon^2$.

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Magnetic Field

Magnets are familiar objects. The word magnetism is derived from the province of Magnesia where the ancient Greek mine magnetite, also known as lodestone, a mineral composed of iron oxide which attracts iron.

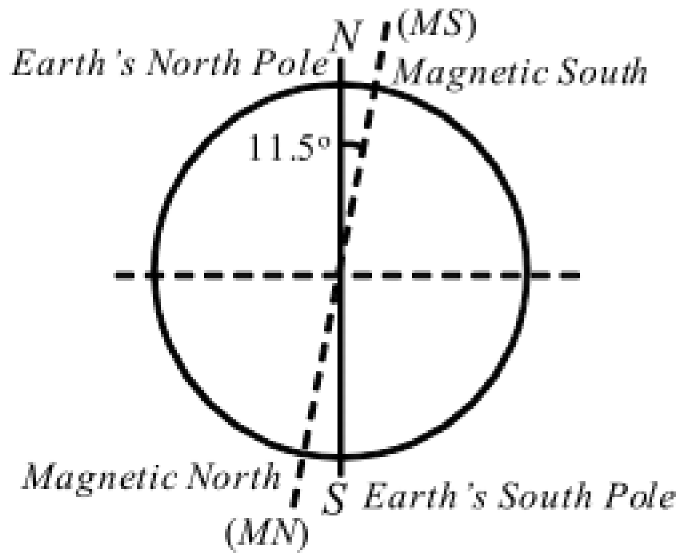
If you ask the average person what "magnetism" is, you will probably be told about the magnets those are used to hold notes on refrigerator door, or keeping paper clips in a holder or may be about lead stone (naturally occurring magnet).

Scholars still dispute about the origin of magnetism. It is believed that magnetism was originally used, not for navigation, but for geomancy ("foresight by earth") and fortune-telling by the Chinese. Chinese fortune tellers used lodestones to construct their fortune telling boards.

From Chinese text, it is known that magnetic compass (used for navigational purpose) is an old Chinese invention . An old Indian literature dates it to as back as 4th century. The compass was used in India was known as the matsya yantra, because of the placement of a metallic fish in a cup of oil.

Earth's magnetic field:

It is understood that a compass needle points along the horizontal component of Earth's magnetic field (a property called declination).



Earth's Geomagnetic North acts as a south pole of a magnet, while its Geomagnetic South acts as a North Pole of a magnet.

As shown in the diagram, the axis of the dipole makes an angle of about 11.5° with earth's rotational axis. The axis of dipole makes an angle of about 11.5° with the earth's rotational axis. And Earth's rotational axis makes an angle of 23.5° with the normal to the plane of earth's orbit about the sun.

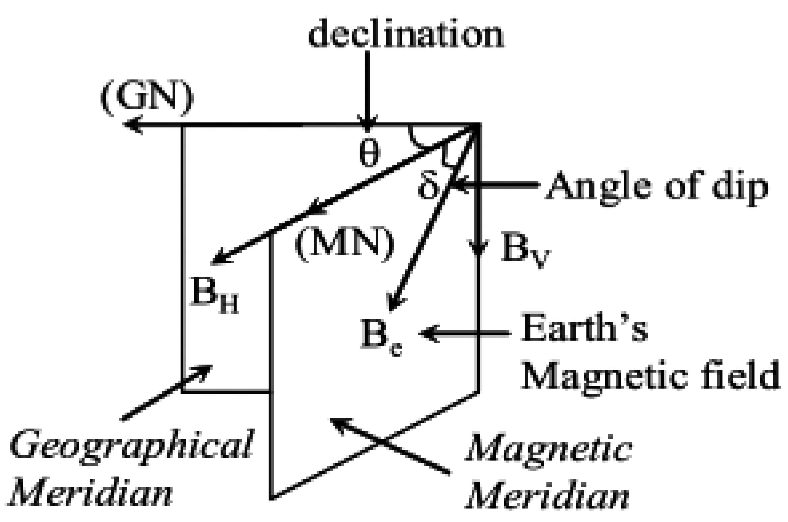
Elements of earth's magnetic field:

The earth's magnetic field is characterized by three quantities:

(a) Declination

(b) Inclination or dip

(c) Horizontal component of the field.



Magnetism due to electricity

Hans Christian Oersted was a professor of science at Copenhagen University. In 1819 he arranged in his home a science demonstration to friends and students. He planned to demonstrate the heating of a wire by an electric current, and also to carry out demonstrations of magnetism, for which he provided a compass needle mounted on a wooden stand.

While performing his electric demonstration, Oersted noted to his surprise that every time the electric current was switched on, the compass needle moved . He kept quiet and finished the demonstrations, but in the months that followed worked hard trying to make sense out of the new phenomenon. And this is what we are going to study now.

We have seen that currents (fundamentally moving charges) are the source of magnetism. This can be readily demonstrated by placing compass needles near a wire. As shown in Figure, all compass needles point in the same direction in the absence of current (in the direction of earth's magnetic field). However, when a strong current passes through (so that earth's magnetic field becomes negligible), the needles will be deflected along the tangential direction of the circular path (Figure).

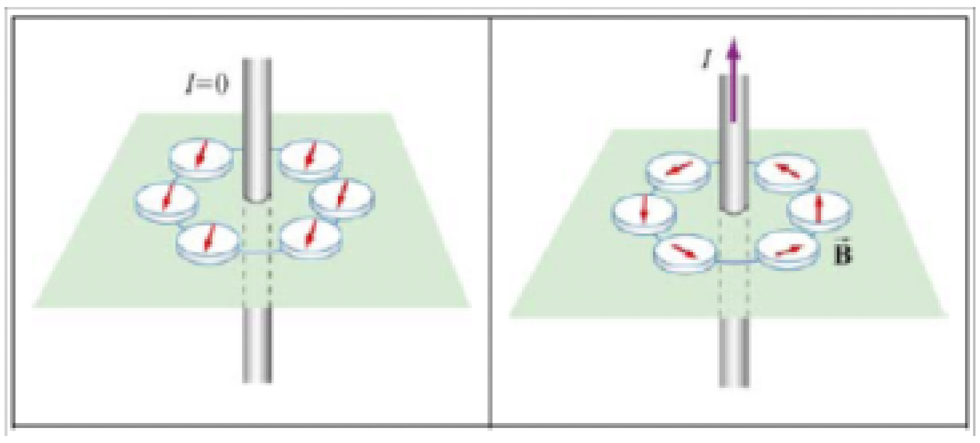


Figure : Deflection of compass needles near a current-carrying wire

Biot-Savart Law

Currents which arise due to the motion of charges are the source of magnetic fields. When charges move in a conducting wire and produce a current I, the magnetic field at any point P due to the current can be calculated by adding up the magnetic field contributions, $d\vec{B}$, from small segments of the wire $d\vec{s}$ (Figure).

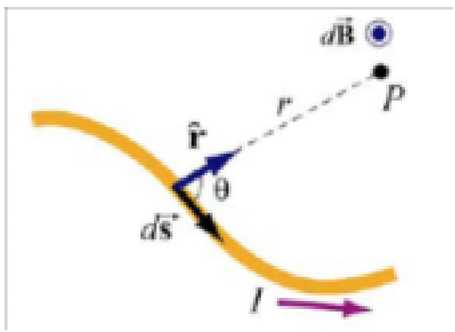


Figure : Magnetic field $d\vec{B}$ at point P due to a current-carrying element $Id\vec{s}$

These segments can be thought of as a vector quantity having a magnitude of the length of the segment and pointing in the direction of the current flow. The infinitesimal current source can then be written as $Id\vec{s}$.

Let r denote as the distance form the current source to the field point P and $\hat{r}$ the corresponding unit vector. The Biot-Savart law gives an expression for the magnetic field contribution, $d\vec{B}$, from the current source, $Id\vec{s}$,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{s}\times\hat{r}}{r^2}$$

where μ_0 is a constant called the *permeability of free space*:

$\mu_0 = 4\pi \times 10^{-7}$ T.m/A here Tesla (T) is SI unit of $\vec{B}$

Notice that the expression is remarkably similar to the Coulomb's law for the electric field due to a charge element dq :

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}$$

Adding up these contributions to find the magnetic field at the point P requires integrating over the current source,

$$\vec{B} = \int d\vec{B} = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{s}\times\hat{r}}{r^2}$$

The integral is a vector integral, which means that the expression for Bis really three integrals, one for each component of $\vec{B}$. The vector nature of this integral appears in the cross product. Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart law.

Magnetic Field due to a Finite Straight Wire.

A thin, straight wire carrying a current I is placed along the x-axis, as shown in Figure. Evaluate the magnetic field at point P due to the segment shown in figure.

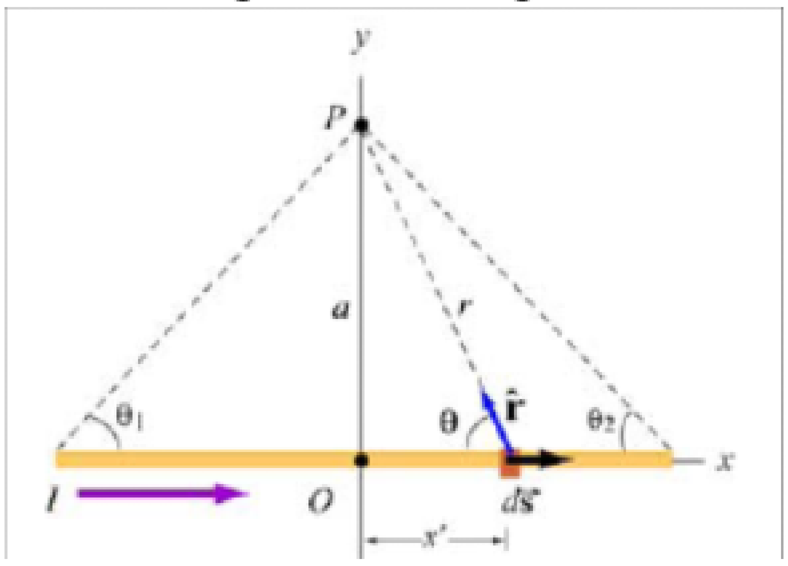


Figure 9.1.3 A thin straight wire carrying a current I.

The contribution to the magnetic field due to $Id\vec{s}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{dx\sin\theta}{r^2}\hat{k}$$

which shows that the magnetic field at P will point in the $+\hat{k}$ direction, or out of the page.

Simplify and carry out the integration

The variables θ, x and r are not independent of each other. In order to complete the integration, let us rewrite the variables x and r in terms of θ. From Figure, we have

$$\begin{cases} r = a/\sin\theta = a\,\mathrm{cosec}\,\theta \\ x = a\cot\theta \Rightarrow dx = -a\,\mathrm{cosec}^2\theta\, d\theta \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is obtained as

$$dB = \frac{\mu_0 I}{4\pi}\frac{(-a\,\mathrm{cosec}^2\theta\, d\theta)\sin\theta}{(a\,\mathrm{cosec}\,\theta)^2} = -\frac{\mu_0 I}{4\pi a}\sin\theta\, d\theta$$

Integrating over all angles subtended from $-\theta_1$ to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x-axis from the origin), we obtain

$$\boxed{B = \frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\sin\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)}$$

The first term involving θ_2 accounts for the contribution from the portion along the $+x$ axis, while the second term involving θ_1 contains the contribution from the portion along the $-x$ axis. The two terms add.

Special cases :

(i) Magnetic field on the perpendicular bisector of a finite straight wire of length 2*L*

In this case where $\theta_2 = \theta_1 = \theta$, the field point *P* is located along the perpendicular bisector. If the length of the rod is 2*L*, then $\cos\theta = L/\sqrt{L^2+a^2}$ and the magnetic field is

$$B = \frac{\mu_0 I}{2\pi a}\cos\theta = \frac{\mu_0 I}{2\pi a}\frac{L}{\sqrt{L^2+a^2}}$$

(ii) Magnetic field due to semiinfinite straight wire

Here $\theta_1=90^0$, $\theta_2=0^0$ or $\theta_1=0^0$, $\theta_2=90^0$

$$B = \frac{\mu_0 I}{4\pi a}$$

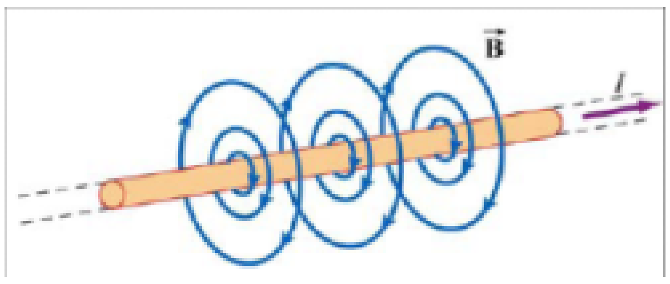
(iii) Magnetic field due to infinite straight wire

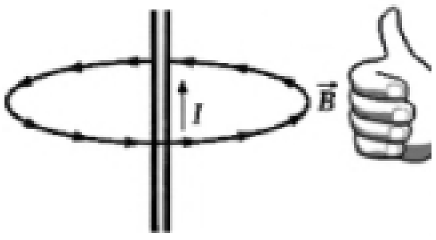
Here $\theta_1=\theta_2=0$

$$B = \frac{\mu_0 I}{2\pi a}$$

Direction of magnetic field of a straight wire

Note that in this limit, the system possesses cylindrical symmetry, and the magnetic field lines are circular, as shown in Figure.





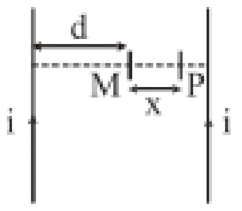
In fact, the direction of the magnetic field due to a long straight wire can be determined by the right-hand rule (Figure). If you direct your right thumb along the direction of the current in the wire, then the fingers of your right hand curl in the direction of the magnetic field.

Illustration :

Two long parallel wires carry equal current *i* flowing in the same direction are at a distance 2d apart. The magnetic field B at a point lying on the perpendicular line joining the wires and at a distance x from the midpoint is

Sol. The magnetic field due

$$B_1 = \frac{\mu_0 i}{2\pi(d+x)}$$



$$B_2 = \frac{\mu_0 i}{2\pi(d-x)}$$

Both the magnetic fields act in opposite direction.

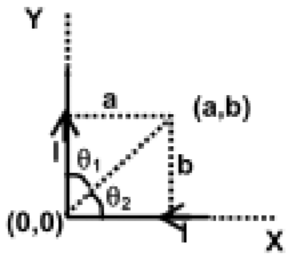
$$\therefore \qquad B = B_2 - B_1 = \frac{\mu_0 i}{2\pi}\left[\frac{1}{d-x} - \frac{1}{d+x}\right]$$

$$= \frac{\mu_0 i}{2\pi}\left[\frac{d+x-d+x}{d^2-x^2}\right]$$

$$= \frac{\mu_0 i x}{\pi(d^2-x^2)}.$$

Illustration :

Two semi-infinitely long straight current carrying conductors are in form of an 'L' shape as shown in the figure. The common end is at the origin. What is the value of magnetic field at a point (a, b), if both the conductors carry the same current I?



Sol. *For the conductor along the X axis, the magnetic field*

$$B_1 = \frac{\mu_0 I}{4\pi b}[\cos\theta_2 + \cos 0]$$ *along the negative Z-axis*

$$= \frac{\mu_0 I}{4\pi b}\left[1 + \frac{a}{\sqrt{a^2+b^2}}\right]$$

For the conductor along Y-axis, the magnetic field is

$$B_2 = \frac{\mu_0 I}{4\pi a}\left[1 + \frac{b}{\sqrt{a^2+b^2}}\right] \quad \text{along the negative } z-\text{axis}$$

$\therefore$ *The net magnetic field is,*

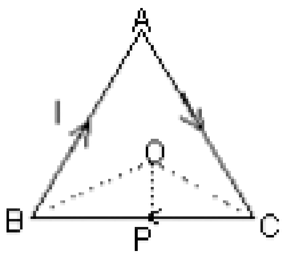
$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

$$= \frac{\mu_0 I}{4\pi}\left[\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{\sqrt{a^2+b^2}}\left(\frac{a}{b} + \frac{b}{a}\right)\right] = \frac{\mu_0 I}{4\pi}\left[\frac{(a+b)}{ab} + \frac{\sqrt{a^2+b^2}}{ab}\right]$$

$$= \frac{\mu_0 I}{4\pi ab}\left[(a+b) + \sqrt{a^2+b^2}\right]$$

Illustration :

A current I is established in a closed loop of an triangle ABC of side ℓ. Find the magnetic field at the centroid 'O'.



Sol. *From geometry*

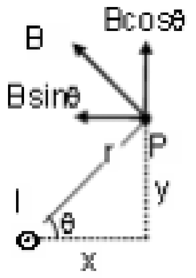
$$OP = \frac{\ell}{2\sqrt{3}}$$

Magnetic fields due to current in all three sides are equal in magnitude and directed into the plane of the paper.

Hence net B $= \frac{3\mu_0 I}{4\pi r}\left[2\times\cos\frac{\pi}{3}\right] = \frac{3\mu_0 I}{4\pi r}\times 2\sin\left(\frac{\pi}{3}\right) = \frac{9\mu_0 I}{2\pi\ell}$

Illustration :

Find the magnetic field due to the wire at point P if the wire is long



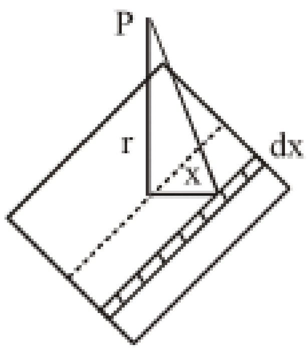
Sol. $\vec{B} = -B\sin\theta\,\hat{i} + B\cos\theta\,\hat{j} = B\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$

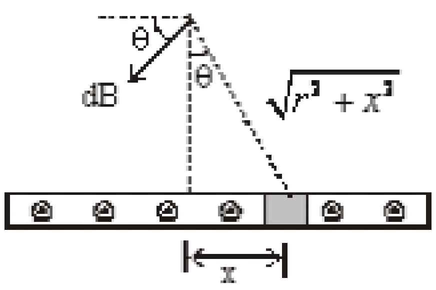
$$\vec{B} = \frac{\mu_0 I}{2\pi r}\left(-\frac{y}{r}\hat{i} + \frac{x}{r}\hat{j}\right) = \frac{\mu_0 I}{2\pi r^2}\left(-y\hat{i} + x\hat{j}\right)$$

Illustraation :

An infinitely large sheet carries current with linear current density i. Find the net magnetic field at a point which is at perpendicular distance r from the sheet.

Sol.





Let us consider a current carrying element idx

$$\therefore \qquad dB_p = \frac{\mu_0 (idx)}{2\pi\sqrt{(r^2 + x^2)}}$$

It has two components one parallel to the plane of the sheet and other perpendicular to it.

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$$dB_x = -dB\cos\theta \qquad and \qquad dB_y = -dB\sin\theta$$

$$\therefore \qquad B_x = \int dB_x = \int_{-\infty}^{+\infty} \frac{\mu_0 idxr}{2\pi(r^2+x^2)} = -\frac{\mu_0 i}{2}$$

$$and \qquad B_y = \int_{-\infty}^{+\infty} dB_y = \int_{-\infty}^{+\infty} \frac{\mu_0 idx\,x}{2\pi(r^2+x^2)} = 0$$

$$\therefore \qquad B = -\frac{\mu_0 I}{2}$$

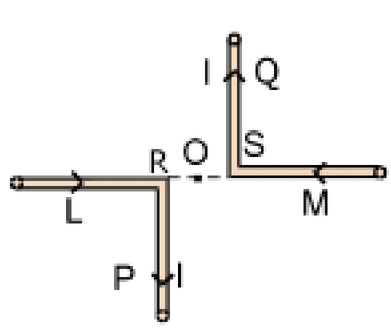
Practice Exercise

Q.1 Find the magnetic field B at the centre of a rectangular loop of length l and width b, carrying a current i.

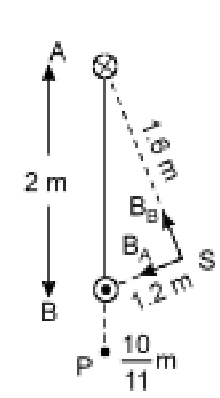
Q.2 A long wire carrying a current i is bent to form a plane angle α. Find the magnetic field B at a point on the bisector of this angle situated at a distance x from the vertex.

Q.3 A pair of stationary and infinitely long bent wires are placed in the x–y plane as shown in **figure**. The wires carry currents of 10 ampere each as shown.

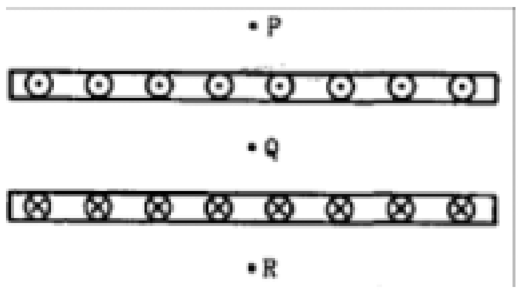
The segments L and M are along the x–axis. The segments P and Q are parallel to the y–axis such that OS = OR = 0.02 m. Find the magnitude and direction of the magnetic induction at the origin O.



Q.4 Two long straight parallel wires are 2 m apart, perpendicular to the plane of the paper. The wire A carries a current of 9.6 ampere, directed into the plane of the paper. The wire B carries a current such that the magnetic field of induction at the point P at a distance (10/11) m from the wire B is zero. Find **(a)** the magnitude and direction of the current in B, **(b)** the magnitude of the magnetic field of induction at the point S,



Q.5 Two large metal sheets carry surface currents as shown in figure . The current through a strip of width *dl* is *Kdl* where *K* is a constant. Find the magnetic field at the points P, *Q* and *R*.



Answers

Q.1 $\dfrac{2\mu_0 i\sqrt{l^2+b^2}}{\pi l b}$ Q.2 $\dfrac{\mu_0 i}{2\pi r}\cot\dfrac{\alpha}{4}$ Q.3 B is 10^{-4} T and out of the page

Q.4 (a) –3 A and opposite to that in A (b) 1.3×10^{-6} T Q.5 0, μ_0 K towards right in the figure, 0

Magnetic Field due to a current carrying Arc at its centre

$$dl = a d\theta$$

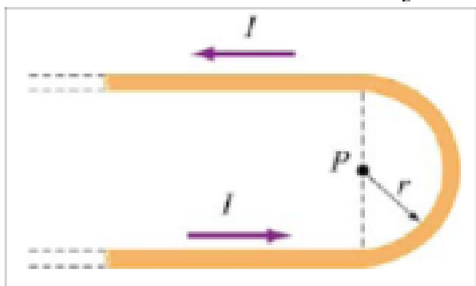
$$dB = \frac{\mu_0}{4\pi}\frac{I(a d\theta)\sin 90^0}{a^2} = \frac{\mu_0 I}{4\pi a} d\theta$$

$$B = \int dB = \frac{\mu_0 I}{4\pi a}\int_0^\beta d\theta$$

$$B = \frac{\mu_0 I}{4\pi R}(\beta\ )$$

Illustration :

An infinitely long current-carrying wire is bent into a hairpin-like shape shown in Figure. Find the magnetic field at the point P which lies at the center of the half-circle.

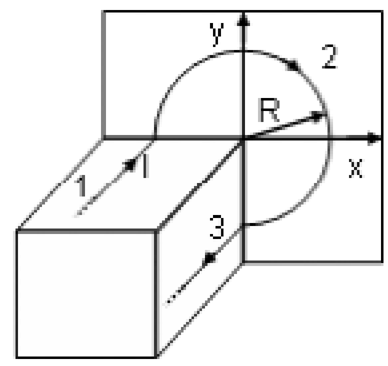


Sol. *The total magnitude of the magnetic field is*

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 = 2\vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{2\pi r}\hat{k} + \frac{\mu_0 I}{4r}\hat{k} = \frac{\mu_0 I}{4\pi r}(2+\pi)\hat{k}$$

Illustration :

Find the magnetic induction at the point O if the wire carrying a current I A has the shape shown in Fig. The radius of the curved part of the wire is R , the linear parts of the wire are very long.



Sol. $\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 = \frac{\mu_0 I}{4\pi R}(-\hat{j}) + \frac{\mu_0 I}{4\pi R}\left(\frac{3\pi}{2}\right)(-\hat{k}) + \frac{\mu_0 I}{4\pi R}(-\hat{i})$

Illustration :

A battery is connected between two points A and B on the circumference of a uniform conducting ring of radius r and resistance R. One of the arcs AB of the ring subtends an angle θ at the centre. Magnetic field due to current at the center of ring is

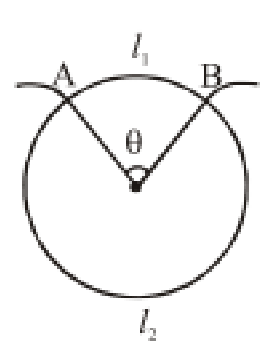
Sol. *For a current flowing into a circular arc, magnetic induction in the centre*

$$B = \frac{\mu_0 I}{4\pi}\int \frac{dl \times r}{r^3} = \frac{\mu_0 I}{4\pi}\int \frac{r^2 d\theta}{r^3} = \left(\frac{\mu_0 I}{4\pi r}\right)\theta$$

The total current is divided into two arcs

$$I_1 = \frac{E}{R_1}$$

$$= \frac{E}{(R/2\pi r)l_1} = \frac{E}{(R/2\pi r)(r\theta)} = \frac{2\pi E}{R\theta}$$



Similarly $I_1\theta = \frac{2\pi E}{R} = constant$

$$I_2 = \frac{E}{R_2}$$

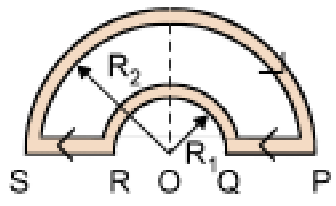
$$= \frac{E}{(R/2\pi r)l_2}$$

$$= \frac{E}{(R/2\pi r)\{r(2\pi - \theta)\}} = \frac{2\pi E}{R(2\pi - \theta)} = \text{cons tan t}$$

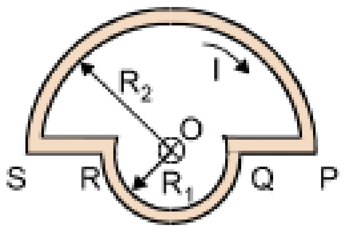
$$B = B_1 - B_2 = \frac{\mu_0}{4\pi r}\left(\frac{2\pi E}{R} - \frac{2\pi E}{R}\right) = 0.$$

Practice Exercise

Q.1 The wire loop PQRSP formed by joining two semi–circular wires of radii R_1 and R_2 carries a current I as shown in **figure**. What is the magnetic induction at the centre O and magnetic moment of the loop in cases **(a)** and **(b)**?

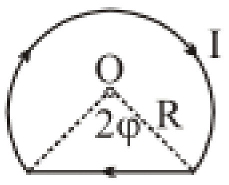


(a)

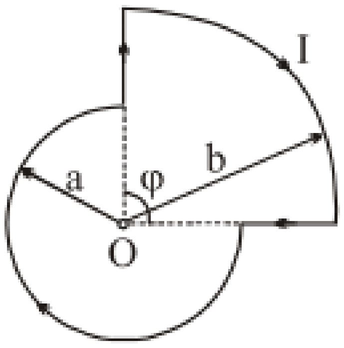


(b)

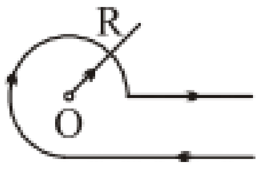
Q.2 A current I flows along a thin wire shaped as shown in Fig. The radius of a curved part of the wire is equal to R , the angle 2φ . Find the magnetic induction of the field at the point O.



Q.3 Find the magnetic induction of the field at the point O of a loop with current I, whose shape is illustrated



Q.4 Find the magnetic induction of the field at the point O if a current-carrying wire has the shape shown in Fig. . The radius of the curved part of the wire is R, the linear parts are assumed to be very long.



Q.5 Two circular coils of radii 5'0 cm and 10 cm carry equal currents of 2A. The coils have 50 and 100 turns respectively and are placed in such a way that their planes as well as the centres coincide. Find the magnitude of the 'magnetic field B at the common centre of the coils if the currents in the coils are (a) in the same sense (b) in the opposite sense.

Q.6 If the outer coil of the previous problem is rotated through 90° about a diameter, what would be the magnitude of the magnetic field B at the centre?

Q.7 A non-conducting thin ring of radius R and charge q rotates about its axis with an angular velocity. Find the magnetic induction at the centre of the ring.

Q.8 A circular disk of radius R with uniform charge density σ rotates with an angular speed ω. Find the magnetic field at the center of the disk .

Answers

Q.1 (a) $\vec{B} = \frac{\mu_0}{4\pi}\pi I\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$ out of the page and $\vec{M} = \frac{1}{2}\pi I\left[R_2^2 - R_1^2\right]$ into the page

(b) $\vec{B} = \frac{\mu_0}{4\pi}\pi I\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$ into the page and $\vec{M} = \frac{1}{2}\pi I\left[R_2^2 + R_1^2\right]$ into the page

Q.2 $B = (\pi - \varphi + \tan\phi)\ \mu_0 I / 2\pi R$

Q.3 $B = \frac{\mu_0 I}{4\pi}\left(\frac{2\pi - \varphi}{a} + \frac{\varphi}{b}\right)$

Q.4 $B = (\mu_0 / 4\pi)\ (1 + 3\pi/2)\ I / R$

Q.5 (a) $8\pi \times 10^{-4}$ T (b) zero

Q.6 1.8 mT

Q.7 $B = \frac{\mu_o q\,\omega}{4\pi\, R}$

Q.8 $B = \frac{1}{2}\mu_0\sigma\omega R$

Magnetic Field due to a Circular Current Loop at a point on its axis

Consider a circular loop of radius a carrying a current i. We have to find the magnetic field at a Point P on the axis of the loop at a distance d from its centre O. In figure

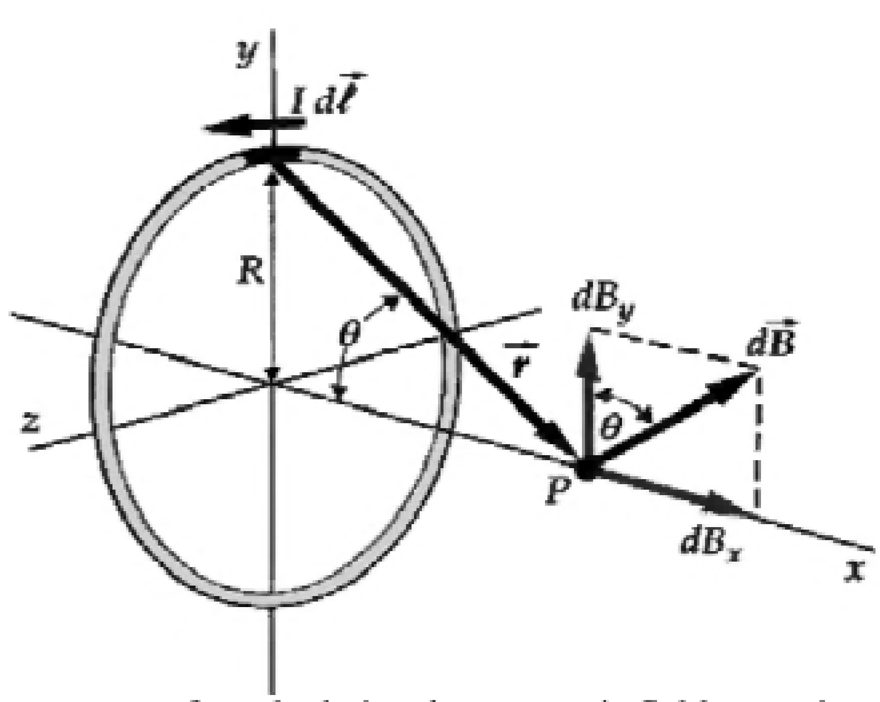


Figure shows the geometry for calculating the magnet ic field at a point on the axis of a circular current loop a distance x from its center. We first consider the current element at the top of the loop. Here, as everywhere around the loop, $Id\vec{\ell}$ is tangent to the loop and perpendicular to the vector $\vec{r}$ from the current element to the field point P. The magnetic field $d\vec{B}$ due to this element is in the direction shown in the figure, perpendicular to $\vec{r}$ and also perpendicular to $Id\vec{\ell}$ The magnitude of $d\vec{B}$ is

$$dB = \frac{\mu_0}{4\pi}\frac{Id\ell \sin 90}{r^2}$$

When we sum around all the current elements in the loop, the components of $d\vec{B}$ perpendicular to the axis of the loop, such as dB_y in Figure sum to zero, leaving only the components dB_x that are parallel to the axis. We thus compute only the x component of the field.
From Figure , we have

$$B_x = \int dB\sin\theta = \int \sin\theta \frac{\mu_0 I}{4\pi r^2} dl = \frac{\mu_0 I}{4\pi r^2}\sin\theta \int dl = \frac{\mu_0 I}{4\pi r^2}\frac{R}{r}(2\pi R) = \frac{\mu_0 I(2\pi R^2)}{4\pi r^3}$$

using the facts that

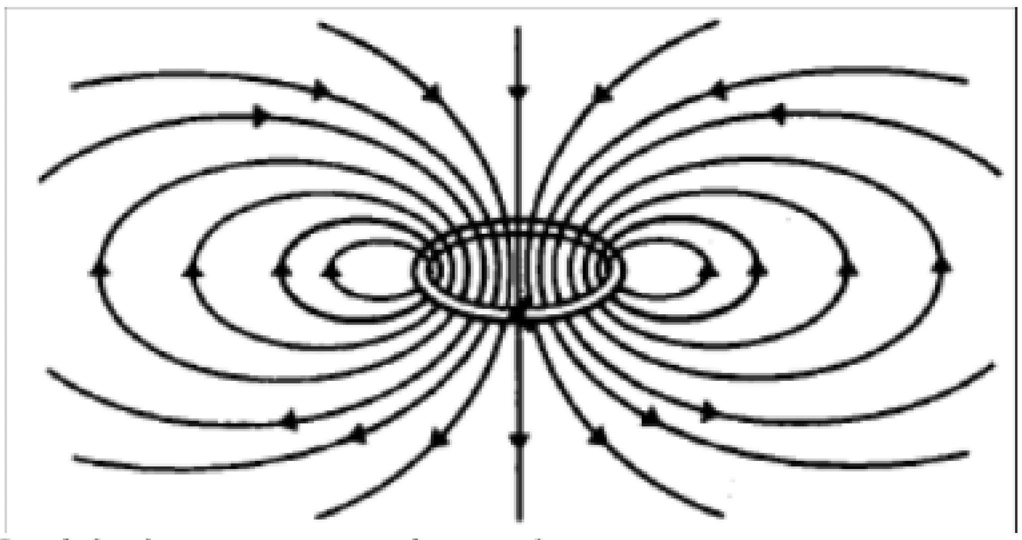
$$r^2 = x^2 + R^2$$

we get

$$B = \frac{\mu_0 I(2\pi a^2)}{4\pi(a^2 + x^2)^{3/2}}$$

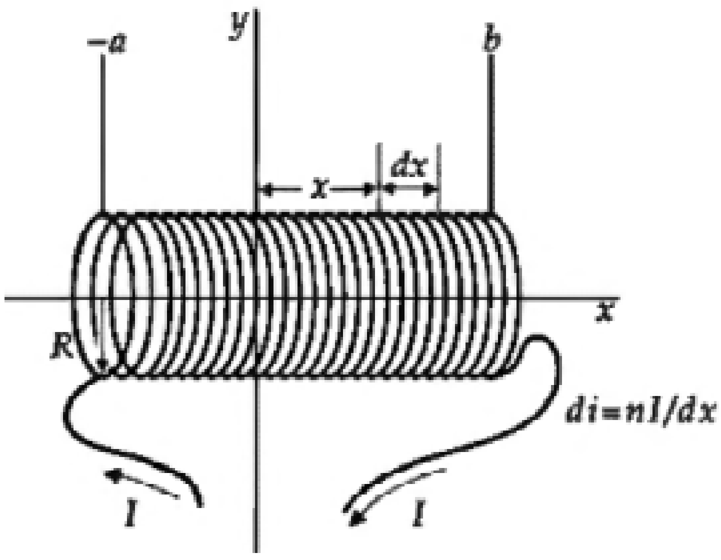
Note: If loop has N close turns then field becomes N times the field due to one turn.

Magnetic field lines due to a circular current



Magnetic Field due to a Solenoid at a point on its axis

A solenoid is a long cylindrical helix, which is obtained by winding closely a large number of turns of insulated copper wire over an in tube of insulating material. When electric current is passed through it, a magnetic field is produced around and within the solenoid.



Consider a solenoid of length L consisting of N turns of wi re carrying a current I. We choose the axis of the solenoid to be along the x axis,with the left end at x = -a and the right end at x = +b as shown in Figure . We will calculate the magnetic field at the origin. The figure shows an element of the solenoid of

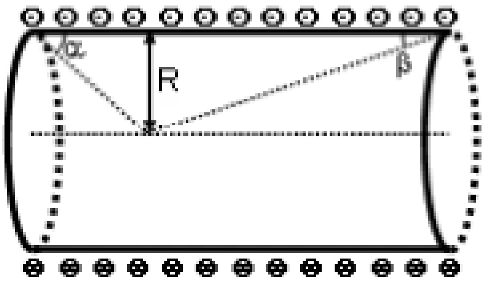
length dx at a distance x from the origin. If n = N/L is the number of turns per unit length, there are ndx turns of wire in this element, with each turn carrying a current I. The element is thus equivalent to a single loop carrying a current di = nIdx. The magnetic field at a point on the x axis due to a loop at the origin carrying a current nI dx is given by Equation with I replaced by nIdx:

$$dB = \frac{\mu_0 (ndx) i R^2}{2(R^2 + x^2)^{3/2}} \text{ along the axis}$$

Net magnetic induction

$$B = \int dB = \int_{-l_2}^{l_1} \frac{\mu_0 n i R^2 dx}{2(R^2 + x^2)^{3/2}}$$

$$\Rightarrow \qquad B = \frac{\mu_0 n i}{2}\left[\frac{a}{\sqrt{R^2 + l_1^2}} + \frac{b}{\sqrt{R^2 + l_2^2}}\right] = \frac{\mu_0 n i}{2}[\cos\theta_1 + \cos\theta_2]$$



$$B = \frac{\mu_0 n i}{2}[\cos\alpha + \cos\beta]$$

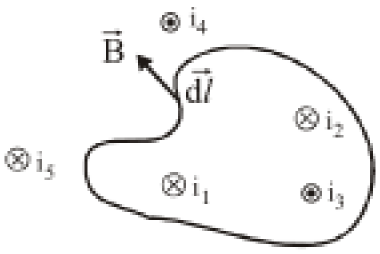
Ampere's Law :

Like Gauss's law in electrostatics, this law provides us a simple method to find magnetic fields in cases of symmetry

Ampere's law gives another method to calculate the magnetic field due to a given current distribution.

Statement: The circulation $\oint \vec{B} \cdot d\vec{l}$ of the resultant magnetic field (of a closed circuit or an infinite wire containing steady current) along a closed path (called amperian path) is equal to μ_0 times the total current crossing the area bounded by the closed curve provided the electric field inside the loop remains constant. Thus,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i.$$



In figure , the positive side is going into the plane of the diagram so that i_1 and i_2 are positive and i_3 is negative. Thus, the total current crossing the area is $i_1 + i_2 - i_3$. Any current outside the area is not included in writing the right - hand side of equation . The magnetic field $\vec{B}$ on the left -hand side is the resultant field due to all the currents existing anywhere.

Ampere's law may be derived from the Biot-Savart law and Bio-Savart law may be derived from the Ampere's law. Thus, the two are equivalent in scientific content. However, Ampere's law is useful under certain symmetrical conditions.

Justification of Ampere's law

Let us consider a long straight wire carrying a current I in upward direction. Now take a circular path of radius r symmetric to the wire.Let us now divide a circular path of radius *r* into a large number of small length vectors $\Delta\vec{s} = \Delta s\hat{\varphi}$, where $\hat{\phi}$ point along the tangential direction with magnitude Δs (Figure).

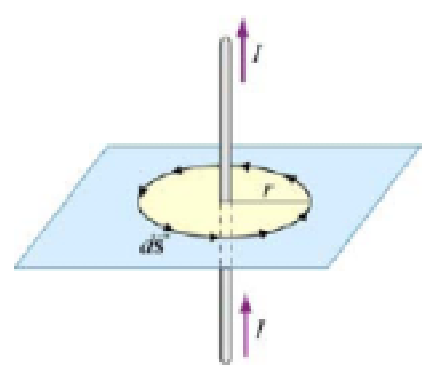


Figure : Amperian loop

In the limit $\Delta\vec{s} \to \vec{0}$, we obtain

$$\oint \vec{B}.d\vec{s} = B\oint ds = \left(\frac{\mu_0 I}{2\pi r}\right)(2\pi r) = \mu_0 I$$

The result above is obtained by choosing a closed path, or an "Amperian loop" that follows one particular magnetic field line. Let's consider a slightly more complicated Amperian loop, as that shown in Figure.

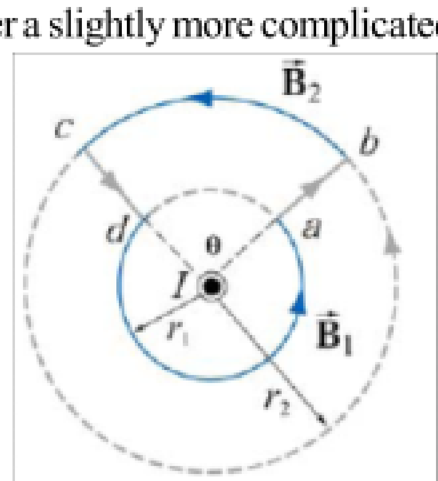


Figure : An Amperian loop involving two field lines

The line integral of the magnetic field around the contour *abcda* is

$$\oint_{abcda} \vec{B}.d\vec{s} = \int_{ab} \vec{B}.d\vec{s} + \int_{bc} \vec{B}.d\vec{s} + \int_{cd} \vec{B}.d\vec{s} + \int_{cd} \vec{B}.d\vec{s} = 0 + B_2(r_2\theta) + 0 + B_1[r_1(2\pi - \theta)]$$

where the length of arc *bc* is $r_2\theta$, and $r_1(2\pi - \theta)$ for arc *da*. The first and the third integrals vanish since the magnetic field is perpendicular to the paths of integration. With $B_1 = \mu_0 I/2\pi r_1$ and $B_2 = \mu_0 I/2\pi r_2$, the above expression becomes

$$\oint_{abcda} \vec{B}\,.\,d\vec{s} = \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1}[r_1(2\pi - \theta)] = \frac{\mu_0 I}{2\pi}\theta + \frac{\mu_0 I}{2\pi}(2\pi - \theta) = \mu_0 I$$

We see that the same result is obtained whether the closed path involves one or two magnetic field lines. As shown above Example, in polar coordinates (r, φ) with current flowing in the $+z$-axis, the magnetic

field is given by $\vec{B} = (\mu_0 I / 2\pi r)\hat{\varphi}$. An arbitrary length element in the polar coordinates can be written as

: $$d\vec{s} = dr\,\hat{r} + r d\phi\,\hat{\phi}$$

which implies

$$\oint_{\text{closed path}} \vec{B} \cdot d\vec{s} = \oint_{\text{closed path}} \left(\frac{\mu_0 I}{2\pi r}\right) r d\varphi = \frac{\mu_0 I}{2\pi} \oint_{\text{closed path}} d\varphi = \frac{\mu_0 I}{2\pi}(2\pi) = \mu_0 I$$

In other words, the line integral of $\oint \vec{B} \cdot d\vec{s}$ around any closed Amperian loop is proportional to I_{enc}, the current encircled by the loop.

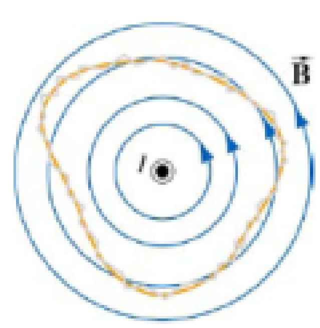


Figure : An Amperian loop of arbitrary shape.

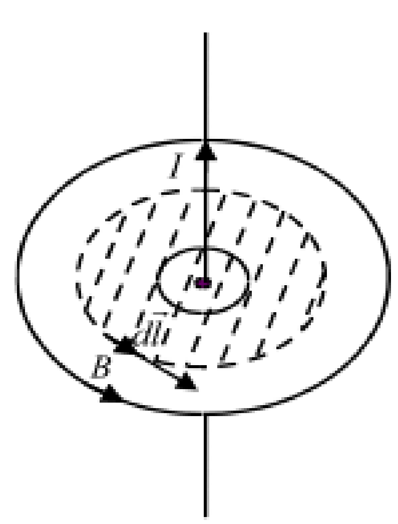
The generalization to any closed loop of arbitrary shape (see for example, Figure) that involves many magnetic field lines is known as Ampere's law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

Calculation of magnetic field by using Ampere's Law :

Calculation of magnetic field due to long straight wire.

Figure shows a long, straight current i. We have to calculate the magnetic field at a point P which is at a distance r from the wire. Figure shows the situation in the plane perpendicular to the wire and passing through P. The current is perpendicular to the plane of the diagram and is coming out of it.



Let us draw a circle passing through the point and with the axis as wire. We put an arrow to show the

positive sense of the circle. The radius of the circle is r. The magnetic field due to the long, straight current at any point on the circle is along the tangent as shown in the figure. Same is the direction of the length-element dl there. By symmetry, all points of the circle are equivalent and hence the magnitude of the magnetic field should be the same at all these points. The circulation of magnetic field along the circle is

$$\oint \vec{B} \cdot \vec{dl} = \oint Bdl = B\oint dl = B(2\pi r)$$

The current crossing the area bounded by the circle is

$$\sum I_{en} = +I$$

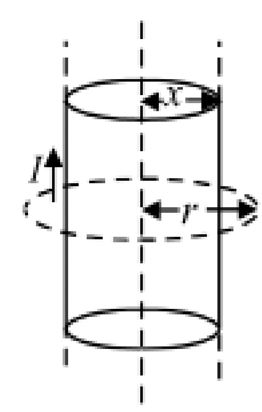
Thus, from Ampere's law,

$$B(2\pi r) = \mu_0 I$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

Magnetic field due to a current carrying thin long pipe.

Case I : $r > R$



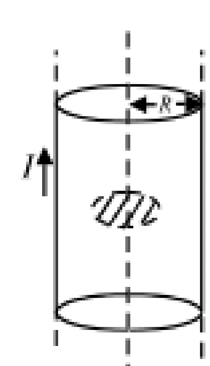
$$\oint \vec{B} \cdot \vec{dl} = \oint B \cdot dl = B\oint dl = B(2\pi r)$$

$$\sum I_{en} = +I$$

$$\Rightarrow B(2\pi r) = \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Case II : $r < R$

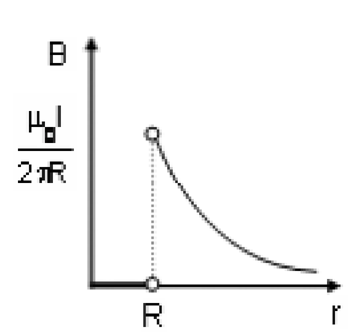


$$\oint \vec{B} \cdot \vec{dl} = \oint Bdl = B\oint dl = B(2\pi r)$$

$$\sum I_{en} = 0$$

$$\Rightarrow B(2\pi r) = \mu_0 (0)$$

$$\Rightarrow B = 0$$



Magnetic field due to current carrying rod having uniform current density.

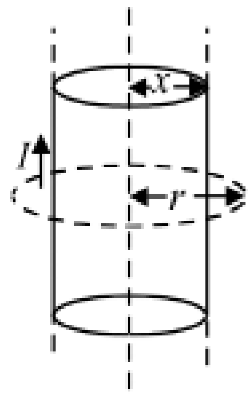
Case I : $r > R$

$$\oint \vec{B} \cdot \vec{dl} = \oint B \cdot dl = B\oint dl = B(2\pi r)$$

$$\sum I_{en} = +I$$

$$\Rightarrow B(2\pi r) = \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$



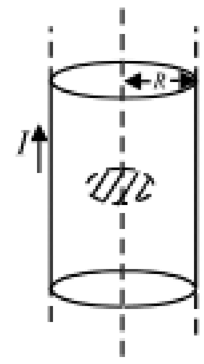
Case II : $r < R$

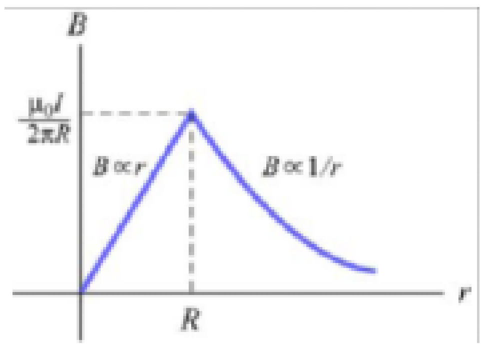
$$\oint \vec{B} \cdot \vec{dl} = \oint Bdl = B\oint dl = B(2\pi r)$$

$$\sum I_{en} = \frac{I}{R^2} r^2$$

$$\Rightarrow B(2\pi r) = \mu_0 \frac{I}{R^2} r^2$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi R^2} r$$





Magnetic Field Due to non uniform current density

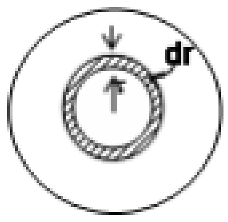
Suppose that the current density in a wire of radius a varies with r according to $J=Kr^2$, where K is a constant and r is the distance from the axis of the wire. We have to find the magnetic field at a point distance r from the axis when (a) r < a and (b) r > a

Choose a circular path centred on the axis of the conductor and apply Ampere's law

(a) To find the current passing through the area enclosed by the path integrate

$$dI = JdA = (Kr^2)(2\pi rdr)$$

$$\Rightarrow I = \int dI = K\int_0^r 2\pi r^3 dr = \frac{K\pi r^4}{2}$$



Since $\int \vec{B} \cdot d\vec{\ell} = \mu_0 I$

$$\Rightarrow \quad B 2\pi r = \mu_0 \cdot \frac{\pi K r^4}{2} \qquad \Rightarrow B = \frac{\mu_0 K r^3}{4}$$

(b) If $r > a$, then net current through the Amperian loop is

$$I = \int_0^a K r^2 2\pi r dr = \frac{\pi K a^4}{2}$$

$$\Rightarrow \quad B = \frac{\mu_0 K a^4}{4r}$$

Practice Exercise

Q.1 Inside a long straight uniform wire of round cross-section there is a long round cylindrical cavity whose axis is parallel to the axis of the wire and displaced from the latter by a distance $\vec{\ell}$. A direct current of density $\vec{j}$ flows along the wire. Find the magnetic induction inside the cavity. Consider, in particular, the case I = 0.

Answers

Q.1 $B = (1/2)\,\mu_0\,[\,\vec{j} \times \vec{\ell}\,]$, i.e. field inside the cavity is uniform.

Magnetic Field due to long Solenoid :

A solenoid is a long coil of wire tightly wound in the helical form. Figure shows the magnetic field lines of a solenoid carrying a steady current *I*. We see that if the turns are closely spaced, the resulting magnetic field inside the solenoid becomes fairly uniform, provided that the length of the solenoid is much greater than its diameter. For an "ideal" solenoid, which is infinitely long with turns tightly packed, the magnetic field inside the solenoid is uniform and parallel to the axis, and vanishes outside the solenoid.

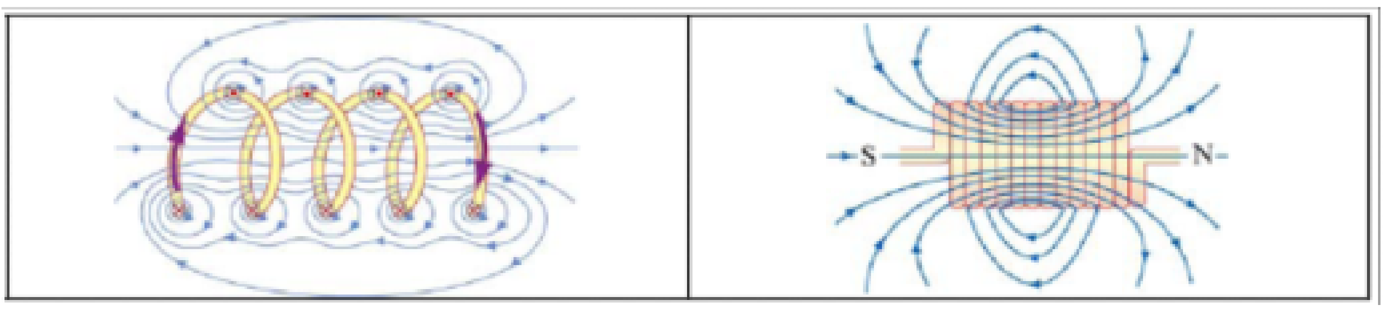


Figure : Magnetic field lines of a solenoid

We can use Ampere's law to calculate the magnetic field strength inside an ideal solenoid. The cross-sectional view of an ideal solenoid is shown in Figure. To compute $\vec{B}$, we consider a rectangular path of length ℓ and width w and traverse the path in a counterclockwise manner. The line integral of $\vec{B}$ along this loop is

$$\oint \vec{B}\,.\,d\vec{s} = \oint_1 \vec{B}\,.\,d\vec{s} + \oint_2 \vec{B}\,.\,d\vec{s} + \oint_3 \vec{B}\,.\,d\vec{s} + \oint_4 \vec{B}\,.\,d\vec{s}$$
$$= 0 \quad + \quad 0 \quad + \quad B\ell \quad + \quad 0$$

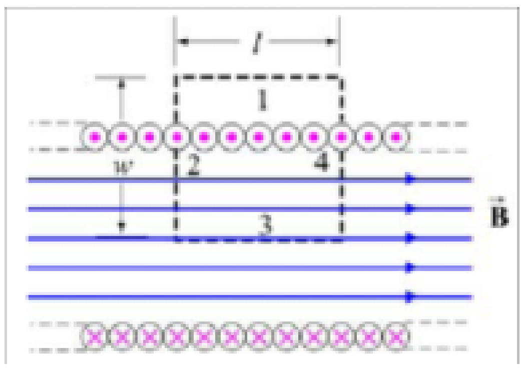


Figure : Amperian loop for calculating the magnetic field of an ideal solenoid.

In the above, the contributions along sides 2 and 4 are zero because $\vec{B}$ is perpendicular to $d\vec{s}$. In addition, $\vec{B} = \vec{0}$ along side 1 because the magnetic field is non-zero only inside the solenoid. On the other hand, the total current enclosed by the Amperian loop is $I_{enc} = nlI$, where *n* is the total number of turns per unit length. Applying Ampere's law yields

$$\oint \vec{B}.d\vec{s} = B\ell = \mu_0 n\ell I \qquad\qquad B = \mu_0 nI$$

Magnetic Field due to Toroid

Consider a toroid which consists of *N* turns, as shown in Figure. Find the magnetic field everywhere.

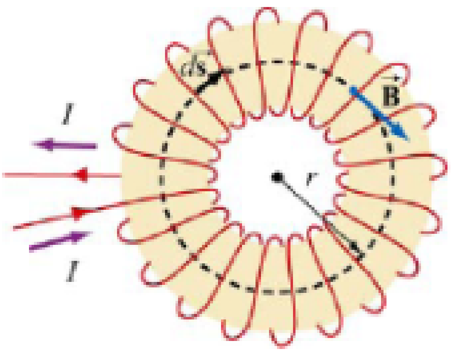


Figure : A toroid with *N* turns

One can think of a toroid as a solenoid wrapped around with its ends connected. Thus, the magnetic field is completely confined inside the toroid and the field points in the azimuthal direction (clockwise due to the way the current flows, as shown in Figure 9.4.5.)

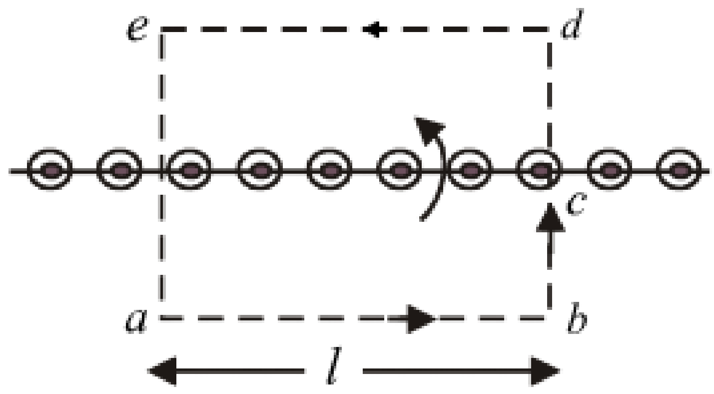
Applying Ampere's law, we obtain

$$\oint \vec{B}.d\vec{s} = \oint Bds = B\oint ds = B(2\pi r) = \mu_0 NI$$

or $$B = \frac{\mu_0 NI}{2\pi r}$$

where *r* is the distance measured from the center of the toroid. Unlike the magnetic field of a solenoid, the magnetic field inside the toroid is non-uniform and decreases as $1/r$.

Magnetic Field thin sheet of infinite dimension carrying a current of uniform linear current density 'i'.



$$\oint \vec{B}\cdot\vec{dl} = \int_a^b \vec{B}\cdot\vec{dl} + \int_b^c B\cdot\vec{dl} + \int_c^d \vec{B}\cdot\vec{dl} + \int_d^e \vec{B}\cdot\vec{dl} + \int_e^f \vec{B}\cdot\vec{dl} + \int_f^g \vec{B}\cdot\vec{dl}$$

$$= \int_a^b B\cdot\cos 0\cdot dl + 0 + 0 + \int_d^e Bdl + 0 + 0 = B\int_a^b dl + B\int_d^e dl = Bl + Bl = 2Bl$$

$$\sum I_{en} = il$$

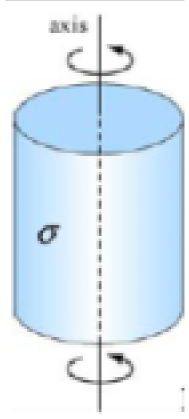
$$\oint \vec{B}\cdot\vec{dl} = \mu_0 \sum I_{en}$$

$$\Rightarrow \quad 2Bl = \mu_0 il$$

$$\Rightarrow \quad B = \frac{\mu_0 i}{2}$$

Practice Exercise

Q.1 We have a long cylindrical shell of non-conducting material which carries a surface charge fixed in place (glued down) of σ C/m^2, as shown in Figure. The cylinder is suspended in a manner such that it is free to revolve about its axis, without friction. Initially it is at rest. We come along and spin it up until the speed of the surface of the cylinder is v_0.

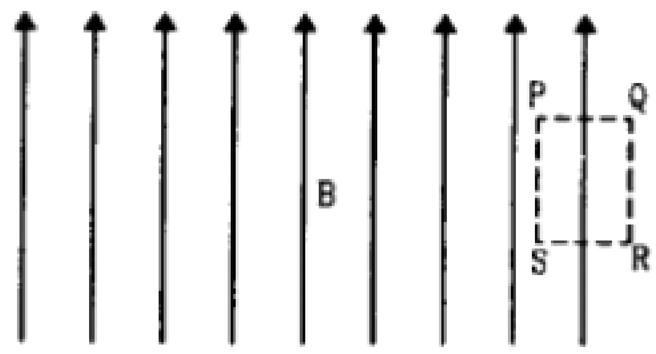


(a) What is the surface current K on the walls of the cylinder, in A/m?

(b) What is magnetic field inside the cylinder?

(c) What is the magnetic field outside of the cylinder? Assume that the cylinder is infinitely long.

Q.2 Sometimes we show an idealised magnetic field which is uniform in a given region and falls to zero abruptly. One such field is represented in figure . Using Ampere's law over the path PQRS, show that such a field is not possible.



Answers

Q.1 (a) $K = \sigma v_0$
(b) $B = \mu_0 K = \mu_0 \sigma v_0$, oriented along axis right-handed with respect to spin (c) 0

Magnetic Field of a Moving Point Charge :

Suppose we have an infinitesimal current element in the form of a cylinder of cross-sectional area A and length ds consisting of n charge carriers per unit volume, all moving at a common velocity $\vec{v}$ along the axis of the cylinder. Let I be the current in the element, which we define as the amount of charge passing through any cross-section of the cylinder per unit time. From Chapter 6, we see that the current I can be written as

$$nAq|\vec{v}| = I$$

The total number of charge carriers in the current element is simply $dN = nAds$, so that the magnetic field $d\vec{B}$ due to the dN charge carriers is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{(nAq|\vec{v}|)d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{(nAds)q\vec{v}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{(dN)q\vec{v}\times\hat{r}}{r^2}$$

where r is the distance between the charge and the field point P at which the field is being measured, the unit vector $\hat{r} = \vec{r}/r$ points from the source of the field (the charge) *to* P. The differential length vector $d\vec{s}$ is defined to be parallel to $\vec{v}$. In case of a single charge, $dN = 1$, the above equation becomes

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

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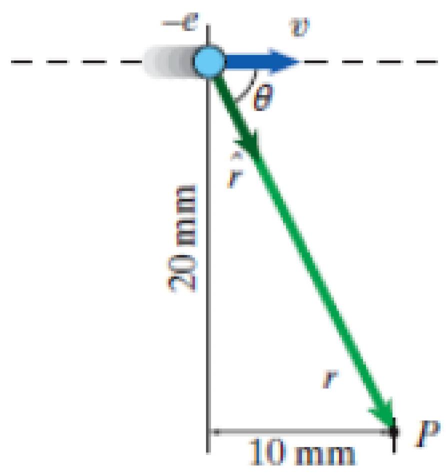
Practice Exercise

Q.1 An electron carrying a charge $e = -1.6 \times 10^{-19}$ C moves in a straight line at a speed $v = 3 \times 10^7$ m/s. What is the magnitude and direction of the magnetic field caused by the electron at a point P, 10 mm ahead of the electron and 20 mm away from its line of motion, as illustrated in figure?



Answers

Q.1 8.8×10^{-14} T

Force on a moving charge in magnetic field :

Consider a particle of charge q and moving at a velocity $\vec{v}$. Experimentally we have the following observations:

(1) The magnitude of the magnetic force $\vec{F}_B$ exerted on the charged particle is proportional to both v and q.

(2) The magnitude and direction of $\vec{F}_B$ depends on $\vec{v}$ and $\vec{B}$.

(3) The magnetic force $\vec{F}_B$ vanishes when $\vec{v}$ is parallel to $\vec{B}$. However, when $\vec{v}$ makes an angle θ with h $\vec{B}$, the direction of $\vec{F}_B$ is perpendicular to the plane formed by and $\vec{v}$ and $\vec{B}$ and the magnitude of $\vec{F}_B$ is proportional to sin θ.

(4) When the sign of the charge of the particle is switched from positive to negative (or vice versa), the direction of the magnetic force also reverses.

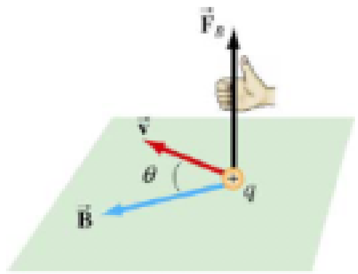


Figure : The direction of the magnetic force

The above observations can be summarized with the following equation:

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

The above expression can be taken as the working definition of the magnetic field at a point in space.

The magnitude of $\vec{F}_B$ is given by

$$F_B = |q|\, vB \sin\theta$$

The SI unit of magnetic field is the tesla (T) :

$$1 \text{ tesla} = 1\text{T} = 1 \frac{\text{Newton}}{(\text{Coulomb})(\text{meter}/\text{second})} = 1 \frac{\text{N}}{\text{C.m}/\text{s}} = 1 \frac{\text{N}}{\text{A.m}}$$

Another commonly used non-SI unit for $\vec{B}$ is the *gauss* (G), where $1\text{T} = 10^4\text{G}$

Note that $\vec{F}_B$ is always perpendicular to $\vec{v}$ and $\vec{B}$, and cannot change the particle's speed v (and thus the kinetic energy). In other words, magnetic force cannot speed up or slow down a charged particle. Consequently, **Magnetic field does no work on charged particle**:

$$dW = \vec{F}_B \cdot d\vec{s} = q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = q(\vec{v} \times \vec{v}) \cdot \vec{B}dt = 0$$

The direction of $\vec{v}$ however, can be altered by the magnetic force

Illustration :

Two very long, straight, parallel wires carry steady currents I and -I respectively. The distance between the wires is d. At a certain instant of time, a point charge q is at a point equidistant from the two wires, in the plane of the wires.Find magnitude of the force due to the magnetic field acting on the charge .

Sol. *Since the currents are flowing in the opposite directions, the magnetic field at a point equidistant from the two wires will be zero. Hence, the force acting on the charge at this instant will be zero.*

Illustration :

An electron with mass m, velocity v and charge e describes half a revolution in a circle of radius r in a magnetic field B,find the energy acquireed by electron.

Sol. *As energy can neither be created nor destroyed, therefore, its energy will remain constant and will acquire no extra energy.*

Practice Exercise

Q.1 A circular loop of radius 20 cm carries a current of 10 A. An electron crosses the plane of the loop with a speed of 2.0 x 106 m/s. The direction of motion makes an angle of 30° with the axis of the circle and passes through its centre. Find the magnitude of the magnetic force on the electron at the instant it crosses the plane.

Q.2 Two protons move parallel to each other with an equal velocity v. Find the ratio of forces of magnetic and electrical interaction of the protons.

Answers

Q.1 $16\pi \times 10^{-19}\ N$ Q.2 $(v/c)^2$

Motion of charged particle in uniform magnetic field

There are three possible paths in which a charged particle may move in presence of uniform magnetic field which is uniform in space.

(a) Straight line path

(b) Circular path

(c) Helical path

We shall see them one by one.

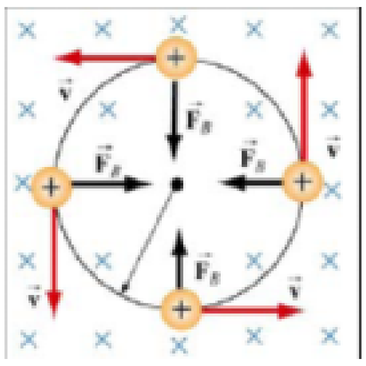
Straight line path

When the charged particle projected in the direction of or opposite to uniform magnetic field, magnetic field exerts no force hence it will travel along straight line with fixed speed.

Circular path

If a particle of mass *m* moves in a circle of radius *r* at a constant speed *v*, what acts on the particle is a radial force of magnitude $F = mv^2/r$ that always points toward the center and is perpendicular to the velocity of the particle.

In previous section, we have also shown that the magnetic force $\vec{F}_B$ always points in the direction perpendicular to the velocity $\vec{v}$ of the charged particle and the magnetic field $\vec{B}$. Since $\vec{F}_B$ can do not work, it can only change the direction of $\vec{v}$ but not its magnitude. What would happen if a charged particle moves through a uniform magnetic field $\vec{B}$ with its initial velocity $\vec{v}$ at a right angle to $\vec{B}$? For simplicity, let the charge be $+q$ and the direction of $\vec{B}$ be into the page. It turns out that $\vec{F}_B$ will play the role of a centripetal force and the charged particle will move in a circular path in a counterclockwise direction, as shown in Figure.



$$qvB = \frac{mv^2}{r}$$

the radius of the circle is found to be $r = \frac{mv}{qB}$

The period *T* (time required for one complete revolution) is $T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB}$

Similarly, the angular speed of the particle can be obtained as $\omega = 2\pi f = \frac{v}{r} = \frac{qB}{m}$

Illustration :

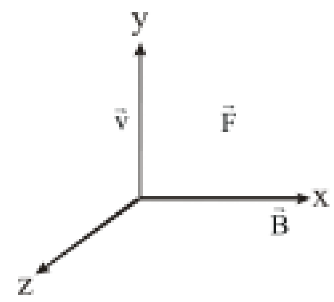
A uniform magnetic field of 30 mT exists in the + X direction. A particle of charge +e and mass 1.67×10^{-27} *kg is projected into the field along the + Y direction with a speed of* 4.8×10^{6} *m/s.*

(a) *Find the force on the charged particle in magnitude and direction*

(b) *Find the force if the particle were negatively charged.*

(c) *Describe the nature of path followed by the particle in the both the case.*

Sol. (a) *Force acting on a charge particle moving in the magnetic field*



$$\vec{F} = q(\vec{v}\times\vec{B})$$

Magnetic field $\vec{B} = 30(\text{mT})\hat{j}$

Velocity of the charge particle $\vec{v} = 4.8\times10^{6}$ *(m/s)* $\hat{j}$

$$\vec{F} = 1.6\times10^{-19}[(4.8\times10^{6}\hat{j})\times(30\times10^{-3})(\hat{i})]$$

$$\vec{F} = 230.4\times10^{-16}(-k)\text{N}\cdot$$

(b) If the particle were negatively charged, the magnitude of the force will be the same but the direction will be along (+z) direction.

(c) *Ad* $v\perp B$, *the path described is a circle*

$$R = \frac{mv}{qB} = (1.67\times10^{-27}).(4.8\times10^{6})/(1.6\times10^{-19}).(30\times10^{-3})$$

$$= 1.67\ m.$$

Illustration :

A positively charged particle of charge q and mass m first accelerated by a voltage v then injected into uniform magnetic field. Find radius of the circle traced by it.

Sol. *Kinetic energy of the particle is*

$$K = qV$$

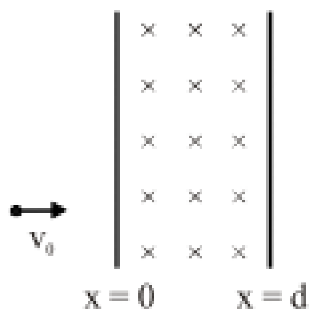
linear momentum of the particle will be

$$p = mv = \sqrt{2mk} = \sqrt{2mqv}$$

$$\therefore \quad r = \frac{p}{qB} = \frac{\sqrt{2mqv}}{qB}$$

Illustration :

A positively particle having chrage q and mass m moving with velocity $v_0\ \hat{i}$ *enters a region in which uniform magnetice field* $\vec{B} = -B_0\hat{k}$ *exist. The magnetic field region extends upto x = 0 to x = d (figrue). Find the time spent by the particle in magentic field if*

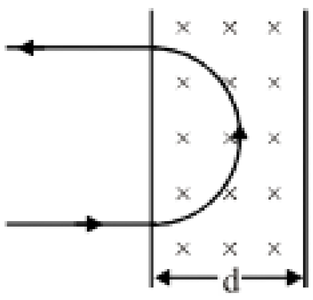


(i) $d = \dfrac{1.5mv}{qB}$ *(ii)* $\dfrac{mv}{qB}$

In second case also find the side ways deflection.

Sol. *Since the velocity of the particle is perpendicular to the magnetic field hence motion inside magenetic field will be circle or part of circle*

(i) *Since* $d = \dfrac{1.5mv}{qB} \Rightarrow d > r$



$\Rightarrow$ *particle has surfficient space to turn back.*

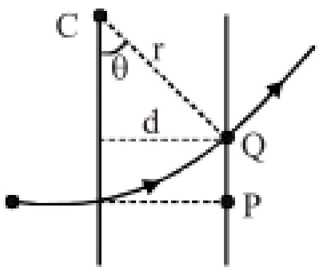
$\Rightarrow$ *particle will complete half circle (figure)*

$\therefore$ *time spent* $= \dfrac{T}{2} = \dfrac{\pi m}{qB}$

(ii) *since* $d = \dfrac{mv}{2qB} \qquad \Rightarrow d < r$

This means particle will not get surficient space between the boundries to turn back. Hence particle will come out of the boundry x = d (figure)

From figure



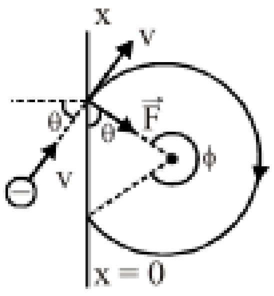
$$\sin\theta = \frac{d}{r} = \frac{\frac{mv}{2qB}}{\frac{mv}{qB}} = \frac{1}{2}$$

$$\Rightarrow \qquad \theta = \frac{\pi}{6}$$

$$\therefore \qquad \text{distance travelled} = r\theta = \left(\frac{mv}{qB}\right)\left(\frac{\pi}{6}\right) = \frac{\pi mv}{6qB}$$

$$\therefore \qquad \text{time spent} = \frac{\text{distance}}{\text{speed}} = \frac{\frac{\pi mv}{6qB}}{v} = \frac{\pi}{6qB}$$

side ways deflection $= PQ = r\,(1 - \cos\theta) = \dfrac{mv}{qB}\left(1 - \dfrac{\sqrt{3}}{2}\right)$



From geometry

$$\phi = \pi + 2\theta$$

$$\therefore \qquad \text{distance travelled} = r\,\phi = \frac{mv}{qB}(\pi + 2\theta)$$

$$\therefore \qquad \text{time} = \frac{\text{distance travelled}}{\text{speed}} = \frac{m}{qB}(\pi + 2\theta)$$

Illustration :

H^+, He^+ and O^{++} all having the same kinetic energy pass through a region in which there is a uniform magnetic field perpendicular to their velocity. The masses of H^+, He^+ and O^{++} are 1 amu, 4 amu and 16 amu respectively. Comment on their amount of deflection

Sol. $$Bqv = \frac{mv^2}{r}$$

$$\Rightarrow \quad Bqr = mv = \text{momentum} = \sqrt{2mE}$$

Where E = Kinetic Energy

$$\therefore \quad r = \frac{\sqrt{2mE}}{Bq}$$

if r_1 and r_2 and r_3 are the radius of circular track of H^+, He^{++} and O^{++}

$$\therefore \quad r_1 : r_2 : r_3 = \frac{\sqrt{2mE}}{Bq} : \frac{\sqrt{2(4m)E}}{Bq} : \frac{\sqrt{2(16m)E}}{B(2q)}$$

1 : 2 : 2

Hence O^{2+} will be deflected most whereas He^+ and O^{2+} will be deflected equally

Illustration :

Two particles X and Y having equal charges, after being accelerated through the same potential difference, enter a region of uniform magnetic field and desribe circular paths of radii R_1 and R_2, respectivley. Find the ratio of the mass of X to that of Y.

Sol. *Let the masses be m_1 and m_2 respectively of X and Y. If E is energy gained by charged particle in electric field.*

$$Bqv = \frac{mv^2}{r} \quad \Rightarrow \quad Bqr = \sqrt{2mE}$$

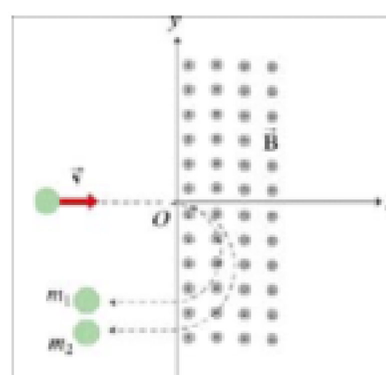
$$R_1 = \frac{\sqrt{2m_1E}}{Bq}; \; R_2 = \frac{\sqrt{2m_2E}}{Bq}$$

$$\frac{R_1}{R_2} = \sqrt{\frac{m_1}{m_2}} \quad \Rightarrow \quad \frac{m_1}{m_2} = \left(\frac{R_1}{R_2}\right)^2$$

Practice Exercise

Q.1 Particle A with charge q and mass m_A and particle B with charge $2q$ and mass m_B, are accelerated from rest by a potential difference ΔV, and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle A and B are R and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particle. What is their mass ratio?

Q.2 Suppose the entire x-y plane to the right of the origin O is filled with a uniform magnetic field $\vec{B}$ pointing out of the page, as shown in Figure.

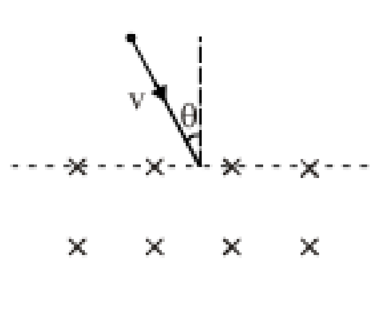


Two charged particles travel along the x axis in the positive x direction, each with speed v, and enter the magnetic field at the origin O. The two particles have the same charge q, but have different masses, m_1 and m_2. When in the magnetic field, their trajectories both curve in the same direction, but describe semi-circles with different radii. The radius of the semi-circle traced out by particle 2 is exactly *twice* as big as the radius of the semi-circle traced out by particle 1.

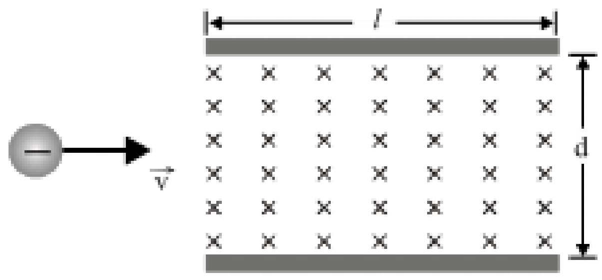
(a) Is the charge q of these particles such that $q > 0$, or is $q < 0$?

(b) What is the ratio m_2 / m_1 ?

Q.3 A particle of mass m and positive charge q, moving with a uniform velocity v, enters a magnetic field B as shown in figure (a) Find the radius of the circular arc it describes in the magnetic field. (b) Find the angle subtended by the arc at the centre. (c) How long does the particle stay inside the magnetic field? (d) Solve the three parts of the above problem if the charge q on the particle is negative.



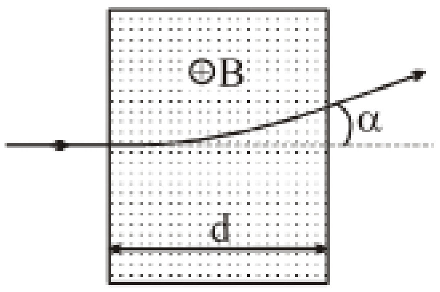
Q.4 A particle of charge is moving with a velocity $\vec{v}$. It then enters midway between two long plates ($\ell \to \infty$) where there exists a uniform magnetic field pointing into the page, as shown in Figure. Assume the space to be electric field and gravity free. Take $d = \frac{mv}{qB}$



(a) Is the trajectory of the particle deflected upward or downward?

(b) Compute the distance between the left end of the plate and where the particle strikes.

Q.5 A proton accelerated by a potential difference V = 500 kV flies through a uniform transverse magnetic field with induction B = 0.51 T. The field occupies a region of space d = 10 cm in thickness (Fig.). Find the angle a through which the proton deviates from the initial direction of its motion.



Answers

Q.1 $\frac{m_A}{m_B} = \frac{1}{8}$ Q.2 (a) – ve (b) 2:1

Q.3 (a) $\frac{mv}{qB}$ (b) $\pi - 2\theta$ (c) $\frac{m}{qB}(\pi - 2\theta)$ (d) $\frac{mv}{qB}, \pi + 2\theta, \frac{m}{qB}(\pi + 2\theta)$

Q.4 (a) downward (b) $d\frac{\sqrt{3}}{2}$ Q.5 $a = \sin^{-1}\left(dB\sqrt{\frac{q}{2mV}}\right) = 30°$

Helical Path:

In this situation velocity of the particle can be resolved into two components.

(1) $v_{||} \rightarrow$ projection of velocity parallel to the magnetic fd.

(2) $v_{\perp} \rightarrow$ projection of velocity perpendicular to the magnetic fd.

Due to $v_{\perp}$ motion of the charged particle will be uniform circular in the plane perpendicular to the fd. whereas

$v_{||}$ remain unaffected as it is perpendicular to magnetic force.

As a whole its motion is helical with constant pitch.

radius of the helix will be

$$r = \frac{mv_{\perp}}{qB}$$

pitch of the helix will be

$$p = \frac{mv_{||}}{qB}$$

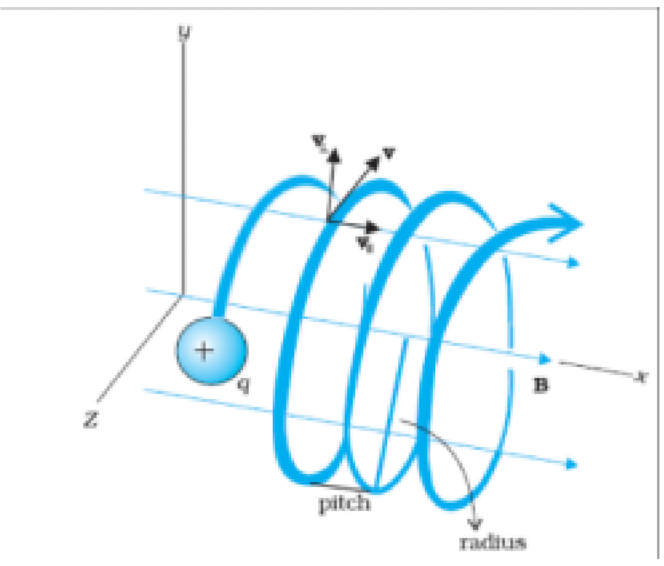


Illustration :

A charged particle P leaves the origin with speed $v = v_0$, at some inclination with the x-axis. There is uniform magnetic field B along the x-axis. P strikes a fixed target T on the x-axis for a minimum value of $B = B_0$. Find the condition so that P will also strike if you can change magnetic field and speed

Sol. *Let d = distance of the tanget T from the point of projection. P will strike T if d an integral multiple of the pitch.*

Pitch

$$\left(2\pi\frac{m}{qB_0}\right)v_0\cos\theta = N\left(2\pi\frac{m}{qB}\right)v\cos\theta$$

Here N is a natural number.

Practice Exercise

Q.1 An electron accelerated by a potential difference V = 1.0 kV moves in a uniform magnetic field at an angle $\alpha = 30°$ to the vector B whose modulus is B = 29 mT. Find the pitch of the helical trajectory of the electron.

Q.2 Can a charged particle be speed up through a uniform magnetic field ?

Q.3 If no work can be done on a charged particle by the magnetic field, how can the motion of the particle be influenced by the presence of a field?

Answers

Q.1 $\Delta l = 2\pi \sqrt{2mV/eB^2} \cos\alpha = 2.0$ cm Q.2 No

Q.3 By changing its direction of motion

Lorentz Force:

In the presence of both electric field $\vec{E}$ and magnetic field $\vec{B}$, the total force on a charged particle is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

This is known as the Lorentz force.

Velocity Selector:

By combining the two fields, particles which move with a certain velocity can be selected. This was the principle used by J. J. Thomson to measure the charge-to-mass ratio of the electrons. In Figure the schematic diagram of Thomson's apparatus is depicted.

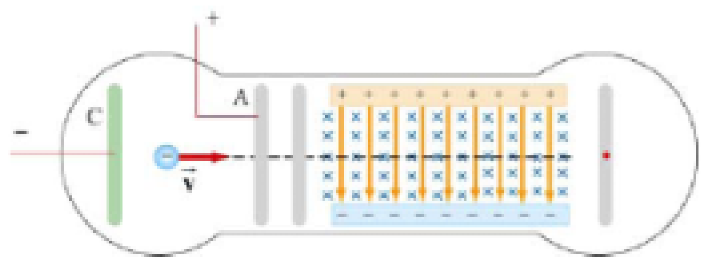


Figure : Thomson's apparatus

The electrons with charge $q = -e$ and mass m are emitted from the cathode C and then accelerated toward slit A. Let the potential difference between A and C be $V_A - V_C = \Delta V$. The change in potential energy is equal to the external work done in accelerating the electrons: $\Delta U = W_{ext} = q\Delta V = -e\Delta V$. By energy conservation, the kinetic energy gained is $\Delta K = -\Delta U = mv^2/2$. Thus, the speed of the electrons is given by

$$v = \sqrt{\frac{2e\Delta V}{m}}$$

If the electrons further pass through a region where there exists a downward uniform electric field, the electrons, being negatively charged, will be deflected upward. However, if in addition to the electric field, a magnetic field directed into the page is also applied, then the electrons will experience an additional downward magnetic force $-e\vec{v} \times \vec{B}$. When the two forces exactly cancel, the electrons will move in a

straight path. From Eq., we see that when the condition for the cancellation of the two forces is given by

$eE = evB$. which implies $\quad \mathbf{v} = \dfrac{E}{B}$

In other words, only those particles with speed $v = E/B$ will be able to move in a straight line. Combining

the two equations, we obtain $\quad \dfrac{e}{m} = \dfrac{E^2}{2(\Delta V)B^2}$

By measuring E, ΔV and B, the charge-to-mass ratio can be readily determined. The most precise measurement to date is $e/m = 1.758820174(71) \times 10^{11}$ C/kg.

Mass Spectrometer:

Various methods can be used to measure the mass of an atom. One possibility is through the use of a mass spectrometer. The basic feature of a *Bainbridge* mass spectrometer is illustrated in Figure. A particle carrying a charge $+q$ is first sent through a velocity selector.

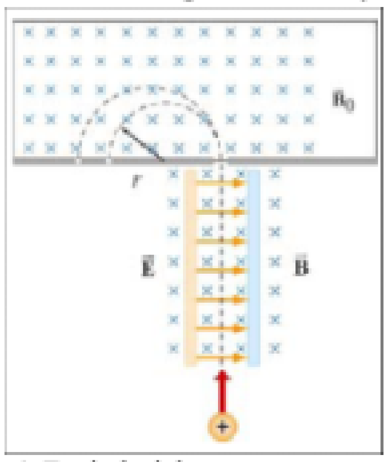


Figure : A Bainbridge mass spectrometer

The applied electric and magnetic fields satisfy the relation $E = vB$ so that the trajectory of the particle is a straight line. Upon entering a region where a second magnetic field $\vec{B}_0$ pointing into the page has been applied, the particle will move in a circular path with radius r and eventually strike the photographic

plate. Using Eq., we have $\quad r = \dfrac{mv}{qB_0}$

Since $v = E/B$, the mass of the particle can be written as $\quad m = \dfrac{qB_0 r}{v} = \dfrac{qB_0 Br}{E}$

Hall's Effect:

In 1879, Edwin H. Hall, then a 24-year-old graduate student at the Johns Hopkins University, showed that they can drift electrons in copper wire in presence of magnetic field. This Hall effect allows us to find

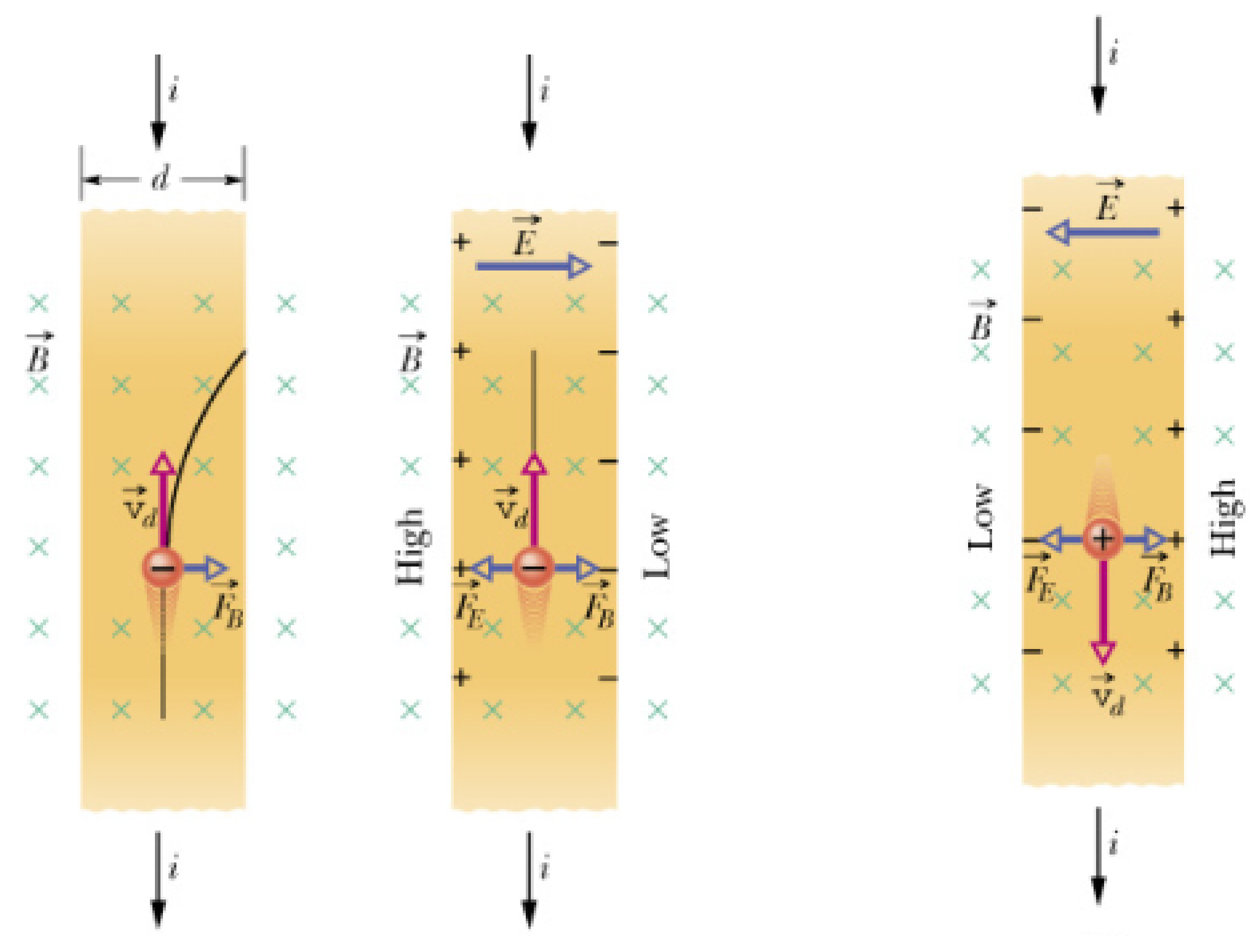
" if charge carriers in a conductor are positively or negatively charged.

" the number of charge carriers per unit volume of the conductor.

Consider a strip of current carrying wire kept in external magnetic field. Let the wire has width d, Cross-sectional area A, and charge carriers per unit volume as n.

Figure (a) shows a copper strip of width d, carrying a current i whose conventional direction is from the top of the figure to the bottom. The charge carriers are electrons and, as we know, they drift (with drift speed vd) in the opposite direction, from bottom to top. At the instant shown in figure, an external

magnetic field B, pointing into the plane of the figure, has just been turned on. We see that a magnetic force will act on each drifting electron, pushing it towards the right edge of the strip.



(a) (b) (c)

As time goes on, electrons move to the right, mostly piling up on the right edge of the strip, leaving uncompensated positive charges in fixed positions at the left edge. The separation of positive charges on the left edge and negative charges on the right edge produces an electric field E within the strip, pointing from left to right in Fig. b. This field exerts an electric force FE on each electron, tending to push it to the left. Thus, this electric force on the electrons, which opposes the magnetic force on them, begins to build up.

Equilibrium quickly develops in which the electric force on each electron has increased enough to match the magnetic force. When this happens, as Fig. b shows, the force due to B and the force due to E are in balance. The drifting electrons then move along the strip toward the top of the page at velocity vd with no further collection of electrons on the right edge of the strip and thus no further increase in the electric field E.

$$eE = ev_dB \qquad ---(1)$$

$$v_d = \frac{J}{ne} = \frac{i}{neA} \qquad ---(2)$$

A Hall potential difference V is associated with the electric field across strip width d.

$$V = E\,d \qquad ...(3)$$

From (1), (2) and (3)

$$\frac{E}{B} = \frac{V}{Bd} = \frac{i}{neA}$$

$$\therefore n = \frac{idB}{eAV} \qquad \& \ v_d = \frac{i}{neV} = \frac{V}{Bd}$$

By connecting a voltmeter across the width, we can measure the potential difference between the two edges of the strip. Moreover, the voltmeter can tell us which edge is at higher potential. For the situation of Fig. b, we would find that the left edge is at higher potential, which is consistent with our assumption that the charge carriers are negatively charged.

It is also possible to use the Hall effect to measure directly the drift speed vd of the charge carriers, which you may recall is of the order of centimeters per hour. In this clever experiment, the metal strip is moved mechanically through the magnetic field in a direction opposite that of the drift velocity of the charge carriers. The speed of the moving strip is then adjusted until the Hall potential difference vanishes.
At this condition, with no Hall effect, the velocity of the charge carriers with respect to the laboratory frame must be zero, so the velocity of the strip must be equal in magnitude but opposite the direction of the velocity of the negative charge carriers.
For a moment, let us make the opposite assumption, that the charge carriers in current i are positively charged (Fig. c).
Convince yourself that as these charge carriers move from top to bottom in the strip, they are pushed to the right edge by F_B and thus that the right edge is at higher potential. Because that last statement is contradicted by our voltmeter reading, the charge carriers must be negatively charged.

Illustration :

A non-relativistic proton beam passes without deviation through the region of space where there are uniform transverse mutually perpendicular electric and magnetic fields with E = 120 kV/m and B = 50 mT. Then the beam strikes a grounded target. Find the force with which the beam acts on the target if the beam current is equal to I = 0.80 mA.

Sol: $F = \frac{dp}{dt} = v\frac{dm}{dt} = v\frac{dm}{dq}\frac{dq}{dt} = \frac{E}{B}\frac{m}{q}I = 20\ \mu N$

Illustration :

A particle of mass m and charge q is released from the origin in a region occupied by electric field E and magnetic field B,

$$B = -B_0\hat{j};\ E = E_0\hat{i}$$

Find the speed of the particle.

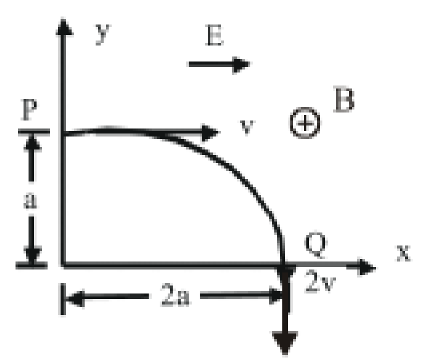
Sol. *Since the magnetic field does not perpform any work, therfore, whatever has been gain in kinetic energy it is only becuase of the work done by electric field. Applying work-energy theorem,*

$$W_E = \Delta K$$

$$qE_0 = \frac{1}{2}mv^2 - o \quad or \quad v = \sqrt{\frac{2qE_0}{m}}$$

Illustration :

A particle of charge +q and mass m moving under the influence of a uniform electric field $E\hat{i}$ *and a uniform magnetic field* $B\hat{k}$ *follows a trajectory from P to Q as shown in figure. The velocities at P and Q are* $v\hat{i}$ *and* $-2v\hat{i}$. *Find (a) E (b)rate of work done by the electric field at P (C) rate of work done by each the fields at Q*



Sol. *Increase in Kinetic energy of particle*

$$= \frac{1}{2}m(2v)^2 - \frac{1}{2}mv^2 = \frac{3}{2}mv^2$$

Work done by the uniform electric field, E, in going from P to Q = (qE) × 2a = 2qEa

Hence, $2qEa = \frac{3}{2}mv^2$

or $\quad E = \frac{3mv^2}{4qa}$

Rate of work done by the electric field at

P_{at}, $P = F.\,v = qE\,.\,v$

$$= qE\hat{i}.v\hat{i} = qEv$$

$$= q.\frac{3mv^2}{4qa}.v = \frac{3}{4}\frac{mv^2}{a}$$

Q is

$$P_{at}, Q = qE\hat{i}.(-2v\hat{j}) = 0$$

At Q, rate of work done by both the fields is zero.

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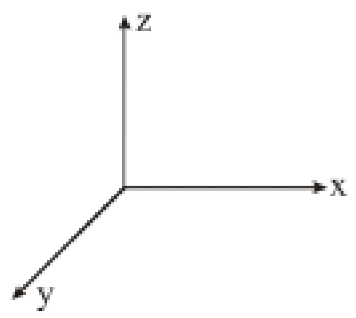
Illustration :

A particle of mass $1\times10^{-26}\,\text{kg}$ *and charge* $1.6\times10^{-19}\,\text{C}$ *traveling with a velocity* $1.28\times10^{6}\,\text{ms}^{-1}$ *in the +x direction enters a region in which a uniform electric field E and uniform magnetic field of induction B are present such that* $E_x = E_y = 0,\ E_z = -102.4\,\text{kVm}^{-1}$ *and* $B_x = B_z = 0$. $B_y = 8\times10^{-2}\,\text{Wbm}^{-2}$. *The particle enters this region at the origin at time t =0. Determine the location (x, y and z cooredinantes) of the particle at* $t = 5\times10^{-6}$ *s. If the electric field is switched off at this instant (with the magnetic field still present), what will be the position of the particle at* $t = 7.45\times10^{-6}\,\text{s}$ *?*

Sol. *Let* $\hat{i}, \hat{j}$ *and* $\hat{k}$ *be unit vector along the positive directions of x, y and z axes. Q = charge on the particle* $= 1.6\times10^{-19}\,\text{C}$ $\vec{v} =$ *velocity of the charged particle* $= (1.28\times10^{6})\text{ms}^{-1}\hat{i}$

$\vec{E} =$ *electric field intensity;*

$$= (-102.4\times10^{3}\,\text{Vm}^{-1})\hat{k}$$



$\vec{B}$ = *magnetic induction of the magnetic field*

$$= (8\times10^{-2}\,\text{Wbm}^{-2})\hat{j}$$

$\therefore$ $\vec{F}_e$ = *electric force on the charge*

$$= q\vec{E} = [1.6\times10^{-19}(-102.4\times10^{3})\text{N}]\hat{k}$$

$$= 163.84\times10^{-16}\,\text{N}(-\hat{k})$$

$\vec{F}_m$ = *magnetic force on the charge* $= q\vec{v}\times\vec{B}$

$$= [1.6\times10^{-19}(1.28\times10^{6})(8\times10^{-2})\text{N}](\hat{i}\times\hat{j})$$

$$= (163.84\times10^{-16}\,\text{N})(\hat{k})$$

The two forces $\vec{F}_e$ *and* $\vec{F}_m$ *are along z-axis and equal, opposite and collinear.*

The net force on the charge is zero and hence the particle does not get deflected and continues to travel along x-axis.

(a)At time $t = 5\times10^{-6}\,\text{s}$

$$x = (5\times10^{-6})(1.28\times10^{6}) = 6.4\text{m}$$

$\therefore$ *Coordinates of the particle = (6.4 m, 0, 0)*

(b) When the electric field is switched off, the particle is in the uniform magnetic field perpendicular to its velocity only and has a uniform circular motion in the x-z plane (i.e. the plane of velocity and magnetic force), anticlockwise as seen along +y axis.

Now, $\frac{mv^2}{r} = qvB$ *where r is the radius of the circle.*

$$\therefore \qquad r = \frac{mv}{qB} = \frac{(1\times10^{-26})(1.28\times10^{6})}{(1.6\times10^{-19})(8\times10^{-2})} = 1$$

The length of the arc traced by the particle in $[(7.45-5)\times10^{-6}\,s]$

$$= (v)(t) = (1.28\times10^{6})(2.45\times10^{-6})$$

$= 3.136\ m = \pi m = \frac{1}{2}$ *circumference*

$\therefore$ *The particle has the coordinates (6.4, 0, 2m) as (x, y, z).*

Practice Exercise

Q.1 A proton goes un-deflected in a crossed electric and magnetic field (the fields are perpendicular to each other) at a speed of 2.0×10^5 m/s. The velocity is perpendicular to both the fields. When the electric field is switched off, the proton moves along a circle of radius 4.0 cm. Find the magnitudes of the electric and the magnetic fields. Take the mass of the proton 1.6×10^{-27} kg.

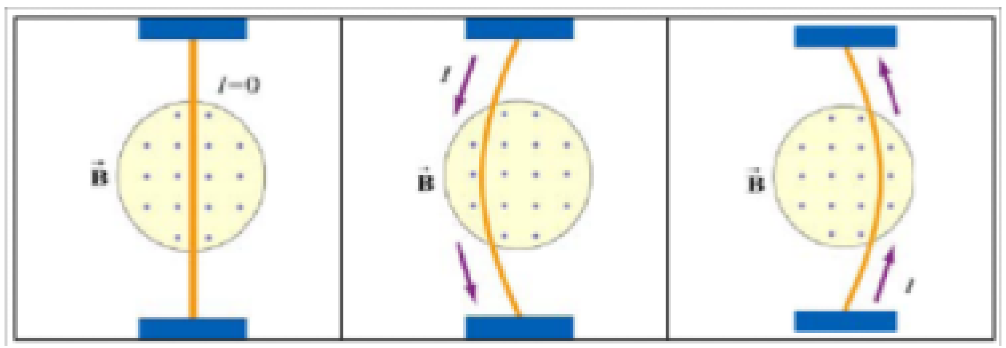
Answers

Q.1 1.0×10^4 N/C, 0.05T

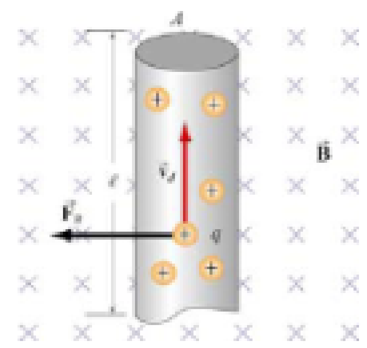
Magnetic Force on a Current-Carrying Wire :

We have just seen that a charged particle moving through a magnetic field experiences a magnetic force $\vec{F}_B$. Since electric current consists of a collection of charged particles in motion, when placed in a magnetic field, a current-carrying wire will also experience a magnetic force.

Consider a long straight wire suspended in the region between the two magnetic poles. The magnetic field points out the page and is represented with dots (•). It can be readily demonstrated that when a downward current passes through, the wire is deflected to the left. However, when the current is upward, the deflection is rightward, as shown in Figure.



To calculate the force exerted on the wire, consider a segment of wire of length ℓ and cross-sectional area A, as shown in Figure. The magnetic field points into the page, and is represented with crosses (X).



The charges move at an average drift velocity $\vec{v}_d$. Since the total amount of charge in this segment is $Q_{tot} = q(nA\ell)$, where n is the number of charges per unit volume, the total magnetic force on the segment is

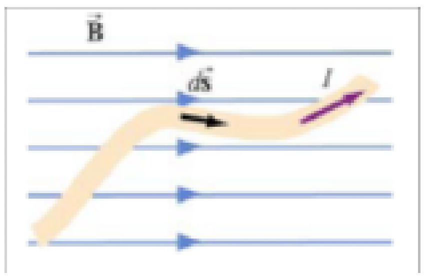
$$\vec{F}_B = Q_{tot}\vec{v}_d \times \vec{B} = qnA\ell(\vec{v}_d \times \vec{B}) = I(\vec{\ell} \times \vec{B})$$

where $I = nqv_dA$, and $\vec{\ell}$ is a *length vector* with a magnitude ℓ and directed along the direction of the electric current.

Special Case-1:

Wire of arbitrary shape placed in uniform magnetic field

For a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces acting on the small segments that make up the wire. Let the differential segment be denoted as $d\vec{s}$ (Figure).

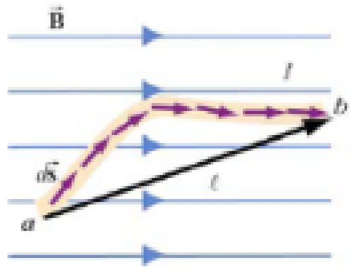


The magnetic force acting on the segment is : $d\vec{F}_B = Id\vec{s} \times \vec{B}$

Thus, the total force is : $$\vec{F}_B = I\int_a^b d\vec{s} \times \vec{B}$$

where a and b represent the endpoints of the wire.

As an example, consider a curved wire carrying a current I in a uniform magnetic field $\vec{B}$, as shown in Figure.



Using the magnetic force on the wire is given by

$$\vec{F}_B = I\left(\int_a^b d\vec{s}\right) \times \vec{B} = I\vec{\ell} \times \vec{B}$$

where $\vec{\ell}$ is the length vector directed from *a* to *b*. However, if the wire forms a closed loop of arbitrary shape (Figure), then the force on the loop becomes

$$\vec{F}_B = I\left(\oint d\vec{s}\right) \times \vec{B}$$

Special Case-2:

Magnetic Force on a closed loop in uniform magnetic field

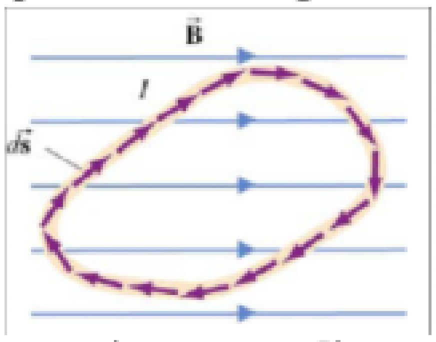
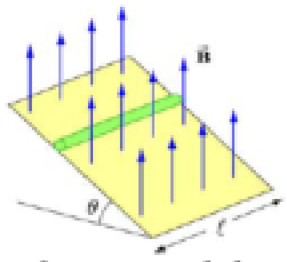


Figure : A closed loop carrying a current *I* in a uniform magnetic field.

Since the set of differential length elements $d\vec{s}$ form a closed polygon, and their vector sum is zero, i.e. $\oint d\vec{s} = 0$. The net magnetic force on a closed loop is $\vec{F}_B = \vec{0}$.

Illustration :

A conducting bar of length is placed on a frictionless inclined plane which is tilted at an angle θ from the horizontal, as shown in Figure.



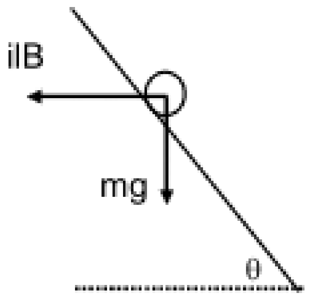
A uniform magnetic field is applied in the vertical direction. To prevent the bar from sliding down, a voltage source is connected to the ends of the bar with current flowing through. Determine the magnitude and the direction of the current such that the bar will remain stationary.

Sol

For equilibrium

$$I\ell B\cos\theta = mg\sin\theta$$

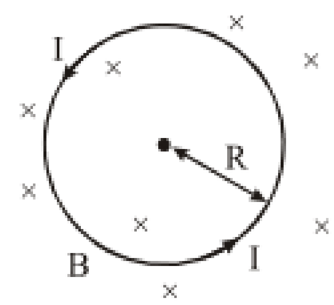
$$\Rightarrow I = \frac{mg\sin\theta}{\ell B\cos\theta}$$



Side view

Illustration :

A current (I) carrying circular wire of radius R is placed in a magnetic field B perpendicular to its plane. Find the tension T along the circumference of the wire



Sol. For small elemental portion

$2T\sin d\theta$

$= 2R\ d\theta\, IB$

$2T d\theta = 2RIB\, d\theta$

$T = IRB$.

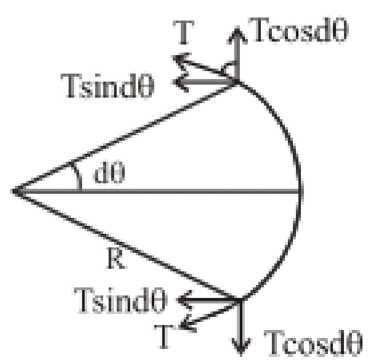
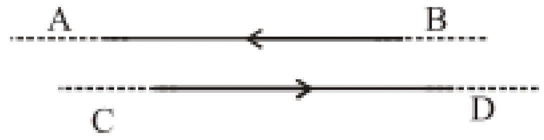


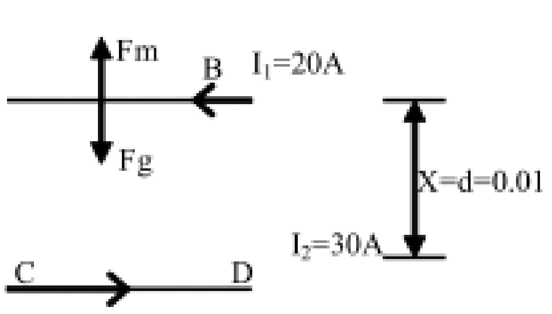
Illustration :

A long horizontal wire AB, which is free to move in a vertical plane and carries a steady current of 20 A, is in equilibrium at a height of 0.01 m over another parallel long wire CD which is fixed in a horizontal plane and carries a steady current of 30 A, as shown in figure. Show that when AB is slightly depressed, it executes simple harmonic motion. Find the period of oscillation.



Sol. *Let m be the mass per unit length of wire AB. At a height x about the wire CD, magnetic force per unit length on wire AB will be given by*

$$F_m = \frac{\mu_0 i_1 i_2}{2\pi x}\text{(upwards)} \qquad ...(i)$$



Wt. per unit of wire AB is

$$F_g = mg\text{(downwards)}$$

At x = d, wire is in equilibrium

i.e., $F_m = F_g \Rightarrow \frac{\mu_0}{2\pi}\frac{i_1 i_2}{d} = mg$

$$\Rightarrow \quad \frac{\mu_0 i_1 i_2}{2\pi d^2} = \frac{mg}{d} \qquad ...(ii)$$

When AB is depressed, x decreases therefore, F_m *will increase, while* F_g *remains the same. Let* ***AB***

is displaced by dx downwards. Differentiating equation (i) w.r.t.x, we get

$$dF_m = -\frac{\mu_0}{2\pi}\frac{i_1 i_2}{x^2}.dx \qquad ...(iii)$$

i.e., restoring force, F = d $F_m \propto - dx$

Hence the motion of wire is simple harmonic. From equation (ii) and (iii), we can write

$$dF_m = -\left(\frac{mg}{d}\right).dx \qquad (x = d)$$

$\therefore$ *Acceleration of wire,* $a = -\left(\frac{g}{d}\right).dx$

Hence period of oscillations

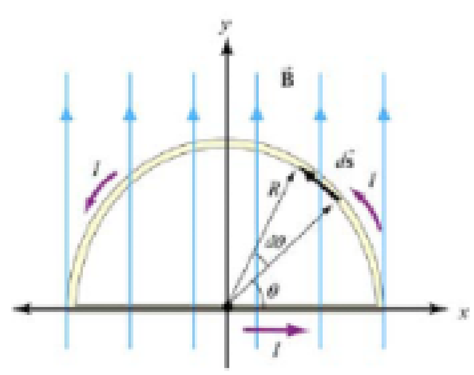
$$T = 2\pi\sqrt{\frac{dx}{a}} = 2\pi\sqrt{\left|\frac{\text{disp.}}{\text{acc.}}\right|}$$

$$\Rightarrow T = 2\pi\sqrt{d/g} = 2\pi\sqrt{\frac{0.01}{9.8}}$$

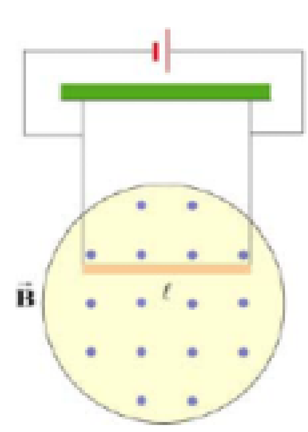
$$\Rightarrow T = 0.2s.$$

Practice Exercise

Q.1 Consider a closed semi-circular loop lying in the *xy* plane carrying a current *I* in the counterclockwise direction, as shown in Figure. Find the magnetic Force on a Semi-Circular Loop

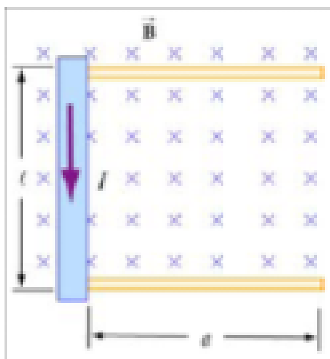


Q.2 A conducting rod having a mass density λ kg/m is suspended by two flexible wires in a uniform magnetic field $\vec{B}$ which points out of the page, as shown in Figure.

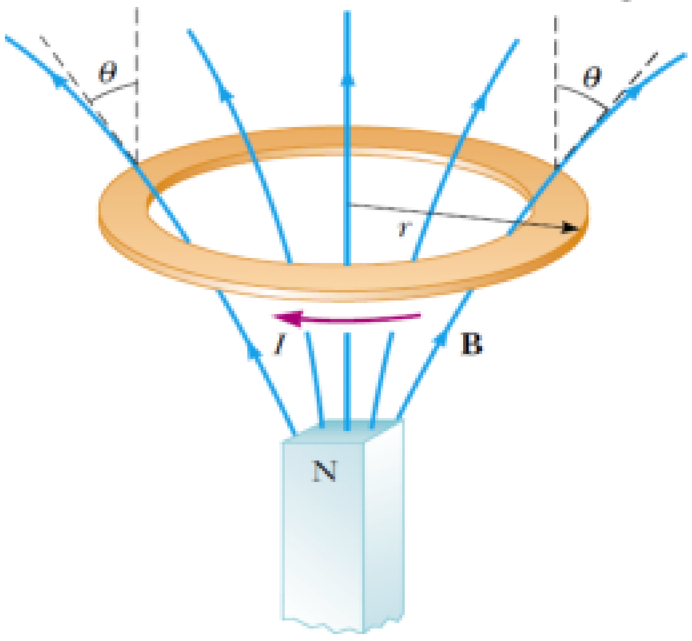


If the tension on the wires is zero, what are the magnitude and the direction of the current in the rod?

Q.3 A rod with a mass m and a radius R is mounted on two parallel rails of length a separated by a distance ℓ, as shown in the Figure. The rod carries a current I and rolls without slipping along the rails which are placed in a uniform magnetic field $\vec{B}$ directed into the page. If the rod is initially at rest, what is its speed as it rolls off the rails?



Q.4 A strong magnet is placed under a horizontal conducting ring of radius r that carries current I as shown in figure. If the magnetic field B makes an angle   with the vertical at the ring's location, what are the magnitude and direction of the resultant force on the ring?



Answers

Q.1 2IRB into the page

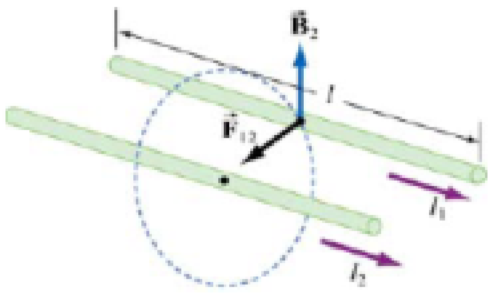
Q.2 $I = \dfrac{mg}{B\ell} = \dfrac{\lambda g}{B}$

Q.3 $v = \sqrt{\dfrac{4I\ell Ba}{3m}}$

Q.4 $F_y = (B\sin\theta)I(2\pi r)$

Force Between Two Parallel Wires :

We have already seen that a current-carrying wire produces a magnetic field. In addition, when placed in a magnetic field, a wire carrying a current will experience a net force. Thus, we expect two current-carrying wires to exert force on each other. Consider two parallel wires separated by a distance a and carrying currents $I_1{}^1$and $I_2{}^2$ in the $+x$-direction, as shown in Figure.



The magnetic force, $\vec{F}_{12}$, exerted on wire 1 by wire 2 may be computed as follows: Using the result from

the previous example, the magnetic field lines due to $I_2{}^2$going in the $+x$-direction are circles concentric with wire 2, with the field $\vec{B}_2$ pointing in the tangential direction. Thus, at an arbitrary point P on wire 1, we have $\vec{B}_2 = -(\mu_0 I_2 / 2\pi a)\hat{j}$, which points in the direction perpendicular to wire 1, as depicted in Figure. Therefore,

$$\vec{F}_{12} = I_1\vec{\ell} \times \vec{B}_2 = (\ell\hat{i}) \times \left(-\frac{\mu_0 I_2}{2\pi a}\hat{j}\right) = -\frac{\mu_0 I_1 I_2 \ell}{2\pi a}\hat{k}$$

Clearly $\vec{F}_{12}$ points toward wire 2. The conclusion we can draw from this simple calculation is that two parallel wires carrying currents in the same direction will attract each other. On the other hand, if the currents flow in opposite directions, the resultant force will be repulsive.

Definition of ampere

Consider two parallel wires separated by 1m and carrying a current of 1A each. Then $i_1 = i_2 = 1A$ and $d = 1m$, so that from equation

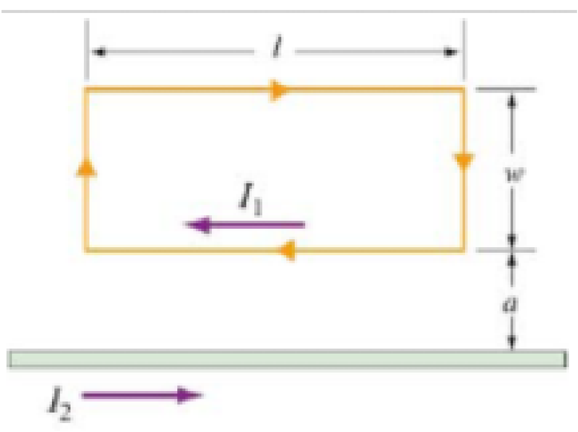
$$\frac{dF}{dl} = 2 \times 10^{-7} \text{ N/m}.$$

This is used to formally define the unit 'ampere' of electric current. If two parallel, long wires, kept 1m apart in vacuum, carry equal currents in the same direction and there is a force of attraction of 2×10^{-7} newton per metre of each wire, the current in each wire is said to be 1 ampere.

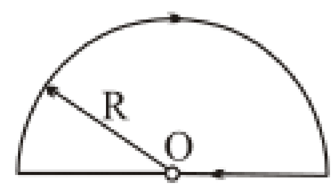
Practice Exercise

Q.1 If a current is passed through a spring, does the spring stretch or compress?

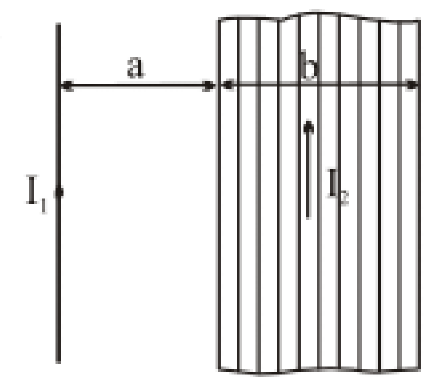
Q.2 A rectangular loop of length ℓ and width w carries a steady current I_1. The loop is then placed near an finitely long wire carrying a current I_2, as shown in Figure. What is the magnetic force experienced by the loop due to the magnetic field of the wire?



Q.3 Find the magnitude and direction of a force vector acting on a unit length of a thin wire, carrying a current I at a point O, if the wire is bent as shown in with curvature radius R



Q.4 Two long thin parallel conductors of the shape shown in Fig. carry direct currents I_1 and I_2. The separation between the conductors is a, the width of the right-hand conductor is equal to b. With both conductors lying in one plane, find the magnetic interaction force between them reduced to a unit of their length.



Answers

Q.1 compress

Q.2 $=\frac{\mu_0 I_1 I_2 \ell}{2\pi}\left[\frac{1}{a}-\frac{1}{(a+w)}\right]$

Q.3 $F_{unit} = m_0 I^2 / 4R$

Q.4 $F_1 = \frac{\mu_0}{4\pi}\frac{2I_1 I_2}{b}\, ln\,(1 + b/a)$

Magnetic Moment

Magnetic field (at large distances) due to current in a circular current loop is very similar in behavior to the electric field of an electric dipole. We know that the magnetic field on the axis of a circular loop, of a radius *R*, carrying a steady current *I*

$$B = \frac{\mu_0 I(2\pi a^2)}{4\pi(a^2 + x^2)^{3/2}}$$

its direction is along the axis and given by the right-hand thumb rule Here, *x* is the distance along the axis from the centre of the loop.

For *x* >> *R*, we may drop the R^2 term in the denominator. Thus

$$B=2\left(\frac{\mu_0}{4\pi}\right)\left(\frac{IA}{x^3}\right)$$

Where $A = \pi R^2$= area of the loop

The expression is very similar to an expression obtained earlier for the electric field of a dipole. The similarity may be seen if we we can define the magnetic dipole moment $\vec{\mu}$ as

$$\vec{\mu} = I\vec{A}$$

The direction of $\vec{\mu}$ is the same as the area vector $\vec{A}$ (perpendicular to the plane of the loop) and is determined by the right-hand rule (Figure). The SI unit for the magnetic dipole moment is ampere-meter^2 $(A\bullet m^2)$..

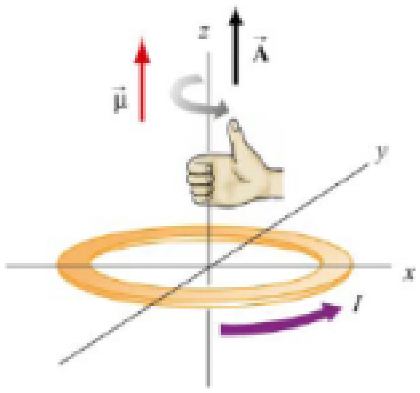


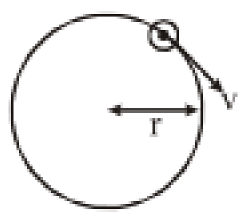
Illustration :

Find the magnetic moment of an electron orbiting in a circular orbit of radius r with a speed v

Sol. *Magnetic moment* $\mu = iA$

I = current; Since the orbiting electron behaves as current loop of current i,

we can write $i = \dfrac{e}{T} = \dfrac{e}{\frac{2\pi r}{v}} = \dfrac{ev}{2\pi r}$



A = area of the loop $= \pi r^2$

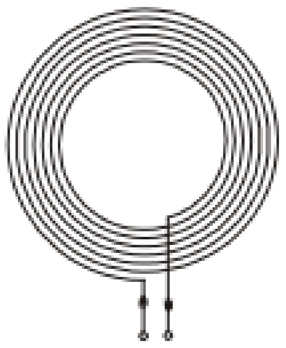
$\Rightarrow \quad \mu = (1)\left(\dfrac{ev}{2\pi r}\right)(\pi r^2)$

$\Rightarrow \quad \mu = \dfrac{evr}{2}$.

Practice Exercise

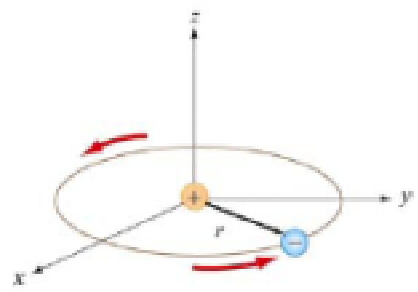
Q.1 Find the magnetic moment of a thin round loop with current if the radius of the loop is equal to $R = 100$ mm and the magnetic induction at its centre is equal to $B = 6.0\ \mu T$.

Q.2 A thin insulated wire forms a plane spiral of $N = 100$ tight turns carrying a current $I = 8$ mA. The radii of inside and outside turns (Fig.) are equal to $a = 50$ mm and $b = 100$ mm.



Find the magnetic moment of the spiral with a given current.

Q.3 We want to estimate the magnetic dipole moment associated with the motion of an electron as it orbits a proton. We use a "semi-classical" model to do this. Assume that the electron has speed v and orbits a proton (assumed to be very massive) located at the origin. The electron is moving in a right-handed sense with respect to the z-axis in a circle of radius $r = 0.53$ Å, as shown in Figure. Note that 1 Å $= 10^{-10}$ m.



(a) The inward force $m_e v^2/r$ required to make the electron move in this circle is provided by the Coulomb attractive force between the electron and proton (m_e is the mass of the electron). Using this fact, and the value of r we give above, find the speed of the electron in our "semi-classical" model.

(b) Given this speed, what is the orbital period T of the electron?

(c) What current is associated with this motion? Think of the electron as stretched out uniformly around the circumference of the circle. In a time T, the total amount of charge q that passes an observer at a point on the circle is just e

(d) What is the magnetic dipole moment associated with this orbital motion? Give the magnitude and direction. The magnitude of this dipole moment is one *Bohr magneton* μ_B

Q.4 A non-conducting thin disc of radius R charged uniformly over one side with surface density σ rotates about its axis with an angular velocity ω. Find:
(b) the magnetic moment of the disc.

Answers

Q.1 $p_m = 2\pi R^3 B / \mu_0 = 30 \text{ mA} \cdot \text{m}^2$ Q.2 $p_m = 1/3\ \pi IN\ (a^2 + ab + b^2)$

Q.3 (a) 2.18×10^6 m/s ; (b) 1.52×10^{-16} s (c) 1.05 mA. Big! ; (d) 9.27×10^{-24} along the z-axis.

Q.4 $p_m = 1/4\ \pi\sigma\omega R^4$

Torque on a Current Loop :

What happens when we place a rectangular loop carrying a current I in the xy plane and switch on a uniform magnetic field $\vec{B} = B\hat{i}$ which runs parallel to the plane of the loop, as shown in Figure (a)?

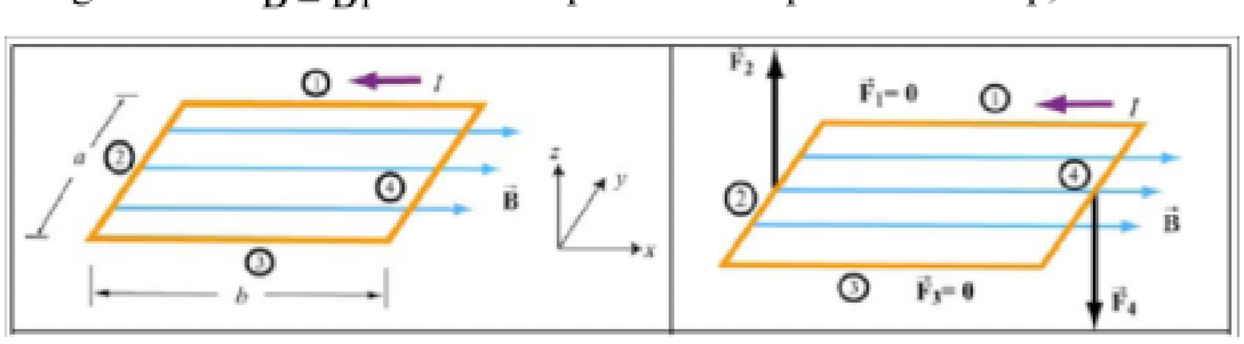


Figure : (a) A rectangular current loop placed in a uniform magnetic field.
(b) The magnetic forces acting on sides 2 and 4.

From Eq., we see the magnetic forces acting on sides 1 and 3 vanish because the length vectors $\vec{\ell}_1 = -b\hat{i}$ and $\vec{\ell}_3 = b\hat{i}$ are parallel and anti-parallel to $\vec{B}$ and their cross products vanish. On the other hand, the magnetic forces acting on segments 2 and 4 are non-vanishing:

$$\begin{cases} \vec{F}_2 = I(-a\hat{j}) \times (B\hat{i}) = IaB\hat{k} \\ \vec{F}_4 = I(a\hat{j}) \times (B\hat{i}) = -IaB\hat{k} \end{cases}$$

with $\vec{F}_2$ pointing out of the page and $\vec{F}_4$ into the page. Thus, the net force on the rectangular loop is

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \vec{0}$$

as expected. Even though the net force on the loop vanishes, the forces $\vec{F}_2$ and $\vec{F}_4$ will produce a torque which causes the loop to rotate about the *y*-axis (Figure). The torque with respect to the center of the loop is

$$\vec{\tau} = \left(-\frac{b}{2}\hat{i}\right) \times \vec{F}_2 + \left(\frac{b}{2}\hat{i}\right) \times \left(IaB\hat{k} + \left(\frac{b}{2}\hat{i}\right) \times \left(-IaB\hat{k}\right)\right)$$

$$= \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\hat{j} = IabB\hat{j} = IAB\hat{j}$$

where A = ab represents the area of the loop and the positive sign indicates that the rotation is clockwise about the *y*-axis. It is convenient to introduce the area vector $\vec{A} = A\hat{n}$ where $\hat{n}$ is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of $\hat{n}$ is set by the conventional right-hand rule. In our case, we have $\hat{n} = +\hat{k}$. The above expression for torque can then be rewritten as

$$\vec{\tau} = I\vec{A} \times \vec{B}$$

Notice that the magnitude of the torque is at a maximum when $\vec{B}$ is parallel to the plane of the loop (or perpendicular to).

Consider now the more general situation where the loop (or the area vector $\vec{A}$) makes an angle θ with respect to the magnetic field.

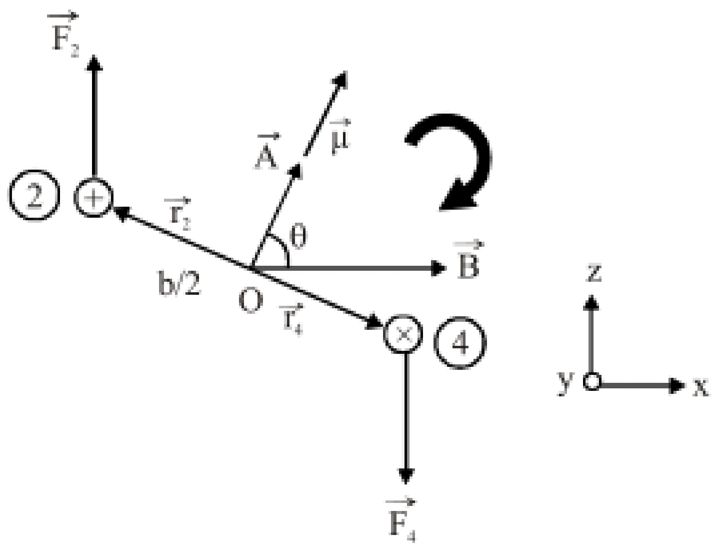


Figure : Rotation of a rectangular current loop

From Figure, the lever arms and can be expressed as:

$$\vec{r}_2 = \frac{b}{2}\left(-\sin\theta\hat{i} + \cos\theta\hat{k}\right) = -\vec{r}_4$$

and the net torque becomes

$$\vec{\tau} = \vec{r}_2 \times \vec{F}_2 + \vec{r}_4 = 2\vec{r}_2 \times \vec{F}_2 = 2.\frac{b}{2}\left(-\sin\theta\hat{i} + \cos\theta\hat{k}\right) \times \left(IaB\hat{k}\right)$$

$$jIabB\sin\theta\,\hat{j} = I\vec{A} \times \vec{B}$$

For a loop consisting of *N* turns, the magnitude of the toque is

$$\tau = NIAB\sin\theta$$

The quantity $NI\vec{A}$ is called the magnetic dipole moment $\vec{\mu}$

$$\vec{\mu} = NI\vec{A}$$

Using the expression for $\vec{\mu}$, the torque exerted on a current-carrying loop can be rewritten as

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

The above equation is analogous to $\vec{\tau} = \vec{p} \times \vec{E}$ in Eq., the torque exerted on an electric dipole moment $\vec{p}$ in the presence of an electric field $\vec{E}$.

Configuratoin energy of current loop in uniform magnetic field.

Recalling that the potential energy for an electric dipole is $U = -\vec{p} \cdot \vec{E}$ [see Eq.], a similar form is expected for the magnetic case. The work done by an external agent to rotate the magnetic dipole from an angle θ_0 to θ is given by

$$W_{ext} = \int_{\theta_0}^{\theta} (\mu B \sin\theta')d\theta' = \mu B(\cos\theta_0 - \cos\theta)$$

$$= \Delta U = U - U_0$$

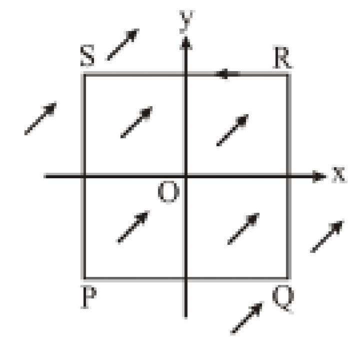
Once again, $W_{ext} = -W$, where W is the work done by the magnetic field. Choosing $U_0 = 0$ at $\theta_0 = \pi/2$, the dipole in the presence of an external field then has a potential energy os

$$U = -\mu B\cos\theta = -\vec{\mu} \cdot \vec{B}$$

The configuration is at a stable equilibrium when $\vec{\mu}$ is aligned parallel to $\vec{B}$, making U a minimum with $U_{min} = -\mu B$. On the other hand, when $\vec{\mu}$ and $\vec{B}$ are anti-parallel, $U_{max} = +\mu B$ is a maximum and the system is unstable.

Illustration :

A uniform, constant magnetic field $\vec{B}$ *is directed at an angle of 45° to the x-axis in the xy-plane. PQRS is a rigid, square wire frame carrying a steady current* I_0*, with its centre at the origin O. At time t =0, the frame is at rest in the position shown in the figure, with its sides parallel to the x and y axes. Each side of the frame is of mass M and length L.*



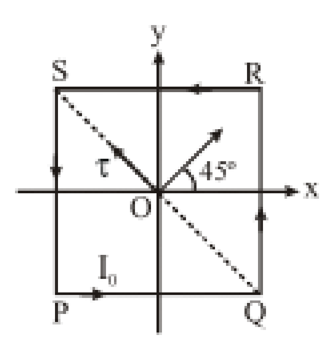
(a) *What is the torque* $\vec{\tau}$ *about O acting on the frame due to the magnetic field?*

(b) *Find the angle by which the frame rotates under the action of this torque in a short interval of time* Δt *, and the axis about which this rotation occurs. (*Δt *is so short that any variation in the torque during this interval may be neglected). Given moment of any variation in the torque during this interval may be neglected). Given moment of inertia of the frame about an axis through its centre perpendicular to its plate is (4/3)* ML^2.

Sol. *(a) As magnetic field* $\vec{B}$ *is in x-y plane and subtends an angle of 45° with x-axis*

$$B_x = B\cos 45^\circ = B/\sqrt{2}$$

And $B_y = B\sin 45 = B/\sqrt{2}$



So in vector form

$$\vec{B} = \hat{i}(B/\sqrt{2}) + j(B/\sqrt{2})$$

and $\vec{M} = I_0 S\hat{k} = I_0 L^2 \hat{k}$

so, $\vec{\tau} = \vec{M}\times\vec{B} = I_0 L^2 \hat{k} \times \left(\dfrac{B}{\sqrt{2}}\hat{i} + \dfrac{B}{\sqrt{2}}\hat{j}\right)$

i.e., $\vec{\tau} = \dfrac{I_0 L^2 B}{\sqrt{2}} \times (-\hat{i} + \hat{j})$

i.e., torque has magnitude $I_0 L^2$ *B and is directed along line QS from Q to S.*

(b) As by theorem of perpendicular axis, moment of inertia of the frame about QS,

$$I_{QS} = \frac{1}{2} I_z = \frac{1}{2}\left(\frac{4}{3}ML^2\right) = \frac{2}{3}ML^2$$

And as $\tau = I\alpha$,

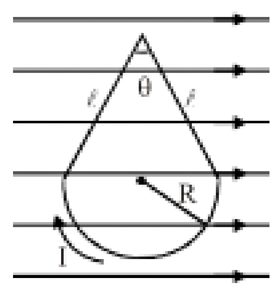
$$\alpha = \frac{\tau}{I} = \frac{I_0 L^2 B \times 3}{2L^2 M} = \frac{3}{2}\frac{I_0 B}{M}$$

As here α *is constant, equations of circular motion are valid and hence from* $\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$ *with* $\omega_0 = 0$ *we have*

$$\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}\left(\frac{3}{2}\frac{I_0 B}{M}\right)(\Delta t)^2 = \frac{3}{4}\frac{I_0 B}{M}\Delta t^2 .$$

Practice Exercise

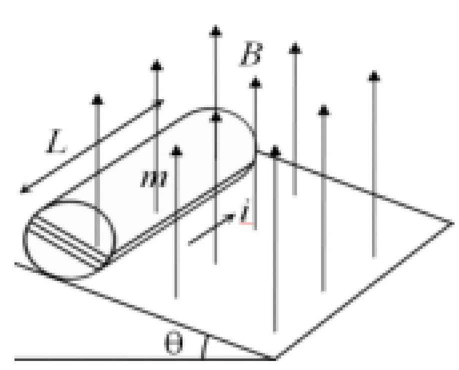
Q.1 A current loop consists of a semicircle of radius R and two straight segments of length ℓ with an angle θ between them. The loop is then placed in a uniform magnetic field pointing to the right, as shown in Figure.



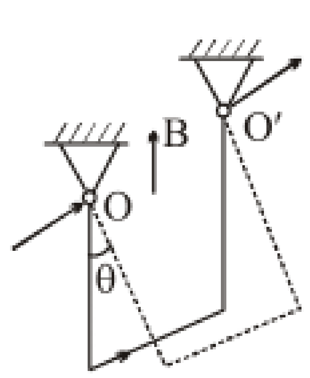
(a) Find the net force on the current loop.

(b) Find the net torque on the current loop.

Q.2 Figure shows a wooden cylinder with a mass m and a length L with N turns of wire wrapped around it longitudinally, so that the plane of the wire loop contains the axis of the cylinder. What is the least current through the loop that will prevent the cylinder from rolling down a plane inclined at an angle to the horizontal, in the presence of a vertical, uniform magnetic field B, if the plane of the windings is parallel to the inclined plane?



Q.3 A copper wire of density ρ with cross-sectional area S bent to make three sides of a square can turn about a horizontal axis OO' (Fig.). The wire is located in uniform vertical magnetic field. Find the magnetic induction if on passing a current I through the wire the latter deflects by an angle θ

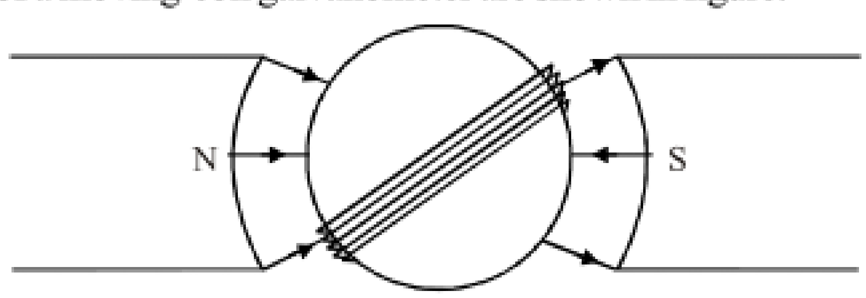


Answers

Q.1 Q.2 $i = \dfrac{mg}{2NBL}$ Q.3 $B = (2rgS/I)\tan\theta$

Moving coil Galvanometer :

The main parts of a moving-coil galvanometer are shown in figure.



The current to be measured is passed through the galvanometer. As the coil is in the magnetic field $\vec{B}$ of the permanent magnet, a torque $\vec{\Gamma} = ni\vec{A} \times \vec{B}$ acts on the coil. Here n = number of turns, i = current in the coil $\vec{A}$ = area-vector of the coil and $\vec{B}$ = magnetic field at the site of the coil. This torque deflects the coil from its equilibrium position.

The pole pieces are made cylindrical. As a result, the magnetic field at the arms of the coil remains parallel to the plane of the coil everywhere even as the coil rotates. The deflecting torque is then $\Gamma = niAB$.

As the upper end of the suspension strip W is fixed, the strip gets twisted when the coil rotates. This produces a restoring torque acting on the coil. If the deflection of the coil is θ and the torsional constant of the suspension strip is k, the restoring torque is $k\theta$. The coil will stay at a deflection θ where

$$niAB = k\theta$$

or,
$$i = \frac{k}{nAB}\theta$$

Hence, the current is proportional to the deflection. The constant $\dfrac{k}{nAB}$ is called the galvanometer constant.

We define the **current sensitivity** of the galvanometer as the deflection per unit current. From Eq. this current sensitivity is.

$$\frac{\phi}{I} = \frac{NAB}{k}$$

A convenient way for the manufacturer to increase the sensitivity is to increase the number of turns N. We choose galvanometers having sensitivities of value, required by our experiment.

We define the **voltage sensitivity** as the deflection per unit volt of applied potential difference

$$\frac{\phi}{I} = \left(\frac{NAB}{k}\right)\frac{I}{V} = \left(\frac{NAB}{k}\right)\frac{1}{R}$$

An interesting point to note is that increasing the current sensitivity may not necessarily increase the voltage sensitivity. If $N \to 2N$, i.e., we double the number of turns, then

$$\frac{\phi}{I} \to 2\frac{\phi}{I}$$

Thus, the current sensitivity doubles. However, the resistance of the galvanometer is also likely to double, since it is proportional to the length of the wire. In eq. $N \to 2N$, and $R \to 2R$, thus the voltage sensitivity,

$$\frac{\phi}{V} \to \frac{\phi}{V}$$

remains unchanged.

Practice Exercise

Q.1 Two moving coil meters, M_1 and M_2 have the following particulars :

$R_1 = 10\Omega, N_1 = 30$
$A_1 = 3.6 \times 10^{-3}\ m^2, B_1 = 0.25\ T$
$R_2 = 14\Omega, N_2 = 42$
$A_2 = 1.8 \times 10^{-3}\ m^2, B_2 = 0.50\ T$

(The spring constants are identical for the two meters)

Determine the ratio of (a) current sensitivity and (b) voltage sensitivity of M_2 and M_1.

Answers

Q.1 (a) 1.4 (b) 1

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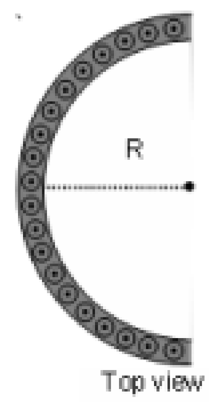
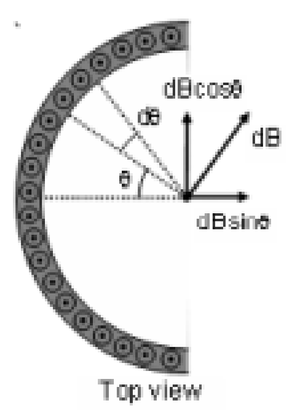
Solved Examples

Q.1 A current I flows in a long straight wire with cross-section having the form of a thin half-ring of radius R (Fig.). Find the induction of the magnetic field at the point O.



Sol. $dI = \frac{1}{\pi R}(Rd\theta) = \frac{1}{\pi}d\theta$

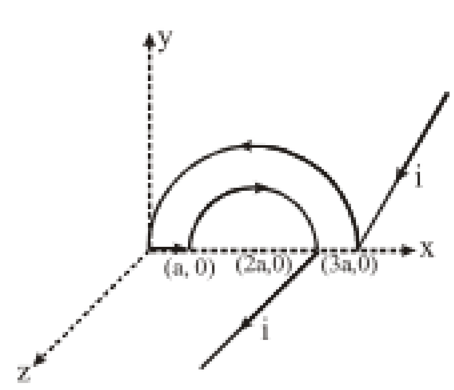
$$dB = \frac{\mu_0 dI}{2\pi R} = \frac{\mu_0 I}{2\pi^2 R}d\theta$$



$$B = \int dB\cos\theta = \frac{\mu_0 I}{2\pi^2 R}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos\theta\, d\theta = \frac{\mu_0 I}{\pi^2 R}$$

$$B = \mu_0 I / \pi^2 R$$

Q.2 In the figure shown the magnetic field at the point P.

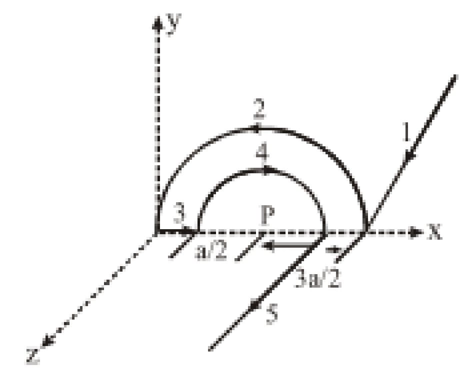


Sol. Consider the figure

$$\vec{B}_p = (\vec{B}_1)_p + (\vec{B}_2)_p + (\vec{B}_3)_p + (\vec{B}_4)_P + (\vec{B}_5)_P$$

where $(\vec{B}_1)_p = \frac{\mu_0 i}{4\pi\left(\frac{3a}{2}\right)}(-\hat{j})$ (semi-infinite wire)

$$(\vec{B}_2)_p = \frac{\mu_0 i}{4\left(\frac{3a}{2}\right)}(+\hat{k})$$



$$(\vec{B}_3)_p = 0$$

$$(\vec{B}_4)_p = \frac{\mu_0 i}{4\left(\frac{a}{2}\right)}(-\hat{k})$$

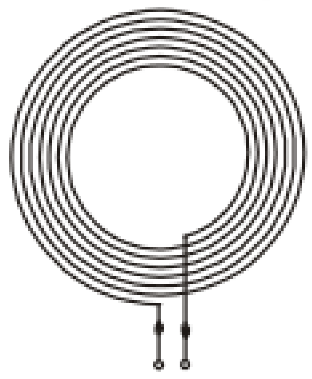
$$(\vec{B}_5)_p = \frac{\mu_0 i}{4\pi\left(\frac{a}{2}\right)}(-\hat{j})$$

$$\Rightarrow \quad \vec{B}_p = \frac{\mu_0 i}{2a}\left[-\left(\frac{1}{3\pi}+\frac{1}{\pi}\right)\hat{j}-\left(1-\frac{1}{3}\right)\hat{k}\right]$$

$$\Rightarrow \quad \vec{B}_p = \frac{2\mu_0 i}{3a}\left[\frac{1}{\pi}\hat{j}-\hat{k}\right]$$

$$\Rightarrow \quad \bar{B}_p = \frac{\mu_0 i}{3\pi a}\sqrt{1+\pi^2}\,.$$

Q.3 A thin insulated wire forms a plane spiral of N tight turns carrying a current I. The radii of inside and outside turns (Fig.) are equal to a and b. Find the magnetic induction at the centre of the spiral;

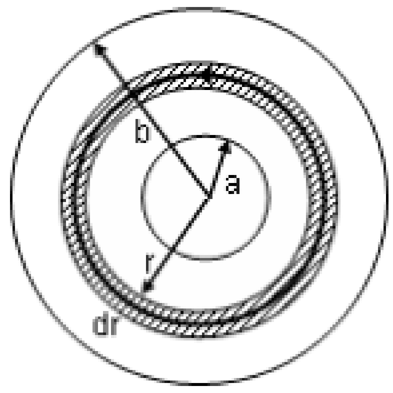


Sol. $B_{one} = \frac{\mu_0 I}{4\pi r}(2\pi) = \frac{\mu_0 I}{2r}$

$dN = \frac{N}{b-a}dr$

$dB = B_{one}\,dN = \frac{\mu_0 N I}{2(b-a)}\frac{dr}{r}$

$B = \int dB = \frac{\mu_0 N I}{2(b-a)}\int_a^b \frac{dr}{r} = \frac{\mu_0 N I}{2(b-a)} \ln\frac{b}{a}$



Q.4 A disc of radius R rotates at an angular velocity ω about the axis perpendicular to its surface and passing through its centre. If the disc has a uniform surface charge density σ, find the magnetic induction on the axis of rotation at a distance x from the centre.

Sol. Consider a ring of radius r and width dr.

Charge on the ring, $dq = (2\pi r\ dr)\sigma$

Current due to ring is $dI = \frac{dq}{T}$

$$= \frac{\omega dq}{2\pi} = \sigma\omega r\ dr$$

Magnetic field due to ring at point P is

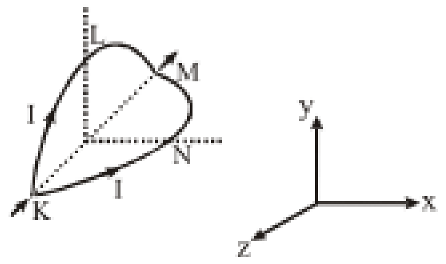
$$dB = \frac{\mu_0\, dI\, r^2}{2(r^2 + x^2)^{3/2}}$$

or $$B = \int dB = \frac{\mu_0 \sigma\omega}{2}\int_0^R \frac{r^3\, dr}{(r^2 + x^2)^{3/2}} \qquad \ldots(i)$$

Putting $r^2 + x^2 = t^2$ and 2r dr = 2t dt and integrating (i) we get

$$B = \frac{\mu_0\sigma\omega}{2}\left[\frac{R^2 + 2x^2}{\sqrt{R^2 + x^2}} - 2x\right].$$

Q.5 A circular loop of radius R is bent along a diameter and given a shape as shown in figure. One of the semi-circles (KNM) lies in the x-z plane and the other one (KLM) in the y-z plane with their centres at origin. Current I is flowing through each of the semi-circles as shown in figure.



A particle of charge q is released at the origin with a velocity $\vec{V} = -V_0\hat{i}$. Find the instantaneous force $\vec{F}$ on the particle. Assuem that space is gravity free.

Sol. Magnetic field at the centre of a circular wire of radius R carrying a current I is given by

$$B = \frac{\mu_0 I}{2R}$$

In this problem, currents are flowing in two semi-circles, KLM in the y-z plane and KNM in the x-z plane. The centres of these semi-circles coincide with the origin of the Cartesian system of axes.

$$\therefore \qquad \vec{B}_{KLM} = \frac{1}{2}\left(\frac{\mu_0 I}{2R}\right)(-\hat{i})$$

$$\vec{B}_{KNM} = \frac{1}{2}\left(\frac{\mu_0 I}{2R}\right)(-\hat{j})$$

The total magnetic field at the origin is

$$B_0 = \frac{\mu_0 I}{4R}(-\hat{i}+\hat{j})$$

It is given that a particle of charge q is released at the origin with a velocity $V = -\ V_0\hat{i}$. The instantaneous force acting on this particle is given by

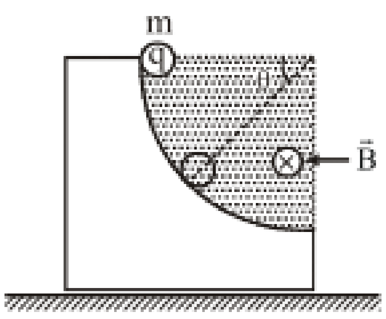
$$f = q[\vec{V}\times\vec{B}]$$

$$= q(-V_0\hat{i})\times\left[\frac{\mu_0 I}{4R}(\hat{i}+\hat{j})\right]$$

$$= \left(\frac{q\,V_0\mu_0 I}{4R}\right)[(-\hat{i})\times(-\hat{i}+\hat{j})]$$

$$= \frac{q\,V_0\mu_0 I}{4R}(-\hat{k}).$$

Q.6 In the figure a charged sphere of mass m and charge q starts sliding from rest on a vertical fixed circular track of radius R from the position shown. There exists a uniform and constant horizontal magnetic field of induction B. The maximum force exerted by the track on the sphere.



Sol. $F_m = qvB$,a and directed radially outward.

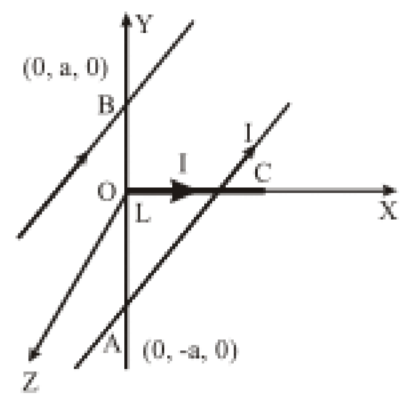
$$\therefore \qquad N - mg\sin\theta + qvB = \frac{mv^2}{R}$$

$$\Rightarrow \qquad N = \frac{mv^2}{R} + mg\sin\theta - qvB$$

Hence at $\theta = \pi/2$

$$\Rightarrow \qquad N_{max} = \frac{2mgR}{R} + mg - qB\sqrt{2gR} = 3mg - qB\sqrt{2gR}\ .$$

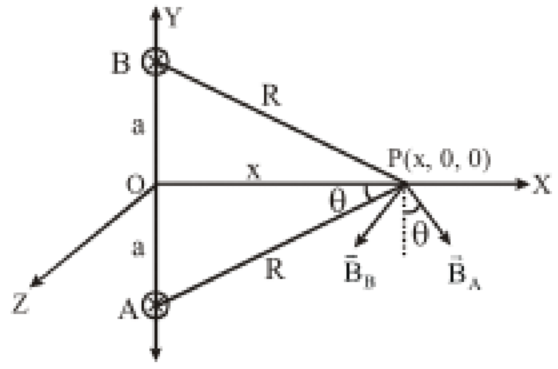
Q.7 A straight segment OC (of length L meter) of a circuit carrying a current 1 amp is placed along the x-axis. Two infinitely long straight wire A and B, each extending $z=-\infty$ to $+\infty$ are fixed at y = – a metre and y = + a metre respectively, as shown in the figure. If the wires A and B each carry a current I amp into the place of the paper, obtain the expression for the force acting on segment OC. What will be the force on OC if the current in the wire B is reversed?



Sol. Magnetic field B_A produced at P(x, 0, 0) due to wire, $B_A = \mu_0 I/2\pi R$, $B_B = \mu_0 I/2\pi R$. Components of B_A and B_B along x-axis cancel, while those along y-axis add up to give total field.

$$B = 2\left(\frac{\mu_0 I}{2\pi R}\right)\cos\theta = \frac{2\mu_0 I}{2\pi R}.\frac{x}{R} = \frac{\mu_0 I}{\pi}\frac{x}{(a^2+x^2)} \text{ (along} - \text{y direction)}$$

The force dF acting on the current element is $d\bar{F} = I(d\bar{I} \times B\bar{B})$

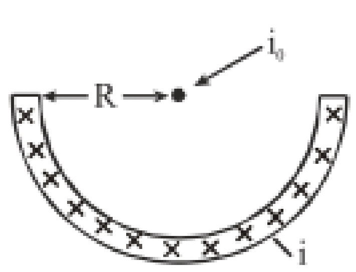


$\therefore \qquad dF = \frac{\mu_0 I^2}{\pi}\frac{x\,dx}{a^2+x^2}[\because \sin 90^o = 1]$

$\Rightarrow \qquad F = \frac{\mu_0 I^2}{\pi}\int_0^L \frac{x\,dx}{a^2+x^2} = \frac{\mu_0 I^2}{2\pi}\ln\frac{a^2L^2}{a^2}$

If the current in B is reversed, the magnetic field due to the two wires would be only along x- direction and the force on the current along x- direction will be zero.

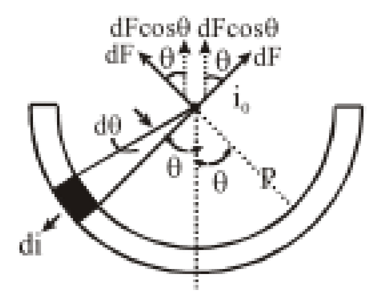
Q.8 Shown in the figure is a very long semicylindrical conducting shell of radius R and carrying a current i along its length. An infinitely long straight current carrying conductor is lying along the axis of the semicylinder. If the current flowing through the straight wire is i_0, then find the froce on the semicylinder.



Sol. The net magnetic force on the conducting wire

$$= F = \int 2dF\cos\theta$$

$$\Rightarrow \quad F = \int 2\left[\frac{\mu_0 (di) i_0}{2\pi R}\right]\cos\theta$$



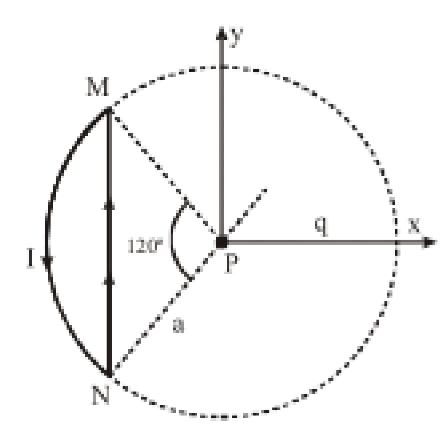
$$\Rightarrow \quad F = \frac{\mu_0 i_0}{\pi R}\int di\cos\theta$$

when $di = \frac{i}{\pi R} \times R d\theta = \frac{i d\theta}{\pi}$

$$\Rightarrow \quad F = \frac{\mu_0 i_0}{\pi R}\int \frac{(i d\theta)\cos\theta}{\pi}$$

$$\Rightarrow \quad F = \frac{\mu_0 i_0 i}{\pi^2 R}\int_0^{\pi/2} \cos\theta d\theta = \frac{\mu_0 i_0 i}{\pi^2 R}.$$

Q.9 A wire loop carrying a current I is placed in the x-y plane as shown in figure. (a) If a particle with charge q and mass m is placed at the centre P and given a velocity v along NP find its instantaneous acceleration. (b) If an external uniform magnetic induction $\vec{B} = B\hat{i}$ is applied, find the force and torque acting on the loop.

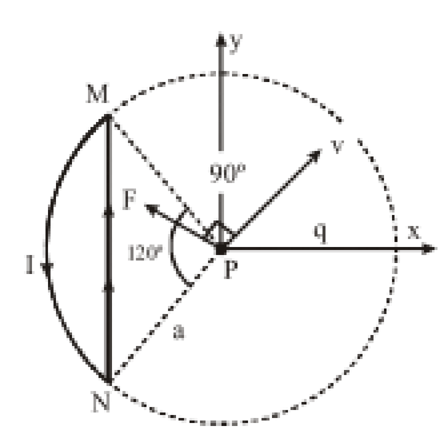


Sol. (a) As in case of current -carrying straight conductor and arc, the magnitude of B is given by

$$B_1 = \frac{\mu_0 I}{4\pi d}(\sin\alpha + \sin\beta)$$

And $$B_2 = \frac{\mu_0 I \phi}{4\pi r}$$

So in accordance with right hand screw rule,



$$\left(\vec{B}_w\right) = \frac{\mu_0}{4\pi}\frac{1}{(a\cos 60)} \times 2\sin 60(-\hat{k})$$

and $$\left(\vec{B}\right)_{MN} = \frac{\mu_0}{4\pi}\frac{I}{a} \times \left(\frac{2}{3}\pi\right)(-\hat{k})$$

and hence net $\vec{B}$ at P due to the given loop

$$\vec{B} = \vec{B}_w + \vec{B}_A \quad \Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{a}\left[\sqrt{3} - \frac{\pi}{3}\right](-\hat{k}) \qquad \ldots(i)$$

Now as force on charged particle in a magnetic fields is given by

$$\vec{F} = q(\vec{v} \times \vec{B})$$

so here, $\vec{F} = qvB\sin 90^\circ$ along PF

i.e. $\vec{F} = \frac{\mu_0}{4\pi}\frac{2qvI}{a}\left[\sqrt{3} - \frac{\pi}{3}\right]$ along PF

and so $\vec{a} = \frac{\vec{F}}{m} = 10^{-7}\frac{2qvI}{a}\left[\sqrt{3} - \frac{\pi}{3}\right]$ along PF

(b) As $d\vec{F} = Id\vec{L} \times \vec{B}$, so $\vec{F} = \int Id\vec{L} \times \vec{B}$

As here I and $\vec{B}$ are constant

$$\vec{F} = I\left[\oint d\vec{L}\right] \times \vec{B} = 0 \qquad \left[\text{as} \oint d\vec{L} = 0\right]$$

Further as area of coil,

$$\vec{S} = \left[\frac{1}{3}\pi a^2 - \frac{1}{2}.2a\sin 60^\circ \times a\cos 60^\circ\right]\hat{k} = a^2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]\hat{k}$$

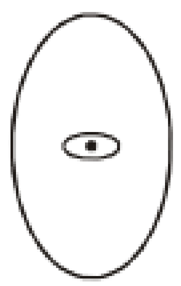
So $\vec{M} = I\vec{S} = Ia^2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]\hat{k}$

and hence $\vec{\tau} = \vec{M} \times \vec{B} = Ia^2B\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right](\hat{k} \times \hat{i})$

i.e. $\vec{\tau} = Ia^2B\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]\hat{j}\,N - m$ as $(\hat{k} \times \hat{i} = \hat{j})$.

Q.10 A coil of radius R carries current I. Another concentric coil of radius ($r << R$) carries current i. Planes of two coils are mutually perpendicular and both the coils are free to rotate about common diameter. Find maximum kinetic energy of smaller coil when both the coils are released, masses of coils are M and m respectively.

Sol. If a magnetic dipole having moment M be rotated through angle θ from equilibrium position in a uniform magnetic field B, work done on it is $W = MB(1 - \cos\theta)$. This work is stored in the system in the form of energy. When system is release, dipole starts to rotate to occupy equilibrium position and teh energy converts into kinetic energy and kinetic energy of the system is maximum when stored energy is completely released.



Magnetic induction, at centres due to current in larger coil is $B = \frac{\mu_0 I}{2R}$

Magnetic dipole moment of smaller coil is $i\pi r^2$.

Initially planes of two coils are mutually perpendicular, therfore θ is $90°$ or energy of the system is

$$U = (i\pi r^2)B(1 - \cos 90°)$$

$$U = \frac{\mu_0 I i\pi r^2}{2R}$$

When coils are release, both the coils start to rotate about their common diameter and their kinetic energies are maximum when they become coplanar.

Moment of inertia of larger coil about axis of rotation is $I_1 = \frac{1}{2}mR^2$

and that of smaller coil is $I_2 = \frac{1}{2}mr^2$

Since, two coils rotate due to their mutual interaction only, therefore, if one coil rotates clockwise then the other rotates anticlockwise.

Let angular velocities of larger and smaller coils be numerically equal to ω_1 and ω_2 respectively when they become coplaner,

According to law of conservation of angular momentum,

$$I_1\omega_1 = I_2\omega_2$$

and according to law of conservation of energy,

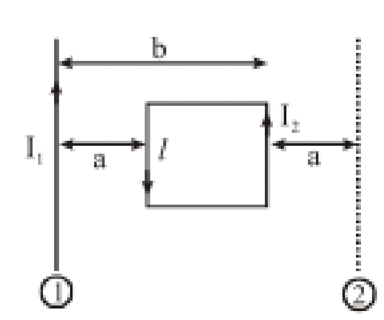
$$\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 = U$$

From above equations, maximum kinetic energy of smaller coil,

$$\frac{1}{2}I_2\omega_2^2 = \frac{UI_1}{I_1 + I_2}$$

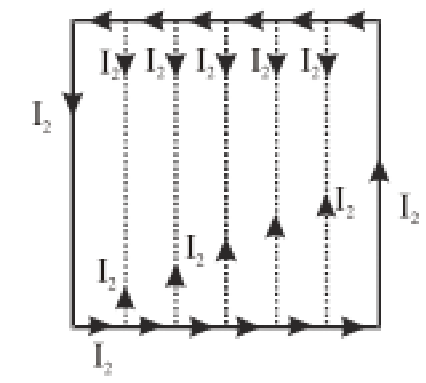
$$= \frac{\mu_0 \pi l i M R r^2}{2(MR^2 + mr^2)}$$

Q.11 What is the work done in transferring the wire from position (1) to position (2)?



Sol. The loop can be considered as the combination of teh number of elementary loops.

The net current in the dotted wires is 0 as current in the neighboring loops flowing through the same wire are opposite in direction.



consider an elementary loop of width dr at a distance r from the wire

The '$d\mu$' magnetic moment of the elemental loop

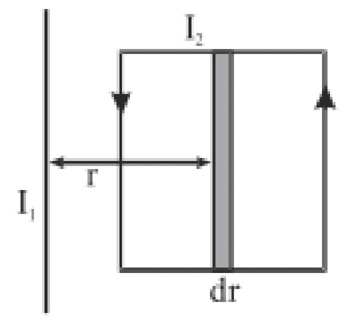
$$= I_2 l \, dr$$

The B at that point due to straight wire

$$= \mu_0 I_1 / 2\pi r .$$

$$dU = -B.d\mu = -\frac{\mu_0 I_1}{2\pi r} I_2 l \, dr(\cos \pi)$$

[As $d\mu$ is anti-parallel to B.]



$$U_1 = \int du = \frac{\mu_0 I_1 I_2 l}{2\pi} \int_a^b \frac{1}{r} dr = \frac{\mu_0 I_1 I_2 l}{2\pi} \ln\left(\frac{b}{a}\right)$$

By symmetry, $U_2 = -U_1$

$$\Rightarrow \qquad -\Delta U = \text{work done} = -(U_2 - U_1) = 2\frac{\mu_0 I_1 I_2 l}{2\pi} \ln\frac{b}{a}.$$

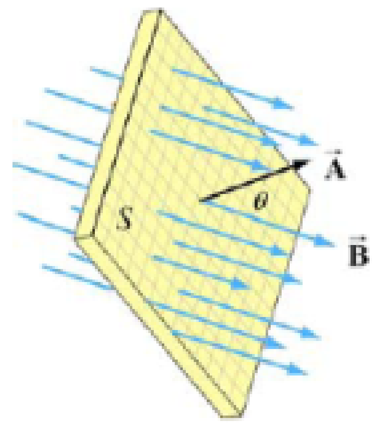
The work done in transferring the wire from position 1 to 2 $= \dfrac{\mu_0 I_1 I_2 l}{\pi} \ln\dfrac{b}{a}$.

ELECTRO MAGNETIC INDUCTION

Magnetic Flux

Consider a uniform magnetic field passing through a surface *S*, as shown in the figure below:

Let the area vector be $\vec{A} = A\hat{n}$, where *A* is the area of the surface $\hat{n}$ and its unit normal. The magnetic flux through the surface is given by $\Phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta$



where θ is the angle between $\vec{B}$ and $\hat{n}$. If the field is non-uniform, Φ_B then becomes

$$\Phi_B = \int_S \vec{B} . d\vec{A}$$

The SI unit of magnetic flux is the weber (Wb): $1\text{Wb} = 1\ \text{T.m}^2$

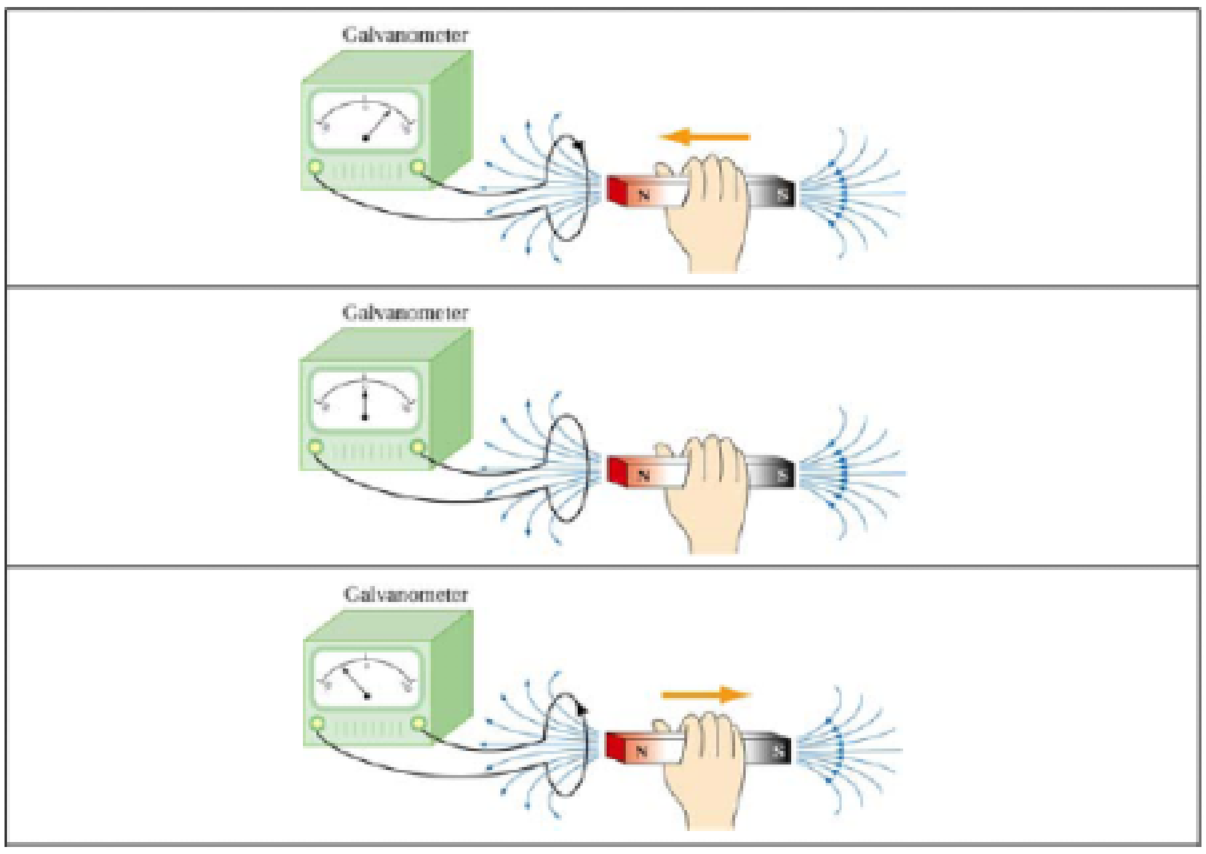
Electromagnetic induction

The electric fields and magnetic fields considered up to now have been produced by stationary charges and moving charges (currents), respectively.

It was observed that when a closed current carrying loop can set up a magnetic field . By nature of symmetry an obvious question comes in mind, can a current be generated with the help of magnetic field?

Answer to this care in 1831 when michacl faraday discovered that a change in magnetic filed can produce a current in a loop . The phenomenon is known as electromagnetic induction. Figure illustrates Faraday's experiments.

Faraday's Experiment



Faraday observed that no current is registered in the galvanometer when bar magnet is stationary with respect to the loop. However, a current is induced in the loop when a relative motion exists between the bar magnet and the loop. In particular, the galvanometer deflects in one direction as the magnet approaches the loop, and the opposite direction as it moves away.

Faraday concluded that whenever the flux of magnetic field through the area bounded by a closed conducting loop changes, an emf is produced in the loop.

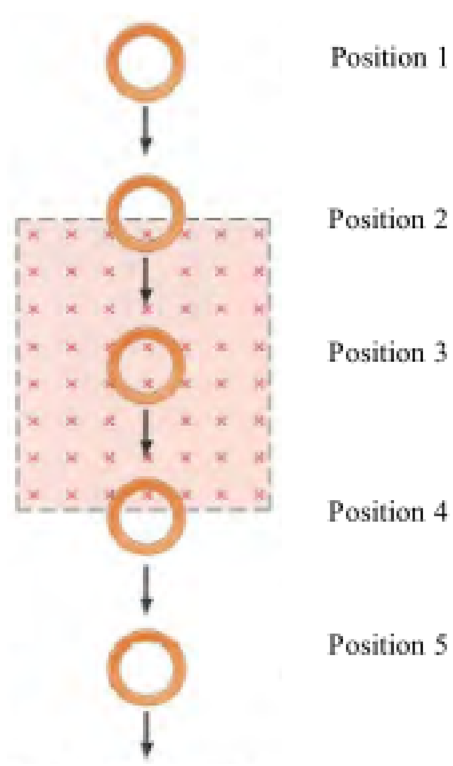
Lenz's Law (Deciding direction of inducted e.m.f.)

The induced emf resulting from a changing magnetic ftux has a polarity that leads to an induced current whose direction is such that the induced magnetic field opposes the original ftux change.We shall discuss later on how it is in accordance with conservation of energy

Illustration

In Figure there is a constant magnetic field in a rectangular region of space. This field is directed perpendicularly into the page. Outside this region there is no magnetic field. A copper ring moves through the region from position 1 to position 5. Find the induced current in the ring as it passes through positions

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5



Sol *(a) Since the field is zero outside the rectangular region, no flux passes through the ring in position 1 , there is no change in the flux through the ring, and there is no induced emf or current in the ring*

(b) In position 2 the flux increases. According to Lenz's law, the induced current must create an induced magnetic field that opposes the increase. To oppose the increase, the induced field must point opposite to the external field and, therefore, must point out of the page. for which the induced current must be counterclockwise .

(c) Here the field is not zero, Hence nonzero flux passes through the ring in position 3 but the flux through the ring is constant, and there is no induced emf or current in the ring

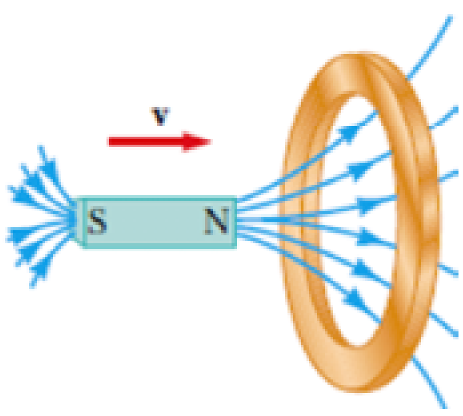
(d) In position 4 the flux decreases. According to Lenz's law, the induced current must create an induced magnetic field that opposes the decrease. To oppose the decrease, the induced field must point in the direction of external field and, therefore, must point into the page. For which the induced current must be clockwise .

(e) Since the field is zero outside the rectangular region, no flux passes through the ring in positions 5, there is no change in the flux through the ring, and there is no induced emf or current in the ring

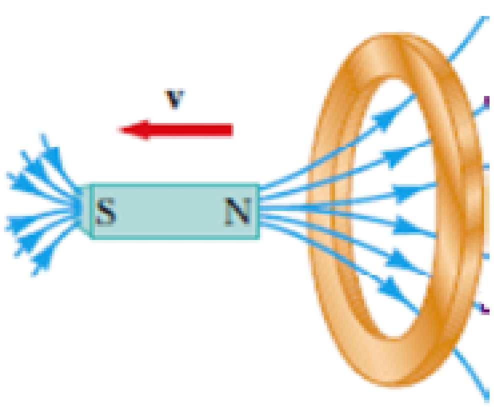
Practice Exercise

Q.1 Determine the direction of the induced current for the following situations.

(a)



(b)

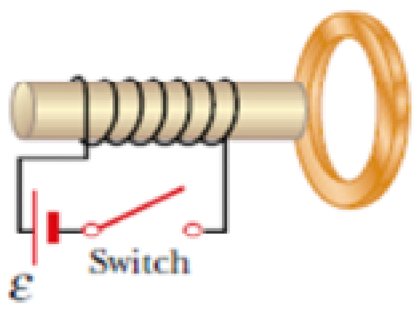


Q.2 A metal ring is placed near a solenoid, as shown in Figure. Find direction of induced current in the ring

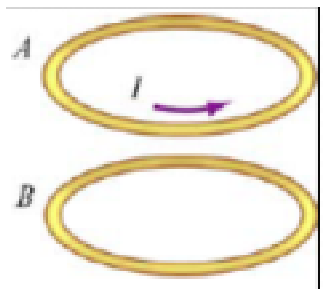
a. at the instant the switch in the circuit containing the solenoid is thrown closed,

b. after the switch has been closed for several seconds, and

c. at the instant the switch is thrown open.



Q.3 Two circular loops A and B have their planes parallel to each other, as shown in Figure.

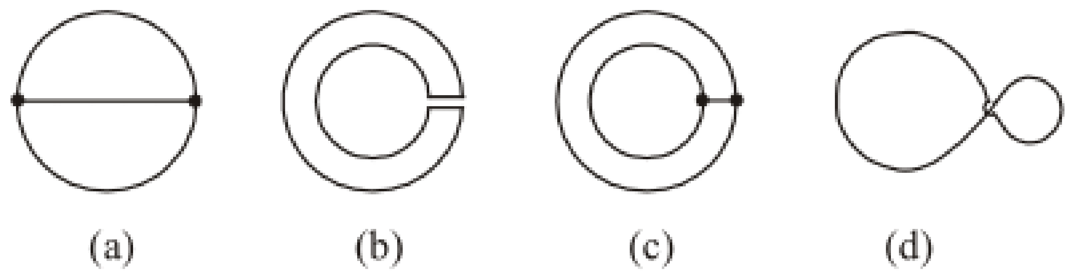


Loop A has a current flowing in the counterclockwise direction, viewed from above.

(a) If the current in loop A decreases with time, what is the direction of the induced current in loop B? Will the two loops attract or repel each other?

(b) If the current in loop A increases with time, what is the direction of the induced current in loop B? Will the two loops attract or repel each other?

Q.4 Fig. illustrates plane figures made of thin conductors which are located in a uniform magnetic field directed away from a reader beyond the plane of the drawing. The magnetic induction starts diminishing. Find how the currents induced in these loops are directed.



Answers

Q.1 (a) anticlockwise when seen from the side of magnet (b) clockwise when seen from the side of magnet

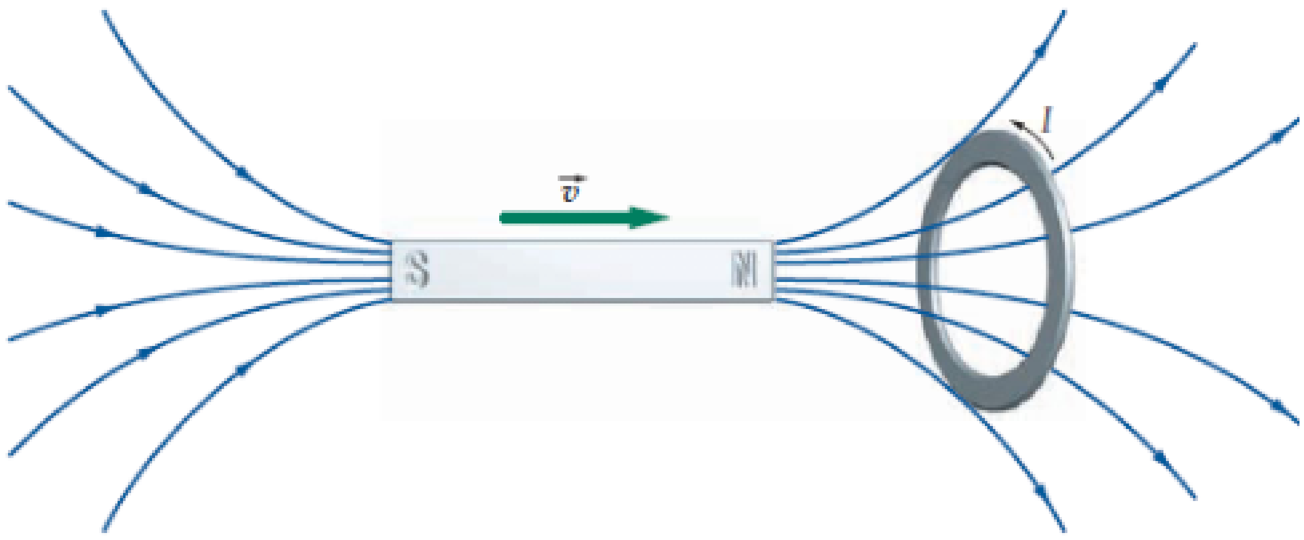
Q.2 (a) clockwise when seen from the side of solenoid (b) zero
(c) anticlockwise when seen from the side of solenoid

Q.3 (a) anticlockwise when seen from above ,attract (b) clockwise when seen from above ,repel

Q.4 (a) In the round conductor the current flows clockwise there is no current in the connector ; (b) in the outside conductor, clockwise ; (c) in both round conductors, clockwise ; no current in the connector, (d) in the left-hand side of the figure eight, clockwise

Lenz's law And conservation of energy

First figure (below) shows a bar magnet moving toward a conducting loop. It is the motion of the bar magnet to the right that induces an emf and current in the loop. Lenz's law tells us that this induced emf and current must be in a direction to oppose the motion of the bar magnet. That is, the current induced in the loop produces a magnetic field of its own, and this magnetic field must exert a force to the left on the approaching bar magnet.



Second figure shows the induced magnetic moment of the current loop when the magnet is moving toward it. The loop acts like a small magnet with its north pole to the left and its south pole to the right. Because like poles repel, the induced magnetic moment of the loop repels the bar magnet; that is, it opposes its motion toward the loop. This result means the direction of the induced current in the loop must be as shown in Second figure .

Suppose the induced current in the loop shown in Second figure was opposite to the direction shown. Then there would be a magnetic force toward the right on the approaching bar magnet, causing the bar magnet to gain speed. This gain in speed would cause an increase in the induced current, which in turn would cause the force on the bar magnet to increase, and so on. This result is too good to be true. Any time we nudge a bar magnet toward a conducting loop it would move toward the loop with ever increasing speed and with no significant effort on our part. Were this situation to occur, it would be a violation of energy conservation. The reality, however, is that energy is conserved, and Lenz's law is consistent with this reality.

Faraday's Law

Whenever the flux of magnetic field through the area bounded by a closed conducting loop changes, an emf is produced in the loop. The emf is given by

: $$\varepsilon = -\frac{d\Phi_B}{dt}$$

Note:

(1) The -ve sign is according to Lenz's law (opposition)

(2) For a coil that consists of *N* loops, the induced emf would be:

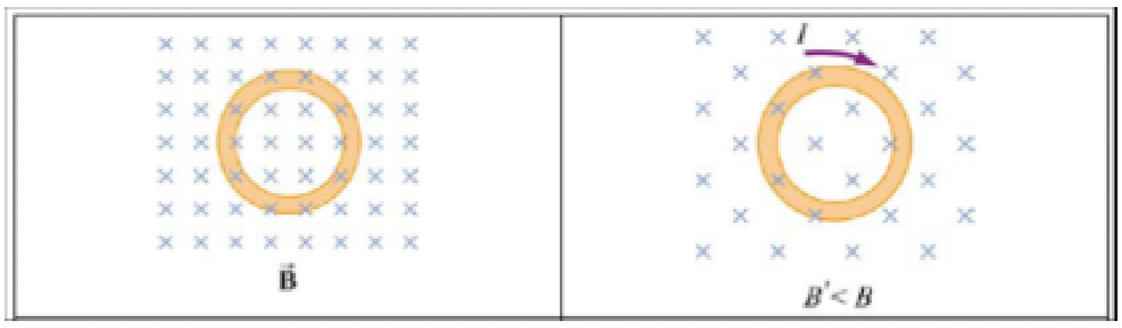
$$\varepsilon = -N\frac{d\Phi_B}{dt}$$

(3) Average emf is is given by

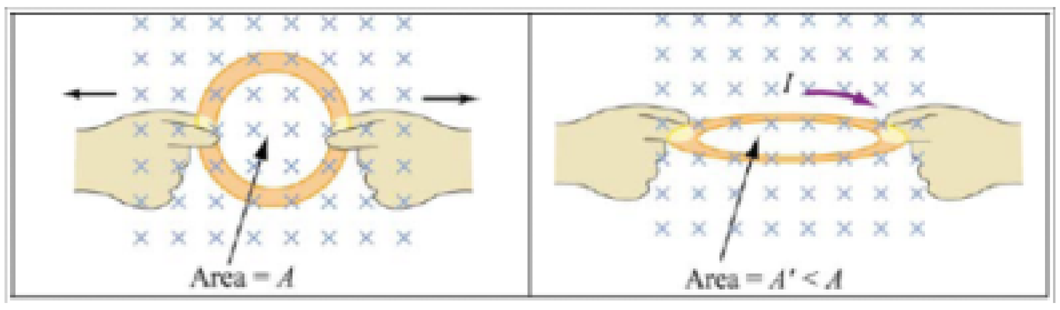
$$\varepsilon = -\frac{\Delta\Phi_B}{\Delta t}$$

(4) We can see that an emf may be induced in the following ways:

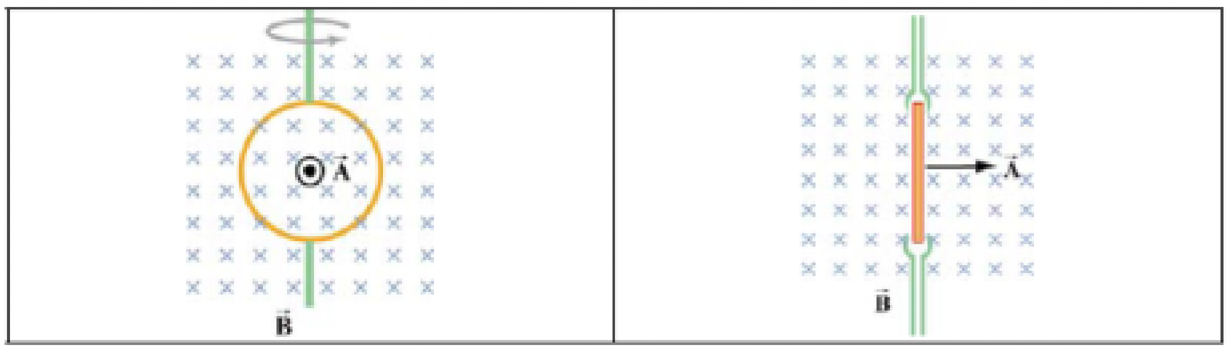
(i) by varying the magnitude of $\vec{B}$ with time (illustrated in Figure)



(ii) by varying the magnitude of $\vec{A}$, i.e., the area enclosed by the loop with time (illustrated in Figure)



(iii) varying the angle between $\vec{B}$ and the area vector $\vec{A}$ with time (illustrated in Figure)



(5) Induced current is given by

$$i = -\frac{1}{R}\frac{d\Phi_B}{dt}$$

(6) Charge flown through a coil

$$\Delta q = \int i dt = \int_{t_1}^{t_2} -\frac{1}{R}\frac{d\Phi}{dt}dt = \frac{1}{R}\int_{\Phi_1}^{\Phi_2} -d\Phi$$

$$\Delta q = \frac{\Phi_1 - \Phi_2}{R}$$

(7) Heat developed will be

$$H = \int i^2 R dt$$

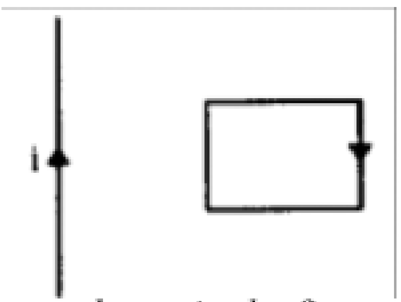
(8) You can also evaluate the direction of ξ from equation $\varepsilon = -\frac{d\Phi_B}{dt}$

The procedure to decide the direction is as follows:

Put an arrow on the loop to choose the positive sense of current. This choice is arbitrary. Using righthand thumb rule find the positive direction of the normal to the area bounded by the loop. If the fingers curl along the loop in the positive sense, the thumb represents the positive direction of the normal. Calculate the flux through the area bounded by the loop. If the flux increases with time, $\frac{d\Phi_B}{dt}$ is positive and ε is negative from equation $\varepsilon = -\frac{d\Phi_B}{dt}$. Correspondingly, the current is negative. It is, therefore, in the direction opposite to the arrow put on the loop. If the flux decreases with time, $\frac{d\Phi_B}{dt}$ is negative, ε is positive and the current is along the arrow.

Illustration :

Figure shows a conducting loop placed near a long, stright wire carrying a current i as shown. If the current increases continuously, find the direction of the induced current in the loop.



Sol. *Let us put an arrow on the loop as shown in the figure. The right-hand thumb rule shows that the positive normal to the loop is going into the plane of the diagram. Also, the same rule shows that the magnetic field at the site of the loop due to the current is also going into the plane of the diagram. Thus $\vec{B}$ and $\vec{A}$ are along the same direction everywhere so that the flux is positive. If i increases, the magnitude of Φ_B increases. Since Φ_B is positive and its magnitude increases, $\frac{d\Phi_B}{dt}$ is positive. Thus, ε is negative and hence , the current is negative. The current is, therefore, induced in the direction opposite to the arrow.*

Illustration :

The magnetic flux through each turn of a 100 turn coil is $(t^3 - 2t) \times 10^{-3}$ Wb, where t is in second.Find the induced emf at t = 2 s .

Sol.

$$\phi = (t^3 - 2t) \times 10^{-3}$$

$$\frac{d\phi}{dt} = (3t^2 - 2) \times 10^{-3}$$

$$\left.\frac{d\phi}{dt}\right|_{t=2} = (3\times4-2) \times 10^{-3} \ . \ Wb/s$$

$$= 10^{-2}\ \text{Wb/s} \quad \Rightarrow \quad e = -N\frac{d\phi}{dt}$$

$$= -100\times10^{-2}\ \text{V} \quad \Rightarrow \quad = -1\ V.$$

Illustration :

A wire in the form of a circular loop of radius 10 cm lies in a plane normal to a magnetic field of 100 T. If this wire is pulled to take a square shape in the same plane in 0.1 s, Find average induced emf in the loop is:

Sol. *According to Faraday's law of electromagnetic induction,*

$$E_{\text{induced}} = \frac{-\Delta\phi}{\Delta t} = -\frac{B(A_f - A_i)}{\Delta t}$$

Let r be the radius of circle; then side of square formed $= \frac{2\pi r}{4} = \frac{\pi r}{2}$

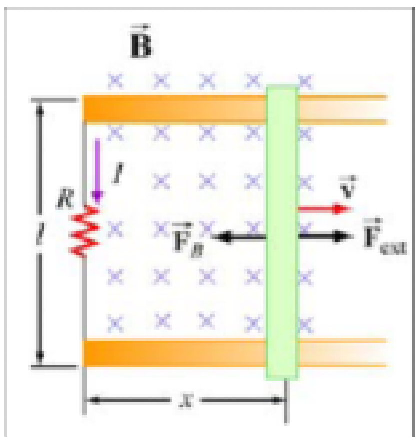
Change is area of loop $= A_i - A_f = \pi r^2 - \left(\frac{\pi r}{2}\right)^2 = \frac{\pi(4-\pi)r^2}{4}$

Hence average emf induced $= \frac{\pi(4-\pi)r^2}{4} \cdot \frac{B}{t}$

$$= \frac{\pi(4-\pi)\times(0.1)^2\times 100}{4\times 0.1} \qquad = 6.75 \text{ volt.}$$

Illustration :

A conducting bar moves through a region of uniform magnetic field $\vec{B} = -B\hat{k}$ (pointing into the page) by sliding along two frictionless conducting rails as shown in figure. An external force $\vec{F}_{ext}$ be applied so that the conductor moves to the right with a constant speed v. Find the



a) emf induced in the coil.
b) curent through the resistor and its direction.
c) External force required to move the rod with constant speed.
d) power delivered by external force
e) power dissipated in resistor.

Sol

(a) *Let us take $\vec{A}$ out of the page.*

Now magnetic flux over the coil.will be

$$\Phi_B = -B\ell x$$

emf induced in the coil.will be

$$\varepsilon = -\frac{d\Phi_B}{dt} = B\ell\frac{dx}{dt} = B\ell v$$

(b) *current through the resistor will be*

$$i = \frac{\varepsilon}{R} = \frac{B\ell v}{R}$$

(c) *Magnetic force on slider will be*

$$\vec{F}_B = i\vec{\ell} \times \vec{B} = \frac{B^2\ell^2 v}{R} \text{ (left)}$$

External force required to move the rod with constant speed. will be

$$\vec{F}_{ext} = -\vec{F}_B = \frac{B^2\ell^2 v}{R} \text{ (right)}$$

d) *power delivered by external force*

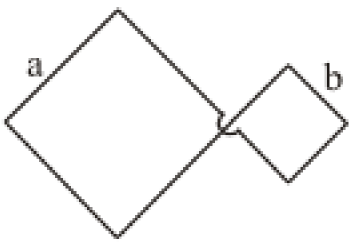
$$P_{ext} = F_{ext}v = \frac{B^2\ell^2 v^2}{R}$$

e) *power dissipated in resistor.*

$$P_R = i^2R = \frac{B^2\ell^2 v^2}{R}$$

Illustration :

. *A plane loop shown in Fig. is shaped as two squares with sides a = 20 cm and b = 10 cm and is introduced into a uniform magnetic field at right angles to the loop's plane out of the page. The magnetic induction varies with time as $B = B_0 \sin \omega t$. Find the amplitude of the current induced in the loop if its resistance per unit length is equal to ρ. The inductance of the loop is to be neglected.*



Sol. *In both the loops e.m.f. are in opposing nature so we have to subtract in writing net flux*
Let us take $\vec{A}$ out of the page for bigger part.
Now magnetic flux over the coil.will be

$$\Phi_B = (a^2 - b^2)B_0 \cos \omega t$$

emf induced in the coil.will be

$$\varepsilon = -\frac{d\Phi_B}{dt} = (a^2 - b^2)\omega B_0 \sin\omega t$$

current through the resistor will be

$$i = \frac{\varepsilon}{R} = \frac{(a^2 - b^2)\omega B_0 \sin\omega t}{(\rho)\{4(a+b)\}} = \frac{(a-b)\omega B_0 \sin\omega t}{4\rho}$$

current amplitude through the resistor will be

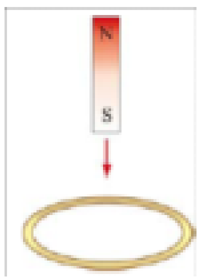
$$i_{max} = \frac{(a-b)\omega B_0}{4\rho}$$

Practice Exercise

Q.1 A bar magnet falls through a circular loop, as shown in Figure.

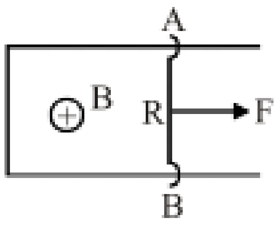
(a) Describe qualitatively the change in magnetic flux through the loop when the bar magnet is above and below the loop.

(b) Make a qualitative sketch of the graph of the induced current in the loop as a function of time, choosing *I* to be positive when its direction is counterclockwise as viewed from above.



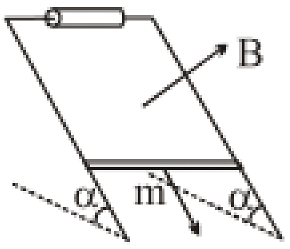
Q.2 A small coil is introduced between the poles of an electromagnet so that its axis coincides with the magnetic field direction. The cross-sectional area of the coil is equal to $S = 3.0\ mm^2$, the number of turns is $N = 60$. When the coil turns through 180° about its diameter, a ballistic galvanometer connected to the coil indicates a charge $q = 4.5\ \mu C$ flowing through it. Find the magnetic induction magnitude between the poles provided the total resistance of the electric circuit equals $R = 40\ \Omega$.

Q.3 A connector AB can slide without friction along a shaped conductor located in a horizontal plane. The connector has a length l, mass m, and resistance R. The whole system is located in a uniform magnetic field of induction B directed vertically. At the moment $t = 0$ a constant horizontal force F starts acting on the connector shifting it translationwise to the right. Find how the velocity of the connector varies with time t.

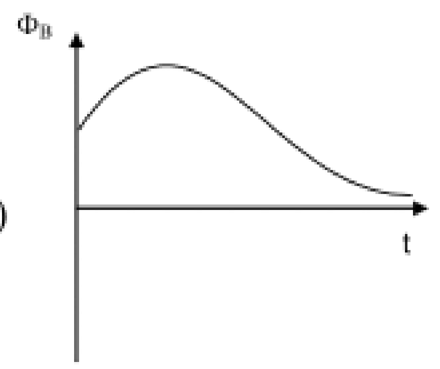


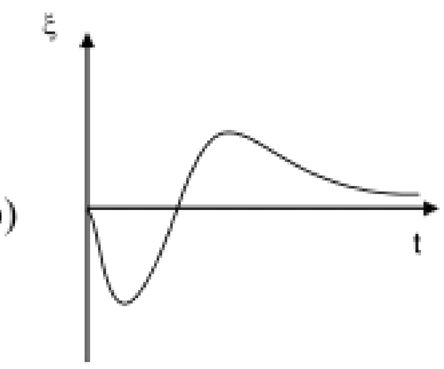
Q.4 In the previous question find the velocity of rod as a function of time, if initially it hasbeen given a velocity vo towards right and F is not acting.

Q.5 A copper connector of mass m slides down on two smooth copper bars, set at an angle α to the horizontal, due to gravity (Fig). At the top the bars are interconnected through a resistance R. The separation between the bars is equal to *l*. The system is located in a uniform magnetic field of induction B, perpendicular to the plane in which the connector slides. The resistances of the bars, the connector and the sliding contacts, as well as the self-inductance of the loop, are assumed to be negligible. Find the steady-state velocity of the connector.



Answers

Q.1 (a) 

(b) 

Q.2 B = 1/2 R/SN = 0.5 T

Q.3 $v = \frac{FR}{B^2l^2}\left(1 - e^{-\frac{B^2l^2}{mR}t}\right)$ Q.4 $v = v_0 e^{-\frac{B^2l^2}{mR}t}$ Q.5 $v = \frac{mg R \sin\alpha}{B^2 l^2}$

Generators :

One of the most important applications of Faraday's law of induction is to generators and motors. A generator converts mechanical energy into electric energy, while a motor converts electrical energy into mechanical energy.

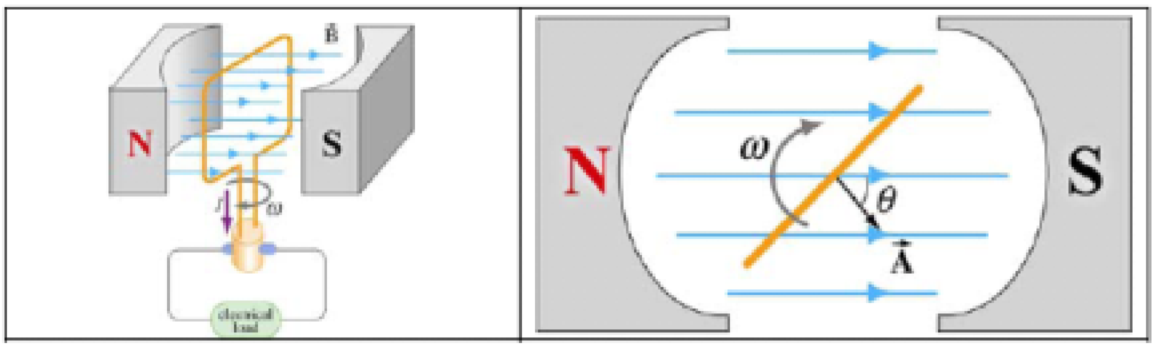


Figure : (a) A simple generator. (b) The rotating loop as seen from above.

Figure (a) is a simple illustration of a generator. It consists of an *N*-turn loop rotating in a magnetic field which is assumed to be uniform. The magnetic flux varies with time, thereby inducing an emf. From Figure (b), we see that the magnetic flux through the loop may be written as

$$\Phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta = BA\cos\omega t$$

The rate of change of magnetic flux is $\frac{d\Phi_B}{dt} = -BA\,\omega\sin\omega t$

Since there are N turns in the loop, the total induced emf across the two ends of the loop is

$$\varepsilon = -N\frac{d\Phi_B}{dt} = NBA\omega\sin\omega t$$

If we connect the generator to a circuit which has a resistance R, then the current generated in the circuit

is given by $$I = \frac{|\varepsilon|}{R} = \frac{NBA\omega}{R}\sin\omega t$$

The current is an alternating current which oscillates in sign and has an amplitude $I_0 = NBA\omega / R$. The power delivered to this circuit is $P = I|\varepsilon| = \frac{(NBA\omega)^2}{R}\sin^2\omega t$

On the other hand, the torque exerted on the loop is

$$\tau = \mu B\sin\theta = \mu B\sin\omega t$$

Thus, the mechanical power supplied to rotate the loop is

$$P_m = \tau\omega = \mu B\,\omega\sin\omega t$$

Since the dipole moment for the N-turn current loop is

$$\mu = NIA = \frac{N^2A^2B\omega}{R}\sin\omega t$$

the above expression becomes

$$P_m = \left(\frac{N^2A^2B\omega}{R}\sin\omega t\right)B\omega\sin\omega t = \frac{(NAB\omega)^2}{R}\sin^2\omega t$$

As expected, the mechanical power in put is equal to the electric power output.

Motional EMF

Consider a conducting bar of length l moving through a uniform magnetic field which points into the page, as shown in Figure. Particles with charge $q > 0$ inside experience a magnetic force $\vec{F}_B = q\vec{v}\times\vec{B}$ which tends to push them upward, leaving negative charges on the lower end.

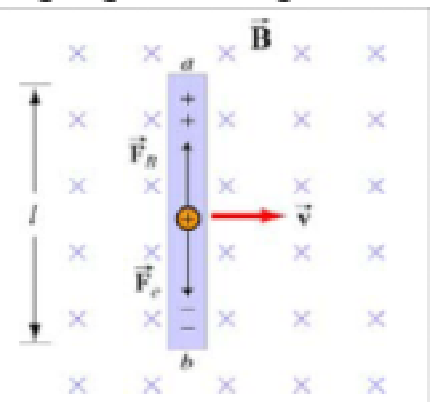


Figure : A conducting bar moving through a uniform magnetic field

The separation of charge gives rise to an electric field $\vec{E}$ inside the bar, which in turn produces a downward electric force $\vec{F}_e = q\vec{E}$. At equilibrium where the two forces cancel,

we have $qvB = qE$ or $E = vB$. Between the two ends of the conductor, there exists a potential difference given by

$$V_{ab} = V_a - V_b = \varepsilon = E\ell = B\ell v$$

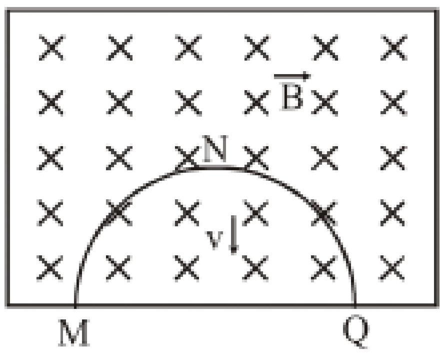
Since ε arises from the motion of the conductor, this potential difference is called the motional emf. In general, motional emf around a closed conducting loop can be written as

$$|\varepsilon| = \left| \oint (\vec{v} \times \vec{B}).d\vec{s} \right|$$

where $d\vec{s}$ is a differential length element.

Illustration :

A thin semi-circular conducting ring of radius R is falling with its plane vertical in a horizontal magnetic induction $\vec{B}$ (see figure). At the position MNQ the speed of the ring is v . Find the potential difference developed across the ring .



Sol. *The induced emf as given by Faraday's law of induction is*

$E = - B\,l\,v$

l = projection of ring perpendicular to the direction of v = 2 R

$= B \times 2\,R \times v$

$= 2\,B\,v\,R.$

Illustration :

An air-plane with 20m wing spread is flying at 250 ms^{-1} straight south parallel to the earth's surface. The earth's magnetic field has a horizontal component of $2 \times 10^{-5}\ \text{Wb}\,\text{m}^{-2}$ and the dip angle is 60°. Calculate the induced emf between the plane tips is:

Sol. *As the plane is flying horizontally it will cut the vertical component of earth's field B_V. So the emf induced between its tips,*

$e = B_V\, vl$

But as by difinition of angle of dip,

$\tan\phi = \dfrac{B_V}{B_H}$ *i.e.,* $B_V = B_H \tan\phi$

So $e = (B_H \tan\phi)vl = 2 \times 10^{-5} \times \sqrt{3} \times 250 \times 20$

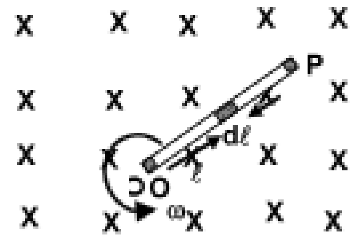
i.e., $e = (\sqrt{3}) \times 10^{-1}\,V = 0.173\,V$.

Illustration :

A copper rod of length 'ℓ' rotates at an angular velocity ω in a uniform magnetic field B as shown in figure. What is the induced emf across its ends?

Solution:

The rod is supposed to be the combination of a number of infinitesimal elements. Speed of each element is different. Consider an element at a distance ℓ from O.



Speed of this element is ωℓ, and is perpendicular to its length

$\Rightarrow \varepsilon = \int vBd\ell$

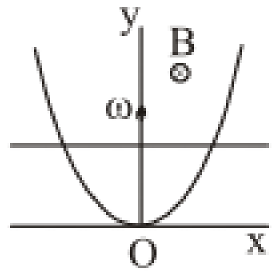
As $\quad v = \omega\ell$

$$\varepsilon = \omega B\int_0^L \ell d\ell = \frac{1}{2}\omega BL^2$$

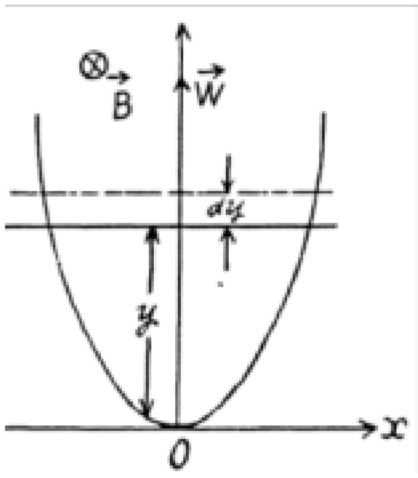
'O' turns out to be '+' and P the – ve terminal

Illustration :

A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field of induction B, the vector B being perpendicular to the x, y-plane. At the moment t = 0 a connector starts sliding translationwise from the parabola apex with a constant acceleration ω (Fig.). Find the emf of electromagnetic induction in the loop thus formed as a function of y.



Sol. Obviously, from Lenz's law, the induced current and hence the induced e.m.f. in the loop is anticlockwise.



From faraday's law of electromagnetic induction.

$$\xi_{in} = \left|\frac{d\phi}{dt}\right|$$

here, $d\phi = \vec{B}.d\vec{S} = -2B\,x\,dy$,

and from

$$y = ax^2, x = \sqrt{\frac{y}{a}}$$

Hence,

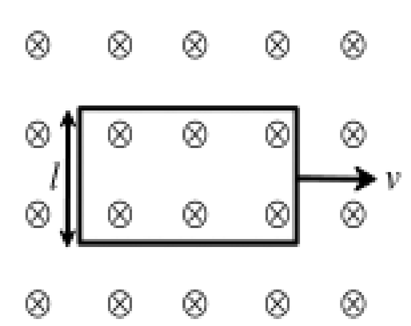
$$\xi_{in} = 2B\sqrt{\frac{y}{a}}\frac{dy}{dt} = By\sqrt{\frac{8\omega}{a}}, \text{ using } \frac{dy}{at} = \sqrt{2\omega y}$$

Alternative sol

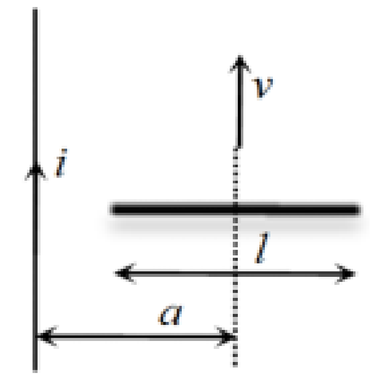
$$\varepsilon = vB\ell = vB(2x) = \left(\sqrt{2\omega y}\right)(B)(2x)$$

Practice Exercise

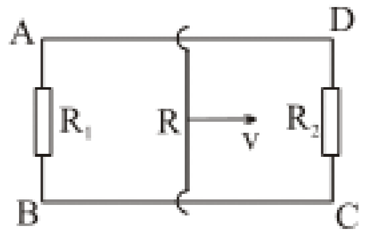
Q.1 For the situation shown, find the emf induced in the loop. Explain your answer by both flux theory and theory of motional emf.



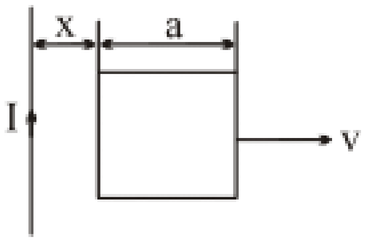
Q.2 Figure shows a straight, long wire carrying a current i and a rod of length l coplanar with the wire and perpendicular to it. The rod moves with a constant velocity v in a direction parallel to the wire. The distance of the wire from the near end of the rod is a. Find the motional emf induced in the rod.



Q.3 A rectangular loop with a sliding connector of length l is located in a uniform magnetic field perpendicular to the loop plane (Fig.). The magnetic induction is equal to B. The connector has an electric resistance R, the sides AB and CD have resistances R_1 and R_2 respectively. Neglecting the self-inductance of the loop, find the current flowing in the connector during its motion with a constant velocity v.

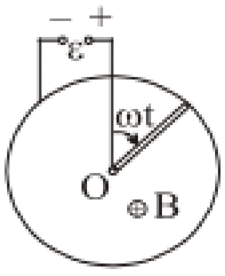


Q.4 A square frame with side a and a long straight wire carrying a current I are located in the same plane as shown in Fig. The frame translates to the right with a constant velocity v. Find the emf induced in the frame as a function of distance x.



Q.5 A spherical conducting shell is placed in a vertical time-varying uniform magnetic field. Is there an induced current along the equator?

Q.6 A metal rod of mass m can rotate about a horizontal axis O, sliding along a circular conductor of radius a (Fig.). The arrangement is located in a uniform magnetic field of induction B directed perpendicular to the ring plane. The axis and the ring are connected to an emf source to form a circuit of resistance R. Neglecting the friction, circuit inductance, and ring resistance, find the law according to which the source emf must vary to make the rod rotate with a constant angular velocity ω.



Answers

Q.1 Zero

Q.2 $\varepsilon = \dfrac{\mu_0 I v}{2\pi} \ln\left(1+\dfrac{\ell}{a}\right)$

Q.3 $I = Bvl/(R + R_m)$, where $R_m = R_1R_2/(R_1 + R_2)$

Q.4 $\xi_i = \dfrac{\mu_0}{4\pi} \dfrac{2Ia^2v}{x(x+a)}$

Q.5 Yes

Q.6 $\xi_i = \dfrac{\omega a^3B^3 + 2mg \sin \omega t}{2aB}$

Induced Electric Field

We have seen that the electric potential difference between two points *A* and *B* in an electric field $\vec{E}$ can be written as

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

When the electric field is conservative, as is the case of electrostatics, the line integral of it is path-independent, which implies $\oint \vec{E} \cdot d\vec{s} = 0$

Faraday's law shows that as magnetic flux changes with time, an induced current begins to flow. What

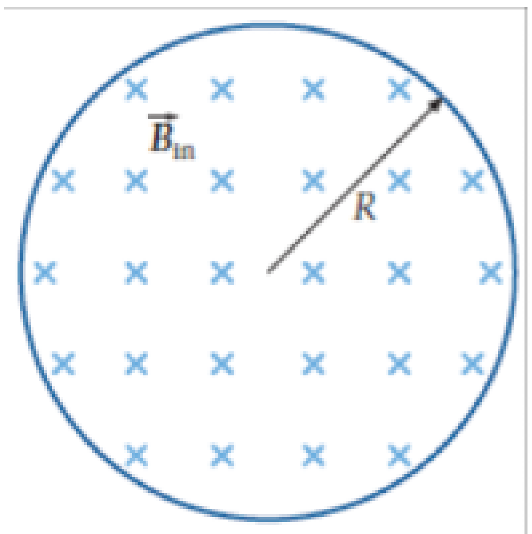
causes the charges to move? It is the induced emf which is the work done per unit charge. However, since magnetic field can do no work, as we have shown, the work done on the mobile charges must be electric, and the electric field in this situation cannot be conservative because the line integral of a conservative field must vanish. Therefore, we conclude that there is a non-conservative electric field $\vec{E}_{nc}$ associated with an induced emf : $\varepsilon = \oint \vec{E}_{nc} \cdot d\vec{s}$

Combining with Faraday's law then yields :

$$\oint \vec{E}_{nc} . d\vec{s} = -\frac{d\Phi_B}{dt}$$

The above expression implies that a changing magnetic flux will induce a non-conservative electric field which can vary with time. It is important to distinguish between the induced, non-conservative electric field and the conservative electric field which arises from electric charges.

As an example, let's consider a uniform magnetic field which points *into* the page and is confined to a circular region with radius *R*, as shown in Figure. Suppose the magnitude of $\vec{B}$ increases with time, *i.e.*, $dB / dt > 0$. Let's find the induced electric field everywhere due to the changing magnetic field.

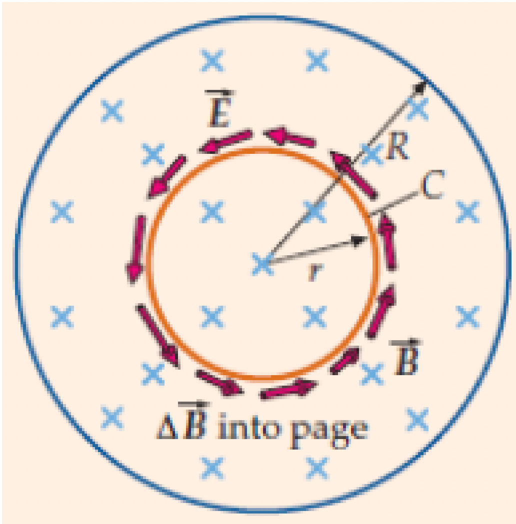


Since the magnetic field is confined to a circular region, from symmetry arguments we choose the integration path to be a circle of radius *r*. The magnitude of the induced field $\vec{E}_{nc}$ at all points on a circle is the same. According to Lenz's law, the direction $\vec{E}_{nc}$ of must be such that it would drive the induced current to produce a magnetic field opposing the change in magnetic flux. With the area vector $\vec{A}$ pointing *out* of the page, the magnetic flux is negative or inward. With $dB / dt > 0$, the inward magnetic flux is increasing. Therefore, to counteract this change the induced current must flow counterclockwise to produce more outward flux. The direction of $\vec{E}_{nc}$ is shown in Figure.

Let's proceed to find the magnitude of $\vec{E}_{nc}$.

In the region r >R

the rate of change of magnetic flux is :



$$\frac{d\Phi_B}{dt} = \frac{d}{dt}(\vec{B}\,.\,\vec{A}) = \frac{d}{dt}(-BA) = -\left(\frac{dB}{dt}\right)\pi r^2$$

Using equation, we have

$$\oint \vec{E}_{nc}.d\vec{s} = E_{nc}(2\pi r) = -\frac{d\Phi_B}{dt} = \left(\frac{dB}{dt}\right)\pi r^2$$

which implies

$$E_{nc} = \frac{r}{2}\frac{dB}{dt}$$

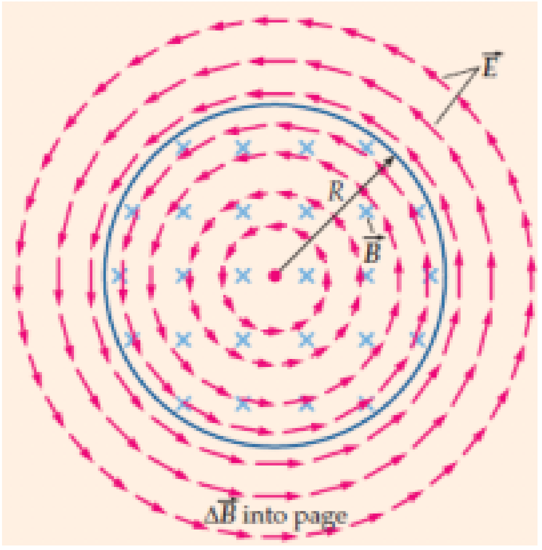
Similarly, for r > R

the induced electric field may be obtained as

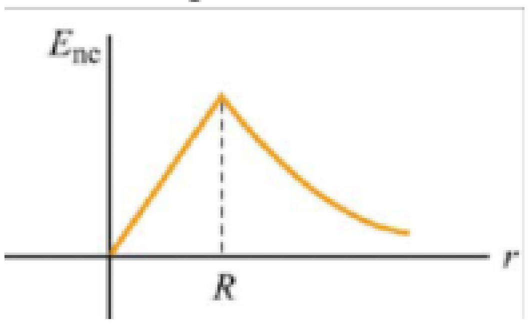
$$E_{nc}\,(2\pi r) = -\frac{d\Phi_B}{dt} = \left(\frac{dB}{dt}\right)\pi R^2$$

or $$E_{nc} = \frac{R^2}{2r}\frac{dB}{dt}$$

The pattern of E_{nc} as a function of r is shown in figure.

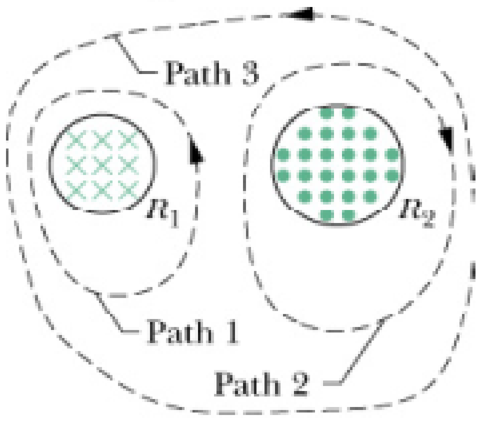


A plot of E_{nc} as a function of r is shown in figure.

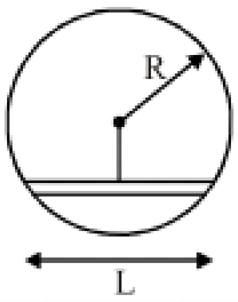


Practice Exercise

Q.1 Figure shows two circular regions of radii R_1 & R_2. (R_2 being the bigger circle).The magnetic field in both are decreasing at a rate of $\frac{dB}{dt}$. Calculate $\oint E.d\vec{l}$ for each of the three paths indicated.



Q.2 A uniform magnetic field fills a cylindrical volume of radius R. If the magnetic field is decreasing at a rate $\frac{dB}{dt}$, find induced emf at rod's end.



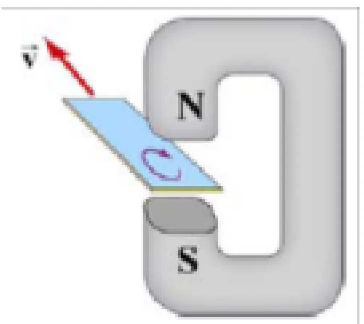
Answers

Q.1 $-\frac{dB}{dt}\pi R_1^2$ for path 1, $-\frac{dB}{dt}\pi R_2^2$ for path 2, $\frac{dB}{dt}\pi\left(R_2^2 - R_1^2\right)$ for path 3

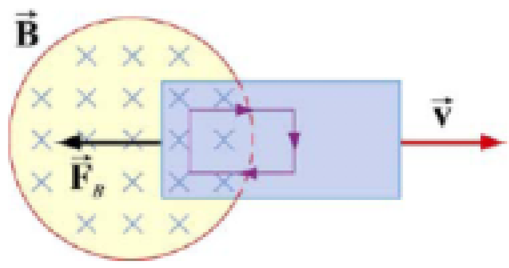
Q.2 $\frac{L}{2}\sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}$

Eddy Currents :

We have seen that when a conducting loop moves through a magnetic field, current is induced as the result of changing magnetic flux. If a solid conductor were used instead of a loop, as shown in Figure, currents can also be induced along any closed loop in the conductor. The induced current are called an *eddy current*.



The induced eddy currents also generate a magnetic force that opposes the motion, making it more difficult to move the conductor across the magnetic field (Figure).



Since the conductor has non-vanishing resistance R, Joule heating causes a loss of power by an amount $P = \varepsilon^2 / R$. Therefore, by increasing the value of R, power loss can be reduced. One way to increase R is to laminate the conducting slab, or construct the slab by using gluing together thin strips that are insulated from one another (see Figure-a). Another way is to make cuts in the slab, thereby disrupting the conducting path (Figure-b).

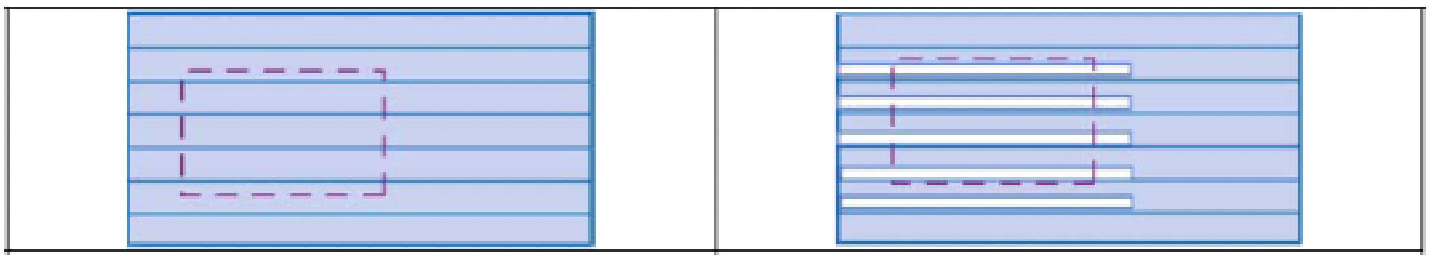
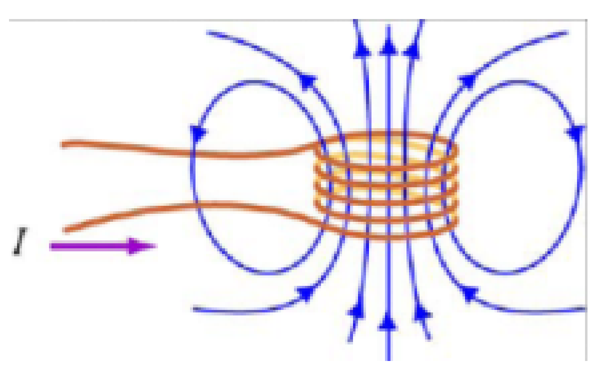


Figure (a) Figure (b)

There are important applications of eddy currents. For example, the currents can be used to suppress unwanted mechanical oscillations. Another application is the magnetic braking systems in high-speed transit cars.

Self-Inductance :

Consider again a coil consisting of *N* turns and carrying current *I* in the counterclockwise direction, as shown in Figure. If the current is steady, then the magnetic flux through the loop will remain constant. However, suppose the current *I* changes with time, then according to Faraday's law, an induced emf will arise to oppose the change. According to lenz law the induced emf will be clockwise if $\frac{dI}{dt} > 0$, and counterclockwise if $\frac{dI}{dt} < 0$. The property of the loop in which its own magnetic field opposes any change in current is called "self-inductance," and the emf generated is called the self-induced emf or back emf, which we denote as ε_L. All current-carrying loops exhibit this property. In particular, an inductor is a circuit element (symbol ) which has a large self-inductance.



Mathematically, the self-induced emf can be written as

$$\varepsilon_L = -N\frac{d\Phi_B}{dt} = -N\frac{d}{dt}\int \vec{B}\,.\,d\vec{A}$$

and is related to the self-inductance *L* by

$$\varepsilon_L = -L\frac{dI}{dt}$$

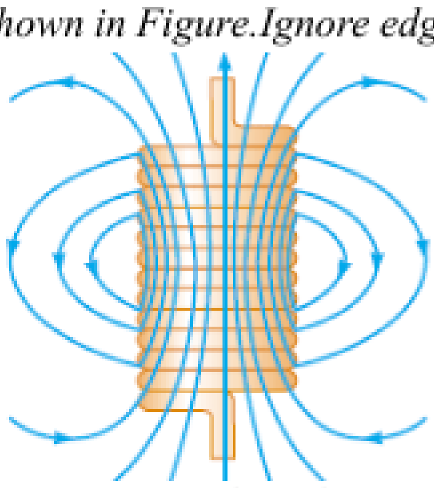
The above two expressions can be combined to yield

$$L = \frac{N\Phi_B}{I}$$

Physically, the inductance *L* is a measure of an inductor's "opposition" to the change of current; larger the value of *L*, lower the rate of change of current.

Illustration :

Compute the self-inductance of a solenoid with N turns, length ℓ, and radius R with a current I flowing through each turn, as shown in Figure.Ignore edge effects



Sol. *Ignoring edge effects and applying Ampere's law, the magnetic field inside a solenoid is given by Eq. :*

$$\vec{B} = \frac{\mu_0 NI}{\ell}\hat{k} = \mu_0 nI\ \hat{k}$$

where $n = N/\ell$ is the number of turns per unit length. The magnetic flux through each turn is

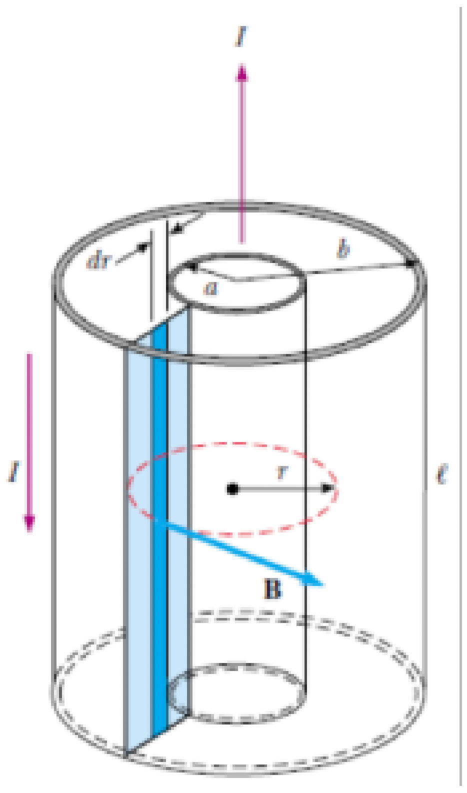
$$\Phi_B = BA = \mu_0 nI\,.\,(\pi R^2) = \mu_0 nI\pi R^2$$

Thus, the self-inductance is $L = \dfrac{N\Phi_B}{I} = \mu_0 n^2 \pi R^2 \ell$

Note We see that L depends only on the geometrical factors (n, R and ℓ) and is independent of the current I.

Illustration :

A long coaxial cable as consisting of two thin concentric cylindrical conducting shells of radii a and b and length l, . The conducting shells carry the same current I in opposite directions. Imagine that the inner conductor carries current to a device and that the outer one acts as a return path carrying the current back to the source. Calculate the self-inductance L of this cable.



Sol. *Imagine a thin radial slice of the coaxial cable, such as the shaded rectangle in Figure. The magnetic field is perpendicular to the rectangle of length l and width b-a, the cross section of interest. Divide this rectangle into strips of width dr. We see that the area of each strip is ldr and that the flux through each strip is*

$$B\,dA = Bl\,dr.$$

Hence, we find the total flux through the entire cross section by integrating

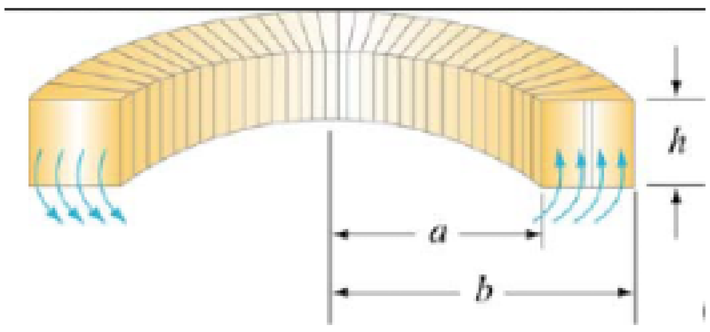
$$\Phi_B = \int B\,dA = \int_a^b \frac{\mu_0 I}{2\pi r}\,\ell\,dr = \frac{\mu_0 I \ell}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 I \ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

Using this result, we find that the self-inductance of the cable is

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 \ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

Practice Exercise

Q.1 Calculate the self-inductance of a toroid which consists of N turns and has a rectangular cross section, with inner radius a, outer radius b and height h, as shown in Figure.



Q.2 How many metres of a thin wire are required to manufacture a solenoid of length $l_0 = 100$ cm and inductance L = 1.0 mH if the solenoid's cross-sectional diameter is considerably less than its length? Ignore edge effects

Ans. $l = \sqrt{4\pi l_0 L/\mu_0} = 100$ m

Practice Exercise

Q.1 $L = \frac{\mu_0 N^2}{\ell}\frac{h}{b-a}$ Q.2 $l = \sqrt{4\pi l_0 L/\mu_0} = 100$ m

Energy Stored in Inductor

Since an inductor in a circuit serves to oppose any change in the current through it, work must be done by an external source such as a battery in order to establish a current in the inductor. From the work-energy theorem, we conclude that energy can be stored in an inductor.

The power, or rate at which an external emf ε_{ext} works to overcome the self-induced emf ε_L and pass current I in the inductor is

$$P_L = \frac{dW_{ext}}{dt} = I\varepsilon_{ext}$$

If only the external emf and the inductor are present, then $\varepsilon_{ext} = -\varepsilon_L$ which implies

$$P_L = \frac{dW_{ext}}{dt} = -I\varepsilon_L = +IL\frac{dI}{dt}$$

If the current is increasing with $dI/dt > 0$, then $P > 0$ which means that the external source is doing positive work to transfer energy to the inductor. Thus, the internal energy U_B of the inductor is increased. On the other hand, if the current is decreasing with $dI/dt < 0$, we then have $P < 0$. In this case, the external source takes energy away from the inductor, causing its internal energy to go down. The total work done by the external source to increase the current form zero to I is then :

$$W_{ext} = \int dW_{ext} \int_0^I LI'dI' = \frac{1}{2}LI^2$$

This is equal to the magnetic energy stored in the inductor :

$$U_B = \frac{1}{2}LI^2$$

We comment that from the energy perspective there is an important distinction between an inductor and a resistor. Whenever a current I goes through a resistor, energy flows into the resistor and dissipates in the form of heat regardless of whether I is steady or time-dependent (recall that power dissipated in a resistor is $P_R = IV_R = I^2R$). On the other hand, energy flows into an ideal inductor only when the current is varying with $dI/dt > 0$. The energy is not dissipated but stored there; it is released later when the current decreases with $dI/dt < 0$. If the current that passes through the inductor is steady, then there is no change in energy since $P_L = LI\,(dI/dt) = 0$. Also note that there is an important distinction between an inductor and a resistor. The potential difference across a resistor depends on I, while the potential difference across an inductor depends on dI/dt. The self-induced emf does not oppose the current itself, but the rate of change of current.

Energy density in a magnetic field

A long solenoid with length ℓ and a radius R consists of N turns of wire. A current I passes through the coil. We have to find the energy density in the magnetic field..
Energy of the solenoid is given by

$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 I^2 \pi R^2 \ell$$

The result can be expressed in terms of the magnetic field strength $B = \mu_0 nI$:

$$U_B = \frac{1}{2\mu_0}(\mu_0 nI)^2(\pi R^2 \ell) = \frac{B^2}{2\mu_0}(\pi R^2 \ell)$$

Since $\pi R^2 \ell$ is the volume within the solenoid, and the magnetic field inside is uniform, the term

Hence magnetic energy density (i.e.the energy per unit volume) of the magnetic field will be given by

$$u_B = \frac{U_B}{\pi r^2 \ell} = \frac{B^2}{2\mu_0}$$

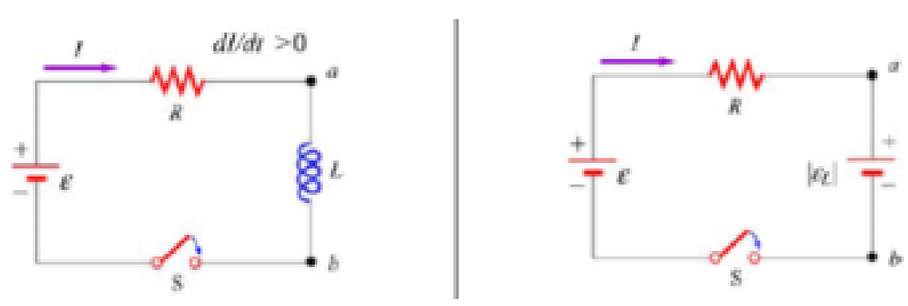
may be identified as the magnetic energy density, or the energy per unit volume of the magnetic field. The above expression holds true even when the magnetic field is non-uniform. The result can be compared with the energy density associated with an electric field:

$$u_E = \frac{\varepsilon_0 E^2}{2}$$

Growth and decay of current in L-R circuit

Growth of Current :

Consider the *RL* circuit shown in Figure. At $t = 0$ the switch is closed. We find that the current does not rise immediately to its maximum value ε / R. This is due to the presence of the self-induced emf in the inductor.



RL circuit is described by the following differential equation:

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

The above equation can be rewritten as

$$\frac{dI}{I - \varepsilon / R} = -\frac{dt}{L/R} \quad \Rightarrow \quad \frac{dI}{I - \varepsilon / R} = -\frac{dt}{\tau}$$

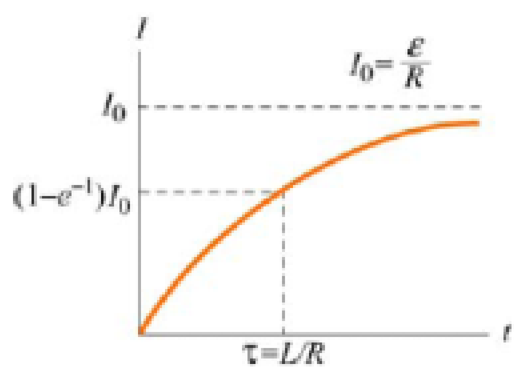
where $\tau = \dfrac{L}{R}$

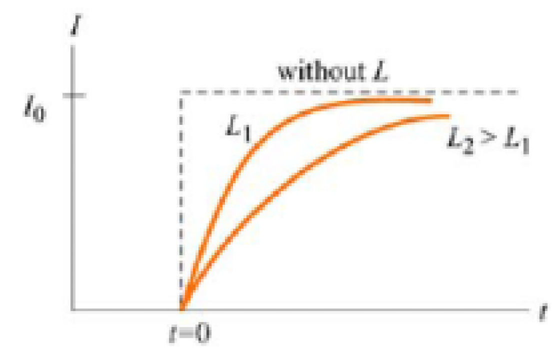
is the time constant of the RL circuit. The qualitative behaviour of the current as a function of time is depicted in figure.

Integrating over both sides with prper limits we get

$$\int_0^I \frac{dI}{I - \varepsilon / R} = -\int_0^t \frac{dt}{\tau}$$

$$I = \frac{\varepsilon}{R}(1 - e^{-t/\tau})$$





Note that after a sufficiently long time, the current reaches its equilibrium value ε/R. The time constant τ is a measure of how fast the equilibrium state is attained; the larger the value of *L*, the longer it takes to build up the current. A comparison of the behavior of current in a circuit with or without an inductor is shown in Figure .

Similarly, the magnitude of the self-induced emf can be obtained as

$$|\varepsilon_L| = \left|-L\frac{dI}{dt}\right| = \varepsilon e^{-t/\tau}$$

which is at a maximum when $t = 0$ and vanishes as t approaches infinity. This implies that a sufficiently long time after the switch is closed, self-induction disappears and the inductor simply acts as a conducting wire connecting two parts of the circuit.

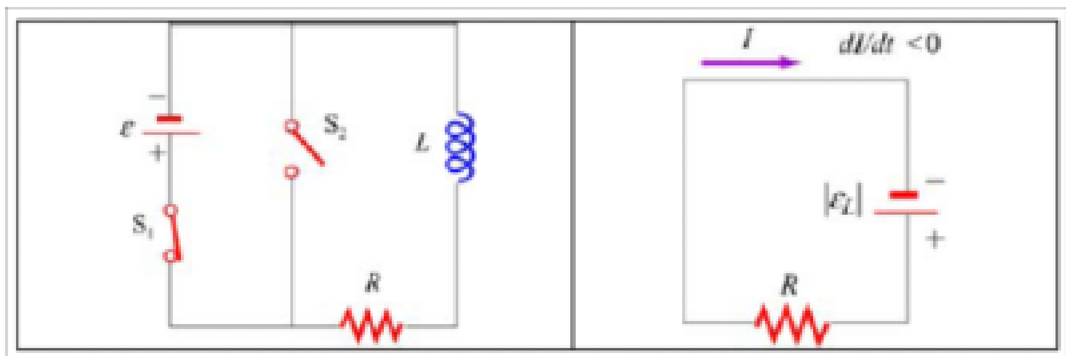
To see that energy is conserved in the circuit, we multiply Eq. by I and obtain

$$P= \qquad I\varepsilon = I^2R + LI\frac{dI}{dt}$$

The left-hand side represents the rate at which the battery delivers energy to the circuit. On the other hand, the first term on the right-hand side is the power dissipated in the resistor in the form of heat, and the second term is the rate at which energy is stored in the inductor. While the energy dissipated through the resistor is irrecoverable, the magnetic energy stored in the inductor can be released later.

Decay of Current :

Next we consider the *RL* circuit shown in Figure. Suppose the switch S_1 has been closed for a long time so that the current is at its equilibrium value ε/R. What happens to the current when at $t = 0$ switches S_1 is opened and S_2 closed?



Writhing loop equation

$$-L\frac{dI}{dt} - IR = 0$$

which can be rewritten as

$$\frac{dI}{I} = -\frac{dt}{L/R} \qquad \Rightarrow \qquad \frac{dI}{I} = -\frac{dt}{\tau}$$

Integrating over both sides with prper limits we get

$$\int_0^I \frac{dI}{I} = -\int_0^t \frac{dt}{\tau}$$

We get

$$I = \frac{\varepsilon}{R} e^{-t/\tau}$$

where $\tau = L/R$ is the same time constant as in the case of rising current. A plot of the current as a function of time is shown in Figure.

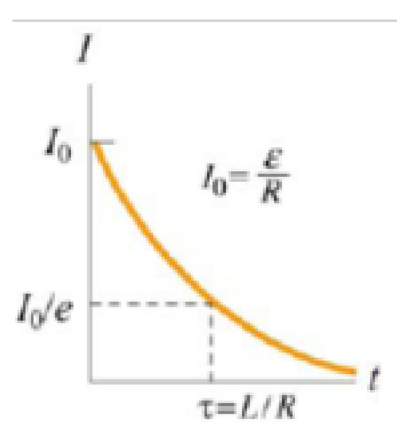


Illustration :

The inductance of a closed-packed coil of 400 turns is 8 mH. A current of 5 mA is passed through it.Find the magnetic flux through the coil .

Sol.

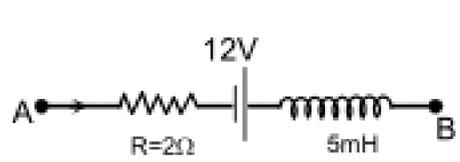
$$L = \frac{N\phi}{i}$$

$$\Rightarrow \qquad 8\times10^{-3} = \frac{400\times\phi}{5\times10^{-3}}$$

$$\Rightarrow \qquad \phi = \frac{40\times10^{-6}}{400}\,\text{Wb} = 10^{-7}\,\text{Wb}$$

Illustration :

A current of I = 10 A is passed through the part of a circuit shown in the figure. What will be the potential difference between A and B when I is decreased at constant rate of 10^2 amp/s, at the beginning?



Sol. *Applying the law of potential between the points A and B we obtain,*

$$V_B - V_A = -IR + E - L\frac{di}{dt}$$

$$\Rightarrow \quad V_B - V_A = -10 \times 2 + 12 - 5 \times 10^{-3} \times (-10^2) = -20 + 12 + 0.5 = -7.5 \text{ volt.}$$

Illustration :

Two *different coils have self inductances $L_1 = 8$ mH and $L_2 = 2$ mH. The current in one coil is increased at a constant rate. The current in the second coil is also increased at the same constant rate. At a certain instant of time, the power given to the two coils is the same. At that time, the current, the induced voltage and the energy stored in the first coil are i_1, V_1 and W_1 respectively. Corresponding values for the second coil at the same instant are i_2, V_2 and W_2 respectively. Then find*

(a) $\frac{i_1}{i_2}$; *(b)* $\frac{W_1}{W_2}$.

Sol. $e_1 = L_1\frac{di_1}{dt}$ and $e_2 = L_2\frac{di_2}{dt} \Rightarrow \frac{di_1}{dt} = \frac{di_2}{dt} \Rightarrow \frac{e_1}{e_2} = \frac{L_1}{L_2} = \frac{8}{2}$

$$\Rightarrow \frac{e_2}{e_1} = \frac{1}{4}.$$

Power given to the coils are same.

So $e_1 i_1 = e_2 i_2 \Rightarrow \frac{i_1}{i_2} = \frac{e_2}{e_1} = \frac{1}{4}$

$$\textit{Energy} = \frac{1}{2}Li^2$$

$$\Rightarrow \frac{W_1}{W_2} = \frac{\frac{1}{2}L_1 i_1^2}{\frac{1}{2}L_2 i_2^2} = \frac{L_1}{L_2}\left(\frac{i_1}{i_2}\right)^2 = 4 \times \frac{1}{16} = \frac{1}{4}.$$

Illustration :

A coil of inductance 8.4 mH and resistance 6 Ω is connected to a 12 V battery. Find the time after which current in the coil is approximately 1.0 A

Sol. *Current developed with time in a coil of inductance*

$$I = \frac{V}{R}(1 - e^{-t/\tau}) \textit{ where } \tau = L/R$$

we have

$$\tau = \frac{8.4\,\text{mH}}{6\Omega} = 1.4\text{ms}$$

Hence

$$1\,A = \left(\frac{12\text{V}}{6\Omega}\right)(1 - e^{-t/1.4\text{ms}})$$

$$\Rightarrow \quad e^{-t/1.4ms} = 1 - \frac{1}{2} = \frac{1}{2} \quad \Rightarrow \quad -t/1.4ms = \ln\left(\frac{1}{2}\right) = -0.693$$

$$\Rightarrow \quad t = (1.4 \times 0.693)ms = 0.97ms \approx 1\ ms.$$

Illustration :

The current in an L – R circuit builds up to $3/4^{th}$ of its steady state value in 4 seconds. Find the time constant of this circuit .

Sol. $I = I_0(1 - e^{-t/\tau})$ *where* $\tau \rightarrow$ *time constant*

$$\therefore \quad \frac{3}{4}I_0 = I_0(1 - e^{-t/\tau})$$

$$\Rightarrow \quad \frac{3}{4} = 1 - e^{-t/\tau} \quad \Rightarrow \quad e^{-t/\tau} = \frac{1}{4} \quad \Rightarrow \quad \frac{-t}{\tau}\ln e = \ln\frac{1}{4}$$

$$\Rightarrow \quad \frac{-4}{\tau} = -2\ln 2 \quad \Rightarrow \quad \tau = \frac{2}{\ln 2}.$$

Practice Exercise

Q.1 Consider the circuit shown in figure below :
Determine the current through each resistor
(a) immediately after the switch is closed.
(b) a long time after the switch is closed.
Suppose the switch is reopened a long time after it's been closed. What is each current
(c) immediately after it is opened?
(d) after a long time?

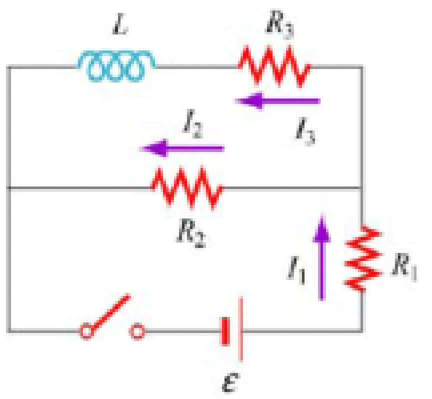
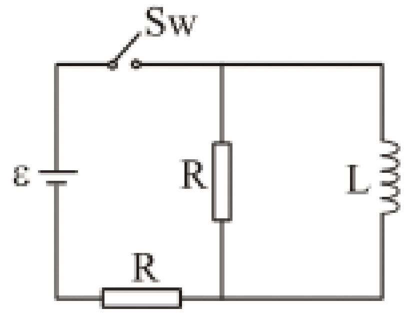


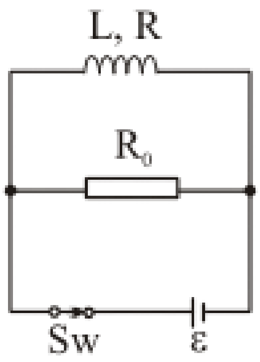
Figure *RL* circuit

Q.2 A closed circuit consists of a source of constant emt ξ and a choke coil of inductance L connected in series. The active resistance of the whole circuit is equal to R. At the moment $t = 0$ the choke coil inductance was decreased abruptly η times. Find the current in the circuit as a function of time t. Instruction. During a stepwise change of inductance the total magnetic flux (flux linkage) remains constant.

Q.3 Find the time dependence of the current flowing through the inductance L of the circuit shown in Fig. after the switch Sw is shorted at the moment $t = 0$.



Q.4 A coil of inductance $L = 2.0\ \mu H$ and resistance $R = 1.0\ \Omega$ is connected to a source of constant emf $\xi = 3.0$ V (Fig.). A resistance $R_0 = 2.0\ \Omega$ is connected in parallel with the coil. Find the amount of heat generated in the coil after the switch Sw is disconnected. The internal resistance of the source is negligible.



Practice Exercise

Q.1 (a) $I_1 = I_2 = \dfrac{\varepsilon}{R_1 + R_2}$

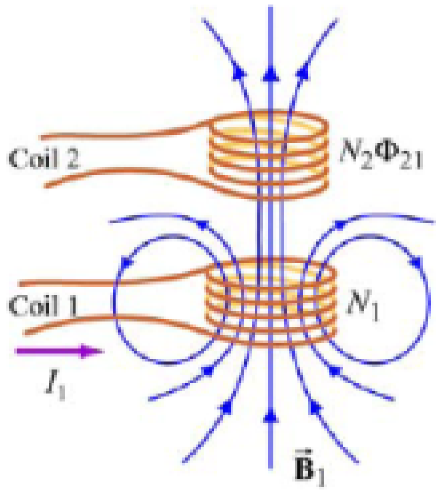
(b) $I_1 = \dfrac{(R_2 + R_3)\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}$, $I_2 = \dfrac{R_3\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}$, $I_3 = \dfrac{R_2\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}$

(c) $I_3 = -I_2 = \dfrac{R_2\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}$ (d) $I_1 = I_2 = I_3 = 0.$

Q.2 $I = \dfrac{\xi}{R}\,[1 + (\eta - 1)\,e^{-t\eta R/L}$ Q.3 $I = \dfrac{\xi}{R}\,(1 - e^{-tR/2L})$ Q.4 $Q = \dfrac{L\xi^2}{2R^2\,(1 + R_0/R)} = 3\mu J$

Mutual Inductance

Suppose two coils are placed near each other, as shown in Figure.



The first coil has N_1 turns and carries a current I_1 which gives rise to a magnetic field $\vec{B}_1$. Since the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let Φ_{21} denote the magnetic flux through one turn of coil 2 due to I_1. Now, by varying I_1 with time, there will be an induced emf associated with the changing magnetic flux in the second coil :

$$e_{21} = -N_2 \frac{d\Phi_{21}}{dt} = -\frac{N_2 d}{dt} \int_{\text{coil 2}} \vec{B}_1 . d\vec{A}_2$$

The time rate of change of magnetic flux Φ_{21} in coil 2 is proportional to the time rate of change of the current in coil 1 :

$$N_2 \frac{d\Phi_{21}}{dt} = M_{21} \frac{dI_1}{dt}$$

where the proportionality constant M_{21} is called the mutual inductance. It can also be written as

$$M_{21} = \frac{N_2\Phi_{21}}{I_1}$$

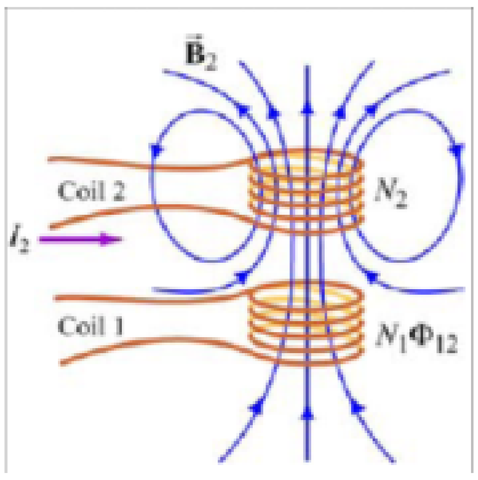
The SI unit for inductance is the henry (H) :

$$1 \text{ henry} = 1 \text{ H} = 1 \text{ T.m}^2/\text{A}$$

We shall see that the mutual inductance M_{21} depends only on the geometrical properties of the two coils such as the number of turns and the radii of the two coils.

In a similar manner, suppose instead there is a current I_2 in the second coil and it is varying with time (Figure). Then the induced emf in coil 1 becomes

$$\varepsilon_{12} = -N_1 \frac{d\Phi_{12}}{dt} = -\frac{N_1 d}{dt} \int_{\text{coil 1}} \vec{B}_2 . d\vec{A}_1$$



This changing flux in coil 1 is proportional to the changing current in coil 2,

$$N_1 \frac{d\Phi_{12}}{dt} = M_{12} \frac{dI_2}{dt}$$

where the proportionality constant M_{12} is another mutual inductance and can be written as

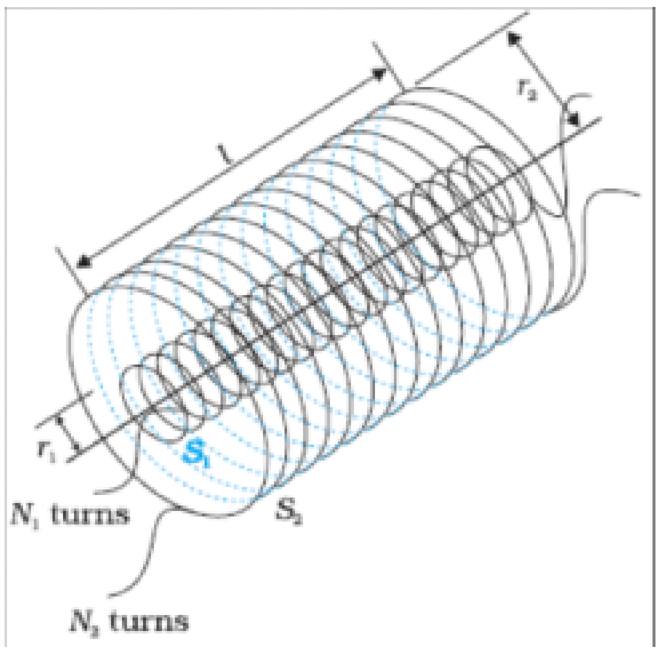
$$M_{12} = \frac{N_1\Phi_{12}}{I_2}$$

According to *reciprocity theorem*

$$M_{12} = M_{21} \equiv M$$

Illustration :

Consider Fig. which shows two long co-axial solenoids each of length l. We denote the radius of the inner solenoid S_1 by r_1 and the number of turns per unit length by n_1. The corresponding quantities for the outer solenoid S_2 are r_2 and n_2, respectively. Let N_1 and N_2 be the total number of turns of coils S_1 and S_2, respectively. Calculate the mutual-inductance M between the solenoids. Neglect the edge effects



Sol *When a current I_2 is set up through S_2, it in turn sets up a magnetic flux through S_1.*
The magnetic field due to the current I_2 in S_2 is
$B=\mu_0 n_2 I_2$.
The resulting flux linkage with coil S_1 is,
$\Phi_1 = (n_1 l)(\pi r_1^2)(\mu_0 n_2 I_2)$
Hence the mutual-inductance M between the solenoids will be
$M=\Phi_1/I_2=\mu_0 \pi n_1 n_2 r_1^2 l$

Illustration :

The coefficient of mutual induction between the primary and secondary of a transformer is 5 H. Calculate the induced emf in the secondary when 3 ampere current in the primary is cut off in 2.5 10^{-4} second.

Sol. *Induced emf in the secondary* $\varepsilon_S = -M\frac{di_P}{dt} = -5\frac{3}{1/4000} = -6 \times 10^4$ V

The negative sign merely indicates that the emf opposes the change.

Illustration :

A small square loop of wire of side l is placed inside a large square loop of wire of side L (L>>l). The loops are co-planar and their centers coincide. Find the mutual inductance of the system .

Sol. *Magnetic field produced by a current in a large square loop of wire at its center*

$$B = \frac{2\sqrt{2}\mu_0 i}{\pi L}$$

The magnetic flux ϕ_{12} *that links big loop with the small square loop of side l (l<<L) is*

$$\phi_{12} = B(l^2) = \frac{2\sqrt{2}\mu_0 i}{\pi}\left(\frac{l^2}{L}\right),$$

$\therefore$ *The mutual inductance*

$$M_{12} = \frac{\phi_{12}}{i} = \frac{2\sqrt{2}\mu_0 i}{\pi}\left(\frac{l^2}{L}\right)$$

Practice Exercise

Q.1 Calculate the mutual inductance of a long straight wire and a rectangular frame with sides a and b. The frame and the wire lie in the same plane, with the side b being closest to the wire, separated by a distance l from it and oriented parallel to it.

Q.2 Two thin concentric wires shaped as circles with radii a and b lie in the same plane. Allowing for a << b, find:

(a) their mutual inductance;

(b) the magnetic flux through the surface enclosed by the outside wire, when the inside wire carries a current I.

Q.3 There are two stationary loops with mutual inductance L_{12}.. The current in one of the loops starts to be vang as $I_1 = \alpha t$, where α is a constant, t is time. Find the time dependence I_2 (t) of the current in the other loop whose inductance is L_2 and resistance R.

Answers

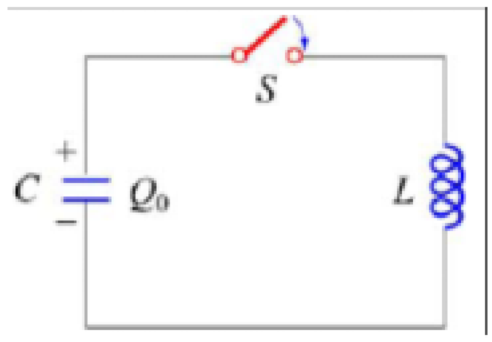
Q.1 $L_{12} = \frac{\mu_0 b}{2\pi}\ ln\left(1 + \frac{a}{l}\right)$

Q.2 (a) $L_{12} \approx \mu_0 \pi a^2 / 2b$; $\Phi_{21} = \mu_0 \pi a^2 I / 2b$

Q.3 $I_2 = \frac{\alpha L_{12}}{R}\ (1 - e^{-t/\tau})$ where $\tau = L_2/R$

LC Oscillations :

Consider an *LC* circuit in which a capacitor is connected to an inductor, as shown in Figure.



Suppose the capacitor initially has charge Q_0. When the switch is closed, the capacitor begins to discharge and the electric energy is decreased. On the other hand, the current created from the discharging process generates magnetic energy which then gets stored in the inductor. In the absence of resistance, the total energy is transformed back and forth between the electric energy in the capacitor and the magnetic energy in the inductor. This phenomenon is called electromagnetic oscillation.

The total energy in the *LC* circuit at some instant after closing the switch is

$$U = U_C + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2$$

The fact that U remains constant implies that

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0$$

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

where

$$I = -\frac{dQ}{dt}$$

and $$\frac{dI}{dt} = -\frac{d^2Q}{dt^2}$$

Notice the sign convention we have adopted here. The negative sign implies that the current I is equal to the rate of decrease of charge in the capacitor plate immediately after the switch has been closed.

The general solution to equation is

$$Q = Q_0 \cos(\omega_0 t + \phi)$$

where Q_0 is the amplitude of the charge and ϕ is the phase. The angular frequency ω_0 is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The corresponding current in the inductor is

$$I = -\frac{dQ}{dt} = \omega_0 Q_0 \sin(\omega_0 t + \phi) = I_0 \sin(\omega_0 t + \phi)$$

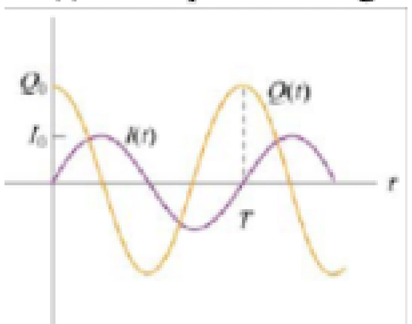
where $I_0 = \omega_0 Q_0$.

From the initial conditions $Q(\text{at } t = 0) = Q_0$ and $I(\text{at } t = 0) = 0$, the phase ϕ can be determined to $\phi = 0$. Thus, the solutions for the charge and the current in our LC circuit are

$$Q(t) = Q_0 \cos \omega_0 t$$

and $$I(t) = I_0 \sin \omega_0 t$$

The time dependence of Q(t) and I(t) are depicted in figure.

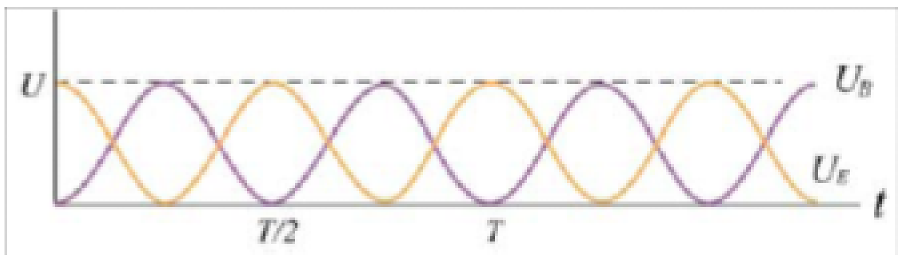


Using Eqs., we see that at any instant of time, the electric energy and the magnetic energies are given by

$$U_E = \frac{Q^2(t)}{2C} = \left(\frac{Q_0^2}{2C}\right)\cos^2 \omega_0 t$$

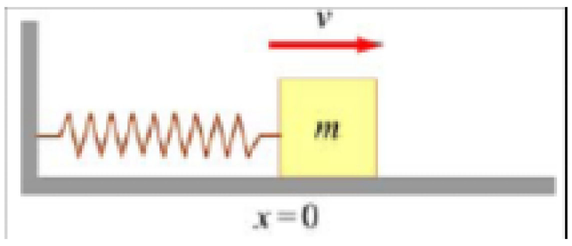
and $$U_B = \frac{1}{2} LI^2(t) = \frac{LI_0^2}{2} \sin^2 \omega t = \frac{L(-\omega_0 Q_0)^2}{2} \sin^2 \omega_0 t = \left(\frac{Q_0^2}{2C}\right)\sin^2 \omega_0 t$$

The electric and magnetic energy oscillation is illustrated in figure.



Mechanical analogy

The mechanical analog of the *LC* oscillations is the mass-spring system, shown in Figure.



If the mass is moving with a speed *v* and the spring having a spring constant *k* is displaced from its

equilibrium by x, then the total energy of this mechanical system is

$$U = K + U_{sp} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

where K and U_{sp} are the kinetic energy of the mass and the potential energy of the spring, respectively. In the absence of friction, U is conserved and we obtain

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

Using $\quad v = \frac{dx}{dt} \quad$ and $\quad \frac{dv}{dt} = \frac{d^2x}{dt^2}$,

the above equation may be rewritten as

$$m\frac{d^2x}{dt^2} + kx = 0$$

The general solution for the displacement is

$$x = x_0 \cos(\omega_0 t + \phi)$$

where $\quad \omega_0 = \sqrt{\frac{k}{m}}$

is the angular frequency and x_0 is the amplitude of the oscillations. Thus, at any instant in time, the energy of the system may be written as

$$U = \frac{1}{2}mx_0^2\omega_0^2 \sin^2(\omega_0 t + \phi) + \frac{1}{2}kx_0^2\cos^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}kx_0^2[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)) = \frac{1}{2}kx_0^2$$

In figure we illustrate the energy oscillations in the LC circuit and the mass spring system (harmonic oscillator).

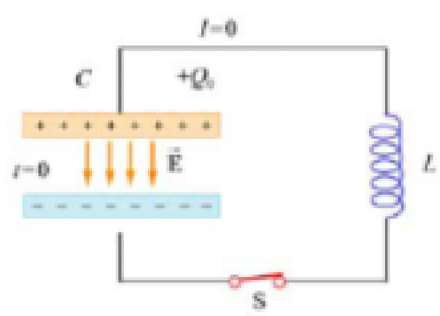
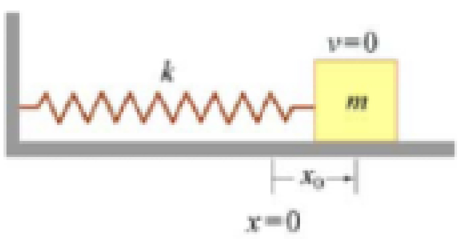
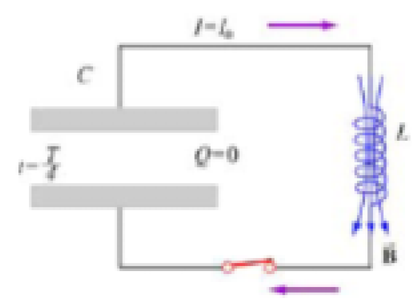
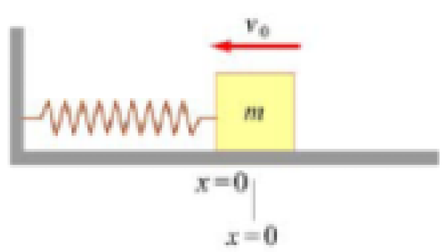
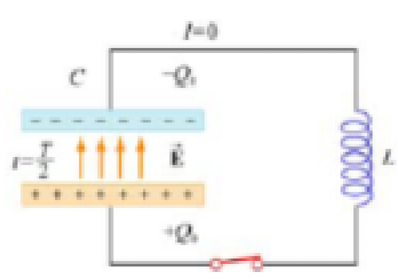
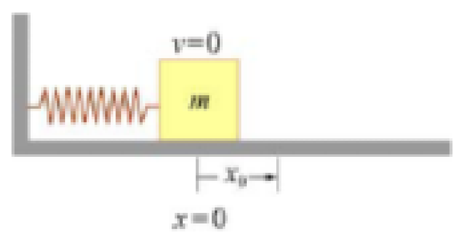
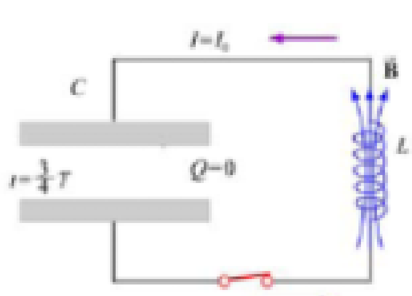
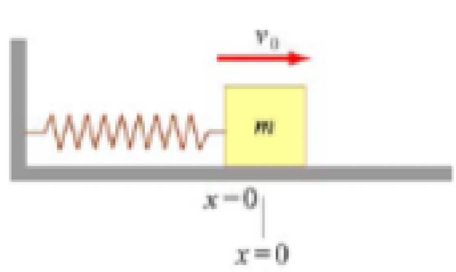
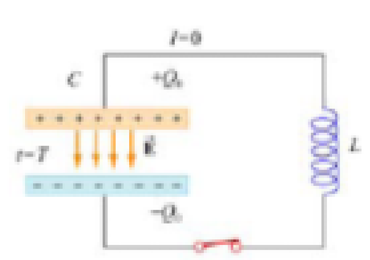
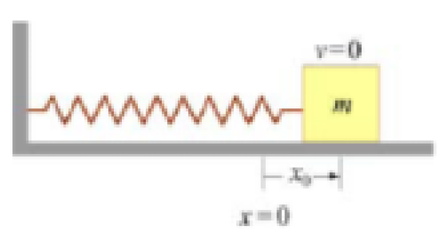
LC Circuit	Mass-spring System	Energy
		 U_E U_B U_{SP} K
		 U_E U_B U_{SP} K
		 U_E U_B U_{SP} K
		 U_E U_B U_{SP} K
		 U_E U_B U_{SP} K

Illustration :

An inductor of inductance 2 mH is connected across a charged capacitor of 5 μF. Let q denote the instantaneous charge on the capacitor, and i the current in the circuit. Maximum value of q is $Q = 200$ μC.

(a) When $q = 100$ μC, what is the value of $|di/dt|$?

(b) When $q = 200$ μC, what is the value of i?

Sol. *(a) Charge stored in the capacitor oscillates simple harmonically as,*

$$Q = Q_0 \sin(\omega t \pm \phi) \text{ and}$$

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2\times10^{-3}\text{H})(5.0\times10^{-6}\text{F})}} = 10^4\,\text{s}^{-1}$$

At $t = 0$, *if* $Q = Q_0$, *then*

$$Q(t) = Q_0 \cos \omega t \text{ and if } Q = Q_0/2, \text{ then } \cos \omega t = ½$$

$$\frac{di(t)}{dt} = -Q_0\omega^2\cos(\omega t) = (200\times10^{-6}\text{C})(10^4\,\text{s}^{-1})^2\left(\frac{1}{2}\right) = 10^4 \text{ A/s}$$

(b) When the energy of the capacitor is maximum, the energy stored in the inductor will be zero.

$$\text{i.e. } \frac{1}{2}Li^2 = 0 \Rightarrow i = 0$$

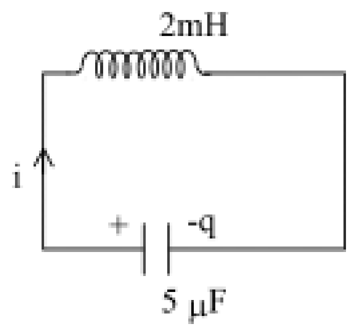


Illustration :

A capacitor of capacitance 2 μF is charged to a potential difference of 12 V. It is then connected across an inductor of inductance 0.6 mH. What is the current in the circuit at a time when the potential difference across the capacitor is 6.0 V?

Sol. *As the capacitor is charged to a potential difference of 12 V, the initial charge on the capacitor is*

$$q_0 = CV_0 = 2\times10^{-6}\times12 \text{ C.} \qquad \text{. . . (1)}$$

At any instant as the capacitor discharges through the inductor (LC circuit), the instantaneous charge on the capacitor is given by

$$q = q_0 \cos \omega t \qquad \text{. . . (2) [because at } t = 0,\ q = q_0]$$

But $q = CV$. . . . (3)

where V is the potential difference at the instant 't'.

From (1) and (3) we obtain $\frac{q}{q_0} = \frac{V}{V_0}$

Putting the value of V and V_0 *we obtain*

$$\frac{q}{q_0} = \frac{1}{2} \quad \Rightarrow \quad \cos\omega t = \frac{1}{2} \quad \Rightarrow \quad \omega t = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \omega t = /3 \text{ rad} \qquad \text{. . . (4)}$$

$$\text{Here } \omega = \frac{1}{\sqrt{LC}} = \frac{1}{\left[0.6\times10^{-3}\times2\times10^{-6}\right]^{\frac{1}{2}}}$$

$$\Rightarrow \quad \omega = \frac{10^5}{2\sqrt{3}}\ rad/sec. \qquad \ldots (5)$$

The current through the circuit at that instant is given by,

$$i = \frac{d}{dt}[q_0 \cos \omega t]$$

$$\Rightarrow \quad i = -q_0 \omega \sin \omega t$$

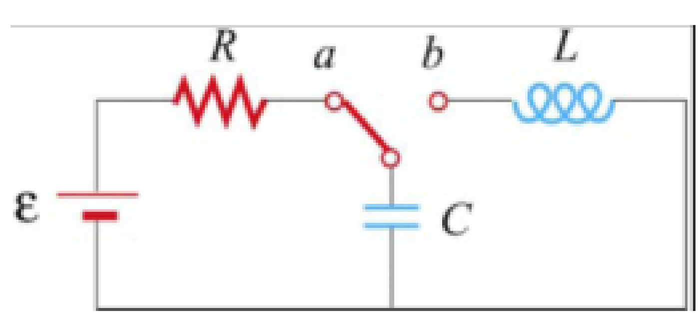
Putting the value of q_0 from (1), ω from (5) and ωt from (4) we obtain.

$$|\, i \,| = 2 \times 10^{-7} \times 12 \times \frac{10^5}{2\sqrt{3}} \sin(\pi/3)$$

$$= 0.6\ A$$

Practice Exercise

Q.1 Consider the circuit shown in Figure. Suppose the switch which has been connected to point *a* for a long time is suddenly thrown to *b* at $t = 0$.



Find the following quantities :

(a) the frequency of oscillation of the *LC* circuit.

(b) the maximum charge that appears on the capacitor.

(c) the maximum current in the inductor.

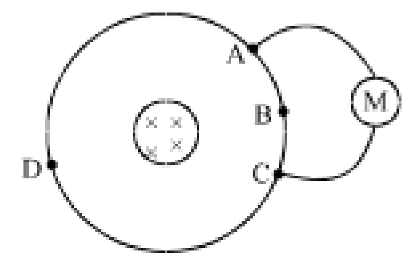
(d) the total energy the circuit possesses at any time *t*.

Answer

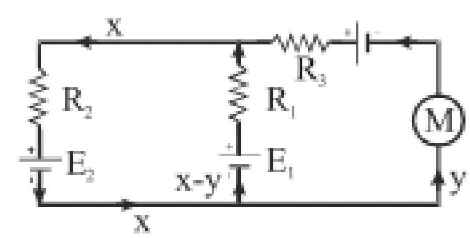
Q.1 (a) $f = \frac{1}{2\pi\sqrt{LC}}$ (b) $Q = C\varepsilon$ (c) $I_0 = \varepsilon\sqrt{\frac{C}{L}}$ (d) $\frac{1}{2}C\varepsilon^2$

Solved Examples

Q.1 A variable magnetic field creates a constant emf E in a conductor ABCDA. The resistances of portion ABC, CDA and AMC are R_1, R_2 and R_3 respectively. What current will be shown by meter M? The magnetic field is concentrated near the axis of the circular conductor.



Sol. Let E_1 and E_2 be the emfs developed in ABC and CDA, respectively. Then $E_1 + E_2 = E$. There is no net emf in the loop AMCBA as it does not enclose the magnetic field. If E_3 is the emf in AMC then $E_1 - E_3 = 0$. The equivalent circuit and distribution of current is shown in figure.



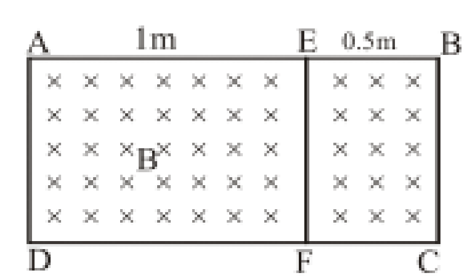
By the loop rule

$$R_1(x-y) + R_2x = E_1 + E_2 = E$$

and $$R_3y - R_1(x-y) = E_3 - E_1 = 0$$

Solving for y, $y = \dfrac{ER_1}{R_1R_2 + R_2R_3 + R_3R_1}$.

Q.2 A rectangular frame ABCD made of a uniform metal wire has a straight connection between E & F made of the same wire as shown in the figure. AEFD is a square of side 1 m & EB = FC = 0.5 m. The entire circuit is placed in a steadily increasing uniform magnetic field directed into the place of the paper & normal to it. The rate of change of the magnetic field is 1 T/s, the resistance per unit length of the wire is 1 Ω/m. Find the current in segments AE, BE & EF.



Sol. Induced e.m.f.

$$= -\frac{d\Phi}{dt} = -\frac{d}{dt}(BA) = -A\frac{dB}{dt}$$

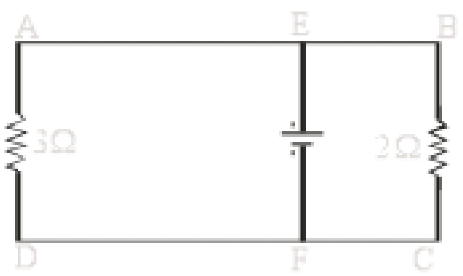
$\therefore$ Induced e.m.f. in AEFD $= 1 \times 1 = 1V$ (area $= 1\ m^2$)

Induced e.m.f. in EBCF $= \frac{1}{2} \times 1 = 0.5V$ (area $= \frac{1}{2}m^2$)

Total induced e.m.f $= 1 + 0.5 = 1.5V$

Given that resistance per unit length of the wire is $1\Omega/m$. Hence the equivalent circuit take the form as shown in figure. The resistances 3Ω and 2Ω are in parallel. Hence their equivalent resistance would

$$\frac{(3\times 2)}{(3+2)}=\left(\frac{6}{5}\right)$$

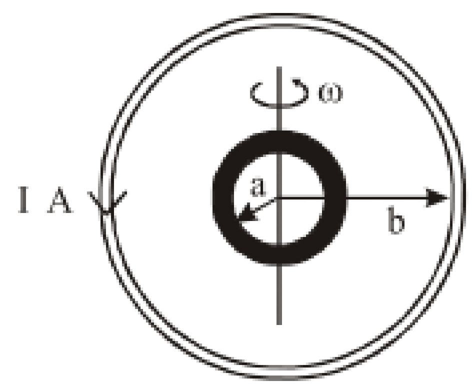


$\therefore$ Current from $EF=\frac{1.5}{(6/5)}=\frac{5}{4}$ amp.

Current in $AE=\frac{5}{4}\times\frac{2}{5}=\frac{1}{2}$ amp.

Current in $BE=\frac{5}{4}\times\frac{3}{5}=\frac{3}{4}$ amp.

Q.3 A very small circular loop of area $5\times 10^{-4}\,m^2$, resistance 2 ohm and negligible inductance is initially coplanar and concentric with a much larger fixed circular loop of radius 0.1 m. A constant current of 1 Amp. is passed in the bigger loop and the smaller loop is rotated with angular velocity ω rad/s about a diameter. Calculate (a) the flux linked with the smaller loop (b) induced emf and induced current in the smaller loop as a function of time.



Sol. (a) The situation is shown in figure. The field at the center of larger loop,

$$B_1=\frac{\mu_0}{4\pi}\frac{2\pi I}{R}=10^{-7}\frac{2\pi\times 1}{0.1}=2\pi\times 10^{-6}\,\frac{Wb}{m^2}$$

is initially along the normal to the area of smaller loop. Now as the smaller loop (and hence normal to its plane) is rotating at angular velocity ω, so in time t it will turn by an angle $\theta=\omega t$ w.r. to $\vec{B}$ and hence the flux linked with the smaller loop at time t,

$$\phi_2=B_1S_2\cos\theta=(2\pi\times 10^{-6})(5\times 10^{-4})\cos\omega t$$

i.e., $\phi_2=\pi\times 10^{-9}\cos\omega t$ Wb

(b) The induced emf in the smaller loop,

$$e_2=-\frac{d\phi_2}{dt}=-\frac{d}{dt}(\pi\times 10^{-9}\cos\omega t)$$

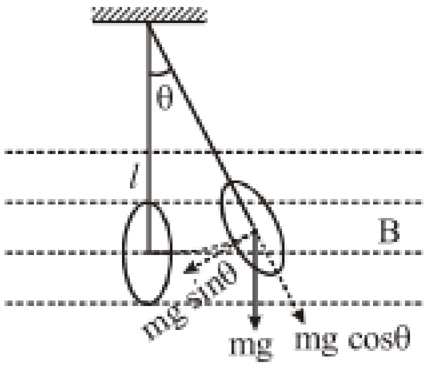
i.e., $e_2=\pi\times 10^{-9}\omega\sin\omega t$ volt

(c) The induced current in the smaller loop,

$$I_2=\frac{e_2}{R}=\frac{1}{2}\pi\omega\times 10^{-9}\sin\omega t$$

Q.4 A wire frame of area A and resistance R is suspended freely from a 0.392 m long thread. There is a uniform magnetic field of B tesla and the plane of wire-frame is perpendicular to the magnetic field. The frame is forcely made to oscillate using external force with small angular amplitude θ_0 along the direction of magnetic field according to the law $\theta = \theta_0 \sin\omega t$. The plane of the frame is always along the direction of thread and does not rotate about it. What is the inducted e.m.f. in wire-frame as a function of time? Also find the maximum current is the frame.

Sol. The situation is shown in figure. The instantaneous flux through the frame when displaced through an angle θ is given by



Equation of S.H.M.

$$\theta = \theta_0 \sin\omega t$$

Now

$$\Phi = BA\cos\theta$$

Instantaneous induced e.m.f.

$$e = -\frac{d\Phi}{dt} = BA\sin\theta\frac{d\theta}{dt} = BA\theta\frac{d\theta}{dt} \qquad (\because \sin\theta = \theta)$$

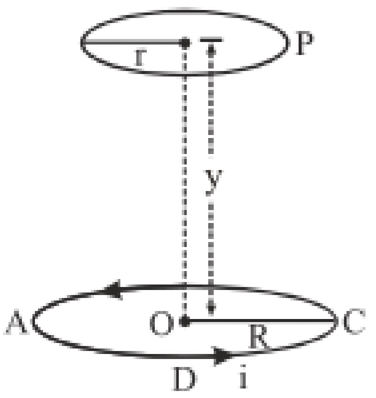
$$e = BA(\theta_0 \sin\omega t)\frac{d}{dt}(\theta_0 \sin\omega t)$$

$$= BA\theta_0 \sin\omega t \quad \theta_0\omega\cos\omega t$$

$$= \frac{1}{2}BA\omega\theta_0^2 \sin 2\omega t$$

and $\quad i_{max} = BA\omega\theta_0^2 \sin 2\omega t / 2R$

Q.5 A coil A C D of radius R and number of turns n carries a current i amp. and is placed in the plane of paper. A small conducting ring P of radius r is placed at a distance y from the centre and above the coil A C D. Calculate the induced e.m.f. produced in the ring when the ring is allowed to fall freely. Express induced e.m.f. in terms of speed of the ring.



Sol. We know that magnetic induction at some point on the axis of a current carrying coil at a distance y from its centre is given by

$$B = \frac{\mu_0}{4\pi} \times \frac{2\pi n i R^2}{(R^2 + y^2)^{3/2}}$$

Now, the magnetic flux linked with ring P is

$$\Phi = BA = \frac{\mu_0}{4\pi} \times \frac{2\pi n i r^2 R^2}{(R^2 + y^2)^{3/2}} \times \pi r^2 = \frac{\mu_0}{2} \times \frac{\pi n i R^2 r^2}{(R^2 + y^2)^{3/2}}$$

Let the ring P falls with velocity v. Now y varies with time as

$$y = y_0 - vt$$

The induced e.m.f. in the ring

$$e = \frac{d\Phi}{dt} = \frac{\mu_0}{2} \times \pi n i R^2 r^2 \frac{d}{dt}(R^2 + y^2)^{-3/2} = -\frac{3}{2}\mu_0 \pi n i R^2 r^2 (R^2 + y^2)^{-5/2} . 2y\left(\frac{dy}{dt}\right)$$

$$= -\frac{3}{2}\frac{\mu_0 \pi n i R^2 r^2}{(R^2 + y^2)^{5/2}} y(-v) = \frac{3}{2}\frac{\mu_0 \pi n i R^2 r^2 y v}{(R^2 + y^2)^{5/2}}.$$

Q.6 An infinitesimally small bar magnet of dipole moment $\overrightarrow{M}$ is pointing and moving with the speed v in the x-direction. A small closed circular conducting loop of radius a and of negligible self-inductance lies in the y-z plane with its center at x = 0, and its axis coinciding with the x-axis. Find the force opposing the motion of the magnet, if the resistance of the loop is R. Assume that the distance x of the magnet from the center of the loop is much greater than a.

Sol. Field due to the bar magnet at distance x (near the loop)

$$B = \frac{\mu_0}{4\pi} \frac{2M}{x^2}$$

$\Rightarrow$ Flux lined with the loop : $\phi = BA = \pi a^2 . \frac{\mu_0}{4\pi}\frac{2M}{x^3}$

emf induced in the loop : $e = -\frac{d\phi}{dt} = \frac{\mu_0}{4\pi}\frac{6\pi M a^2}{x^4}\frac{dx}{dt}$

$$= \frac{\mu_0}{4\pi}\frac{6\pi M a^2}{x^4} v.$$

$\Rightarrow$ Induced current : $i = \frac{e}{R} = \frac{\mu_0}{2\pi}\frac{3\pi M a^2}{R x^4}.v. = \frac{3\mu_0 M a^2}{R x^4}$

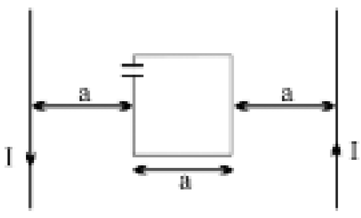
Let F = force opposing the motion of the magnet

Power due to the opposing force = Heat dissipated in the coil per second.

$$\Rightarrow \quad Fv = i^2 R \Rightarrow \quad F = \frac{i^2 R}{v} = \left(\frac{\mu_0}{4\pi}\right)^2 \times \left(\frac{6\pi M a^2}{R x^4}\right)^2 \times v^2 \times \frac{R}{v}$$

$$F = \frac{9}{4}\frac{\mu_0^2 M^2 a^4 v}{R x^4}.$$

Q.7 A square loop of side 'a' with a capacitor of capacitance C is located between two current carrying long parallel wires as shown. The value of I in the wire is given as $I = I_0 \sin\omega t$.

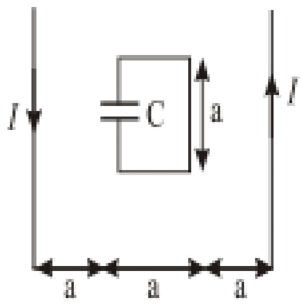


(a) calculate maximum current in the square loop.

(b) Draw a graph between charge on the lower plate of the capacitor v/s time. [JEE 2003]

Sol. (a) $I = I_0 \sin \omega t$

Flux linked with square loop



$$\int d\phi = \int_a^{2a} \frac{\mu_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{3a - x}\right] a dx$$

$$= \frac{\mu_0 Ia}{2\pi}[\ln x - \ln(3a - x)\,|_a^{2a}$$

$$= \frac{\mu_0 Ia}{2\pi}[\ln 2a - \ln a - \ln a + \ln 2a]$$

$$= \frac{\mu_0 Ia}{2\pi} 2\ln 2 = \frac{\mu_0 Ia}{\pi}\ln 2$$

Charge an capacitor $= \left|C\dfrac{d\phi}{dt}\right| = C\dfrac{\mu_0 a}{\pi}\ln 2\dfrac{dI}{dt}$

$$= \frac{\mu_0 Ca}{\pi}\ln 2(I_0\omega)\cos\omega t \qquad \ldots(i)$$

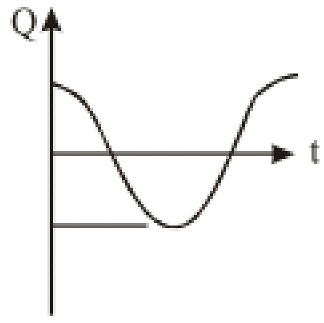
$$= \frac{\mu_0 C(I_0\omega)a}{\pi}\ln 2\cos\omega t \qquad \ldots(ii)$$

Maximum current $I = \left|\dfrac{dq}{dt}\right|_{max} = \dfrac{\mu_0 C I_0 \omega^2 a}{\pi}\ln 2$

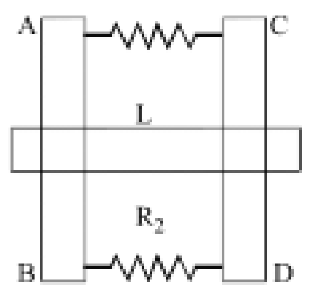
(b) The graph for charge and time can be drawn from equation (i) as shown in figure.

$\dfrac{\mu_0 C I_0 \omega a}{\pi}\ln 2$

$-\dfrac{\mu_0 C I_0 \omega a}{\pi}\ln 2$.



Q.8 Two parallel vertical metallic rails AB & CD are separated by 1 m. They are connected at the two ends by resistance R_1 & R_2 as shown in the figure. A horizontally metallic bar L of mass 0.2 kg slides without friction, vertically down the rails under the action of gravity. There is a uniform horizontal magnetic field of 0.6T perpendicular to the plane of the rails, it is observed that when the terminal velocity is attained, the power dissipated in R_1 & R_2 are 0.76 W & 1.2 W respectively. Find the terminal velocity of bar L & value R_1 & R_2.



Sol. At terminal velocity

$$mg = I\ell B \quad \Rightarrow \quad 0.2 \times 9.8 = I \times 1 \times 0.6 \quad \Rightarrow \quad I = \frac{1.96}{0.6}\ \text{Amp.} \qquad \text{..... (1)}$$

$$Ie = (0.76 + 1.2) = 1.96 \qquad \text{..... (2)}$$

From (1) and (2)

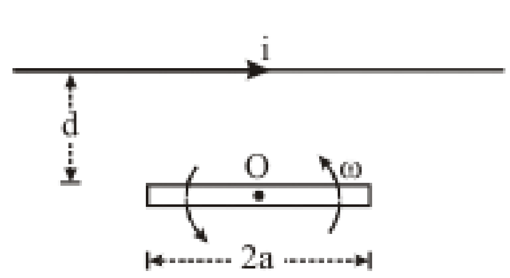
$$\frac{1.96}{0.6}e = 1.96 \quad \Rightarrow \quad e = 0.6\ \text{volt}$$

$$P_1 = \frac{e^2}{R_1} = 0.76 \quad \Rightarrow \quad R_1 = \frac{e^2}{0.76} = \frac{0.36}{0.76} = 0.47\ \Omega$$

$$P_2 = \frac{e^2}{R_2} = 1.2 \quad \Rightarrow \quad R_2 = \frac{e^2}{1.2} = 0.3\ \Omega$$

$$P = Fv \quad \Rightarrow \quad 1.56 = mg\,v \quad \Rightarrow \quad 1.96 = 1.96\ v \quad \Rightarrow \quad v = 1\ \text{m/s}$$

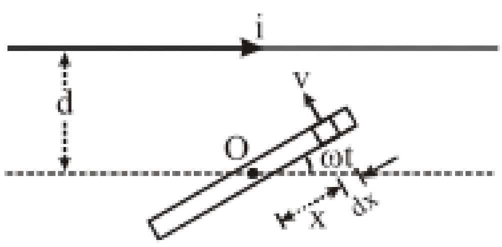
Q.9 A rod of length 2 a is free to rotate in a vertical plane, about a horizontal axis O passing through its midpoint. A long straight, horizontal wire is in the same plane and is carrying a constant current i as shown in figure. At initial moment of time the rod is horizontal and starts to rotate with constant angular velocity ω. Calculate e.m.f. induced in rod as a function of time.



Sol. The rotated position of the rod after a time t is shown in figure. Consider a small element of length dx of the rod at a distance x from the centre.

The velocity of the element $v = x\omega$ and its distance from the wire is $r = (d - x\sin\omega t)$. Magnetic induction at this position

$$B = \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 i}{2\pi(d - x\sin\omega t)}$$



The induced e.m.f in this element

$$de = Bv\,dx = \frac{\mu_0 i(x\omega)\,dx}{2\pi(d - x\sin\omega t)}$$

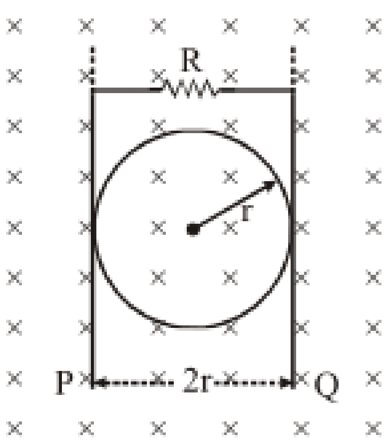
In order to obtain the resultant e.m.f., we integrate this expression from (–a) to +a. Hence

$$e = \frac{\mu_0 i\,\omega}{2\pi}\int_{-a}^{a}\frac{x\,dx}{(d - x\sin\omega t)}$$

$$= \frac{\mu_0 i\,\omega}{2\pi}\left(\frac{-1}{\sin^2\omega t}\right)\left[2a\sin\omega t + d\log\left(\frac{d - a\sin\omega t}{d + a\sin\omega t}\right)\right]$$

$$= \frac{\mu_0 i\omega}{2\pi\sin^2\omega t}\left[-2a\sin\omega t + d\log\left(\frac{d + a\sin\omega t}{d - a\sin\omega t}\right)\right].$$

Q.10 P and Q are two infinite conducting plates kept parallel to each other and separated by a distance 2r. A conducting ring of radius r falls vertically between the plates such that planes are always tangent to the ring. Both the planes are connected by a resistance R. There exists a uniform magnetic field of strength B perpendicular to the plane of ring. The arrangement is shown in figure. Plane Q is smooth and friction between the plane P and the ring is enough to prevent slipping. At $t = 0$, the planes and the ring. Find



(a) the current through R as a function of time

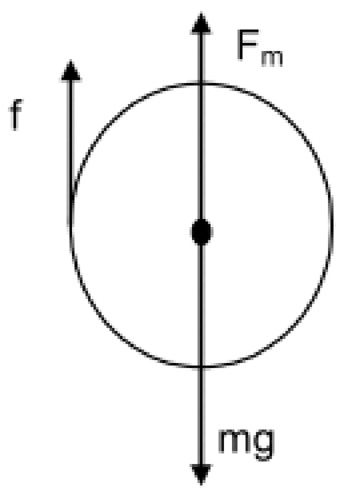
(b) terminal velocity of the ring (assume g to be constant).

Sol. Let v be the velocity of C.M. of the ring at any time t.

e.m.f. across the diameter of ring, e = 2 B v r.

$\therefore$ Current through R is I = (2 B v r/R) …(i)

The different force on the ring are shown in figure.



For rotational motion

$$fr = I\alpha$$

$$f = \frac{I\alpha}{r} = \frac{Ia}{r^2} \qquad (\because \alpha = a/r) \qquad \ldots(ii)$$

where a = acceleration of C.M.

Now $$F_m = BiL = \frac{4B^2r^2}{R}v \qquad (\because L = 2r) \qquad \ldots(iii)$$

For translational motion

$$mg - f - F_m = ma \qquad \ldots(iv)$$

Substituting the value of f and F_m from eqs. (ii) and (iii) in equation (iv), we get

$$mg - \frac{Ia}{r^2} - \frac{4B^2r^2}{R}v = ma$$

$$mg - \frac{4B^2r^2}{R}v = ma + \frac{Ia}{r^2}$$

or $$mg - \frac{4B^2r^2}{R}v = a\left[m + \frac{I}{r^2}\right]$$

or $$\left(mg - \frac{4B^2r^2}{B}v\right) = \frac{dv}{dt}\left[m + \frac{I}{r^2}\right]$$

or $$\frac{dv}{\left[mg - \frac{4B^2r^2}{R}v\right]} \times \left(m + \frac{I}{r^2}\right) = dt$$

for a ring $I = mr^2$

$\therefore$ $$\frac{dv}{\left[mg - \frac{4B^2r^2}{R}v\right]}(2m) = dt \quad \ldots(v)$$

Integrating equation (iv) with proper limits, we get

$$-\frac{mR}{2B^2r^2}\log_e\left[mg - \frac{4B^2r^2}{R}v\right]_0^v = t$$

or $$\frac{-mR}{2B^2r^2}\log_e\left[\frac{mg - \frac{4B^2r^2}{R}v}{mg}\right] = t$$

$\therefore$ $$v = \frac{mgR}{4B^2r^2}\left[1 - e^{-\frac{2B^2r^2}{mR}.t}\right] \quad \ldots(vi)$$

For equation (i) and (vi), we get

$$i = \frac{2B\,v\,r}{R} = \frac{mg}{2Br}\left[1 - e^{-\frac{2B^2r^2}{mR}.t}\right].$$

Q.11 A rectangular conducting loop in the vertical x-z plane has length L, width W, mass M and resistance R. It is dropped lengthwise from rest. At $t = 0$ the bottom of the loop is at a height h above the horizontal x-axis. There is a uniform magnetic field B perpendicular to the x-z plane, below the x-axis. The bottom and top of the loop cross this axis at $t = t_1$ and t_2 respectively. Obtain the expression for the velocity of the loop for the time $t_1 \le t \le t_2$

Sol. For time t_1, the loop is freely falling under gravity, so velocity attained by loop at

$$t = t_1$$

$$\upsilon_1 = gt_1 = \sqrt{2gh}$$

During the time $t_1 \le t \le t_2$, flux linked with the loop is changing, so induced emf

$$e = -\frac{d\phi}{dt} = -B\upsilon W$$

and Induced current

$$I = -\frac{B\upsilon W}{R}\text{clockwise}$$

Magnetic Force

$$F = WIB = -\frac{B^2\upsilon W^2}{R}$$

So, $$m\frac{d\upsilon}{dt} = mg - \frac{B^2\upsilon W^2}{R}$$

$$dt = \frac{m d\upsilon}{\left[\left(mg - \frac{B^2W^2\upsilon}{R}\right)\right]}$$

Integrating,

$$t = -\frac{mR}{B^2W^2}\log_e\left[mg - \frac{B^2\upsilon W^2}{R}\right] + A$$

At $t = t_1, \upsilon = \upsilon_1 = gt_1$

$\therefore$ $$A = t_1 + \frac{mR}{B^2W^2}\log_e\left[mg - \frac{B^2\upsilon_1 W^2}{R}\right]$$

Substituting for A,

$$e^{-\frac{B^2W^2}{mR}(t-t_1)} = \log_e\left[\frac{mg - \frac{B^2\upsilon W^2}{R}}{mg - \frac{B^2\upsilon_1 W^2}{R}}\right]$$

Gives the velocity v of the loop in the interval $t_1 \le t \le t_2$.

Q.12 A coil of inductance $L = 50 \times 10^{-6}$ henry and resistance $= 0.5\ \Omega$ is connected to a battery of emf $= 5.0$ V. A resistance of $10\ \Omega$ is connected parallel to the coil. Now at some instant the connection of the battery is switched off. Find the amount of heat generated in the coil after switching off the battery.

Sol Total energy stored in the inductor $= \frac{1}{2} L i_0^2$

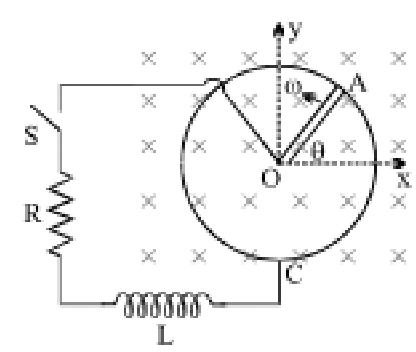
$$E_L = \frac{1}{2} L \left(\frac{V}{r}\right)^2$$

$\therefore$ Fraction of energy lost across inductor

$$= E_L \cdot \frac{r}{(R+r)}$$

$$= \frac{LV^2}{2r(R+r)} = \frac{50 \times 10^{-6} \times 5^2}{2 \times 0.5(10+0.5)} = 1.19 \times 10^{-4}\,\text{J}$$

Q.13 A metal rod OA of mass m & length r is kept rotating with a constant angular speed ω in a vertical plane about a horizontal axis at the end O. The free end A is arranged to slide without friction along a fixed conducting circular ring in the same plane as that of rotation. A uniform & constant magnetic induction $\vec{B}$ is applied perpendicular & into the plane of rotation as shown in figure. An inductor L and an external resistance R are connected through a switch S between the point O & a point C on the ring to form an electrical circuit. Neglect the resistance of the ring and the rod. Initially, the switch is open.



(a) What is the induced emf across the terminals of the switch ?

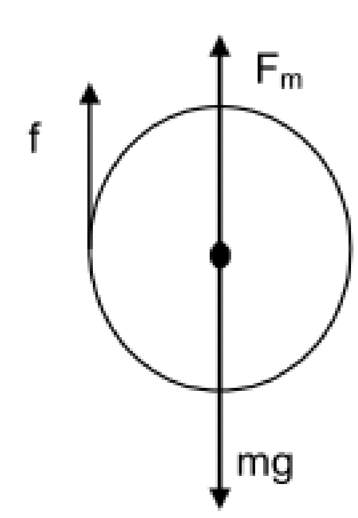
(b) (i) Obtain an expression for the current as a function of time after switch S is closed.

(ii) Obtain the time dependence of the torque required to maintain the constant angular speed, given that the rod OA was along the positive X-axis at $t = 0$.

Sol. Let v be the velocity of C.M. of the ring at any time t.

e.m.f. across the diameter of ring, $e = 2\,B\,v\,r$.

$\therefore$ Current through R is $I = (2\,B\,v\,r/R)$ ...(i)



The different force on the ring are shown in figure.

For rotational motion

$$fr = I\alpha$$

$$f = \frac{I\alpha}{r} = \frac{Ia}{r^2} \qquad (\because \alpha = a/r) \qquad \text{...(ii)}$$

where a = acceleration of C.M.

Now $F_m = B\,i\,L = \frac{4B^2r^2}{R} v \qquad (\because L = 2r) \qquad \text{...(iii)}$

For translational motion

$$mg - f - F_m = ma \qquad ...(iv)$$

Substituting the value of f and F_m from eqs. (ii) and (iii) in equation (iv), we get

$$mg - \frac{Ia}{r^2} - \frac{4B^2r^2}{R}v = ma$$

$$mg - \frac{4B^2r^2}{R}v = ma + \frac{Ia}{r^2}$$

or $$mg - \frac{4B^2r^2}{R}v = a\left[m + \frac{I}{r^2}\right]$$

or $$\left(mg - \frac{4B^2r^2}{B}v\right) = \frac{dv}{dt}\left[m + \frac{I}{r^2}\right]$$

or $$\frac{dv}{\left[mg - \frac{4B^2r^2}{R}v\right]} \times \left(m + \frac{I}{r^2}\right) = dt$$

for a ring $I = mr^2$

$$\therefore \quad \frac{dv}{\left[mg - \frac{4B^2r^2}{R}v\right]}(2m) = dt \qquad \ldots(v)$$

Integrating equation (iv) with proper limits, we get

$$-\frac{mR}{2B^2r^2}\log_e\left[mg - \frac{4B^2r^2}{R}v\right]_0^v = t$$

or $$\frac{-mR}{2B^2r^2}\log_e\left[\frac{mg - \frac{4B^2r^2}{R}v}{mg}\right] = t$$

$$\therefore \quad v = \frac{mgR}{4B^2r^2}\left[1 - e^{-\frac{2B^2r^2}{mR}.t}\right] \qquad \ldots(vi)$$

For equation (i) and (vi), we get

$$i = \frac{2Bvr}{R} = \frac{mg}{2Br}\left[1 - e^{-\frac{2B^2r^2}{mR}.t}\right].$$

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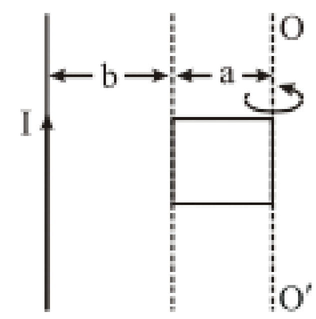
Q.14 A square loop of side a and a straight, infinite conductor are placed in the same plane with two sides of the square parallel to the conductor. The inductance and resistance are equal to L and R respectively. The frame is turned through 180° about the axis OO'. Find the electric charge that flows in the square loop.



Sol. By circuit equation $i = \left(\varepsilon - L\dfrac{di}{dt}\right)/R$ where $\varepsilon =$ induced emf and $L\dfrac{di}{dt} =$ self-induced emf

$$\Rightarrow \quad Ri = \varepsilon - L\frac{di}{dt} \quad \Rightarrow \quad \int Ri\,dt = \int \varepsilon dt - \int L\frac{di}{dt}dt$$

$$\Rightarrow \quad Rq = \int -\frac{d\varphi}{dt}dt - L[i]_i^f = \varphi_i - \varphi_f \quad (\because i_{initial} = 0, i_{final} = 0)$$

$$\Rightarrow \quad q = (\varphi_i - \varphi_f)/R$$

Consider a strip at a distance x in the initial position. Then $B = (\mu_0/4\pi)(2I/x)$ along the inward normal to the plane.

$$\therefore \quad d\varphi_i = (\mu_0 I/2\pi x)a\ dx\cos 0 = \frac{\mu_0 Ia}{2\pi}\frac{dx}{x}$$

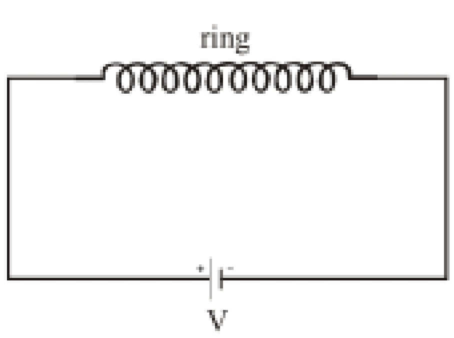
$$\Rightarrow \quad \varphi_i = \frac{\mu_0 Ia}{2\pi}\int_b^{a+b}\frac{dx}{x} = \frac{\mu_0 Ia}{2\pi}\ln\frac{a+b}{b}$$

Similarly $\varphi_f = \dfrac{\mu_0 Ia}{2\pi}\ln\dfrac{2a+b}{a+b}$

$$\therefore \quad |\varphi_i - \varphi_f| = \frac{\mu_0 Ia}{2\pi}\ln\frac{2a+b}{b}$$

$$\therefore \quad |q| = \frac{\mu_0 Ia}{2\pi R}\ln\frac{2a+b}{b}.$$

Q.15 A thin wire ring of radius a and resistance r is located inside a long solenoid so that their axes coincide. The length of the solenoid is equal to l its cross-sectional radius to b. At a certain moment, the solenoid was connected to a source of constant voltage V. The total resistance of the circuit is equal to R. Assuming the inductance of the ring to be negligible, find the maximum value of the radial force acting per unit length of the ring.



Sol. The inductance L of the solenoid

$$L = \mu_0 n^2 \pi b^2 l$$

The current through the solenoid varies as

$$I(t) = \frac{V}{R}(1 - e^{-tR/L})$$

B(t) = magnetic induction

$$= \mu_0 n \; I(t)$$

$$e = \frac{d\phi}{dt} \;\; \& \;\; \vec{B}.\vec{A}$$

flux through ring $= \mu_0 n.\pi a^2$

$$e = \frac{d\phi}{dt}, \;\; \mu_0 n \pi a^2 \frac{dI}{dt}$$

where $\quad$ i (t) is $\dfrac{dI(t)}{dt}$

Force unit per length on the ring $= \dfrac{F}{\ell} = BI(t)$

$$\therefore \qquad F(t) = \mu_0 n \, I(t).\mu_0 \, n \, \dot{I}(t) \, \pi a^2 / r$$

$$= \frac{\mu_0^2.\pi a^2}{r}.n^2 I(t)\dot{I}(t)$$

$$F(t) = \frac{\mu_0^2 \pi a^2}{r}.n^2 \left[\frac{V}{R}(1 - e^{-tR/L})\right].\left[\frac{V}{R}.\frac{Re^{-tR/L}}{L}\right]$$

$$= \frac{\mu_0^2 \pi a^2 V^2}{r}.e^{-t/RL}(1 - e^{-t/RL})\left(\frac{n^2}{RL}\right)$$

Initial conditions are $F(t) = 0$ as $t = 0$

$\quad$ F (t) = 0 at $t = \infty$

For finding maximum value of force, we differentiate F(t) w.r.t. t and put to zero. This gives the time at which this occurs.

$$e^{-t/RL} = \frac{1}{2}$$

$$\therefore \qquad F_{max} = \frac{\mu_0^2 \pi a^2 V^2}{4rR}\frac{n^2}{L} = \frac{\mu_0 a^2 v^2}{4\pi R/b^2}$$

Alternating current

Introduction

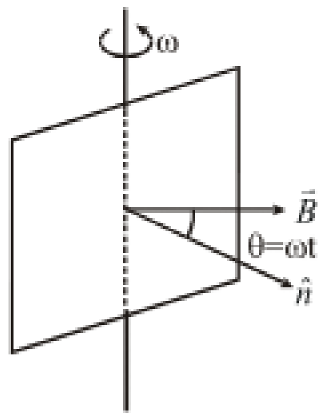
Voltages and currents that vary symmetrically in magnitude and direction with time are very common. The electric mains supply in our homes and offices is a voltage that varies like a sine function with time. Such a voltage is called alternating voltage (ac voltage) and the current driven through the appliances is called the alternating current (ac current)

Basic Principle of AC Generation:

Alternating voltage is generated by rotating a coil of conducting wire in a strong magnetic field. The magnetic flux linked with the coil changes with time and an alternating emf is thus induced. Instantaneous flux linked with coil is

$$\phi = (\vec{A}.\vec{B})n$$

$$= ABn\cos(\omega t + \theta_0)$$



where A = area of the coil (in m^2)

B = magnetic field (in tesla)

n = number of turns

ω = angular frequency $= \frac{2\pi}{T} = 2\pi f$ (in rad s^{-1})

f = frequency (in hertz)

θ_0 = initial phase angle.

With the change of time $\cos(\omega t + \theta_0)$ changes consequently an emf V is induced. According to Faraday's law

$$V = \frac{d\varphi}{dt}$$

$$= -\frac{d}{dt}[ABn\cos(\omega t + \theta_0)]$$

$$= ABn\ \omega\sin(\omega t + \theta_0)$$

$$V = V_m \sin(\omega t + \theta_0)$$

Here V_m = voltage amplitude of sinusoidal voltage or the peak value of ac voltage

where $V_m = ABn\omega$

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Average Values of ac Voltage and ac Current:

average volue of voltage (from $t = t_1$ to $t = t_2$) $= \dfrac{\int_{t_1}^{t_2} V(t)\,dt}{t_2 - t_1}$

AC voltage or currrents are commonly sinusoidal (sine or cosine function) and their mean values for complete cycle is zero.

(i) **Average value for positive half cycle (or rectified average value):**

$V = V_m \sin \omega t$

$$\therefore \quad (V)_{av} = \frac{\int_0^{T/2} V_0 \sin \omega t\,dt}{\frac{T}{2} - 0} = \frac{2}{T}\int_0^{T/2} V_0 \sin \omega t\ dt = \frac{2}{\pi} V_0 = 0.637 V_0 .$$

This is also known as the **rectified average value of a sinusoidal voltage**

(ii) ***Root Mean Square Value*** $(V_{rms}$ or $I_{rms})$ **:**

$$V_{rms}\,(\text{from } t = t_1 \text{ to } t = t_2) = \sqrt{\int_{t_1}^{t_2} V^2(t)\,dt / t_2 - t_1}$$

rms value for a complete cycle of sinusoidal voltage

$\therefore \quad V = V_0 \sin \omega t$

$$V_{rms} = \sqrt{\frac{\int_0^T V_0^2 \sin^2 \omega t\,dt}{T}} = \frac{V_0}{\sqrt{2}}$$

rms value for a complete cycle of sinusoidal current

$I = I_0 \sin \omega t$

$$\text{and} \quad I_{rms} = \sqrt{\frac{\int_0^T I_0^2 \sin^2 \omega t\,dt}{T}} = \frac{I_0}{\sqrt{2}}$$

$$\text{or RMS value} = \frac{\text{Peak value}}{\sqrt{2}}$$

How do we measure sinusoidally varying voltage or current?

Moving coil galvanometer measure steady currents but if we pass sinusoidal current through them, the needle may wiggle at low fequency but its average deflection is zero. Hot wire instruments are commonly used to measure the rms values (also known as **virtual values**).

It must be emphasied here that meters used for ac voltage and current measurement are always calibrated to read rms values, not peak (or maximum) or rectified average values. The usual domestic power supply "220 - volt ac" has rms voltage of 220 V. The voltage amplitude or peak value is

$$V_0 = \sqrt{2}V_{rms} = \sqrt{2}(220V) = 311V.$$

Illustration:

If a domestic appliance draws 2.5 A from a 220-V, 60- Hz power supply, find

(a) The average current for full cycle

(b) The average of the square of the current

(c)The current amplitude

(d) The supply voltage amplitude.

Sol. *(a)* *The average of sinusoidal AC values over any whole number of cycles is zero.*

(b) *RMS value of current* $= I_{rms} = 2.5\ A$

$\therefore \quad (I^2)_{av} = (I_{rms})^2 = 6.25 A^2$

(c) $I_{rms} = \dfrac{I_m}{\sqrt{2}}$

$\therefore$ *Current amplitude* $= \sqrt{2} I_{rms} = \sqrt{2}(2.5\ A)$

$= 3.5\ A$

(d) $V_{rms} = 220V = \dfrac{V_m}{\sqrt{2}}$

$\therefore$ *Supply voltage amplitude*

$V_m = \sqrt{2}(V_{rms}) = \sqrt{2}(220V)$

$= 311\ V.$

AC Voltage Applied to a Resistor :

A resistor connected to a source ε of ac voltage as shown in the ciruit digram . The symbol for an ac source in a circuit diagram is $\circleddash$. For simplicity, we consider a source which produces sinusoidally varying potential difference across its terminals. Let this potential difference, also called ac voltage, be given by

$$V = V_0 \sin \omega t \qquad \ldots(i)$$

where V_0 is the amplitude of the sinusoidal voltage and ω is its angular frequency.

AC voltage applied to a resistor

The instantaneous potential drop across the resistor R is

$$V_0 \sin \omega t = IR$$

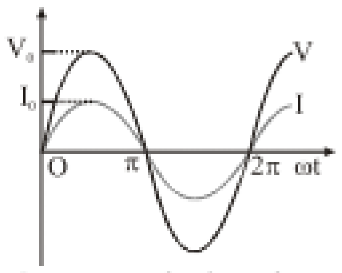
or $$I = \frac{V_0}{R} \sin \omega t$$

$$I = I_0 \sin \omega t \qquad \ldots(ii)$$

where I is the instantaneous current and the current amplitude I_0 is given by

$$I_0 = \frac{V_0}{R} \qquad \ldots(iii)$$

Equation (iii) is just Ohm's law which for resistors work equally well for both ac and dc voltages. The voltage across a pure resistor and the current through it, given by equation (i) and (ii) are plotted as a function of time in figure. Note, in particular that both V and I reach zero, minimum and maximum values at the same time. Clearly, the voltage and current are in phase for a circuit containing pure resistance.



In a pure resistor, the voltage and current are in phase. The minima with zero and maxima occur at the same respective times.

We see that, like the applied voltage, the current varies sinusoidally and has corresponding positive and negative values during each cycle. Thus, **the sum of the instantaneous current values over one complete cycle is zero, and the average current is zero**. The fact that the average current is zero, however, does not mean that the average power is zero and that there is no dissipation of electrical energy. As you know, joule heating is given by I^2R and depends on I^2 (which is always positive whether I is positive or negative) and not on I. Thus there is Joule heating and dissipation of electrical energy when an ac current passes through a resistor.

The instantaneous power dissipated in the resistor is

$$P = I^2 R = I_0^2 R \sin^2 \omega t \qquad \ldots[iv]$$

The average value of Power P over a cycle is

$$\overline{P} = \frac{1}{2} I_0^2 R = I_{rms}^2 R \qquad \ldots[v]$$

Where the bar over a letter(here, P) denotes its average value.

To express ac power in the same form as dc power $(P = I^2R)$, as special value of current is used. It is called, root mean square (rms) or **effective current** and is denoted by I_{rms}.

Similarly, we define the rms voltage or **effective voltage**

From equation [iii], we have

$$V_0 = I_0 R \qquad ...[vi]$$

or $$\frac{V_0}{\sqrt{2}} = \frac{I_0}{\sqrt{2}} R \qquad ...[vii]$$

or $$V_{rms} = I_{rms} R \qquad ...[ix]$$

In terms of rms values, the equation for power and relation between current and voltage in ac circuits are essentially the same as those for the dc case.

In fact, the I_{rms} or rms current is the equivalent dc current that would produce the same average power loss as the alternating current. Equation [v] can also be written as

$$\overline{P} = V_{rms}^2 / R = I_{rms}^2 R \qquad (\text{since } V_{rms} = I_{rms} R) \qquad ...[ix]$$

Illustration:

A bulb is rated 60 W at 220 V/50 Hz. Find the maximum value of instantaneous current through the filament?

Sol. $V_{max} = 220\sqrt{2} = 311\text{V}$

$$R = \frac{220^2}{P} = \frac{220 \times 220}{60} = \frac{2420}{3} = 806.67\,\Omega$$

$$I = \frac{V_{max}}{R} = \frac{311}{806.67} = 0.39\text{A}$$

Illustration:

A light bulb is rated at 200 W for a 220 V supply. Find

(a) The resistance of the bulb;

(b) The peak voltage of the source; and

(c) The rms current through the bulb.

Sol. *(a) We are given P = 100 W and V = 220 V. The resistance of the bulb is*

$$R = \frac{V_{rms}^2}{P} = \frac{(220\text{V})^2}{200\text{W}} = 242\,\Omega$$

(b) *The peak voltage of the source is*

$$V_m = \sqrt{2} V_{rms} = 311\text{V}$$

(c) *Since,* $P = I_{rms} V_{rms}$

$$I_{rms} = \frac{P}{V_{rms}} = \frac{200\text{W}}{220\text{V}} = 0.90\text{A}.$$

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Practice Exercise

Q.1 If E_0 represents the peak value of the voltage in an ac circuit. Find the rms voltage.

Q.2 In an ac circuit, the rms value of current I_{rms} is related to the peak current I_0 by the relation :

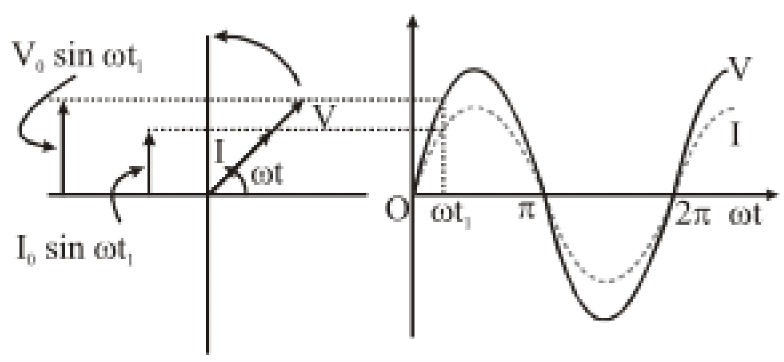
Q.3 The electric current in a circuit is given by $i = i_0\, t/\tau$ for some time. What is the rms current for the period $t = 0$ to $t = \tau$

Answers

Q.1 $E_0/\sqrt{2}$ Q.2 $I_{rms} = (I_0/\sqrt{2})$ Q.3 $i_0/\sqrt{3}$

Representation of AC Current and Voltage by Rotating Vectors – Phasors:

In the previous section, we saw that the current through a resistor is in phase with the ac voltage. But this is not so in the case of an inductor, a capacitor or a combination. In order to show phase relationship between voltage and current in an ac circuit, we use the motion of PHASORS. The analysis of an ac circuit is facilitated by the use of a phasor diagram. A phasor is a vector which rotates about the origin with angular speed ω, as shown in figure. The vertical components of phasors V and I represent the sinusoidally varying quantities V and I. The magnitudes of phasors V and I represent the amplitudes or the peak values V_0 and I_0 of these oscillating quantities. Figure (a) shows the voltage and their relationship at time t_1 i.e., corresponding to the circuit show in figure for the case of an ac source connected to a resistor. The projection of voltage and current phasors on vertical axis, i.e., $V_0 \sin \omega t$ and $I_0 \sin \omega t$, respectively represent the instaneous value of voltage and current at that instant. As they rotate with frequency ω, curves in figure(b) are generated which respresent the sinusoidal variation of voltage and current with time.



(a) (b)

(a) A phasor diagram for the circuit in figure
(b) Graph of V and I versus tω

From figure (a) we see that phasors V and I for the case of a resistor are in the same phase. This is so for all times. This means that the phase angle between the voltage and the current is zero.

AC Voltage Applied to an Inductor:

An ac source connected to an inductor as shown in the circuit below. Usually, inductors have appreciable resistance in their windings, but we shall assume that this is ideal inductor (having zero resistance). Thus, the circuit is a purely inductive ac circuit. Let the voltage across the source be $V = V_0 \sin \omega t$. Using the loop equation. $\sum \varepsilon(t) = 0$, and since there is no resistor in the circuit.

$$V - L\frac{dI}{dt} = 0 \qquad \ldots[x]$$

An AC source connected to an inductor

where the second term is the self-induced emf in the inductor; and L is the self-inductance of the coil. Combining equation [i] and [x], we have

$$\frac{dI}{dt} = \frac{V}{L} = \frac{V_0}{L}\sin \omega t \qquad \ldots[xi]$$

$$dI = \frac{V_0}{L}\sin \omega t \, dt$$

$$I = -\frac{V_0}{\omega L}\cos(\omega t)$$

Using $-\cos(\omega t) = \sin\left(\omega t - \frac{\pi}{2}\right)$, we have

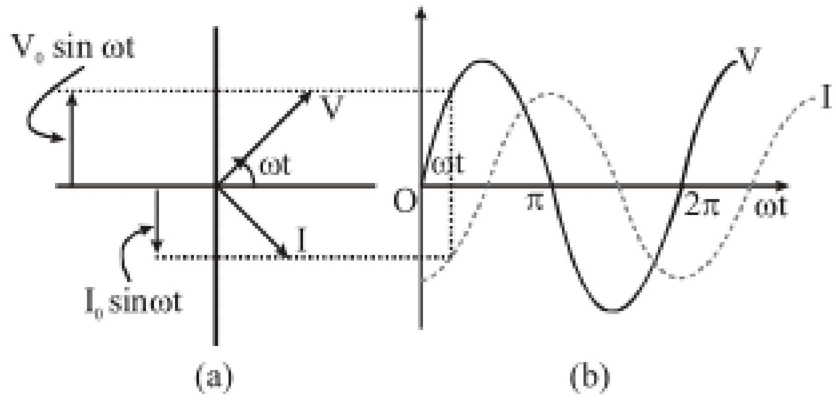
$$i = I_0 \sin\left(\omega t - \frac{\pi}{2}\right) \qquad \ldots(xii)$$

where $I_0 = \frac{V_0}{\omega L}$ is the amplitude of the current. The quantity ωL is analogous to the resistance and is called **inductive reactance**, denoted by $\mathbf{X_L}$:

$$X_L = \omega L = 2\pi f L \qquad \ldots(xiii)$$

The dimension of inductive reactance is the same as that of resistance and its SI unit is ohm (Ω). The inductive reactance limits the current in a pure inductive circuit in the same way as does the resistance in a pure resistive circuit. The inductive reactance is directly proportional to the inductance frequency of the voltage source.

A comparison of equation (i) and (ii) for the source voltage and the current in an inductor shows that the current lags the voltage by $\frac{\pi}{2}$ or one-quarter $\left(\frac{1}{4}\right)$ cycle. Figure a shows the voltage and the current phasors in the present case at instant t. The current phasor I is $\frac{\pi}{2}$ behind the voltage phasor V. When rotated with frequency ω counter-clockwise, they generate the voltage and current given by equation [i] and [xii], respectively and as shown in figure (b)



(a) A phasor diagram for the circuit in figure
(b) Graph of V and I versus ωt.

We see that the current reaches its maximum value later than the voltage by one-fourth of a period $\left[\frac{T}{4}=\frac{\pi/2}{\omega}\right]$. You have seen that an inductor has reactance that limits current similar to resistance in a dc circuit. Does it also consume power like a resistance? Let us try to find out.
The instantaneous power supplied to the inductor is

$$P_L = IV = I_m \sin\left(\omega t - \frac{\pi}{2}\right) V_0 \sin(\omega t) = -I_0 V_0 \cos(\omega t).\sin(\omega t) = - -\frac{I_0 V_0}{2}\sin(2\omega t) \quad ...[xiv]$$

So, the average power over a complete cycle is zero

since the average of $\sin(2\omega t)$ over a complete cycle is zero.

Thus, the average power supplied to an inductor over one complete cycle is zero.

Physically, this result means the follows. During the first quarter of each current cycle, the flux through the inductor builds up and sets up a magnetic field and energy is stored in the inductor. In the next quarter of cycle, as the current decrease, the flux decreases and the stored energy is returned to the source. Thus, in each half cycle, the energy which is withdrawn from the source is returned to it without any dissipation of power.

Illustration:

A pure inductor of 50.0 mH is connected to a source of 220 V. Find the inductive reactance and rms current in the circuit if the frequency of the source is 50 Hz.

Sol. The inductive reactance.

$$X_L = 2\pi f L = 2 \times 3.14 \times 50 \times 50 \times 10^{-3}\,\Omega$$

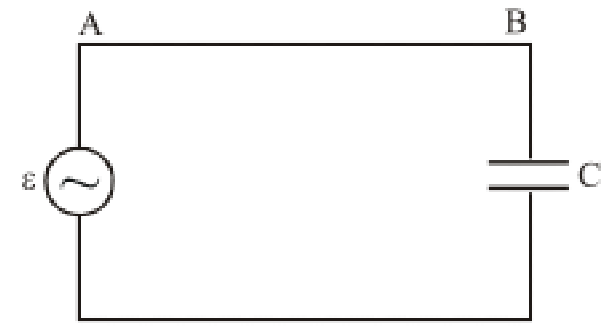
$$= 15.7\,\Omega$$

The rms current in the circuit is

$$I_{rms} = \frac{V_{rms}}{X_L} = \frac{220\text{V}}{15.7\Omega} = 14.01\text{A}\,.$$

AC Voltage Applied to A Capacitor:

An ac source ε connected to a capacitor only, a purely capacitive ac circuit is as shown.



An ac source connected to a capacitor

When the capacitor is connected to an ac source, as in figure, it limits or regulates the current, but does not completely prevent the flow of charge. The capacitor is alternately charged and discharged as the current reverses each half cycle. Let q(t) be the charge on the capacitor at any time t. The instantaneous voltage V(t) across the capacitor is

$$V(t) = \frac{q(t)}{C} \qquad \text{...(xv)}$$

From the Kirchhoff's loop rule, the voltage across the source and the capacitor are equal,

$$V_0 \sin \omega t = \frac{q}{C}$$

To find the current, we use the relation $I = \frac{dq}{dt}$

$$I = \frac{d}{dt}(V_0 C \sin \omega t) = \omega C V_0 \cos(\omega t)$$

Using the relation, $\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right)$, we have

$$I = I_0 \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \text{...(xvi)}$$

where the amplitude of the oscillating current is

$$I_0 = \frac{V_0}{(1/\omega C)}$$

Comparing it to $I_0 = \frac{V_0}{R}$ for a purely resistive circuit, we find that $(1/\omega C)$ plays the role of resistance.

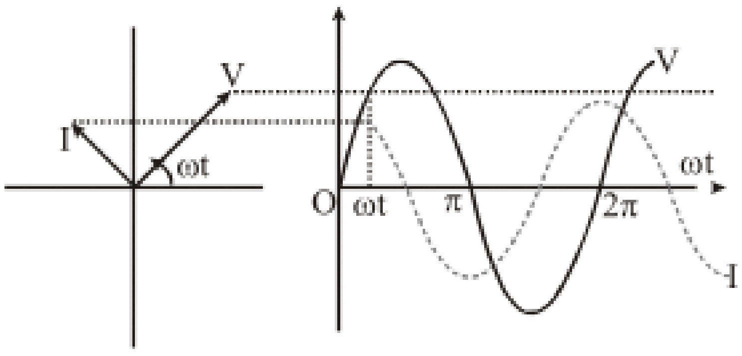
It is called **capacitive reactance** and is denoted by $\mathbf{X_C}$,

$$\mathbf{X_C = 1/\omega C = 1/2\pi fC} \qquad \ldots\text{(xvii)}$$

So that the amplitude of the current is

$$I_0 = \frac{V_0}{X_C} \qquad \ldots\text{(xviii)}$$

The dimension of capacitive reactance is the same as that of resistance and its SI unit is Ohm (Ω). The capacitive reactance limits the amplitude of the current in a purely capacitive circuit in the same way as does the resistance in a purely resistive circuit. But it is inversely proportional to the frequency and the capacitance.



A comparison of equation (xvii) with the equation of source voltage equation (i) shows that the current in a capacitor leads the voltage by $\pi/2$. Figure shows the phasor diagram at an instant t. Here the current phasor I is $\frac{\pi}{2}$ rad ahead of the voltage phasor V as they rotate counter clockwise. Figure shows the variation of voltage and current with time. We see that the current reaches its maximum value earlier than the voltage by one-fourth of a period.

The instantaneous power supplied to the capacitor is

$$P_C = IV = I_0 \cos(\omega t).\ V_0 \sin(\omega t)$$

$$= I_0 V_0 \cos(\omega t) \sin(\omega t)$$

$$= \frac{I_0 V_0}{2} \sin(2\omega t) \qquad \ldots[\text{xix}]$$

So, as in the case of an inductor, the average power

Since average of sin $2\omega t$ over a complete cycle is zero. As discussed in the case of an inductor, the energy stored by a capacitor in each quarter period is returned to the source in the next quarter period.

Thus, we see that in the case of an inductor. the current lags the voltage by 90° and in the case of a capacitor, the current leads the voltage by 90°.

Illustration:

30.0 μF capacitor is connected to a 220 V, 50 Hz source. Find the capacitive reactance and the current (rms and peak) in the circuit. If the frequency is doubled, what happens to the capacitive reactance and the current.

Sol. *The capacitive reactance is*

$$X_C = \frac{1}{2\pi fC} = 106\Omega$$

The rms current is

$$I_{rms} = \frac{V_{rms}}{X_C} = 2.08\text{A}$$

The peak current is

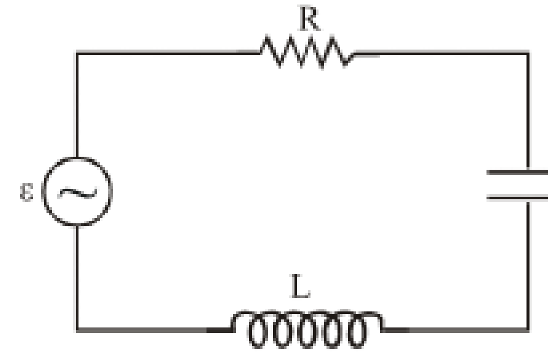
$$I_0 = \sqrt{2} I_{rms} = 2.96\text{A}$$

This current oscillates between 2.96A and – 2.96A and is ahead of the voltage by 90º.

If the frequency is doubled, the capacitive reactance is halved and consequently, the current is doubled.

AC Voltage Applied to a Series LCR Circuit:

Figure shows a series LCR circuit connected to an ac source ε. As usual, we take the voltage of the source to be $V = V_0 \sin\omega t$.



A series LCR circuit connected to an ac source

If q is the charge on the capacitor and I the current, at time t, we have, from Kirchhoff's loop rule:

$$L\frac{dI}{dt} + IR + \frac{q}{C} = V \qquad \ldots(xx)$$

We want to determine the instantaneous current I and its phase relationship to the applied alternating voltage V. We shall use the technique of phasors to solve equation [xx] to obtain the time –dependence of I.

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Phasor-Diagram Solution:

From the circuit shown in figure we see that the resistor, inductor and capacitor are in series. therefore, the ac current in each element is the same, having the same amplitude and phase. Let it be

$$I = I_0 \sin(\omega t + \phi) \qquad \ldots \text{(xxi)}$$

where ϕ is the phase difference between the voltage across the source and the current in the circuit. On the basis of that we construct a phasor diagram for the present case.

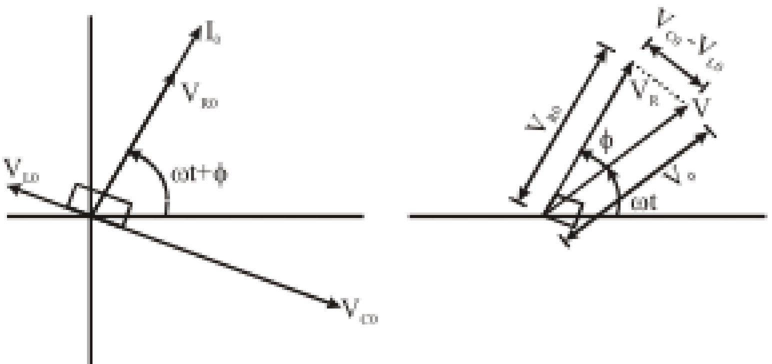
Let I be the phasor representing the current in the circuit as given by equation [xxi]. Further, let V_L, V_R, V_C, and V represent the voltage across the inductor, resistor, capacitor and the source, respectively. From previous section, we know that V_R is in phase with I, V_C is $\frac{\pi}{2}$ rad behind I and V_L is $\frac{\pi}{2}$ rad ahead of I. V_R, V_C and I are shown in figure (a) with appropriate phase-relations.

The length of these phasors or the amplitude of V_R, V_C and V_L are :

$$V_{R0} = I_0 R\ ,\ V_{C0} = I_0 X_C,\ V_{L0} = I_0 X_L \qquad \ldots \text{(xxii)}$$

The voltage equation (xx) for the circuit can be written as

$$V = V_L + V_R + V_C \qquad \ldots \text{(xxiii)}$$



(a) (b)

(a) Relation between the phasors. (b) Relation between the phasors for the circuit in

This relation is represented in figure (b). Since $\vec{V}_{C0}$ and $\vec{V}_{L0}$ are always along the same line and in opposite direction, they can be combined vectorilly into a single phasor $(\vec{V}_C + \vec{V}_L)$ which has a magnitude $|\vec{V}_{C0} - \vec{V}_{L0}|$. Since, $\vec{V}_0$ is represented as the hypotenuse of a right- angle triangle

$$V_0^2 = V_{R0}^2 + (V_{C0} - V_{L0})^2 \qquad \ldots\text{[xxiv]}$$

Substituting the values of $V_{R0,}$ V_{C0} and V_{L0} from equation [xxii] into the above equation, we have

$$V_0^2 = (I_0 R)^2 + (I_0 X_C - I_0 X_L)^2$$
$$= I_0^2 [R^2 + (X_C - X_L)^2]$$

or $$I_0 = \frac{V_0}{\sqrt{R^2 + (X_C - X_L)^2}} \qquad \ldots \text{(xxv (a))}$$

By analogy to the resistance in a circuit, we introduce the impedance Z in ac circuit :

$$I_0 = \frac{V_0}{Z} \quad \ldots \text{(xxv (b))}$$

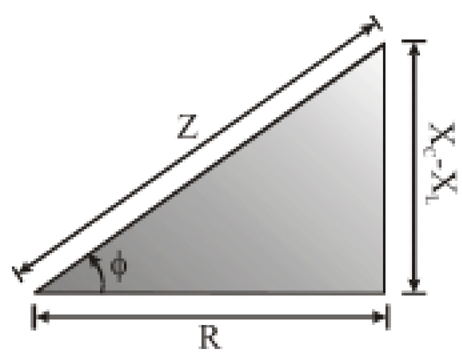
where $Z = \sqrt{R^2 + (X_C - X_L)^2}$ ... (xxvi)

Since phasor I is always parallel to phasor V_R, the phase angle ϕ is the angle between V_R and V and can be determined from figure:

$$\tan\phi = \frac{V_{C0} - V_{L0}}{V_{R0}}$$

from the impedance triangle

$$\tan\phi = \frac{X_C - X_L}{R} \quad \ldots \text{(xxvii)}$$



impedance triangle

Equations (xxvi) and (xxvii) are shown in figure. This is called **Impedance triangle** which is a right triangle with Z as its hypotenuse.

Equation (xxv (a)) gives the amplitude of the current and figure gives the phase angle. With these, equation (xxi) is completely specified.

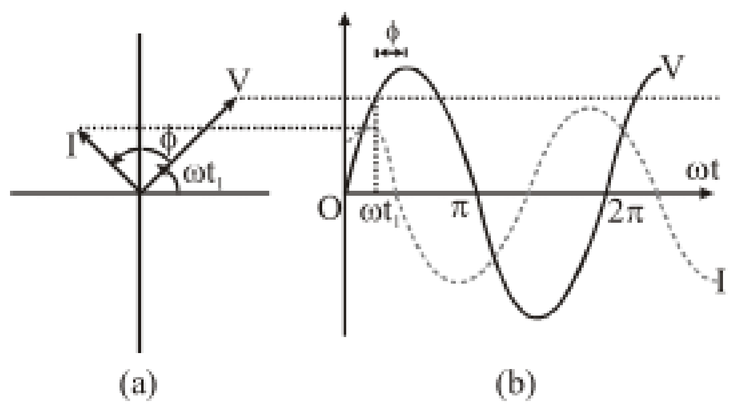
If $X_C > X_L, \phi$ is positive and the circuit is capacitive. Consequently, the voltage across the source lags the current.

If $X_C < X_L, \phi$ is negative and the circuit is inductive.

Consequently, the voltage across the source leads the current.

Figure shows the phasor diagram and variation of V and I with ωt for the case $X_C > X_L$.

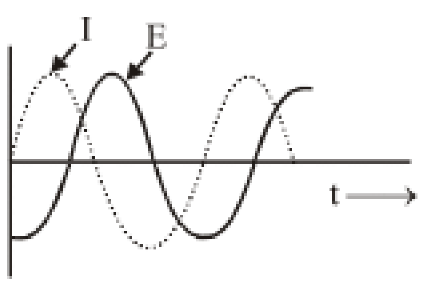
Thus, we have obtained the amplitude and phase of current for an LCR series circuit using the technique of phasors. But this method of analyzing ac circuits suffers from certain disadvantages. First, the phasor diagram states nothing about the initial condition. One can take any arbitrary value of t and draw different phasors which show the relative angle between different phasors. The solution so obtained is called the **steady-state solution.**



(a) Phasor diagram of V and I. (b) Graphs of V and I versus ωt for a series LCR circuit where $X_C > X_L$.

Illustration :

When an AC source of e.m.f. $E = E_0 \sin(100t)$ is connected across a circuit, the phase difference between the e.m.f. E and the current I in the circuit is observed to be $\pi/4$, as shown in the figure. If the circuit consists possibly only of R-C in series. What will be the relation between the two elements of the circuit ?



Sol. *Given $E = E_0 \sin(100\,t)$. Comparing this with $E = E_0 \sin \omega t$, we have $\omega = 100\,\text{rad}\,\text{s}^{-1}$. It follows from the figure that the current leads the e.m.f. which is true only for R-C circuit, and not for R-L circuit. Hence the circuit does not contain an inductor. Thus choices (c) and (d) are not possible. For R-C circuit, the phase difference between E and I is given by*

$$\tan\phi = \frac{1}{\omega RC} \;.....$$

Given $\phi = \pi/4$. Also $\omega = 100\,\text{rad}\,\text{s}^{-1}$. Using these values in, we get

$$\tan\left(\frac{\pi}{4}\right) = \frac{1}{100\,RC} \text{ or } RC = \frac{1}{100}$$

Illustration :

An LCR series circuit with 100 Ω resistance is connected to an ac source of 200 V and angular frequency 300 rad/s. When only the capacitance is removed, the current lags behind the voltage by 60°. When only the inductance is removed, the current leads the voltage by 60°. Calculate the current in the LCR circuit.

Sol. *When capacitance is removed, the circuit becomes L-R with,*

$$\tan\phi = \frac{X_L}{R} \qquad \text{i.e.,} \qquad X_L = R\tan\phi = 100\sqrt{3}\ \Omega$$

and when inductance is removed the circuit becomes C-R with,

$$\tan\phi = \frac{X_C}{R} \qquad \text{i.e.,} \qquad X_C = R\tan\phi = 100\sqrt{3}\ \Omega$$

as here $X_L = X_C$ so the circuit is in series resonance and hence as $X = X_L - X_C = 0$,

i.e., $Z=\sqrt{R^2+X^2}=R$.

So, $I_{rms}=\frac{V_{rms}}{Z}=\frac{V_{rms}}{R}=\frac{200}{100}=2\ A$

Resonance:

An interesting characteristic of the series RLC circuit is the phenomenon of resonance. The phenomenon of resonance is common among systems that have a tendency to oscillate at a particular frequency. This frequency is called the system's natural frequency. If such a system is driven by an energy source at a frequency that is near the natural frequency, the amplitude of oscillation is found to be large.

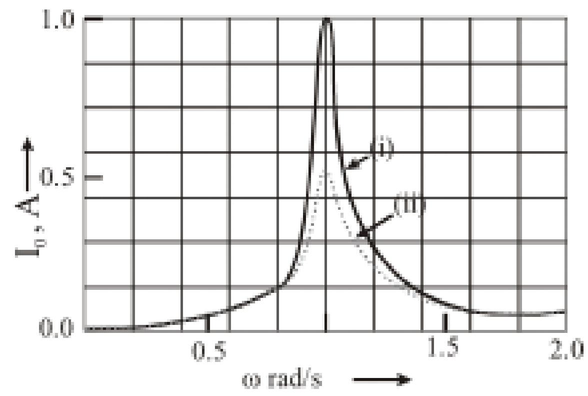
For an RLC circuit driven with voltage of amplitude V_0 and frequency ω, we found that the current amplitude is given by $I_0=\frac{V_0}{Z}=\frac{V_0}{\sqrt{R^2+(X_C-X_L)^2}}$

with $X_C=\frac{1}{\omega C}$ and $X_L=\omega L$. So if ω is varied, then at a particular frequency $\omega_0, X_C=X_L$, the impendance is minimum. This frequency is called the **resonant frequency** :

then $\frac{1}{\omega_0 C}=\omega_0 L$ or $\omega_0=\frac{1}{\sqrt{LC}}$ ... (xxxv)

At resonant frequency, the current amplitude is maximum; $I_0=\frac{V_0}{R}$

The variation of I_0 with ω in a RLC series circuit with $L=1.00$ mH, $C=1.00$ nF for two values of R: (i) $R=100\ \Omega$ and (ii) $R=200\,\Omega$ are shown in the figure. For the source applied $V_0=100$ V. ω_0 for this case is $\left(\frac{1}{\sqrt{LC}}\right)=1.00\times10^6$ rad/s.



Variation of I_0 with ω for two cases: (i) $R=100\Omega$, (ii) $R=200\Omega$, $L=1.00$mH, $C=1.00$nF and $V_0=100$V for both cases.

We see that the current amplitude is maximum at the resonant frequency.

Illustration :

Resonance frequency of a circuit is f. If the capacitance is made 4 times the initial value, then find the resonance frequency ?

Sol. $f=\frac{1}{2\pi\sqrt{LC}}$ i.e. $f\propto\frac{1}{\sqrt{C}}\rightarrow\frac{1}{\sqrt{4}}=\frac{1}{2}$ time

Power in AC Circuits: The Power Factor:

We have seen that a voltage $V = V_0 \sin \omega t$ applied to a series RLC circuit drives a current in the circuit given by $I = I_0 \sin(\omega t + \phi)$ where

$$I_0 = \frac{V_0}{Z} \text{ and } \phi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)$$

Therefore, the instantaneous power P supplied to the source is

$$P = VI = (V_0 \sin \omega t) \times [I_0 \sin(\omega t + \phi)]$$

$$= \frac{V_0 I_0}{2} [\cos \phi - \cos(2\omega t + \phi)] \qquad \ldots(xxix)$$

The average power over a cycle is given by the average of the two terms in R.H.S. of equation [xxxvii]. Second term average is zero, therefore,

$$\overline{P} = \frac{V_0 I_0}{2} \cos\phi = \frac{V_0}{\sqrt{2}} \frac{I_0}{\sqrt{2}} \cos\phi$$

$$= V_{rms} I_{rms} \cos\phi \qquad \ldots(xxx)$$

This can also be written as,

$$\overline{P} = I_{rms}^2 Z \cos\phi \qquad \ldots(xxxi)$$

So, the average power dissipated depends not only on the voltage and current but also on the cosine of the phase angle ϕ between them. The quantity $\cos\phi$ is called the **power factor**. Let us discuss the following cases: from impedence triangle $\cos\phi = \frac{R}{z}$

Case (I) Resistive circuit:

If the circuit contains only pure R, it is called resistive. In that case $\phi = 0$, $\cos\phi = 1$. There is maximum power dissipation.

Case (II) Purely Inductive or Capacitive Circuit :

If the circuit contains only an inductor or capacitor, we know that the phase difference between voltage and current is $\frac{\pi}{2}$. Therefore, $\cos\phi = 0$, and no power is dissipated even though a current is flowing in the circuit. This current is sometimes referred to as **wattless current**.

Case(III) LCR Series Circuit:

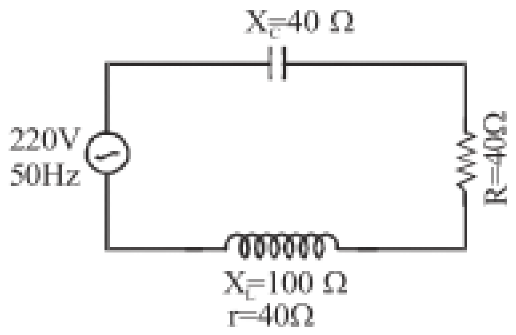
In an LCR series circuit, power dissipated is given by equation [xxxviii] where $\phi = \tan^{-1}\frac{X_C - X_L}{R}$. So, ϕ may be non-zero (except $\frac{\pi}{2}$) in a RL or RC or RCL circuit. Even in such cases, power is dissipated only in the resistor.

Case (IV) Power Dissipated at Resonance in LCR Circuit:

At resonance $X_C - X_L = 0$, and $\phi = 0$. Therefore, $\cos\phi = 1$, and $P = I_0^2 z = I_0^2 R$. That is, maximum power is dissipated in a circuit (through R) at resonance.

Illustration:

Find the power factor of the circuit shown in figure ?



Sol. $R = 40 + 40 = 80\,\Omega$,

$$X_L - X_C = 100 - 40 = 60\,\Omega$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{80^2 + 60^2} = 100$$

Power factor, $\cos\phi = \frac{R}{Z} = \frac{80}{100} = 0.8$.

Illustration :

A current of 4 A flows in a coil when connected to a 12 V dc source. If the same coil is connected to a 12 V, 50 rad/s ac source a current of 2.4 A flows in the circuit. Determine the inductance of the coil. Also find the power developed in the circuit if a 2500 μF *capacitor is connected in series with the coil.*

Sol. *In case of a coil as* $Z = \sqrt{R^2 + \omega^2 L^2}$

i.e., $I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + \omega^2 L^2}}$

so when dc is applied

$$I = \frac{V}{R} \quad \text{i.e.,} \quad R = \frac{12}{4} = 3\,\Omega \qquad \text{... (i)}$$

and when ac is applied,

$$I = \frac{V}{Z} \quad \text{i.e.,} \quad Z = \left(\frac{V}{I}\right) = \left(\frac{12}{2.4}\right) = 5\,\Omega$$

or, $R^2 + X_L^2 = 5^2 \quad (\text{as } Z = \sqrt{R^2 + X_L^2})$

so, $X_L^2 = 5^2 - R^2 = 5^2 - 3^2 = 4^2$ i.e., $X_L = 4\,\Omega$

but as $X_L = \omega L$, $L = \frac{X_L}{\omega} = \frac{4}{50} = 0.08$

Now when the capacitor is connected to the above circuit in series,

As $$X_C = \frac{1}{\omega C} = \frac{1}{50 \times 2500 \times 10^{-6}} = \frac{10^3}{125} = 8\ \Omega$$

So, $$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{3^2 + (4-8)^2} = 5\ \Omega$$

and hence $$I = \frac{V}{Z} = \frac{12}{5} = 2.4\text{ A}$$

so, $$P_{av} = V_{rms} I_{rms} \cos\phi = (I_{rms} \times Z) \times I_{rms} \times \left(\frac{R}{Z}\right)$$

i.e., $$P_{av} = I_{rms}^2 R = (2.4)^2 \times 3 = 17.28\text{ W}.$$

Illustration:

A 750 Hz., 20 V source is connected to a resistance of 100 ohm, an inductance of 0.1803 henry and a capacitance of 10 microfarad all in series. Calculate the time in which the resistance (thermal capacity 2 J/°C) will get heated by 10°C.

Sol. *As in this problem,*

$$X_L = \omega L = 2\pi f L = 2\pi \times 750 \times 0.1803 = 849.2\Omega$$

and, $$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 750 \times 10^{-5}} = 21.2\Omega$$

so, $$X = X_L - X_C = 849.2 - 21.2 = 828\Omega$$

and hence, $$Z = \sqrt{R^2 + X^2} = \sqrt{(100)^2 + (828)^2} = 834\Omega$$

but as in case of ac,

$$P_{av} = V_{rms} I_{rms} \cos\phi = V_{rms} \times \frac{V_{rms}}{Z} \times \frac{R}{Z}$$

i.e., $$P_{av} = \left(\frac{V_{rms}}{Z}\right)^2 \times R = \left(\frac{20}{834}\right)^2 \times 100 = 0.0575\text{W}$$

and as, $U = P \times t = mc\Delta\theta = (TC)\Delta\theta$

$$t = \frac{(TC) \times \Delta\theta}{P} = \frac{2 \times 10}{0.0575} = 348\text{s}$$

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Practice Exercise

Q.1 An ac circuit having resistance (R) and inductance (L) connected in series with a source of ω angular frequency. Find the power factor.

Q.2 In an LCR series ac circuit, the voltage across each of the components, L, C and R is 50V. Find the voltage across the LC combination.

Q.3 If in a series L-C-R circuit, the voltage across R, L and C are V_R, V_L and V_C respectively. Find the voltage of applied AC source.

Q.4 In a L R circuit the A.C. source has voltage 220 V and the potential difference across the inductance is 176 V. Find the potential difference across the resistance.

Answers

Q.1 $R/(R^2+\omega^2L^2)^{1/2}$ Q.2 Zero Q.3 $\sqrt{[(V_R)^2+(V_L-V_C)^2]}$

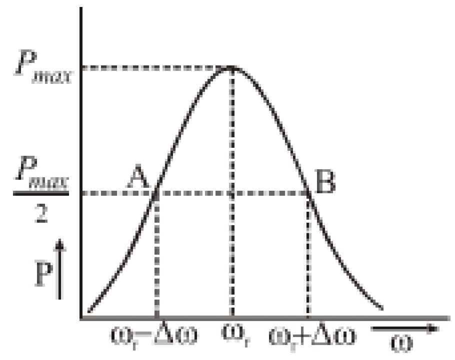
Q.4 132 V

Band Width and Q-Factor:

Angular frequency variation of power in LCR series circuit.

$$P = P_{max}\sqrt{\frac{R^2}{\left[R^2+\left(L\omega-\frac{1}{C\omega}\right)^2\right]}}$$

Graph between P & ω as shown in the figure.



$$\omega_1 = +\frac{R}{2L}+\left(\omega_r^2+\frac{R^2}{4L^2}\right)^{1/2} \text{ and } \omega_2 = -\frac{R}{2L}+\left(\omega_r^2+\frac{R^2}{4L^2}\right)^{1/2}$$

Now, $\omega_1-\omega_2 = \frac{R}{L}$

or $(\omega_r+\Delta\omega)-(\omega_r-\Delta\omega) = \frac{R}{L}$ or $2\Delta\omega = \frac{R}{L}$.

The frequency interval between half maximum power points is known as **band width**.

The ratio of resonance frequency and band width is known as **quality factor (Q)**.

$\therefore \quad Q = \frac{\omega_r}{2\Delta\omega} = \frac{\omega_r L}{R}$.

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Q factor is a measure of the sharpness of resonance. Resonance will be sharp if the value of bandwidth $(2\Delta\omega)$ is small. This is of course possible only when the power-frequency curve fall steeply around $\omega = \omega_r$.

Illustration:

A resistance R and inductance L and a capacitor C all are connected in series with an AC supply. The resistance of R is 16 ohm and for a given frequency, the inductive reactance of L is 24 ohm and capacitive reactance of C is 12 ohm. If the current in the circuit is 5 amp., find

(a) The potential difference across R, L and C

(b) The impedance of the circuit

(c) The voltage of AC supply

(d) Phase difference between applied source voltage and current drawn

Sol. *(a) Potential difference across the resistance*

$$V_R = iR = 5 \times 16 = 80 \text{ Volt}$$

Potential difference across the inductor

$$V_L = iX_L = 5 \times 24 = 120 \text{ Volt}$$

Potential difference across the capacitor

$$V_C = iX_C = i \times (1/\omega C) = 5 \times 12 = 60 \text{ Volt}$$

(b) $Z = \sqrt{R^2 + (X_L - X_C)^2}$

$$= \sqrt{[(16)^2 + (24-12)^2]} = 20 \text{ ohm}$$

(c) *The voltage of AC supply is given by*

$$E = IZ = 5 \times 20 = 100 \text{ volt}$$

(d) *Phase angle*

$$\phi = \tan^{-1}\left[\frac{\omega L - (1/\omega C)}{R}\right] = \tan^{-1}\left[\frac{24-12}{16}\right] = \tan^{-1}(0.75) = 36^\circ 46'.$$

Illustration:

Obtain the resonant freqency and Q-factor of a series LCR circuit with L = 3.0 H, C = 27μF and $R = 7.4\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its full width at half maximum by a factor of 2. Suggest a suitable way.

Sol. *L = 3 henry,* $C = 27 \times 10^{-6} F$, $R = 7.4\Omega$

Resonant angular frequency is given by $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{3 \times 27 \times 10^{-6}}} \, rad\ s^{-1}$

$$= \frac{1}{9 \times 10^{-3}} \, rad\ s^{-1} = \frac{1000}{9} \, rad\ s^{-1} = 111 \, rad\ s^{-1}$$

$$Q = \frac{\omega_0 L}{R} = \frac{111 \times 3}{7.4} = 45$$

To double Q without changing ω_0, *reduce R to* 3.7Ω.

Illustration:

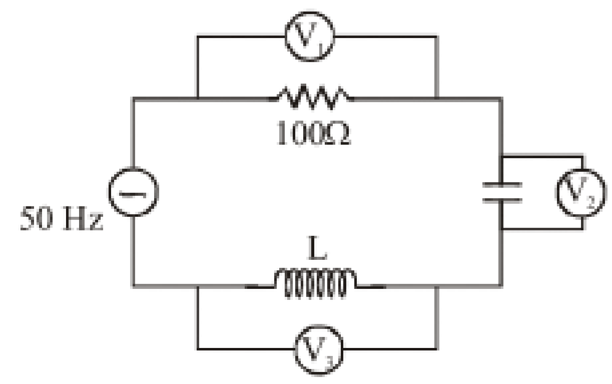
A series L-C-R circuit is connected to an AC source of 50 Hz. as shown in figure. If the readings of the three voltmeters V_1, V_2 and V_3 are 65 V, 415 V and 204 V respectively, calculate,

(i) The current in the circuit

(ii) The value of inductor

(iii) The value of the capacitor C and

(iv) The value of C (for the same L) required to produce resonance.



Sol. ***(i)*** *Here $V_R = i_{rms} R$,*

where i_{rms} is the rms value of current in the circuit.

$$\therefore \qquad i_{rms} = \frac{V_R}{R} = \frac{65}{100} = 0.65 \text{ amp}$$

(ii) $V_L = i_{rms} \times X_L$ or $X_L = \dfrac{V_L}{i_{rms}}$ $\therefore$ $\quad X_L = \dfrac{204}{0.65} = 313.85\Omega$

Now, $X_L = \omega L = 2\pi f L$ or $L = \dfrac{X_L}{2\pi f}$ $\therefore$ $\quad L = \dfrac{313.85}{2\pi \times 50} = 1.0$ henry

(iii) $\quad X_C = \dfrac{V_C}{i_{rms}} = \dfrac{415}{0.65} = 638.46\Omega$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \therefore C = \frac{1}{2\pi \times 50 \times 638.46} = 5 \times 10^{-6} = 5\mu F$$

(iv) *Let 'C' be the capacitance of capacitor that will produce resonance with inductor L = 1.0 henry. Then*

$$f = \frac{1}{2\pi\sqrt{(LC)}} \text{ or } C \frac{1}{4\pi^2 f^2 L}$$

$$= \frac{1}{4\pi^2 \times (50)^2 \times 1.0} = 10.1 \times 10^{-6} F = 10.1\mu F.$$

Solved Examples

Q.1. A circuit consists of a series combination of a 50 mH inductor and a $20\mu F$ capacitor connected to 220 V, 50 Hz supply. The circuit has negligible ohmic resistance. (a) Find the amplitude and rms value of current. (b) Find the rms values of voltage across inductor and capacitor. (c) What is the average power transferred to (i) the inductor, and (ii) the capacitor ? (d) What is the total power absorbed by the circuit

Sol. Given L = 50 mH = 50 × 10^{-3} H, C = $20\mu F = 20\times10^{-6}\,F, V_{rms} = 220V, V = 50Hz$ and

$\omega = 2\pi v = 100\pi\,rads^{-1}$. Therefore, peak value of voltage is

$$V_0 = \sqrt{2}\,V_{rms} = \sqrt{2}\times 220 = 311V$$

The peak value of the current in a series LCR circuit is

$$I_0 = \frac{V_0}{\left[R^2\left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} = \frac{V_0}{Z} \quad ..... (1)$$

Putting R = 0 we have

$$I_0 = \frac{V_0}{\omega L - \frac{1}{\omega C}} \quad ..... (2)$$

(a) Substituting the values of V_0, ω, L and C in equation (2) and solving we get $I_0 = -2.17A$. The amplitude of current is the magntidue of modulus of current. Therefore, amplitude of curernt is $|I_0| = 2.17A$.

$$\therefore \quad I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{2.17}{\sqrt{2}} = 1.53A$$

(b) Voltage drop across inductor is

$$(V_L)_{rms} = I_{rms}\times\omega L = 1.53\times100\pi\times50\times10^{-3} = 23.1\,V$$

Voltage drop across capictor is

$$(V_C)_{rms} = \frac{I_{rms}}{\omega L} = \frac{1.53}{100\pi\times20\times10^{-6}} = 243\,V$$

(c) Since in an inductor the voltage leads the current by a phase angle of 90°, the power transferred to it is 0. Thus $P_L = 0$. We know that in a capacitor the current leads the voltage by 90°. Hence power transfered to capcitor is 0. Thus $P_C = 0$.

(d) Total power absorbed by the circuit is $P = P_L + P_C = 0$

Q.2 A bulb is rated 55 W/110 V. It is to be connected to a 220 V/50 Hz with inductor in series. What should be the value of inductance so that bulb gets correct voltage.

Sol. We want $\frac{V_{app}R}{\sqrt{R^2+L^2\omega^2}}=110$ or $\frac{220R}{\sqrt{R^2+L^2\omega}}=110$

or $4R^2=R^2+L^2\omega^2$ or $L\omega=\sqrt{3}\,R$

$L(100\pi)=220\sqrt{3}$

$L=\frac{2.2\sqrt{3}}{3.14}=\frac{2.2(1.732)}{3.14}=1.2H$; $R=\frac{110\times110}{55}=220\Omega$

Q.3 A circuit draws a power of 550 watt from a source of 220 volt, 50 Hz. The power factor of the circuit is 0.8 and the current lags in phase behind the potential difference. To make the power factor of the circuit as 1.0, what capacitance will have to be connected with it.

Sol. As the current lags behind the potential difference, the circuit contains resistance and inductance.

Power, $P=v_{rms}\times i_{rms}\times\cos\phi$

Here, $i_{rms}=\frac{V_{rms}}{Z}$, where $Z=\sqrt{[(R^2+(\omega L)^2)]}$

$\therefore$ $P=\frac{V_{rms}^2\times\cos\phi}{Z}$ or $Z=\frac{v_{rms}^2\times\cos\phi}{P}$

So, $Z=\frac{(220)^2\times0.8}{550}=70.4\,\text{ohm}$

Now, power factor $\cos\phi=\frac{R}{Z}$ or $R=Z\cos\phi$

$\therefore$ $R=70.4\times0.8=56.32$ ohm

Further, $Z^2=R^2+(\omega L)^2$ or $(\omega L)=\sqrt{(Z^2-R^2)}$

Or $\omega L=\sqrt{(70.4)^2-(56.32)^2}=42.2\,\text{ohm}$

When the capacitor is connected in the circuit,

$$Z=\sqrt{\left[R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right]}$$

and $\cos\phi=\frac{R}{\sqrt{\left[R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right]}}$

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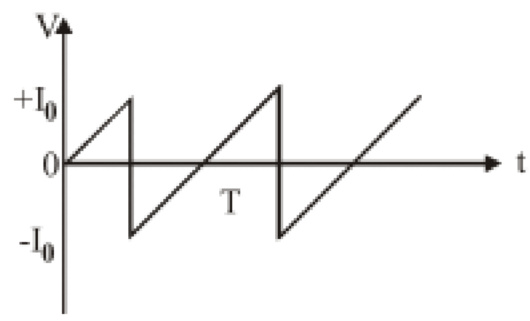
when $\cos\phi = 1, \omega L = \frac{1}{\omega C}$

$$\therefore \qquad C = \frac{1}{\omega(\omega L)} = \frac{1}{2\pi f(\omega L)}$$

$$= \frac{1}{(2\times 3.14\times 50)\times(42.2)}$$

$$= 75\times 10^{-6}\,F = 75\mu F.$$

Q.4 Find the average current in terms of I_0 for the waveform shown.



Sol. $I = 2I_0\frac{t}{T_0};\ 0 < t < \frac{T_0}{2}$;

$$I = 2I_0\left(\frac{t}{T_0} - 1\right);\frac{T_0}{2} < t < T_0$$

$$I_{av} = \frac{2}{T}\int_0^{T_0} I.dt = \frac{2}{T_0}\left[\int_0^{T/2}\frac{2I_0 t}{T_0}dt\right]$$

$$= \frac{2}{T_0^2}\left[\frac{2I_0T_0^2}{2\times 4}\right] = \frac{I_0}{2}$$

Q.5 A sinusoidal voltage of peak value 283 V and frequency 50 Hz is applied to a series LCR circuit in which $R = 3\Omega$, $L = 25.48$ mH, and $C = 796\ \mu F$. Find (a) the impedance of the circuit; (b) the phase difference between the voltage across the source and the current; (c) the power dissipated in the circuit; and (d) the power factor.

Sol. (a) To find the impedance of the circuit, we first calculate X_L and X_C.

$$X_L = 2\pi f L$$

$$= 2\times 3.14\times 50\times 25.48\times 10^{-3}\Omega = 8\Omega$$

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\times 3.14\times 50\times 796\times 10^{-6}} = 4\Omega$$

Therefore,

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{3^2 + (8-4)^2} = 5\Omega$$

(b) Phase difference, $\phi = \tan^{-1}\frac{X_C - X_L}{R} = \tan^{-1}\left(\frac{4-8}{3}\right) = -53.1^\circ$

since ϕ is negative, the current in the circuit lags the voltage across the source.

(c) The power dissipated in the circuit is

$P = I_{rms}^2 R$

Now, $I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{283}{5}\right) = 40A$

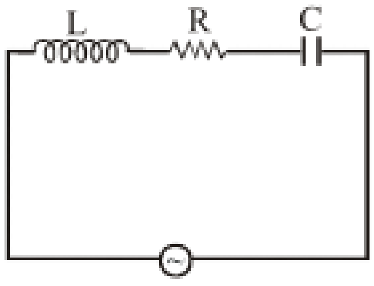
Therefore, $P = (40A)^2 \times 3\Omega = 4800W$

(d) Power factor $= \cos\phi = \cos 53.1^\circ = 0.6$.

Q.6 An inductor of 20×10^{-3} henry, a capacitor $100\ \mu F$ and a resistor $50\ \Omega$ are connected in series across a source of emf $V = 10\sin(314t)$. Find the energy dissipated in the circuit in 20 minutes. If resistance is removed from the circuit and the value of inductance is doubled, then find the variation of current with time in the new circuit.

Sol. Here, time of 1 cycle $T = \frac{2\pi}{\omega} = \frac{2\pi}{314} = 1/50$ s. So, we have to calculate the average energy as time >> T.

Energy consumed in time $t = (V_{rms} I_{rms} \cos\phi)t$



$$= \frac{I_0}{\sqrt{2}}\cdot\frac{V_0}{\sqrt{2}}\cdot\frac{R}{Z}t$$

$$E = \frac{V_0^2 R}{2Z^2}t \qquad \left(\because I_0 = \frac{V_0}{Z}\right)$$

Now, $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega c}\right)^2}$

$$= \sqrt{(50)^2 + \left(314\times20\times10^{-3} - \frac{1}{314\times100\times10^{-6}}\right)^2}$$

$$= \sqrt{3153.6} \approx 56 \text{ ohm}$$

$\therefore$ Energy consumed $= \frac{10^2\times50\times20\times60}{2\times3153.6}$ joule

When resistance is removed,

$$\cos\phi = \frac{R}{Z'} = 0 \text{ or } \qquad \phi = \pi/2$$

$$Z' = \frac{1}{\omega c} - \omega L' = \frac{1}{314 \times 10^{-4}} - 314 \times 40 \times 10^{-3} \ \Omega$$

$$= 19.3\,\Omega$$

$$I = \frac{E_0}{Z'}\sin(\omega t + \phi)$$

$$= \frac{10}{19.3}\sin(314t + \pi/2)$$

$$= 0.52 \cos 314\,t \text{ Amp.}$$

Q.7 A resistor of 200Ω and a capacitor of $15.0\ \mu F$ are connected in series to a 220 V, 50 Hz ac source. (a) Calculate the current in the circuit; (b) Calculate the voltage (rms) across the resistor and the capacitor. Is the algebraic sum of these voltage more than the source voltage? If yes, resolve the paradox.

Sol. Given

$$R = 200\Omega,\ C = 15.0\mu F = 15.0 \times 10^{-6} F$$

$$V_{rms} = 220 V,\ f = 50 Hz$$

(a) In order to calculate the current, we need the impedance of the circuit. It is

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + (2\pi f C)^{-2}}$$

$$= \sqrt{(200\Omega)^2 + (2 \times 3.14 \times 50 \times 15 \times 10^{-6}\,\Omega)^{-2}}$$

$$= \sqrt{(200\Omega)^2 + (212.3\Omega)^2}$$

$$= 291.5\Omega.$$

Therefore, the current in the circuit is

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{220V}{291.5\Omega} = 0.755A$$

(b) Since the current is the same throughout the circuit, we have

$$V_R = I_{rms}R = (0.755A)(200\Omega) = 151V$$

$$V_C = I_{rms}X_C = (0.755\,A)(212.3\Omega) = 160.3V$$

The algebraic sum of the two voltage, V_R and V_C is 311.3V which is more than the source voltage of 220 V. How to resolve this paradox? As you have learnt in the text, the two voltages are not in the same phase. Therefore, they cannot be added like ordinary numbers. The two voltages are out of phase by ninety degrees. Therefore, the total of these voltages must be obtained using the Pythagorean theorem:

$$V_{R+C} = \sqrt{V_R^2 + V_C^2} = 220\text{ V}$$

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Thus, if the phase difference between two voltage is properly taken into account, the total voltage across the resistor and the capacitor is equal to the voltage of the source.

Q.8 An LCR circuit has L = 10 mH, R = 3 Ω and C = 1 μF connected in series to a source of 15 cos ωt V. Calculate the current amplitude and the average power dissipated per cycle at a frequency that is 10% lower than the resonance frequency.

Sol. As here resonance frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-2}\times 10^{-6}}} = 10^4 \frac{\text{rad}}{\text{s}}$$

so, $$\omega = \omega_0 - \frac{10}{100}\omega_0 = \frac{9}{10}\omega_0 = 9\times 10^3 \frac{\text{rad}}{\text{s}}$$

and hence, $X_L = \omega L = 9\times 10^3 \times 10^{-2} = 90\Omega$

$$X_C = \frac{1}{\omega C} = \frac{1}{9\times 10^3 \times 10^{-6}} = 111.11\Omega$$

so, $$X = X_C - X_L = 111.11 - 90 = 21.11\Omega$$

and hence, $Z = \sqrt{R^2 + X^2} = \sqrt{3^2 + (21.11)^2}$

i.e., $$Z = \sqrt{9 + 445.63} = 21.32\Omega$$

and as here E = 15 cos ωt, i.e., E_0 = 15 V,

$$I_0 = \frac{E_0}{Z} = \frac{15}{21.32} = 0.704 \text{ A}$$

The average power dissipated,

$$P_{av} = V_{rms} I_{rms} \cos\phi = (I_{rms}\times Z)\times I_{rms}\times \frac{R}{Z}$$

i.e., $$P_{av} = I^2_{rms} R = \frac{1}{2} I_0^2 R \quad \left[\text{as } I_{rms} = \frac{I_0}{\sqrt{2}}\right]$$

so, $$P_{av} = \frac{1}{2}\times(0.704)^2 \times 3 = 0.74 \text{ W}$$

Now as $f = \frac{\omega}{2\pi} = \frac{9\times 10^3}{2\pi}\frac{\text{cycle}}{\text{s}}$

So, $$\frac{P_{av}}{\text{cycle}} = \frac{0.74\,\text{J/s}}{(9\times 10^3/2\pi)\,\text{cycle/s}} = \frac{2\pi\times 0.74}{9\times 10^3}\frac{\text{J}}{\text{cycle}}$$

i.e., $$\frac{P_{av}}{\text{cycle}} = 5.16\times 10^{-4}\frac{\text{J}}{\text{cycle}}.$$

Q.9 A series LCR circuit containing a resistance of 120 Ω has angular resonance frequency 4×10^5 rad s^{-1}. At resonance the voltages across resistance and inductance are 60 V and 40 V respectively. Find the values of L and C. At what frequency the current in the circuit lags the voltage by 45°?

Sol. At resonance as $X=0,\ I=\dfrac{V}{R}=\dfrac{60}{120}=\dfrac{1}{2}\text{A}$

and as $V_L = IX_L = I\omega L \qquad L=\dfrac{V_L}{I\omega}$

so, $$L=\frac{40}{(1/2)\times4\times10^5}=0.2\,\text{mH}$$

and as $\omega_0=\dfrac{1}{\sqrt{LC}}, \qquad C=\dfrac{1}{L\omega_0^2}$

i.e., $$C=\frac{1}{0.2\times10^{-3}\times(4\times10^5)^2}=\frac{1}{32}\mu\text{F}$$

Now in case of series LCR circuit,

$$\tan\phi=\frac{X_L-X_C}{R}$$

so current will lag the applied voltage by 45° if,

$$\tan 45=\frac{\omega L-\dfrac{1}{\omega C}}{R}$$

i.e., $$1\times120=\omega\times2\times10^{-4}-\frac{1}{\omega(1/32)\times10^{-6}}$$

i.e., $$\omega^2-6\times10^5\omega-16\times10^{10}=0$$

i.e., $$\omega=\frac{6\times10^5\pm\sqrt{(6\times10^5)^2+64\times10^{10}}}{2}$$

i.e., $$\omega=\frac{6\times10^5+10\times10^5}{2}=8\times10^5\,\frac{\text{rad}}{\text{s}}.$$

Q.10 In a series RC circuit $R=500\Omega, C=2\mu F, V=282\sin(377t)$. The power consumed is

(A) 14100 W (B) 141 W (C) 10 W (D) 14.1 W

Sol. $$P=\frac{V_P^2R}{2Z^2}=\frac{282\times282\times500}{2\times\left(\sqrt{(500)^2+\left(\dfrac{10^6}{2\times377}\right)^2}\right)^2}=\frac{282\times282\times500}{2\times1410\times1410}=10\,\text{W}$$

Q.11 The value of L, C and R in an LCR series circuit are 4 mH, 40 pF and $100\,\Omega$ respectively. The quality factor of the circuit is

(A) 10,000 (B) 100 (C) 1000 (D) 10

Sol. $\omega = \frac{1}{\sqrt{LC}}$ $Q = \frac{L\omega}{R} = \frac{L}{R\sqrt{LC}} = \frac{4\times10^{-3}}{100\sqrt{4\times10^{-3}\times40\times10^{-12}}} = 100$

Q.12 In an LR series AC circuit the angular frequency of applied emf is $2\times10^{4}\,\text{rads}^{-1}$ and the value of resistance is $20\,\Omega$. The instant at which the value of emf is maximum E_0, the value of current is $\frac{i_{max}}{\sqrt{2}}$. The inductance in the circuit will be

(A) 1 mH (B) 40 mH (C) 8 mH (D) cannot be predicted

Sol. $\phi = \pi/2 - \pi/4 = \pi/4,\ \tan\pi/4 = 1 = \frac{L\omega}{R}$ $\therefore L = 1\text{ m H}$

Q.13 The self inductance of the motor of an electric fan is 10 Henry. In order to impart maximum power at 50 Hz it should be connected to a capacitance of

(A) 3×10^{-6} Farad (B) 2×10^{-6} Farad (C) 10^{-6} Farad (D) 10^{-4} Farad

Sol. For maximum power to be transferred

$$X_L = X_C \text{ or } L\omega = \frac{1}{C\omega}$$

$$\text{or } C = \frac{1}{L\omega^2} = \frac{1}{10\times(100\pi)^2} = 10^{-6}\text{F}$$

Q.14 A coil of resistance 200 ohms and self inductance 1.0 henry has been connected to an a.c. source of frequency $200/\pi$ Hz. The phase difference between voltage and current is :

(A) 30° (B) 63° (C) 45° (D) 75°.

Sol. $\tan\phi = \frac{X_L}{R} = \frac{2\pi fL}{R} = \frac{2\pi(200/\pi)1}{200} = 2$

$\therefore\quad \phi = 63°$.

Q.15 The p.d. across an instrument in an a.c. circuit of frequency f is V and the current flowing through it is I such that $V = 5\cos\pi ft$ (B) volt and $I = 2\sin(2\pi ft)$ amp. The power dissipate in the instrument is :

(A) zero (B) 10 watt (C) 5 watt (D) 2.5 watt.

Sol. As $V = 5\cos\pi ft(2) = 5\sin(2\pi ft + \pi/2)$

And $I = 2\sin(2\pi ft)$

$\therefore$ phase difference between V and I is $\phi = \dfrac{\pi}{2}$

Average power $P = V_{rms} I_{rms} \times \cos\phi = 0$

Q.16 In an L-R circuit, the value of L is $(0.4/\pi)$ henry and the value of R is 30 ohm. If in the circuit, an alternating emf of 200 volt at 50 cycles per second is connected, the impedance of the circuit and current will be :

(A) 11.4 ohm, 17.5 ampere
(B) 30.7 ohm, 6.5 ampere
(C) 40.4 ohm, 5 ampere
(D) 50 ohm, 4 ampere.

Sol. Here $X_L = \omega L = 2\pi f L$

$$= 2\pi \times 50 \times \frac{0.4}{\pi} = 40\Omega$$

$$R = 30\,\Omega$$

$$\therefore \quad Z = \sqrt{R^2 + X_L^2} = \sqrt{30^2 + 40^2} = 50\Omega$$

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{200}{50} = 4A.$$

Q.17 In an a.c. circuit, V & I are given by
$V = 100 \sin(100\,t)$ volt.
$I = 100 \sin(100\,t + \pi/3)$ mA.
The power dissipated in the circuit is :

(A) 10^4 watt
(B) 10 watt
(C) 2.5 watt
(D) 5 watt.

Sol. $P = V_{rms} I_{rms} \cos\phi$

$$= \frac{100}{\sqrt{2}} \cdot \left(\frac{100}{\sqrt{2}} \times 10^{-3}\right) \cos 60^\circ$$

$$= \frac{10}{2} \times \frac{1}{2} = 2.5 \text{ watt}.$$

Q.18 An alternating voltage (in volts) varies with time t (in seconds) as $V = 200 \sin(100\,\pi t)$

(A) The peak value of the voltage is 200 V
(B) The rms value of the voltage is 220 V
(C) The rms value of the voltage is $100\sqrt{2}$ V
(D) The frequency of the voltage is 50 Hz

Sol. $V = 200 \sin(100\,\pi t)$

Compare this equation with $V = V_0 \sin\omega t$

$$V_0 = 200\ \text{V}, \quad V_{rms} = \frac{V_0}{\sqrt{2}} = \frac{200}{\sqrt{2}} = 100\sqrt{2}$$

$$\omega = 50$$

Q.19 A 50 Ω electric heater is connected to 100 V, 60 Hz ac supply.

(A) The peak value of the voltatge is 100 V

(B) The peak value of the current in the circuit is $2\sqrt{2}$ A

(C) The rms value of the voltage is 100 V

(D) The rms value of the current is 2 A

Sol. V_{rms} = 100 V. Peak value of voltage $=100\sqrt{2}\,V$. Peak value of current $=\dfrac{100\sqrt{2}}{50}=2\sqrt{2}\,A$

$$I_{rms}=\frac{2\sqrt{2}}{\sqrt{2}}=2A$$

Q.20 L, C and R respectively represent inductance, capacitance and resistance. Which of the following combinations have the dimensions of frequency ?

(A) R/L

(B) 1/RC

(C) $R/\sqrt{LC}$

(D) $1/\sqrt{LC}$

Sol. The dimensions of ωL and $1/\omega C$ are the same as those of resistance, where $\omega = 2\pi v$.

Q.21 In a series LCR circuit

(A) the voltage V_L across the inductance leads the current in the circuit by a phase angle of $\pi/2$

(B) the voltage V_C across the capacitance lags behind the current by a phase angle of $\pi/2$

(C) the voltage V_R across the resistance is in phase with the current

(D) the votage across the series combination of L, C and R is $V_{max}=V_{L\,max}+V_{C\,max}+V_{R\,max}$.

Sol. The voltage across the combination is given by $V=\left[(V_R)^2+(V_C-V_L)^2\right]^{1/2}$.

Q.22 If $V=V_p\sin(\omega t+\pi/3)$ when will the voltage be maximum for the first time ?

(A) T/6

(B) T/12

(C) T/3

(D) None of these

Sol. $\sin(\omega t+\pi/3)=1$ or $\omega t=\pi/6$

$$\text{or } \frac{2\pi t}{T}=\pi/6 \text{ or } t=\frac{T}{12} \text{ seconds}$$

Q.23 If $i=t^2,\ 0<t<T$ then rms value of current is

(A) $\dfrac{T^2}{\sqrt{2}}$

(B) $\dfrac{T^2}{2}$

(C) $\dfrac{T^2}{\sqrt{5}}$

(D) None of these

Sol. $i_{rms}\sqrt{\dfrac{1}{T}\int_0^T i^2dt}=\dfrac{T^2}{\sqrt{5}}$

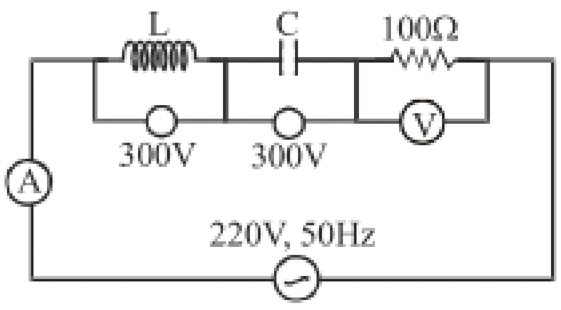
Q.24 In the circuit shown in figure, what will be the readings of voltmeter and ammeter ?

(A) 800 V, 2 A

(B) 220 V, 2.2 A

(C) 300 V, 2 A

(D) 100 V, 2 A.



Sol. As $V_L = V_C = 300\,V$,

and $V = \sqrt{V_R^2 + (V_L - V_C)^2}$

$\therefore$ $V_R = V = 220\,V$

Also $I = \dfrac{V}{R} = \dfrac{220}{100} = 2.2$ A.

Q.25 An AC voltmeter in an LCR circuit reads 30 V across resistance 80 V across inductance and 40 V across capacitance. The value of applied voltage will be

(A) 25 V (B) 50 V (C) 70 V (D) 150 V

Sol. $V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{30^2 + 40^2} = 50V$

Q.26 In an AC circuit, a resistance of 3Ω, an inductance coil of 4Ω and a condenser of 8Ω are connected in series with an AC source of 50 V (rms). The average power loss in the circuit will be

(A) 300 W (B) 600 W

(C) 400 W (D) 500 W

Sol. $P = \dfrac{V_{rms}^2 \times R}{|Z|^2} = \dfrac{50^2 \times 3}{5^2} = 300W$

Q.27 The current flowing in a coil is 3 A and the power consumed is 108 W. If the a.c. source is of 120 V, 50 Hz, the resistance of the circuit is :

(A) 24Ω (B) 10Ω

(C) 12Ω (D) 6Ω.

Sol. From $P = I^2R, R = \dfrac{P}{I^2} = \dfrac{108}{3^2} = 12\Omega$

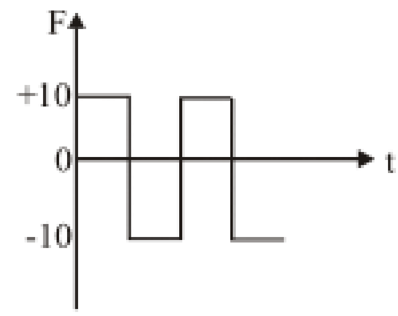
Q.28 If the phase difference between voltage and current is $\pi/6$ and the resistance in the circuit is $\sqrt{300}\,\Omega$, then the impedance of the circuit will be

(A) 40Ω (B) 20Ω

(C) 50Ω (D) 13Ω

Sol. $\cos\phi = \frac{R}{|Z|}$ or $\frac{\sqrt{3}}{2} = \frac{\sqrt{300}}{|Z|}$ or $Z = 20\ \Omega$

Q.29 The rms voltage of the wave form shown is



(A) 10 V (B) 7 V

(C) 6.37 V (D) None of these

Sol. $V_{rms} = \sqrt{\frac{1}{T}\int_0^T 10^2\,dt} = 10V$

Question No. 30 to 32

A 100Ω resistance is connected in series with a 4H inductor. The voltage across the resistor is, $V_R = (2.0V)\sin(10^3 t)$.

Q.30 Find the expression of circuit current

(A) $(2\times10^{-2}\,A)\sin(10^3 t)$ (B) $(2\times10^{-3}\,A)\sin(10^2 t)$

(C) $(2\times10^{-3}\,A)\sin(10^3 t)$ (D) None of these

Q.31 Find the inductive reactance

(A) 2×10^3 ohm (B) 3×10^3 ohm

(C) 4×10^3 ohm (D) 5×10^3 ohm

Q.32 Find amplitude of the voltage across the inductor.

(A) 40 V (B) 60 V

(C) 80 V (D) 90 V

Sol. 30 to 32

30 $i=\frac{V_R}{R}=\frac{(2.0V)\sin(10^3t)}{100}=(2.0\times10^{-2}A)\sin(10^3t)$

31 $X_L=\omega L=(10^3)\times(4H)=4.0\times10^3\,\text{ohm}$

32 Amplitude of voltage across inductor,

$V_0=I_0X_L=(2.0\times10^{-2}A)(4.0\times10^3\,\text{ohm})=80$ volts

Question No. 33 to 35

If various elements, i.e., resistance, capcitance and inductance which are in series and having values $1000\ \Omega$, $1\mu F$ and 2.0 H respectively. Given emf as, $V=100\sqrt{2}\sin 1000t$ volts

Q.33 Voltage across the resistor is

(A) 70.7 Volts (B) 100 Volts

(C) 141.4 Volts (D) 270.7 Volts

Q.34 Voltage across the inductor is

(A) 70.7 Volts (B) 100 Volts

(C) 141.4 Volts (D) 270.7 Volts

Q.35 Voltage across the capacitor is

(A) 70.7 Volts (B) 100 volts

(C) 141.4 Volts (D) 270.7 Volts

Sol. 33 to 35

33 rms value of voltage across the source

$$V_{rms}=\frac{100\sqrt{2}}{\sqrt{2}}=100\text{ Volt}$$

From question, $\omega=1000\,\text{rad/s}$

$$i_{rms}=\frac{V_{rms}}{|Z|}=\frac{V_{rms}}{\sqrt{R^2+(X_L-X_C)^2}}=\frac{V_{rms}}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$$

$$=\frac{100}{\sqrt{(1000)^2+\left(1000\times2-\frac{1}{1000\times1\times10^{-6}}\right)^2}}=0.0707\,\text{amp}$$

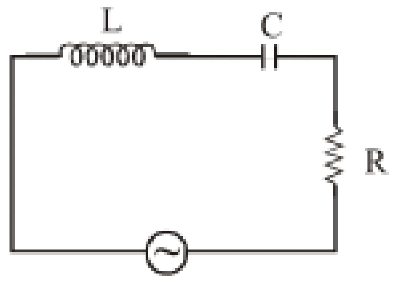
Since the curernt will be same every where in the circuit, therefore

P.D. across resistor $V_R=i_{rms}R=0.0707\times1000=70.7$ volts

34 P.D. across inductor $V_L = i_{rms} XL = 0.0707 \times 1000 \times 2 = 141.4$ volt

35 P.D. across capalitor $V_C = i_{rms} X_C = 0.0707 \times \frac{1}{1 \times 1000 \times 10^{-6}} = 70.7$ volts

Q.36 Figure shows a series LCR circuit connected to a variable frequency 200 V source. $L = 5$ H, $C = 80\,\mu F$ and $R = 40\,\Omega$.



Column I		Column II
(A) The impedance of the circuit at resonance (in ohm)	(P)	1250
(B) The current amplitude at resonance (in A)	(Q)	200
(C) The rms potential drop across the inductor at resonance (in volt)	(R)	40
(D) The rms potential drop across the resistor at resonance (in V)	(S)	$5\sqrt{2}$

Sol.

$$Z = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}$$

$\omega = \omega_r = 1/\sqrt{LC}$ (resonance)

$Z = R = 40\,\Omega$

$$I_0 = V_0 / Z = \frac{\sqrt{2} V_{rms}}{R} = 5\sqrt{2}\text{ A}$$

$$I_{rms} = \frac{V_{rms}}{R} = 5\text{ A}$$

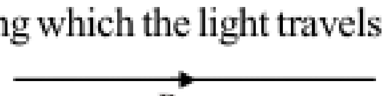
The rms potential drop across L is 1250 V

The rms potential across R is $= I_{rms} \times R = 5 \times 40 = 200$V

Reflection of light at plane surface

Some Basic terms

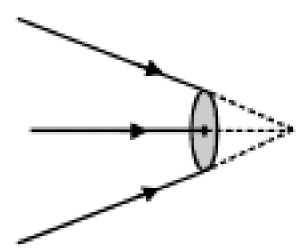
Ray : The straight line path along which the light travels in a homogeneous medium is called a ray.



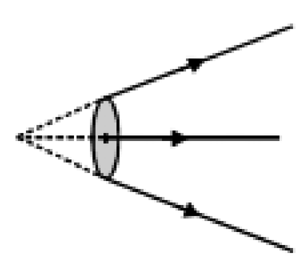
Ray

Beam of light : A bundle or bunch of rays is called a beam. It is of following three types :

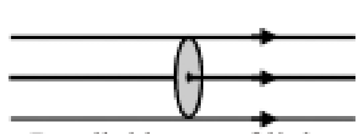
(a) Convergent beam : In this case diameter of beam decreases in the direction of ray.



(b) Divergent beam : It is a beam in which all the rays meet at a point when produced backward and the diameter of beam goes on increasing as the rays proceed forward.



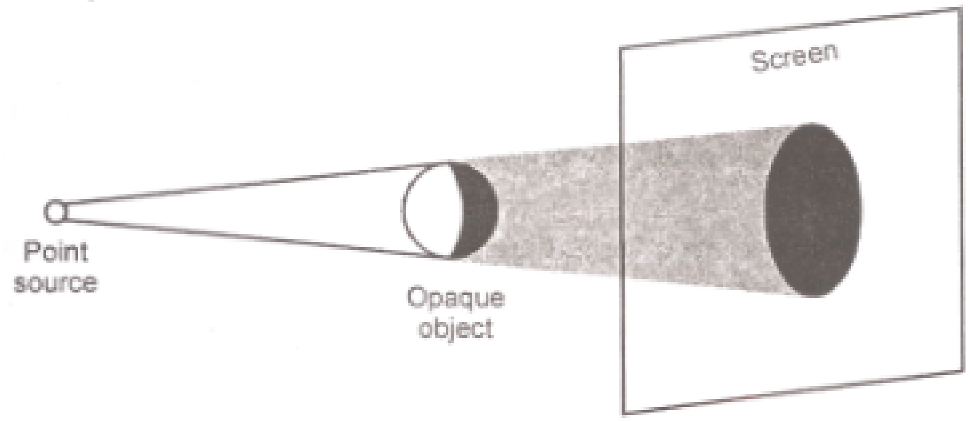
(c) Parallel beam : It is a beam in which all the rays constituting the beam move parallel to each other and diameter of beam remains same.



Parallel beam of light

Shadow formation

Shadow formation is explained by the law of rectilinear propagation of light which state that in a homogeneous medium light travels along straight paths. Thus, an opaque object placed between a point source of light and screen will cast a shadow with a sharply defined boundary.



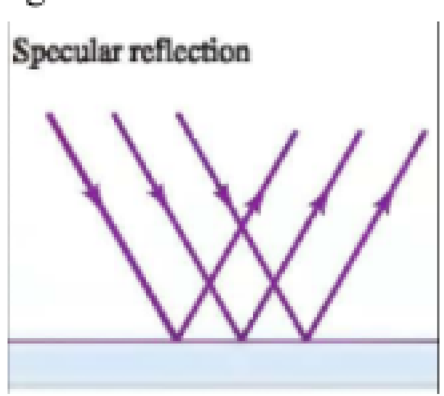
Reflection of light

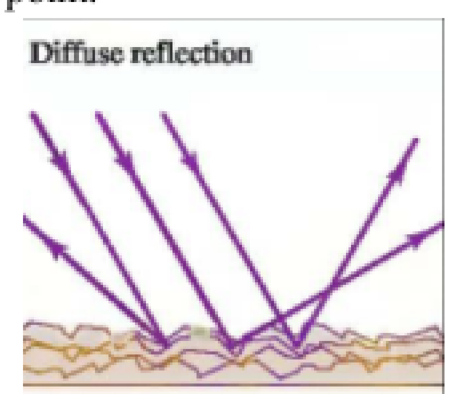
When a beam of light is incident on a boundary separating two media then some part of it may be transmitted and some is turned back in the medium from which it became incident (reflection of light).

Specular and diffuse reflection

The type of reflection which is usually invoked in a discussion of reflection at plane and spherical mirrors is known as specular or regular reflection. An incident parallel beam of light is reflected as a parallel beam in figure. The energy in the incident light is confined to one direction only on reflection.

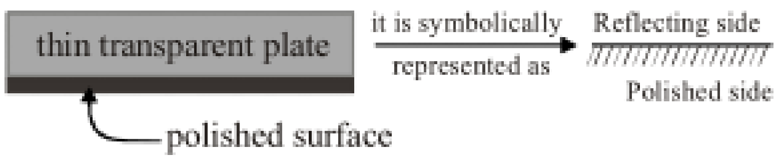
Diffuse or irregular reflection is the most common type of reflection and no image formation takes place as the reflected light can not intersect at a common point.



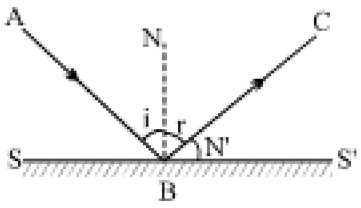


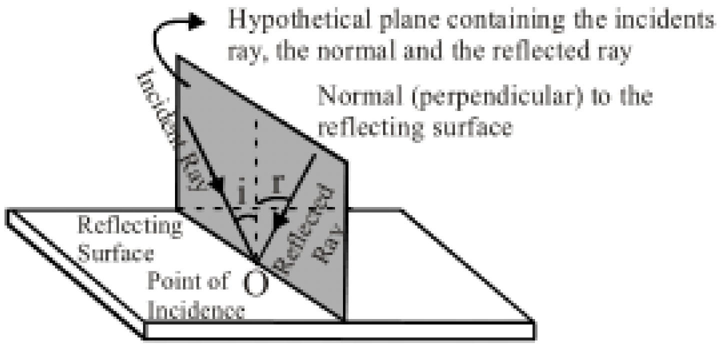
Plane Mirror

A highly polished smooth surface is a mirror. To form a good mirror a thin layer of silver is chemically deposited on a glass surface for high reflectivity.



Laws of Reflection





Reflection from a plane surface

(i) The incident ray (AB), the reflected ray (BC) and normal (NN') to the surface (SC') of reflection at the point of incidence (B) lie in the same plane. This plane is called the plane of incidence (also plane of reflection).

(ii) The angle of incidence (the angle between normal and the incident ray) and the angle of reflection (the angle between the reflected ray and the normal) are equal

$$\angle i = \angle r$$

Laws of Reflection in vector form

Let $\hat{e}_1$ = unit vector along incident ray

$\hat{n}$ = unit vector along normal

$\hat{e}_2$ = unit vector along reflected ray

Now $\vec{e}_{\parallel}$ = component of $\hat{e}$ parallel to mirror

$= \hat{e}_1 - (\hat{e}_1 \cdot \hat{n})\hat{n}$

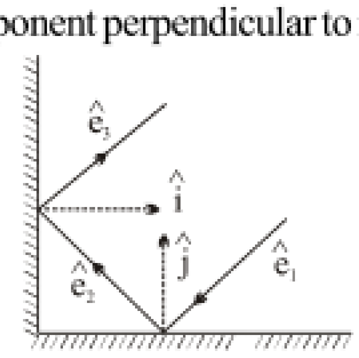
and $\vec{e}_{\perp}$ = component of $\hat{e}_1$ perpendicular to mirror

$= (\hat{e}_1 \cdot \hat{n})\hat{n}$

Hence $\hat{e}_2 = \vec{e}_{\parallel} - \vec{e}_{\perp} = \hat{e}_1 - 2\hat{n}\,(\hat{e}_1 \cdot \hat{n})$

Note :

Whenever reflection takes place, the component of incident ray parallel to reflecting surface remains unchanged, while component perpendicular to reflecting surface (i.e., along normal) reverses in direction.

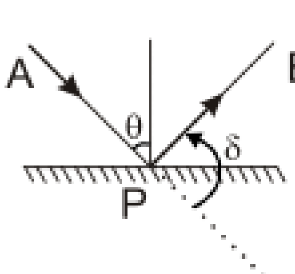


Consider incident ray along unit vector $\hat{e}_1$ given $\hat{e}_1 = -x\hat{i} - y\hat{j}$ unit vector along reflected ray will be given by $\hat{e}_2 = -x\hat{i} + y\hat{j}$ similarly $\hat{e}_3 = x\hat{i} + y\hat{j}$ diverge.

Principle of reversibility of light :

According to this principle if the path of light is reversed then it will retrace its path. i.e., if BP would incident ray then PA would be corresponding reflected ray.

Deviation produced by plane mirror :



$\delta = 180^\circ - 2\theta$

Angle of deviation produced by a single surface depends on the angle of incidence, Greater the angle of incidence lower will be the deviation and vice–versa.

Rotation of Mirror and Incident ray :

(i) If incident ray is rotated (Mirror is kept fixed) in the plane of incidence by angle θ then reflected ray rotates by the same angle in the same angle in plane of incidence but in opposite sense.

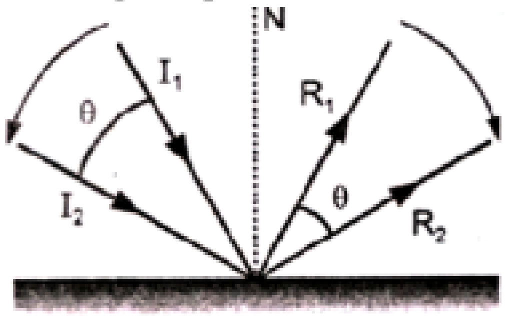


Fig. Rotation of incident ray

(ii) If mirror is rotated (taking position of incident ray same) by angle θ such that normal at the point of incidence rotates in the plane of incidence then reflected ray rotates by 2θ and in same sense.

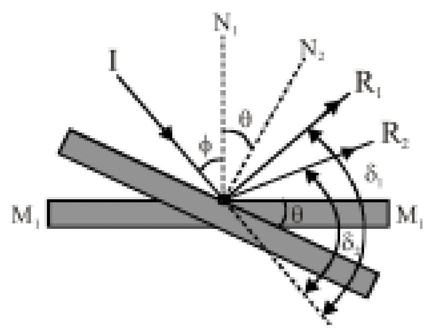
δ_1 = Deviation in first position of mirror

$= \pi - 2\phi$

δ_2 = Deviation in second position of mirror

$= \pi - 2(\theta + \phi)$

$\therefore \quad \delta_1 - \delta_2 = \pi - 2\phi \{\pi - 2(\theta + \phi)\} = 2\theta$

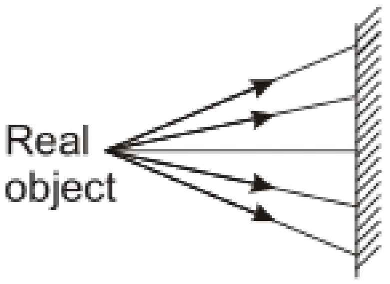


Rotation of plane mirror

Image formed by Plane mirror

Object : The point of intersection of incident beam is called point object.

Real object : If the incident beam is diverging then its intersection point is called real object . It can be seen by human eye and can be photographed by camera.



Virtual object point :

If the incident beam is converging then its intersection point is called virtual object. It cannot be seen by human eye and photographed by a camera.

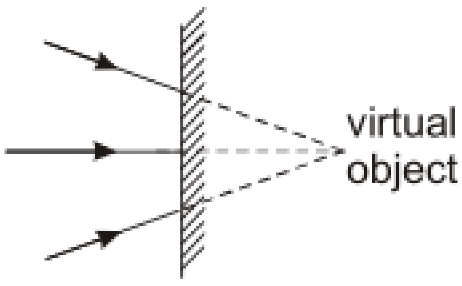
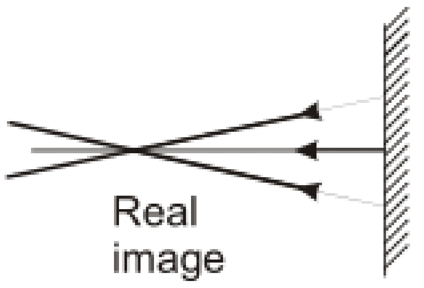


Image : The point of intersection of reflected or refracted beam is called image.

Real image: If the reflected or refracted beam is converging then its intersection point is called real

image. It can be seen by eye, photographed by a camera and can be taken on screen.



Virtual image: If the reflected or refracted beam is diverging then its intersection point is called virtual image. It can be seen by eye, photographed by a camera but can't be taken on screen.

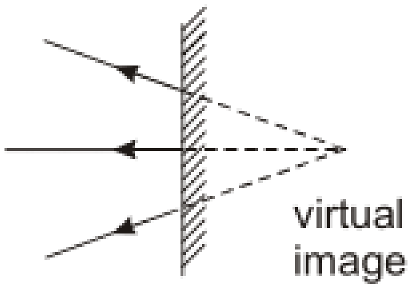
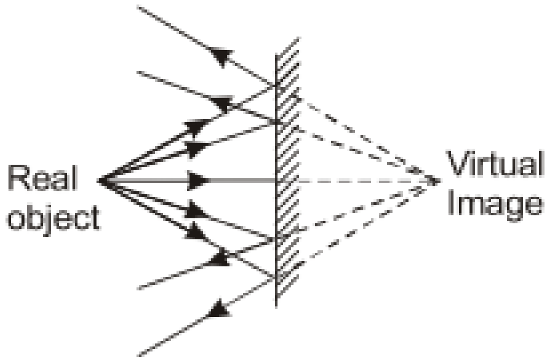
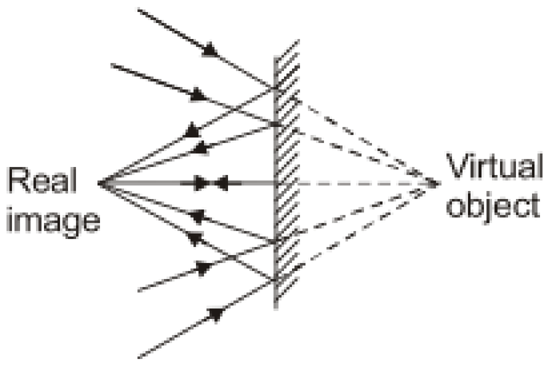


Image of a point object formed by plane mirror

Case : I For real object



Case : II For virtual object



Features :

(i) Distance of object from mirror = Distance of image from the mirror.

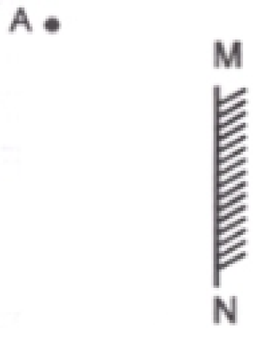
(ii) The line joining a point object and its image is normal to the reflecting surface.

(iii) The size of the image is the same as that of the object.

(iv) For a real object the image is virtual and for a virtual object the image is real.

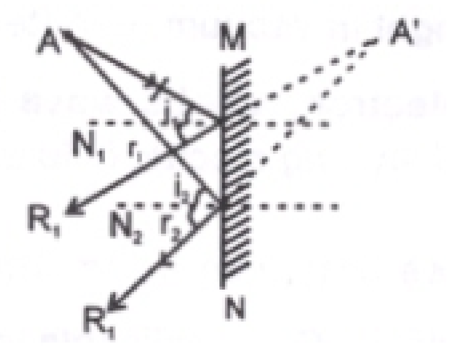
Illustration :

Figure shows a point object A and a plane mirror MN. Find the position of image of object A, in mirror MN, by drawing ray diagram. Indicate the region in which observer's eye must be present in order to view the image. (This region is called field of view.)



Sol. *See figure, consider any two rays emanating from the object N_1 and N_2 are normals ;*

$i_1 = r_1$ and $i_2 = r_2$



The meeting point of reflected rays R_1 and R_2 is image A′. Though only two rays are considered it must be understood that all rays from A reflect from A reflect from mirror such that their meeting point is A′. To obtain the region in which reflected rays are present, join A′ with the ends of mirror and extend. The following figure shows this region as shaded. In figure there are no reflected rays beyond the rays 1 and 2, therefore the observers P and Q cannot see the image because they do not receive any reflected ray.

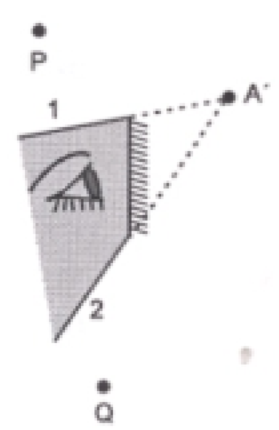
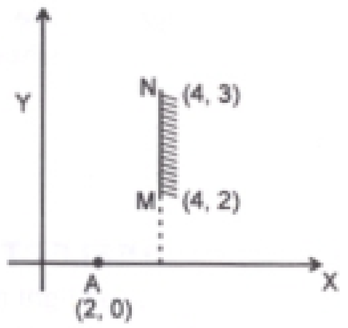
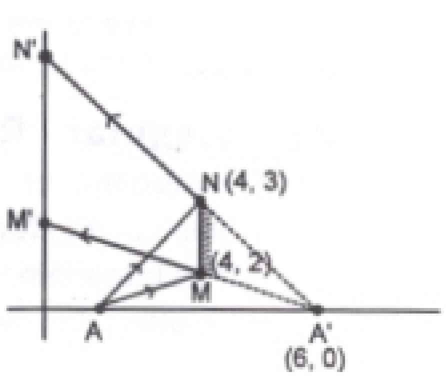


Illustration :

Find the region on Y axis in which reflected rays are present. Object is at A (2, 0) and MN is a plane mirror, as shown.



Sol. *The image of point A, in the mirror is at A′ (6, 0). Join A′M and extend to cut Y axis at M′ (Ray originating from A which strikes the mirror at M gets reflected as the ray MM′ which appears to come from A′). Join A′N and extend to cut Y axis at N′ (Ray originating from A which strikes the mirror at N get reflected as the ray NN′ which appears to come from A′).*



From Geometry.

$M' \equiv (0, 6)$

$N' \equiv (0, 9)$. *M′N′ is the region on Y axis in which reflected rays are present.*

Practice Exercise

Q.1 See the following figure. Which of the object(s) shown in figure will not from its image in the mirror.

Answers

Q.1 O_3.

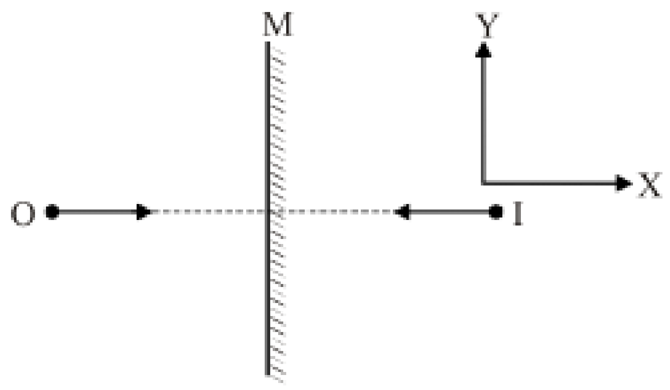
Motion of object and mirror :

Let, $X_{O/m}$ = X co-ordinate of object w.r.t. mirror

$X_{I/m}$ = X co-ordinate of image w.r.t. mirror

$Y_{O/m}$ = Y co-ordinate of object w.r.t. mirror

$Y_{I/m}$ = Y co-ordinte of image w.r.t. mirror



For plane mirror

$$X_{o/m} = -X_{I/m}$$

Differentiating both sides w.r.t. 't'

$$\frac{d}{dt}(X_{o/m}) = \frac{-d}{dt}(X_{I/m})$$

$$[\vec{V}_{o/m}]_x = -[\vec{V}_{IX} - \vec{V}_{mX}]$$

$$\therefore \quad \vec{V}_{IX} = 2\vec{V}_{mX} - \vec{V}_{oX}$$

Similarly, $Y_{I/m} = Y_{o/m}$

Differentiating both side w.r.t. 't' we get

$$(\vec{V}_{I/m})_Y = (\vec{V}_{o/m})_Y$$

In nutshell, for sloving numerical problems involving calculation of velocity of image of object with respect to any observer, always calculate velocity of image first with respect to mirror using following points.

$$(\vec{V}_{I/M})_{||} = (\vec{V}_{o/m})_{||}$$

$$(\vec{V}_{I/M})_{\perp} = (\vec{V}_{O/M})_{\perp}$$

$$\vec{V}_{I/M} = (\vec{V}_{I/M})_{||} + (\vec{V}_{I/M})_{\perp}$$

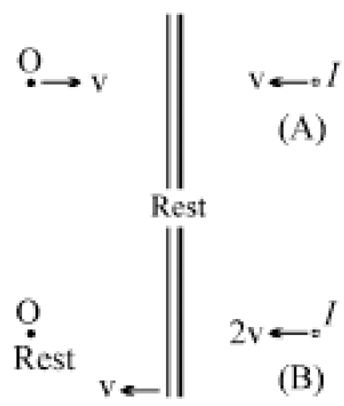
Velocity of image with respect to required observer is then calculated using basic equation for relative

motion.

$$\vec{V}_{A/B} = \text{Velocity of A with respect B}$$

$$= \vec{V}_A - \vec{V}_B$$

(i) If an object moves towards (or away from) a plane mirror at speed v, the image will also approach (or recede) at the same speed v, and the relative velocity of image with respect to object will be 2v, as shown in figure.



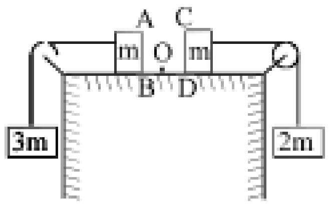
The object and its image move towards each other with same speed v.

The image of a stationary object moves with 2v if the mirror moves toward it with v

(ii) If the mirror is moved toward (or away from) the object with speed v, the image will also move toward (or away from) the object with a speed 2v, as shown in figure.

Illustration :

Two blocks each of mass m lie on a smooth table. They are attached to two other masses as shown in the figure. The pulleys and strings are light. An object O is kept at rest on the table. The sides AB & CD of the two blocks are made reflecting. The acceleration of two images formed in those two reflecting surfaces w.r.t. each other is:



(A) 5g / 6 *(B) 5g / 3* *(C) g / 3* *(D) 17g / 6*

Sol. *We know that*

$$V_I = 2\,V_m + V_0$$

differentiating $a_I = 2a_m + a_0$

$$a_0 = 0$$

$$a_I = 2a_m$$

$$a_A = \frac{3}{4}g \qquad a_C = \frac{2g}{3}$$

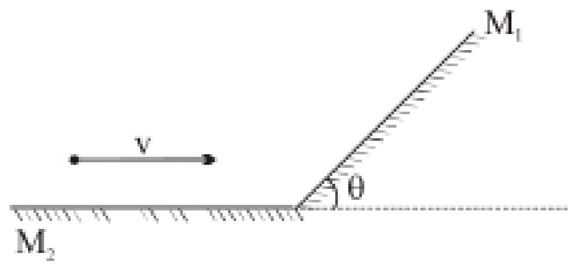
accelerate of image in AB $= 2a_A = \dfrac{3g}{2}$

accelerate of image in CD $= 2a_C = \dfrac{4g}{3}$

acceleration of image in AB w.r.t. that CD $= \dfrac{3g}{2} + \dfrac{4g}{3} = \dfrac{17g}{6}\ m/s^2$

Practice Exercise

Q.1 A point object is moving with a speed v before an arrangement of two mirrors as shown in figure. Find the velocity of image in mirror M_1 with respect to image in mirror M_2.



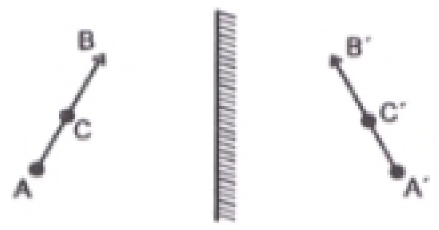
Q.2 A person walks at a velocity v in a straight line forming an angle α with the plane of a mirror. What is the velocity v_{rel} at which he approaches his image assuming that the object and its image are symmetric relative to the plane of the mirror?

Answers

Q.1 $2v\sin\theta$ Q.2 $2v\sin\alpha$

Image of extended object :

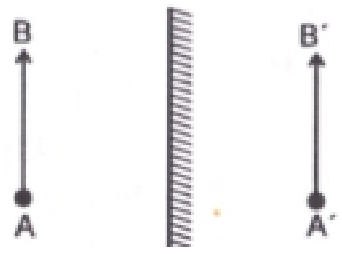
An extended object like AB shown in figure is combination of infinite number of point objects from A to B. Image of every point object will be formed individually and thus infinite images will be formed A′ will be images of A, C′ will be image of C, B′ will be image of B etc. All point images together form extended image. Thus extended image is formed of an extended object.



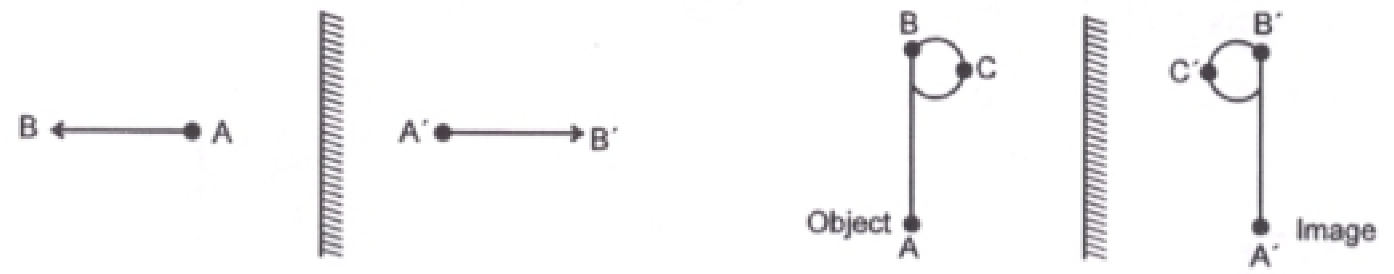
Properties of image of an extended object, formed by a plane mirror :

(1) Size of extended object = size of extended image.

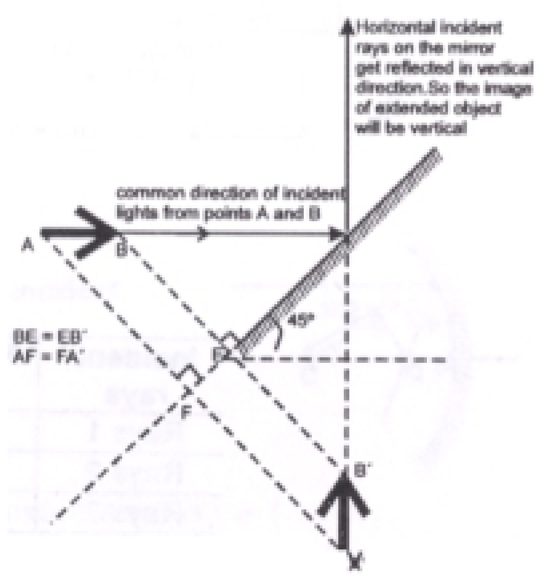
(2) The image of erect, if the extended object is placed parallel to the mirror.



(3) The image is inverted if the extended object lies perpendicular the plane mirror.

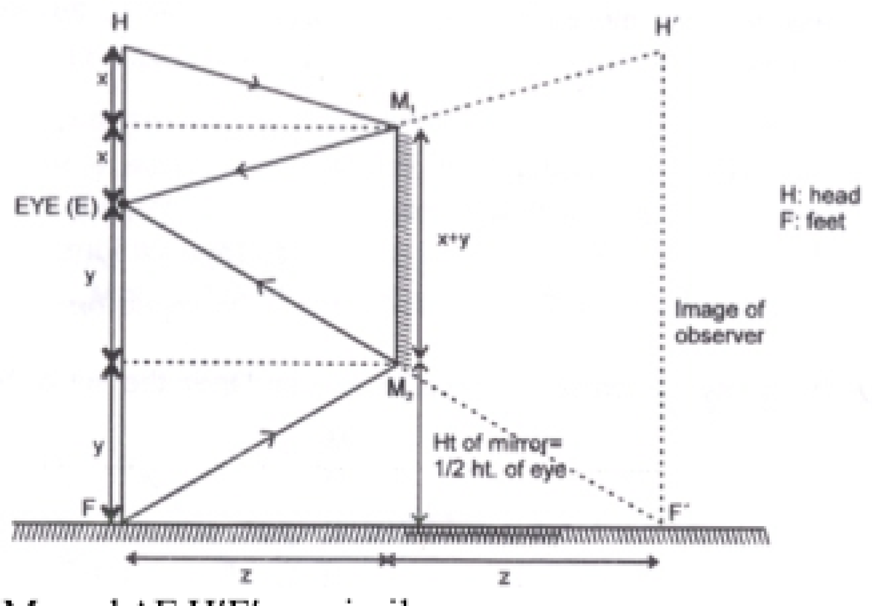


Note: If an extended horizontal object is placed infront of a mirror inclined 45° with the horizontal, the image formed will be vertical. See fig.



Calculation of minimum height of mirror :

(i) Minimum height of a single mirror required for a man to see its complete image



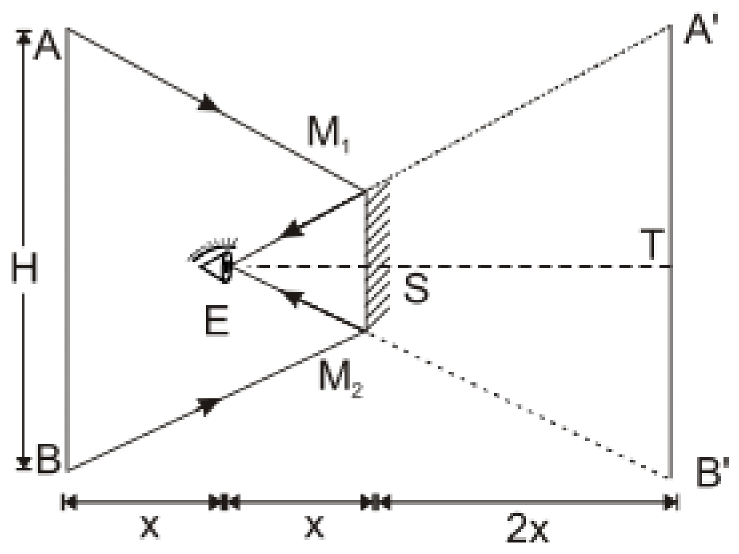
$\Delta E\, M_1, M_2$ and $\Delta E\, H'F'$ are similar

$$\therefore \quad \frac{M_1M_2}{H'F'} = \frac{z}{2z}$$

or $\quad M_1M_2 = H'F' / 2 = HF / 2$

$\therefore$ the minimum size of a plane mirror, required to see the full image of an observer is half the size of that observer.

(ii) Minimum height of a mirror required to see top as well as bottom of wall when man is mid of wall and mirror.



$\Delta E\ M_1, M_2$ and $\Delta E\ H'F'$ are similar

$\therefore \quad \dfrac{M_1M_2}{A'B'} = \dfrac{x}{3x}$

or $\quad M_1M_2 = \dfrac{A'B'}{3} = \dfrac{AB}{3} = \dfrac{H}{3}$

$\therefore$ the minimum size of a plane mirror, required to see the full image of an observer is one third the size of wall if observer is standing extactly between wall and mirror.

Reflection at two mirrors

Calculation of deviation :

In this case net derviation surffered by incident ray is algebraic sum of deviation due to individeula reflection.

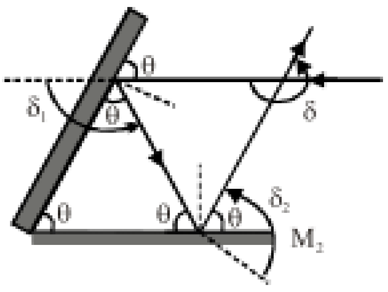
$$\delta_{net} = \Sigma\delta_i$$

where δ_i = Devidation due to single reflection

Note : While summing up. sense of rotation is taken inot account

Illustration :

Two plane mirrors are inclined to each othere such that a ray of light incident on the first mirror and parallel to the second is reflected from the second mirror parallel to the first mirror. Determine the angle between the two mirror. Also determine the totaldeviation produced in the incidnet ray due to the two reflections.



Sol. *Form figure* $3\theta = 180^o$

$\theta = 60^o$

$\delta_1 = 180^o - 2 \times 30^o$

$= 120^o$

$\delta_2 = 180^o - 2 \times 30^o = 120^o$

$\therefore \quad$ *Total deviation* $= \delta_1 + \delta_2$

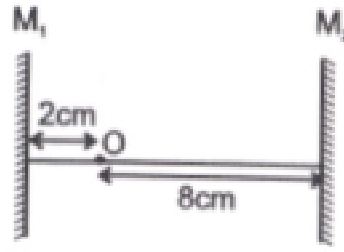
$= 240^o$

Formation of Multiple image by two parallel mirrors:

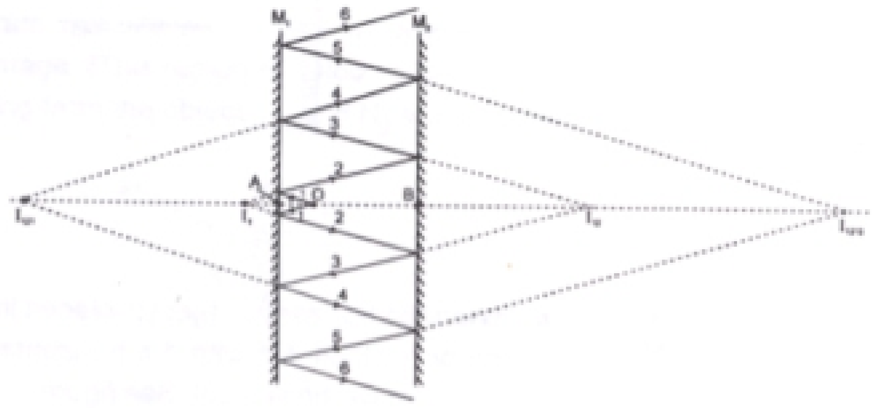
If rays after getting reflected from one mirror strike second mirror, the image formed by first mirror will function as an object for second mirror, and this process will continue for every successive reflection.

Illustration :

Figure shows a point object placed between two parallel mirrors. Its distance from M_1 is 2cm and that from M_2 is 8 cm. Find the distance of image from the two mirrors considering reflection on mirror M_1 first.



Sol. *Let us start forming image from M_1. O is an object for M_1 which from image I_1 behind it. Now I_1 act an object for M_2 which from image I_{12} behind it. Again I_{12} act as object for M_1 and M_1 form image I_{121} behind it and so on. Here we get a series of image.*



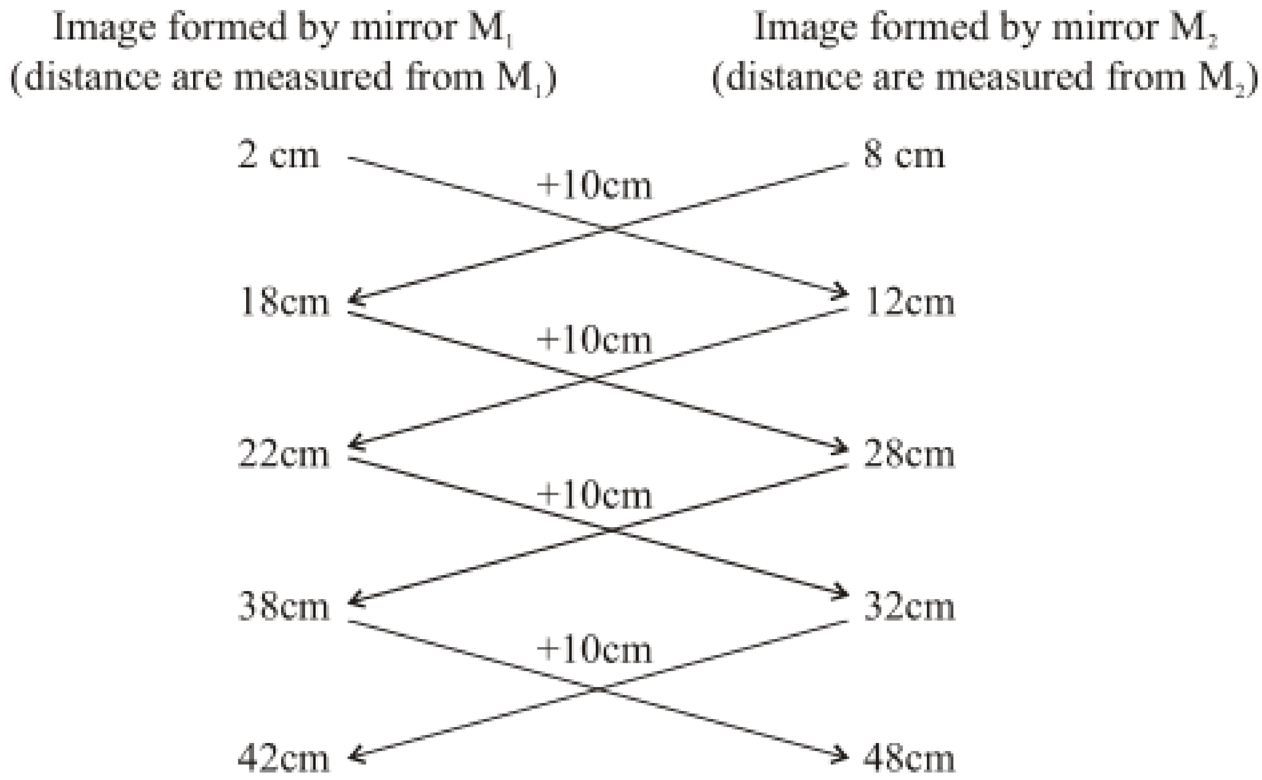
Incident Ray	Reflected by	Reflected ray	Object	Image	Object Distance	Image Distance
Ray 1	M_1	Ray 2	O	I_1	AO = 2cm	AI_1 = 2cm
Ray 2	M_2	Ray 3	I_1	I_{12}	BI_1 = 12cm	BI_{12} = 12cm
Ray 3	M_1	Ray 4	I_{12}	I_{121}	AI_{12} = 22cm	AI_{121} = 22cm
Ray 4	M_2	Ray 5	I_{121}	I_{1212}	BI_{121} = 32cm	BI_{1212} = 32cm

And so on ....

Similarly a series of images will be formed by the rays striking mirror M_2 first. Total number of image = ∞.

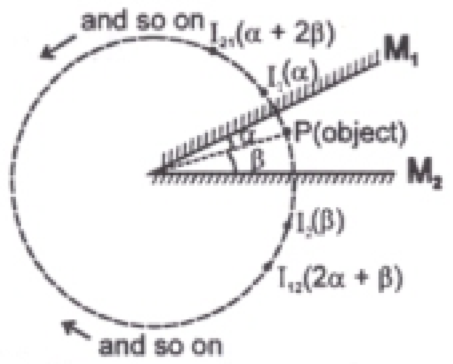
Incident Ray	Reflected by	Reflected ray	Object	Image	Object Distance	Image Distance
Ray 1	M_2	Ray 2	O	I_2	BO = 8cm	BI_2 = 8cm
Ray 2	M_1	Ray 3	I_2	I_{21}	AI_2 = 18cm	AI_{21} = 18cm
Ray 3	M_2	Ray 4	I_{21}	I_{212}	BI_{21} = 28cm	BI_{212} = 28cm
Ray 4	M_1	Ray 5	I_{212}	I_{2121}	AI_{212} = 38cm	AI_{2121} = 38cm

A scheme is given in which both series of images are covered.



Locating all the images formed by two inclined plane Mirrors

Consider two plane mirrors M_1 and M_2 inclined at an angle $\theta = \alpha + \beta$ as shown in figure.



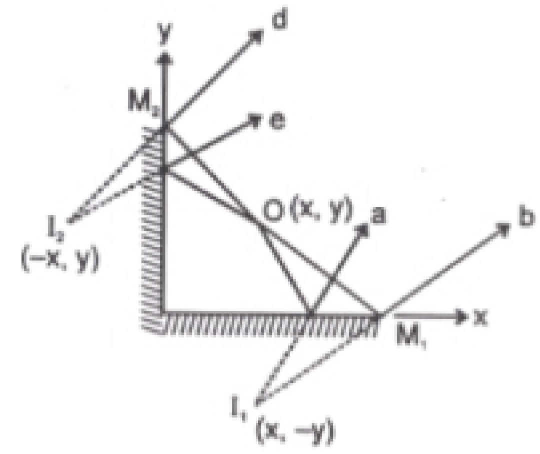
Point P is an object kept such that it makes angle α with mirror M_1 and angle β with mirror M_2. Image of object P formed by M_1, denoted by I_1, will be inclined by angle α on the other side of mirror M_1. This angle is written in bracket in the figure besides I1. Similarly image of object P formed by M_2, denoted by I_2, will be inclined by angle β on the other side of mirror M_2. This angle is written in bracket in the figure besides I2.

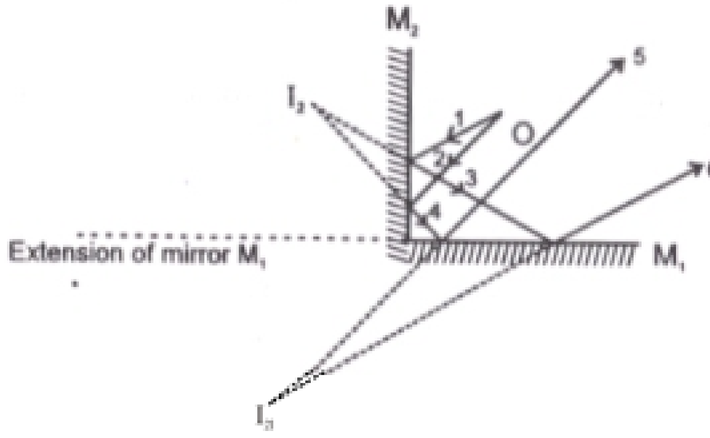
Now I_2 will act as an obect for M_1 which is at an angle $(\alpha + 2\beta)$ from M_1. Its image will be formed at $(\alpha + 2\beta)$ on the opposite side of M_1. This image will be denoted as I_{21}, and so on. Think when hsi will process stop. Hint : The virtual image formed by a plane mirror must not be in front of the mirror or its extension.

Illustration :

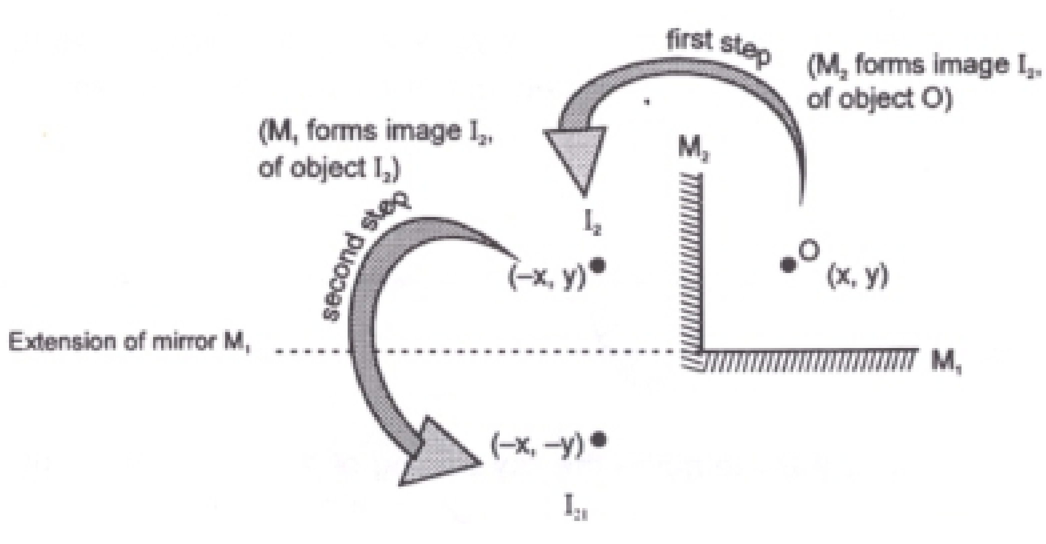
Consider two perpendicular mirrors. M_1 and M_2 and a point object O. Taking origin at the point of intersection of the mirrors and the coordinate of object as (x, y), find the position and number of images.

Sol. *Rays 'a' and 'b' strike mirror M_1 only and these rays will from image I_1 at $(x, -y)$, such that O and I_1 are equidistant from mirror M_1. These rays do not form further image because they do not strike any mirror again. Similarly rays 'd' and 'e' strike mirror M_2 only and these rays will form image I_2 at $(-x, y)$,*
such that O and I_2 are equidistant from mirror M_2.
Now consider those rays which strike mirror M_2 first and then the mirror M_1.





For incident rays 1, 2 object is O, and reflected rays 3, 4 form image I_2.
Now rays 3, 4 incident on M_1 (object is I_2) which reflect as rays 5, 6 and form image I_{21}. Rays 5, 6 do not strike any mirror, so image formation stops.
I_2 and I_{21}, are equidistant from M_1. To summarize see the following figure.
Now rays 3, 4 incident on M_1 (object is I_2) which reflect as rays 5, 6 and form image I_{21}. Rays 5,6 do not strike any mirror, so image formation stops.



For rays reflecting first from M_1 and then from M_2, first image I_1 (at $(x, -y)$) will be formed and this will function as object for mirror M_2 and then its image I_{12} (at $(-x, -y)$) will be formed.

I_{12} and I_{21} coincide.

$\therefore$ *Three images are formed*

Shortcut :

When 360º is exactly divisible by θ.

Here two cases may arise

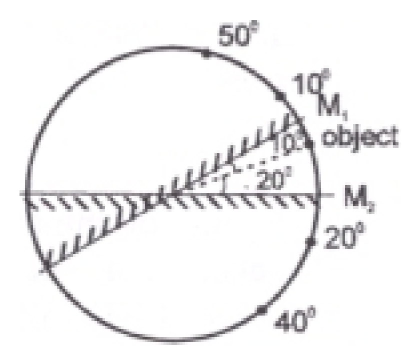
(a) If $\frac{360^o}{\theta}$ is even integer then number of images = $\frac{360^o}{\theta} - 1$. What ever the location of object (symmetric or unsymmetric)

(b) If $\frac{360^o}{\theta}$ is odd integer then number of images = $\frac{360^o}{\theta}$ for unsymmetric placement

$= \frac{360^o}{\theta} - 1$ for symmetric placement.

Illustration:

Two mirrors are inclined by an angle 30°. An object is placed making 10° with the mirror M_1. Find the positions of first two images formed by each mirror. Find the total number of images using (i) direct formula and (ii) counting the images.



Sol. Figure is self explanatory

Number of images

(i) Using direct formula ; $\frac{360°}{30°} = 12$ (even number)

$\therefore$ number of images = 12 – 1 = 11

(ii) By counting. See the following table

Image formed by Mirror M_1 (angles are measured from the mirror M_1.)		Image formed by Mirror M_2 (angles are measured from the mirror M_2.)
10°	+30°	20°
50°	+30°	40°
70°	+30°	80°
110°	+30°	100°
130°	+30°	140°
170°		160°
Stop because next angle will be more than 180°		Stop because next angle will be more than 180°

To check whether the final images made by the two mirrors coincide or not : add the last angles and the angle between the mirrors. If it comes out to be exactly 360°, it implies that the final images formed by the two mirrors coincide. Here last angles made by the mirrors + the angle between the mirrors = 160° + 170° + 30° = 360°. Therefore in this case the last images coincide. Therefore the number of images = number of images formed by mirror M_1 + number of images formed by mirror M_2 -1 (as the last images coincide)= 6 + 6 –1 = 11.

Practice Exercise

Q.1 Two plane mirrors forms an angle of $120°$. The distance between the two image of a point source formed in them is 20 cm and the point source lies on the bisector of the angle formed by the mirrors. What is the distance of the light source from the point where the mirrors touch?

Q.2 To get three images of a single object, what should be the angle between two plane mirrors?

Q.3 Two plane mirrors are inclined at an angle of $60°$ to each other. If an object is placed between them,find the number of images produced.

Answers

Q.1 11.55 cm Q.2 90^o Q.3 5

Reflection at spherical mirror

Aperture : The edge of a spherical mirror is a circle. Part of the plane of circle, enclosed by the circle is called its aperture.

Paraxial Ray: A light ray incident on the mirror at very small angle then the ray is called paraxial ray.

Marginal Ray: A light ray incident on the mirror at finite angle then the ray is called marginal ray.

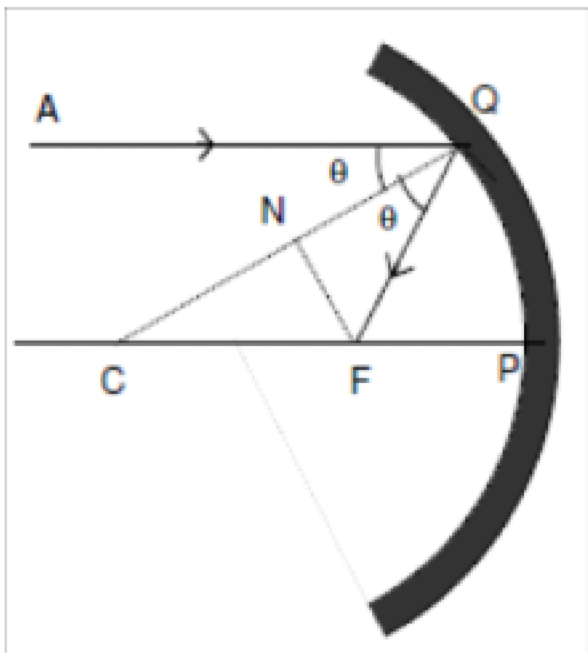
focus : Suppose a light ray AQ parallel to x axis become incident on a concave mirror at angle of incidence θ (fig). After reflection we have reflected ray QF at angle of reflection θ which intersects x axis at F. We want to calculate PF

In triangle CFQ

$$\angle QCF = \angle AQC = \theta \qquad \text{(alternate angle)}$$

$\Rightarrow$ triangle CFQ is an isoscless triangle (CF=QF).

$\Rightarrow$ CN=QN=CQ/2=R/2



In triangle NFQ

$$\text{COS}\theta = \frac{\text{QN}}{\text{QF}} = \frac{\text{R}/2}{\text{QF}}$$

$$\Rightarrow \qquad \text{QF} = \frac{\text{R}}{2\text{COS}\theta}$$

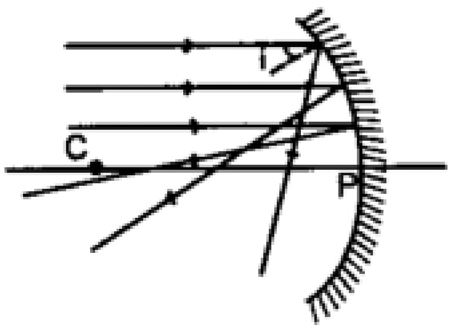
$$\Rightarrow \qquad \text{CF} = \text{QF} = \frac{\text{R}}{2\text{COS}\theta}$$

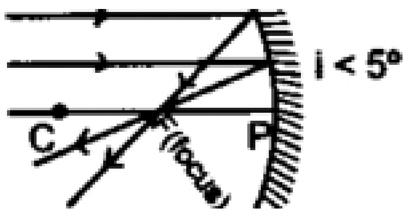
$$\therefore \qquad \text{PF} = \text{PC} - \text{CF} = \text{R} - \text{CF} = \text{R} - \frac{\text{R}}{2\text{COS}\theta}$$

For marginal rays θ is not small. Hence different light rays intersects x-axis at different points. But if we consider paraxial beam $(\theta \to 0)$

$$\Rightarrow \qquad \text{PF} = \frac{\text{R}}{2} \qquad (\text{As } \theta \to 0 \Rightarrow \cos\theta \to 1)$$

i.e all the light rays intersects x-axis at single point. This single point is called focus of the spherical mirror





As i increases cos i decreases.

Hence CQ increases

If i is a small angle cos i ≈ 1

$\therefore$ CQ = R/2

Principal axis : A line passing through focus and centre of curvature.

Pole : Point of intersection of principal axis and mirror.

Focal length (f) : The distance between focus and pole is called focal length .

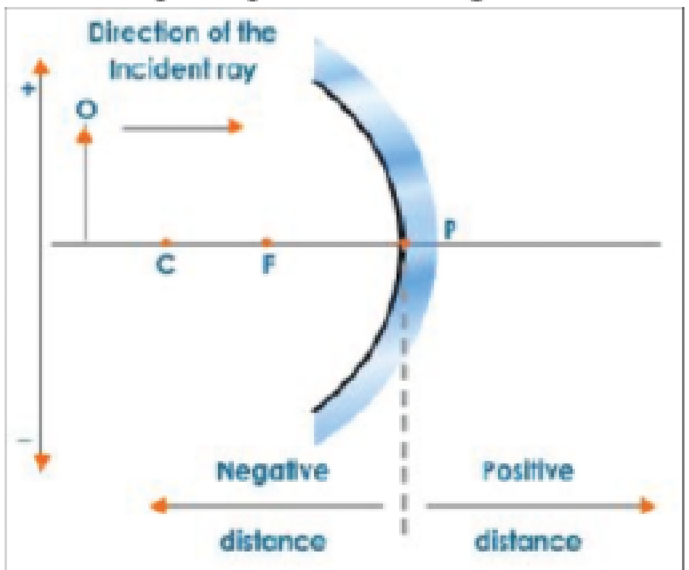
Sign Convention :

The following sign convention is used for measuring various distances in the ray diagrams of spherical mirrors:

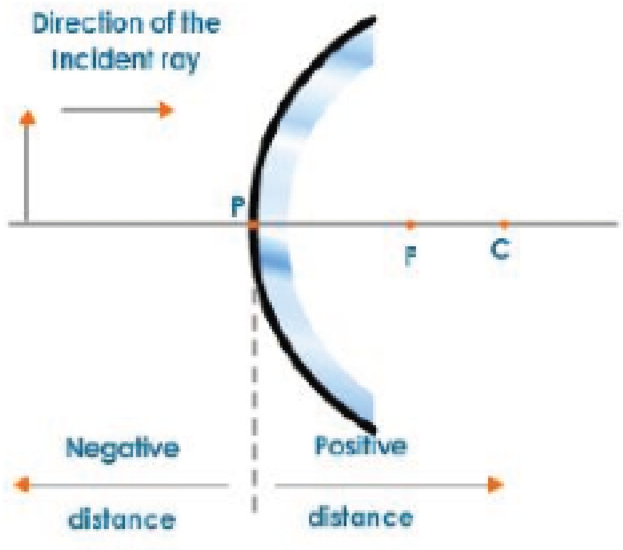
All distances are measured from the pole of the mirror.

Distances measured in the direction of the incident ray are positive and the distances measured in the direction opposite to that of the incident rays are negative.

Distances measured along y-axis above the principal axis are positive and that measured along y-axis below the principal axis are negative.



Concave Mirror

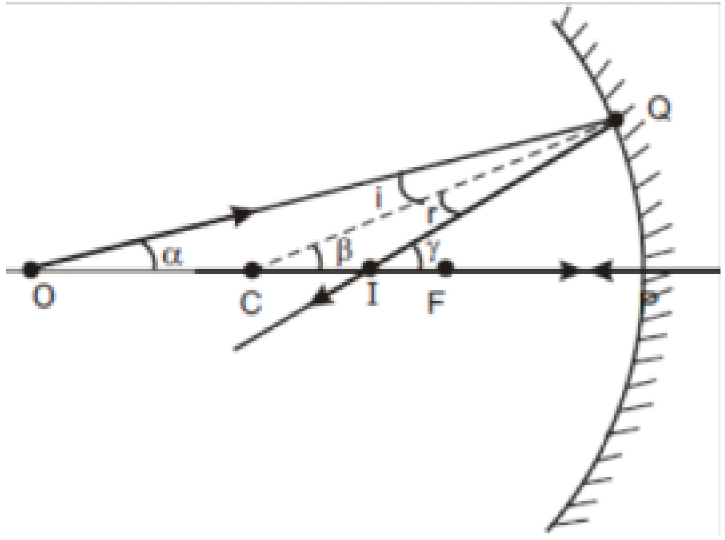


Convex Mirror

	Spherical Mirrors	Lenses
Focal Length	+ for concave mirrors	+ for a converging lens
	- for convex mirrors	- for a diverging lens
Object Distance	+ if object is in front of the mirror (real object)	+ if the object is to the left of the lens (real object)
	- if object is behind the mirror (virtual object)*	- if the object is to the right of the lens (virtual object)*
Image Distance	+ if the image is in front of the mirror (real image)	+ for an image (real) formed to the right of the lens by a real object
	- if the image is behind the mirror (virtual image)	- for an image (virtual) formed to the left of the lens by a real object
Magnification	+ for an image that is upright with respect to the object	+ for an image that is upright with respect to the object
	- for an image that is inverted with respect to the object	-for an image that is inverted with respect to the object.

Mirror formula

In this section we describe quantitatively where images are formed when light rays are reflected at spherical mirror. Consider two transparent media having indices a spherical mirror of radius R (Fig.). We assume that the object at O. Let us consider the paraxial rays leaving O. As we shall see, all such rays are reflected at the spherical surface and focus at a single point I, the image point.Figure shows a single ray leaving point O and reflecting to point I.



Now we use the fact that an exterior angle of any triangle equals the sum of the two opposite interior angles. Applying this rule to triangles OQC and CQI in Figure gives

$$i=\beta-\alpha \qquad\qquad r=\gamma-\beta$$

From law of reflection

$$i = r$$

$$\Rightarrow \quad \beta - \alpha = \gamma - \beta$$

$$\Rightarrow \quad \alpha + \gamma = 2\beta$$

$$\Rightarrow \quad \frac{QP}{OP} + \frac{QP}{IP} = \frac{QP}{CP} \qquad \text{(paraxial ray approximation)}$$

Taking sign convention

$$u = -OP \qquad v = -IP \qquad R = -CP$$

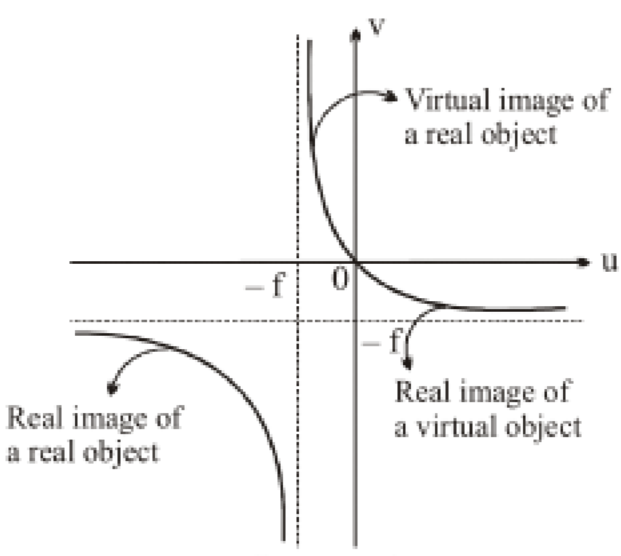
we get

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

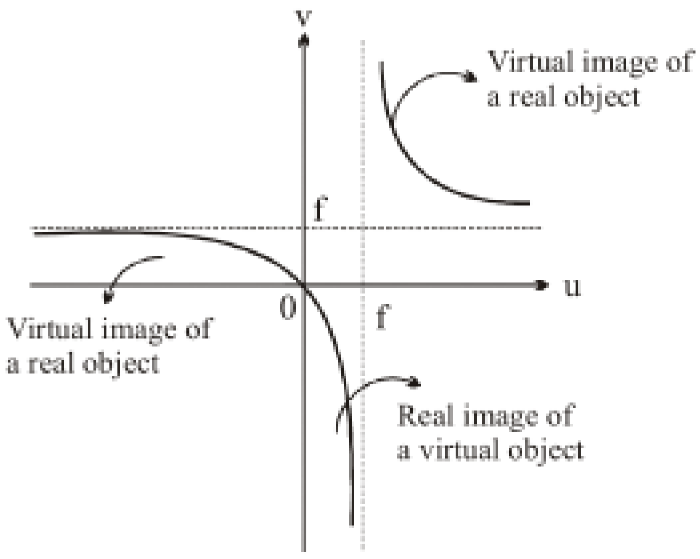
Note : This relation is same in all cases no matter object or image point is real or virtual, mirror is concave or convex.

Graph : v vs u :

(a) For concave mirror **(b) For convex mirror**



Concave mirror



Concave mirror

Image Tracing for Transverse Extended Object

In tracing image of a transverse extended object we should keep in mind following :

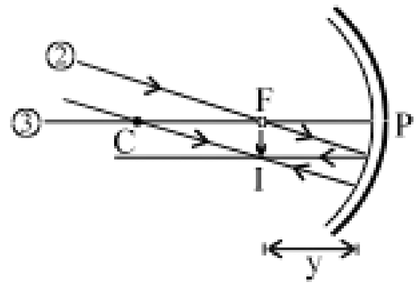
(1) A ray parallel to principal axis after reflection from the mirror passes or appears to pass through its focus

(2) A ray passing through or directed towards centre of curvature, after reflection from the mirror, retraces its path (as for it $\theta_1 = 0$ and so $\theta_2 = 0$).

(3) Ray drawn from the top of the object toward the focal point on the back side of the mirror and is reflected parallel to the principal axis.

Image Tracing in some cases

(i) When the object is placed at infinity, a real, inverted and very small image is formed at the focus.

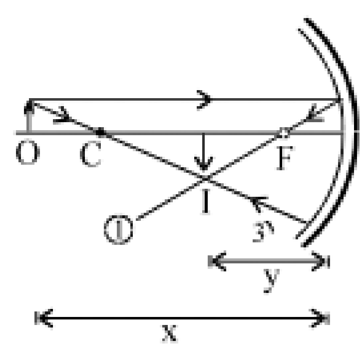


For a distance object image is formed at the focus

$x = \infty$

$v = -y$	where	$y = f_0$
$m = -\delta$	where	$\delta << 1$

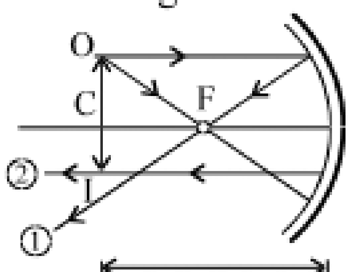
(ii) When the object is placed beyond C ($2f_0 < x < \infty$), a real, inverted and diminished image is formed between F acid C.



For an object placed beyond C, image is formed between C and F

$v = -y$	where	$f_0 < y < 2f_0$
$m = -\delta$	where	$0 < \delta < 1$

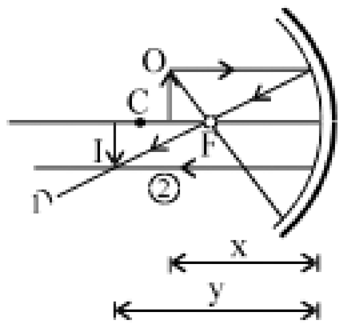
(iii) When the object is placed at C.
($x = 2f_0$), a real, inverted and equal size image is formed at C.



At C both object and image coincide

$v = -y$	where	$y = 2f_0$
$m = -\delta$	where	$\delta = 1$

(iv) When the object is placed between F and C ($f_0 < x < 2f_0$), a real, inverted and large image is formed beyond C.

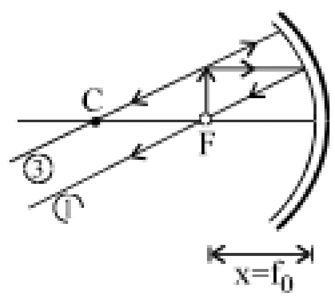


For an object placed between F and C image is formed beyond C.

$v = -y$ where $y < 2f_0$

$m = -\delta$ where $\delta > 1$

(v) When the object is placed at focus F, a real, inverted and very large image is formed at infinity.



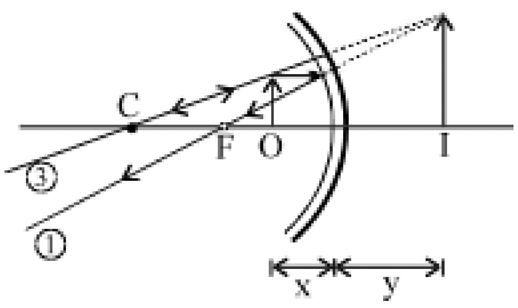
For an object placed at focus, image is formed at infinity

$v = -y$ where $y = \infty$

$m = -\delta$ where $\delta >> 1$

Note virtual object

When the object is placed between F and P, a virtual, erect and enlarged image is formed behind the mirror.



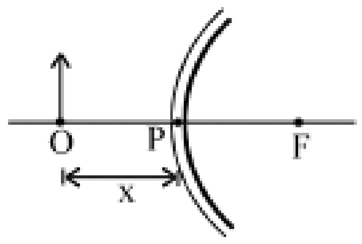
A virtual image is formed for an object placed within focus

$v = +y$

$m = +\delta$ where $\delta > 1$

Convex mirror

The fig. shows a convex mirror of focal length f_0 in front of which an object O is placed at a distance x from the pole P.



An object O placed in front of a convex mirror

According to Cartesian sign convention, the formulae may be modified as

$$u = -x \quad \text{and} \quad f = +f_0$$

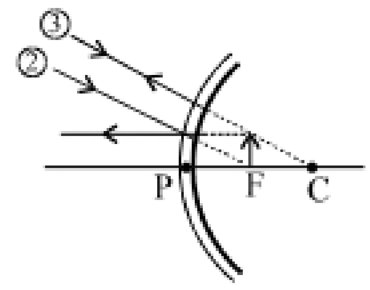
Thus $\quad v = \dfrac{xf_0}{f_0 + x}$

The above expression shows that whatever may be the value of x, v is always positive and its value is always less than or equal to f_0.

The magnification formula may be modified as

$$m = \frac{f_0}{f_0 + x}$$

When the object is placed at infinity, a virtual, erect and very diminished image is formed at the focus.



For a distance object image is formed at the focus

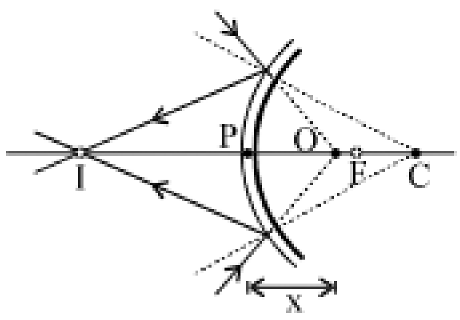
$x = \infty \qquad v = f_0 \qquad m << 1$

Illustration :

Can a convex mirror form real images ?

Sol. *Yes, only when the object is virtual and is placed between F and P.*

The fig. shows a convex mirror exposed to a converging beam which converges to a point that lies between F and P.



A real image formed by convex mirror

$$v = \frac{-xf_0}{f_0 - x}$$

v becomes negative (real image) only when $x < f_0$.

FOR CONCAVE MIRROR

Position of object (real)	Position of image	Characteristics of image
At infinity	At. F	Real, inverted, highly diminished
Between infinity and C	Between C and F	Real, inverted, diminished
At C	At C	Real, inverted, same size as that of object
Between C and F	Between infinity and C	Real, inverted, magnified
At F	At infinity	Real, inverted, highly magnified
within F	Behind the mirror	Virtual, erect, magnified

FOR CONVEX MIRROR

Position of object (real)	Position of image	Characteristics of image
At infinity	At F	Virtual, erect, highly diminished
At a finite distance	Between P and F	Virtual, erect, diminished

Transverse or Lateral or linear magnification

It is defined as

$$m_T = \frac{\text{height of image}}{\text{height of object}} = \frac{h_I}{h_O}$$

After using geometry we get

$$m_T = -\frac{v}{u} = \frac{f}{f-u}$$

Note :

Sign of m_T states orientation of image w.r. to object and its magnitude compares size of image with size of object

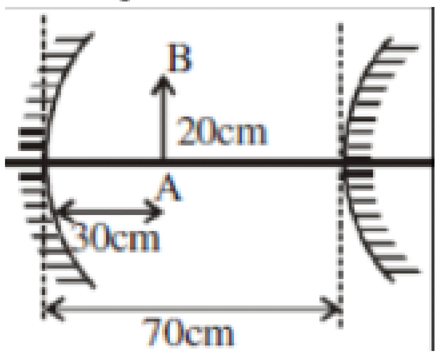
Newtonian formula for mirrors

here distances are measured from focus

$$uv = f^2 \quad \& \quad m_T = -\frac{f}{u}$$

Illustration :

A concave and convex mirror of focal length 10 cm and 15 cm are placed at distance 70 cm. An object *AB* of height 2 cm is placed at distance 30 cm from concave mirror. First ray is incident on concave mirror then on convex mirror. Find size position and nature of image.



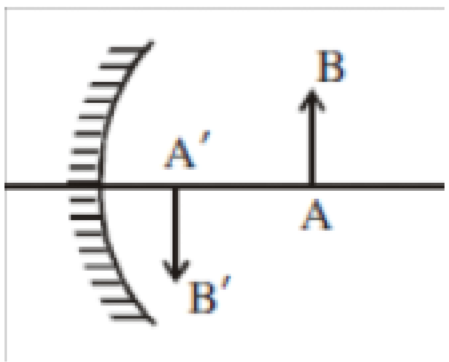
Sol. For concave mirror,

$$u = -30\text{cm}, \quad f = -10\text{cm}$$

Using

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v} - \frac{1}{30} = \frac{-1}{10} \Rightarrow \quad v = -15\,cm$$



Now,

$$\frac{A'B'}{AB} = \frac{-V}{u} = \frac{(-15)}{(-30)} \quad \Rightarrow \quad A'B' = -1\,cm$$

Image formed by first reflection will be real inverted and diminished

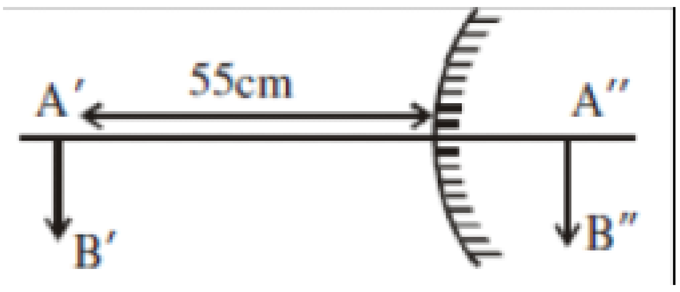
For convex mirror

$$u' = -55\text{cm}, \quad f' = +15\text{cm}$$

Using

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v'} - \frac{1}{55} = \frac{1}{15} \quad \Rightarrow \quad v' = \frac{165}{14}\,cm$$



Now,

$$\frac{A''B''}{A'B'} = -\frac{V'}{u'} = -\frac{\left(+\frac{165}{14}\right)}{(-55)} \Rightarrow \quad A''B'' = \left(-\frac{3}{14}\right)(-1) = -0.2\ cm$$

Image for Longitudinal Object

If an object is placed along principal axis then it is called longitudinal object.

If object is of very small size then longitudinal magnification is defined as.

$$m_L = \frac{\text{length of image}}{\text{length of image}} = \frac{dv}{du}$$

Now using

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

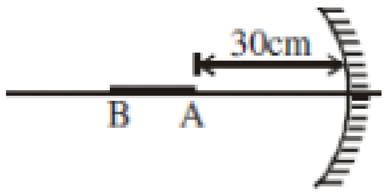
Differentiating w.r.to. u we get

$$\frac{dv}{du}=-\frac{v^2}{u^2}=-m_T^2$$

$$\Rightarrow \quad m_L=-m_T^2$$

Illustration :

*An object AB is placed on the axis of concave mirror of focal length 10 cm end A of the object is at 30 cm from mirror. Find the length of the image **(a)** If length of object is 5 cm **(b)** If length of object is 1 mm*



Sol. *For point A,* $u=-30\text{cm}, \quad f=-10\text{cm}$

Using $$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v'}-\frac{1}{30}=-\frac{1}{10} \Rightarrow \quad v=-15\text{cm}$$

Similarly for point B $u=-35\text{cm}$

Using $$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

we will get $v'=-14\text{cm}$

Now size of image $|A'B'|=|v-v'|=|(-15)-(-14)|=1\text{cm}$

(b) *here* $u=-30\text{cm}, \quad f=-10\text{cm}$

Using $$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v'}-\frac{1}{30}=-\frac{1}{10} \Rightarrow \quad v=-15\text{cm}$$

Now, $$\frac{dv}{du}=-\frac{(-15)^2}{(-30)^2} \Rightarrow \quad |dv|=\frac{(15)^2}{(30)^2}|du|$$

$$\Rightarrow \quad |dv|=\left(\frac{225}{900}\right)(10^{-3})=2.5\times10^{-4}\,m$$

i.e.the length of the image is 2.5 × 10^{-4} m.

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Relation between velocity of object and image if object is moving on principal axis:

we can write

velocity of object relative to mirror= $v_{O/M} = \frac{du}{dt}$

velocity of image relative to mirror= $v_{I/M} = \frac{dv}{dt}$

Now using

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

Differentiating w.r.to. t we get

$$\frac{dv}{dt}=-\frac{v^2}{u^2}\frac{du}{dt}$$

$$\Rightarrow \qquad v_{I/M}=m_L v_{O/M}$$

Relation between velocity of object and image if object is moving perpendicular to principal axis:

we can write

velocity of object relative to mirror = $v_{O/Mirror} = \frac{dh_O}{dt}$

velocity of image relative to mirror = $v_{I/Mirror} = \frac{dh_I}{dt}$

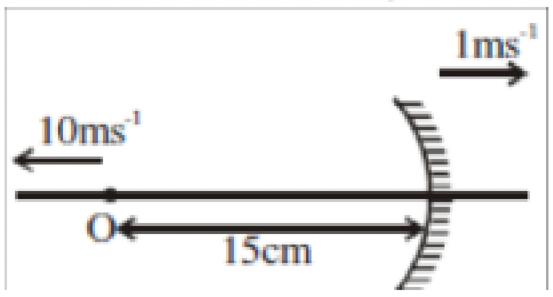
dividing we get

$$\frac{v_{I/Mirror}}{v_{O/Mirror}}=\frac{dh_I}{dh_O}=\frac{h_I}{h_O}=m_T$$

$$\Rightarrow \qquad v_{I/Mirror}=m_T v_{O/Mirror}$$

Illustration :

A mirror of radius of curvature 20 cm and an object which is placed at distance 15 cm both are moving with velocity 1 ms^{-1} and 10 ms^{-1} as shown in figure. Find the velocity of image.



Sol. Using

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{R}$$

$$\Rightarrow \qquad \frac{1}{v}-\frac{1}{15}=-\frac{1}{10} \qquad \Rightarrow \qquad v=-30\text{cm}$$

Now, using

$$V_{im}=-\frac{v^2}{u^2}V_{OM}$$

$$\Rightarrow \quad (V_i - V_m) = -\frac{v^2}{u^2}(V_0 - V_m)$$

$$\Rightarrow \quad V_i - (1) = -\frac{(-30)^2}{(-15)^2}[(-10)-(1)]$$

$$\Rightarrow \quad V_i = 45\,\text{cm/s}$$

Practice Exercise

Q.1 A beam of light converges towards a point O, behind a convex mirror of focal length 20 cm. Find the nature and position of image if the point O is (a) 10 cm behind the mirror (b) 30 cm behind the mirror.

Q.2 A plane mirror is placed 22.5 cm in front of a concave mirror of focal length 10 cm. Find where an object can be placed between the two mirrors, so that the first image in both the mirror coincides.

Q.3 A concave mirror of focal length f produces a real image n times the size of the object. What is the distance of the object from the mirror?

Q.4 The focal length of a concave mirror is 30 cm. Find the position of the object in front of the mirror, so that the image is three times the size of the object.

Q.5 A short linear object of length b lies along the axis of a concave mirror of focal length f at a distance u from the pole. What is the size of the image?

Q.6 If a luminous point is moving at a speed V_0 towards a spherical mirror, along its axis, show that the speed at which the image of this object is moving will be given by :

$$V_i = \left[\frac{f}{u-f}\right]^2 V_0$$

Answers

Q.1 (a) –20 (b) +60 Q.2 15 cm from the concave mirror Q.3 (n+1)f/n

Q.4 Case I If the image is inverted (i.e., real) : 40 cm in front of the mirror.
Case II If the image is erect (i.e., virtual) : 20 cm in front of the mirror

Q.5 $b\left[\frac{f}{(u-f)}\right]^2$

Refraction of Light At plane Surface

Refractive Index

The refractive index (denoted by μ or n) of a material for a given colour at light is defined as

$$\mu = \frac{c}{v}$$

Where c = speed of light in vacuum and

v = speed of light in the medium for that colour of light.

Note :

(i) $\mu_{vac} = 1$

(ii) We can also say refractive index as absolute refractive index.

(iii) The refractive index of a material depends on wavelength of light and given by (Cauchy's equation)

$$\mu_{\lambda}^{2} = \mu_0 + \frac{A}{\lambda_0^2}$$

Where μ_{λ} = refractive index of a material for light ray of wave length λ.

μ_0 = constant depending on nature of material.

A = constant depending on nature of material.

It is well known that in visible range the wavelength violet light is minimum of red light is maximum. So using Cauchy's equation we can say that in visible range the refractive index of a material is maximum for violet and minimum for red and hence (i) speed of light in a material is maximum for red and minimum for violet in visible range (ii) deviation of violet ray is maximum and red colour of light ray is minimum in visible range for same material and same angle incidence.

(iv) When light passes from one medium to another then speed and wavelength changes such that frequency remains unchanged (hence colour also).

$$\lambda = \frac{\lambda_0}{\mu}$$

(v) **Relative Refractive Index :**

The ratio μ_2/μ_1 is called refractive index of second medium relative to first and represented by μ_{21} or ${}_1\mu_2$.

${}_1\mu_2 = \frac{\mu_2}{\mu_1} = \frac{v_1}{v_2}$ (for given colour of light and different two media). Note : ${}_1\mu_2 = \frac{1}{{}_2\mu_1}$

Illustration:

How long will the light take in travelling a distance of 500 metre in water? Given that μ for water is 4/3 and the velocity of light in vacuum is 3×10^{10} cm/sec.

Sol. *We know that*

$$\mu = \frac{\text{velocity of light in vaccum}}{\text{velocity of light in water}}$$

$$\frac{4}{3} = \frac{3 \times 10^{10}}{\text{velocity of light in water}}$$

or *velocity of light in water* $= 2.25\times10^{10}$ *cm/sec.*

$$\text{Time taken} = \frac{500\times100}{2.25\times10^{10}} = 2.22\times10^{-6}\ \text{sec}.$$

Illustration :

(a) Find the speed of light of wave length λ = 780 nm (in air) in a medium of refractive index μ = 1.55

(b) What is the wavelength of this light in the given medium ?

Sol. (a) $v = \frac{c}{\mu} = \frac{3.0\times10^8}{1.55} = 1.94\times10^8$ m/s (b) $\lambda_{medium} = \frac{\lambda_{air}}{\mu} = \frac{780}{1.55} = 503$ nm

Practice Exercise

Q.1 Light of wavelength 6000Å enters from air into water [a medium of index of refraction(4/3)]. What is the (a) speed (b) wavelength (c) frequency and (d) colour of light in water? [Speed of light in free space is 3×10^8 m/s]

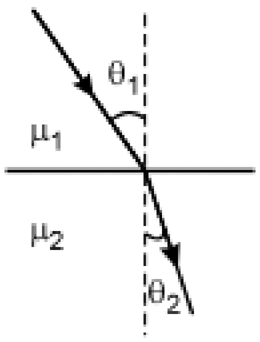
Q.2 If the wavelength of light in vacuum is 5800 Å, then calculate the wavelength in glass [${}_a\mu_g = 3/2$]

Answers

Q.1 (a) 2.25×10^8 m/s, (b) 4500Å (c) 5×10^{14} Hz and (d) yellow   Q.2 3866A

Refraction of Light

If a light ray passes from one transparent medium to another medium but having oblique incidence then it deviates from its path (either towards normal or away from the normal) This bending of light ray due to change of medium is called refraction of light.



Laws of Refraction

Ist Law : Incident ray, refracted ray and normal at the point of incidence are coplanar i.e. refracted ray must lie in the plane of incidence.

IInd Law (Snell's law) : The ratio of sine of angle of incidence with sine of angle of refraction is constant for given two media and given colour of light which is equal to refractive index of second medium relative to first for that colour of light.

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

In vector form :

$$\mu_1(\hat{u}_i \times \hat{u}_n) = \mu_2(\hat{u}_r \times \hat{u}_n)$$

Deductions :

(1) (a) If there is no change of medium light ray passes undeviated.

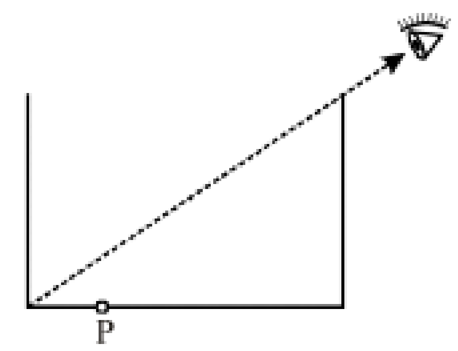
(b) If light ray incident normally then transmitted light ray is also normal.

(c) If light ray passes from rarer to denser then it bends towards the normal.

(d) If light passes from denser to rarer then it bends away from the normal.

Illustration :

A cylindrical vessel, whose diameter and height both are equal to 30cm is placed on a horizontal surface and a small particle P is placed in it at a distance of 5.0 cm from the centre. An eye is placed at a position such that the edge of the bottom is just visible. The particle P is in the plane of drawing. Up to what minimum height should water be poured in the vessel to maks the particle P visible ? Given : refractive index of water = 4/3.



Sol. *From figure*

$$x = 30 - h$$

$$PA = 20 - x$$

$$= 20 - (30 - h) = h - 10$$

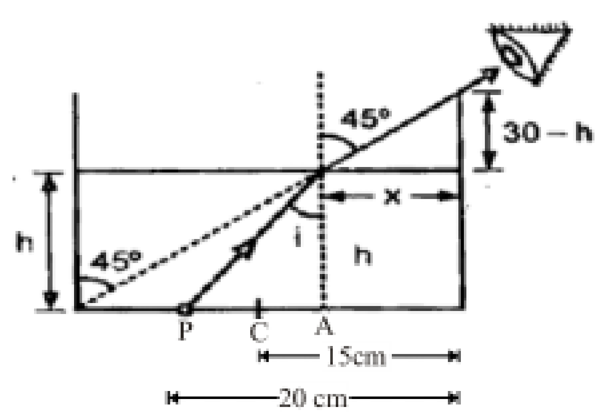
Using Snell's law we get

$$i \times \sin r = \frac{4}{3} \sin i$$

$$1 \times \sin 45^{o} = \frac{4}{3} \times \frac{h-10}{\sqrt{h^2+(h-10)^2}}$$

$$23\,(h-10)^2 = 9h^2$$

$$h = 26.7 \text{ cm}$$

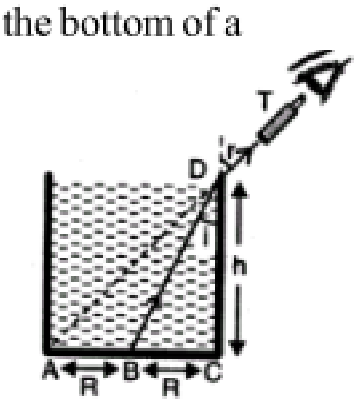


Practice Exercise

Q.1 If one face of a prism of prism angle 30° and $\mu = \sqrt{2}$ is silvered, the incident ray retraces its initial path. What is the angle of incidence?

Q.2 A ray of light falls on a transparent glass slab with refractive index (relative to air) μ find the angle of incidence for which the reflected and refracted rays are mutually perpendicular.

Q.3 A person looking through the telescope T just sees the point A on the rim at the bottom of a cylindrical vessel when the vessel is empty. When the vessel is completely filled with a liquid of refractive index 1.5, he observes a mark at the centre B of the bottom, without moving the telescope or the vessel. What is the height of the vessel if the diameter of its cross–section is 10 cm?



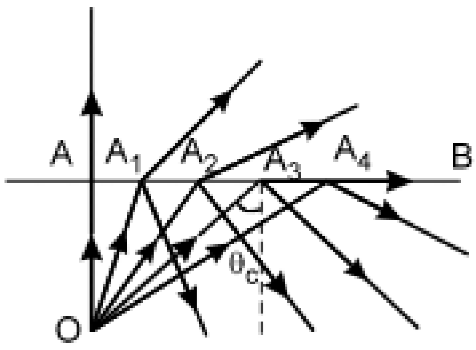
Q.4 A pole standing in a clear water pond stands 1 m above the water surface, the pond is 2 m deep. What are the lengths of the shadows thrown by the pole on the surface and bottom of the pond if the sun is 30^o over the horizon?[Refractive index of water is 4/3]

Answers

Q.1	45^o	Q.2	$\tan^{-1}\mu$	Q.3	8.45 cm	Q.4	$2\sqrt{3}$ m

Critical Angle of incidence

Here AB is refracting surface, O is a point source of light, placed in denser medium. A number of rays become incident from O to the surface AB. The ray OA_1 being normally incident, travels undeviated. As we consider the rays having larger angle the rays will be deviated more and more. For a particular value of angle of incidence (called critical angle of incidence) light ray grazes in the surface.



For $\theta_1 = \theta_c \quad \Rightarrow \quad \theta_2 = 90^0$

Using snell's law,

$$\mu_{denser} \sin\theta_c = \mu_{rarer} \sin 90^0$$

$$\Rightarrow \sin\theta_c = \frac{\mu_{rarer}}{\mu_{denser}}$$

If the light ray is incident having angle of incidence greater than critical angle of incidence then the light is totally reflected and obeys the laws of reflection. This phenomenon is called total internal reflection.

Illustration :

A ray of light traveling in a transparent medium falls on a surface separating the medium from air at an angle of incidence of 45°. The ray undergoes total internal reflection. Find the refractive index of the medium

Sol. *Here* $45^o > \theta_c$

$\sin 45^o > \sin\theta_c$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} > \frac{1}{n}$$

$$\Rightarrow \quad n > \sqrt{2}$$

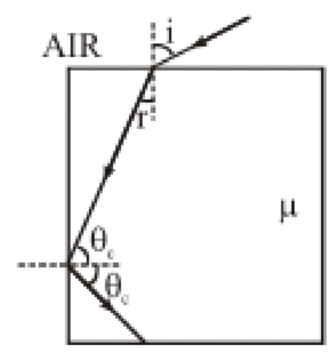
Illustration :

In the figure shown, for an angle of incidence i at the top surface, what is the minimum refractive index needed for total internal reflection at the vertical face?

Sol. *Applying Snell's law at the top surface*

$$\mu \sin r = \sin i \qquad ...(i)$$

For total internal reflection the vertical face



$$\mu \sin \theta_c = 1$$

Using geometry, $\theta_c = 90^o - r$

$$\therefore \qquad \mu \sin(90 - r) = 1$$

or $\qquad \mu \cos r = 1 \qquad ...(ii)$

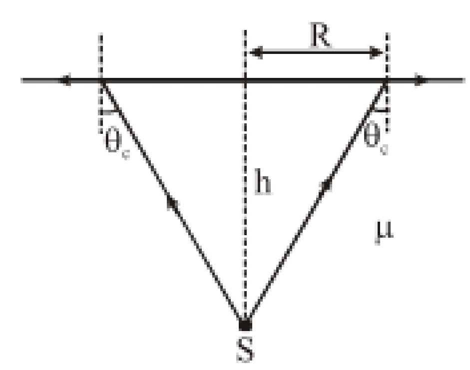
On squaring and adding equation (i) and (ii), we get

$$\mu^2 \sin^2 r + \mu^2 \cos^2 r = 1 + \sin^2 i$$

or $\qquad \mu = \sqrt{1 + \sin^2 i}\,.$

Illustration :

A point source of light is placed at the bottom of a tank containing a liquid (refractive index = μ) upto a depth h. A bright circular spot is seen on the surface of the liquid. Find the radius of this bright spot.



Sol. *Rays coming out of the source and incident at an angle greater than* θ_c *will be reflected back into the liquid therefore, the corresponding region on the surface will appear dark. As is obvious from the figure, the radius of the bright spot is given by*

$$R = h \tan \theta_c = \frac{h \sin \theta_c}{\cos \theta_c} \qquad or \qquad R = \frac{h \sin \theta}{\sqrt{1 - \sin^2 \theta_c}}$$

Since $\quad \sin \theta_c = \dfrac{1}{\mu}$

$$\therefore \qquad R = \frac{h}{\sqrt{\mu^2 - 1}}.$$

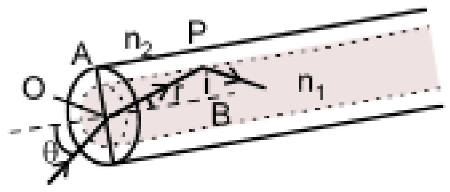
Practice Exercise

Q.1 A ray of light from a denser medium strike a rarer medium at an angle of incidence i. If the reflected and refracted rays are mutually perpendicular to each other, what is the value of critical angle?

Q.2 A beam of parallel rays of width 20 cm propagates in glass at an angle 60^o to its plane face. Find the beam width after it goes over to air through this face. The refractive index of glass is 1.8.

Q.3 A ray of light travelling in glass ($\mu = 3/2$) is incident on a horizontal glass–air surface at the critical angle θ_c. If a thin layer of water ($\mu = 4/3$) is now poured on the glass–air surface, at what angle will the ray of light emerge into air at the water–air surface?

Q.4 A particular optical fibre consists of a glass core (index of refraction n_1) surrounded by a cladding (index of refraction $n_2 < n_1$). Suppose a beam of light enters the fibre from air at an angle θ with the fibre axis as shown in **figure**. Show that the greatest possible value of θ for which a ray can be propagated down in fibre is given by $\theta = \sin^{-1}[\sqrt{n_1^2 - n_2^2}]$

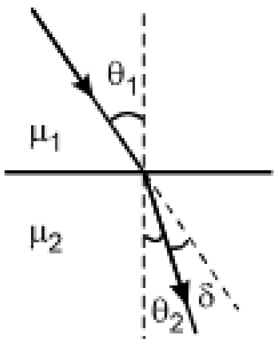


Answers

Q.1 $\sin^{-1}(\tan i)$ Q.2 20 cm Q.3 90^o

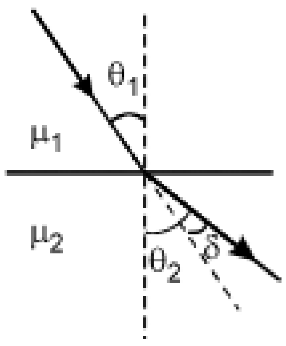
(3) **Calculation of angle of deviation :**

When $\mu_1 < \mu_2$



$\delta = \theta_1 - \theta_2$

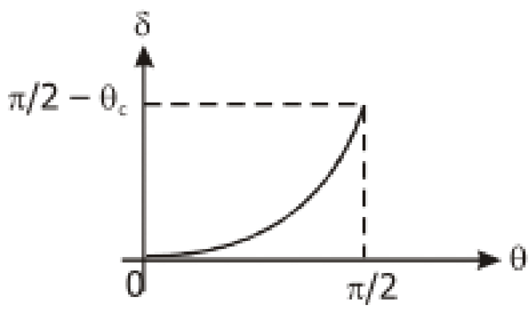
When $\mu_1 > \mu_2$



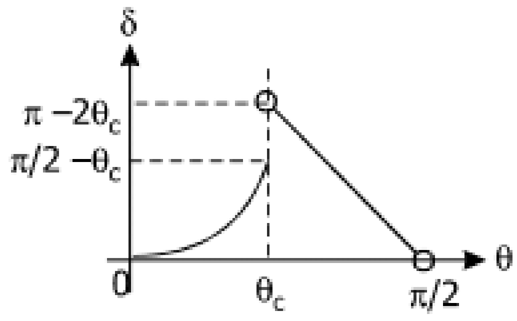
$\delta = \theta_2 - \theta_1$

Plot of deviation vs angle of incidence

light travells from rarer to denser



light travells from rarer to denser



Practice Exercise

Q.1 A ray of light is incident on the surface of a spherical glass paper weight making an angle α with the normal and is refracted in the medium at an angle β. Calculate the deviation.

Answers

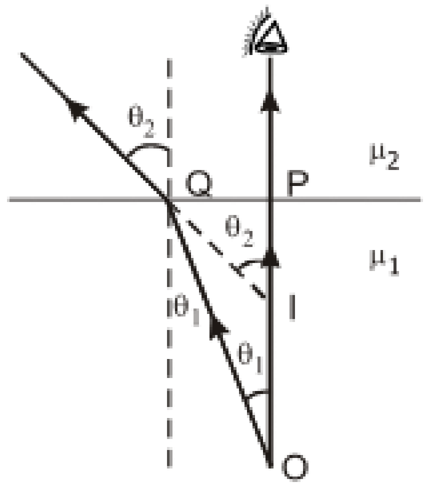
Q.1 $2(\alpha-\beta)$

Shifting of image :

Here we will encounter two cases

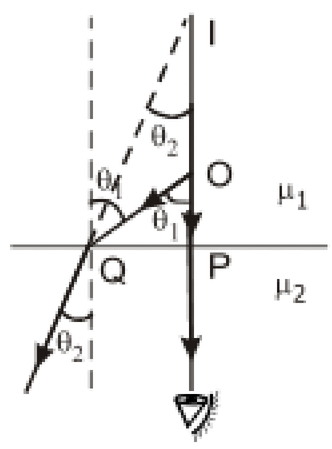
(a) When object is in denser medium with respect to observer

Here $\mu_1 > \mu_2$



(b) when object is in rarer medium with respect to observer.

Here $\mu_1 < \mu_2$



From Snell's law

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

for paraxial rays $\theta \to 0$

$\Rightarrow \qquad \mu_1\theta_1 = \mu_2\theta_2$

$\Rightarrow \qquad \mu_1 \tan\theta_1 = \mu_2 \tan\theta_2$

$\Rightarrow \qquad \mu_1 \dfrac{PQ}{OP} = \mu_2 \dfrac{PQ}{IP}$

$\Rightarrow \qquad \dfrac{IP}{OP} = \dfrac{\mu_2}{\mu_1}$

Relation between the velocities of object and image if object is moving perpendicular to the surface:

$$v_{O/surface} = \frac{d}{dt}(OP) \qquad \text{and} \qquad v_{I/surface} = \frac{d}{dt}(IP)$$

but

$$\frac{IP}{OP} = \frac{\mu_2}{\mu_1} \qquad \Rightarrow \qquad IP = \frac{\mu_2}{\mu_1} OP$$

differentiating both sides with respect to time we get

$$\frac{d}{dt} IP = \frac{\mu_2}{\mu_1} \frac{d}{dt} OP$$

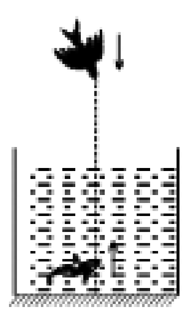
$$\Rightarrow \qquad v_{I/surface} = \frac{\mu_2}{\mu_1} v_{O/surface}$$

Practice Exercise

Q.1 A fish rising vertically to the surface of water in a lake uniformly at the rate of 3 m/s observes a king–fisher (bird) diving vertically towards the water at a rate of 9 m/s vertically above it. If the refractive index of water is (4/3), find the actual velocity of the dive of the bird.

Q.2 A concave mirror of radius of curvature one metre is place at the bottom of a tank of water. The mirror forms an image of the sun when it is directly overhead. Calculate the distance of the image from the mirror for (a) 80 cm and (b) 40 cm of water in the tank.

Q.3 A bird in air is diving vertically over a tank with speed 6 cm/s. Base of the tank is silvered. A fish in the tank is rising upward along the same line with speed 4 cm/s. [Take: $\mu_{water} = 4/3$]



(A) Speed of the image of fish as seen by the bird directly

(B) Speed of image of bird relative to the fish looking upwards

Answers

Q.1 4.5 m/s

Q.2 (a) 50cm from the mirror (b) 47.5 cm

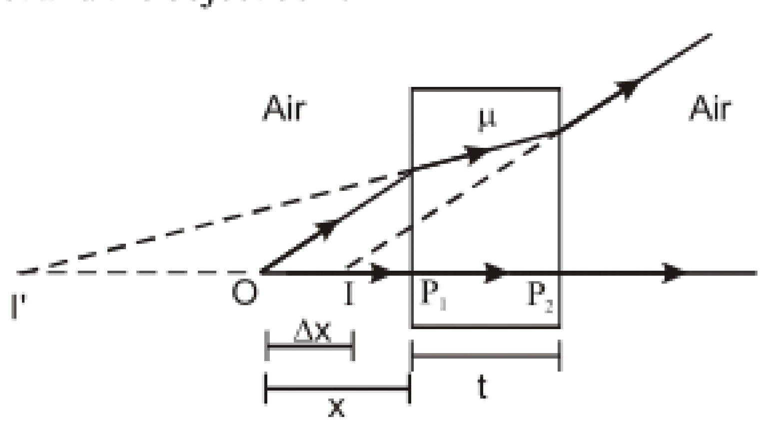
Q.3 (A) Velocity of fish in air $= 4 \times \frac{3}{4} = 3\uparrow$, Velocity of fish w.r.t bird $= 3 + 6 = 9\uparrow$

(B) Velocity of bird in water $= 6 \times \frac{4}{3} = 8\downarrow$, w.r.t fish $= 8 + 4 = 12\downarrow$

Refraction Through Slab

Shifting of image :

Let an object O be placed at one of the slab (thickness–t, R. I.–μ) and the distance between the surface closer to the object and the object be x.



At first surface

$$\frac{I'P_1}{OP_1} = \frac{\mu}{1} \qquad \Rightarrow \qquad I'P_1 = \mu OP_1 = \mu x$$

At second surface

$$\frac{IP_2}{I'P_2} = \frac{1}{\mu} \qquad \Rightarrow \qquad \frac{IP_2}{\mu x + t} = \frac{1}{\mu} \qquad \Rightarrow \qquad IP_2 = x + \frac{t}{\mu}$$

Now, shifting will be given by

$$\Delta x = OP_2 - IP_2 = (x + t) - \left(x + \frac{t}{\mu}\right)$$

$$\Rightarrow \qquad \Delta x = t\left(1 - \frac{1}{\mu}\right)$$

Note that shifting occurs in the direction of propagation of light

Practice Exercise

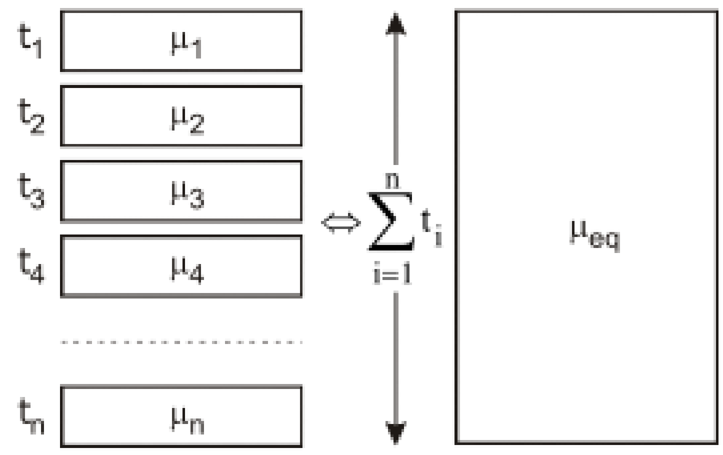
Q.1 An object is placed 21 cm infront of a concave mirror of radius of curvature 10 cm. A glass slab of thickness 3 cm and refractive index 1.5 is then placed close to the mirror in the space between the object and the mirror. Find the position of the final image formed. (You may take the distance of the near surface of the slab from the mirror to be 1 cm).

Q.2 A 20 cm thick glass slab of refractive index 1.5 is kept in front of a plane mirror. Find the position of the image (relative to mirror) as seen by an observer through the glass when a point object is kept in air at a distance of 40 cm from the mirror.

Answers

Q.1 7.67 cm in front of mirror Q.2 80/3 cm

Combination of slabs :



At first surface

$$\frac{I'P_1}{OP_1} = \frac{\mu}{1} \qquad \Rightarrow \qquad I'P_1 = \mu OP_1 = \mu x$$

At second surface

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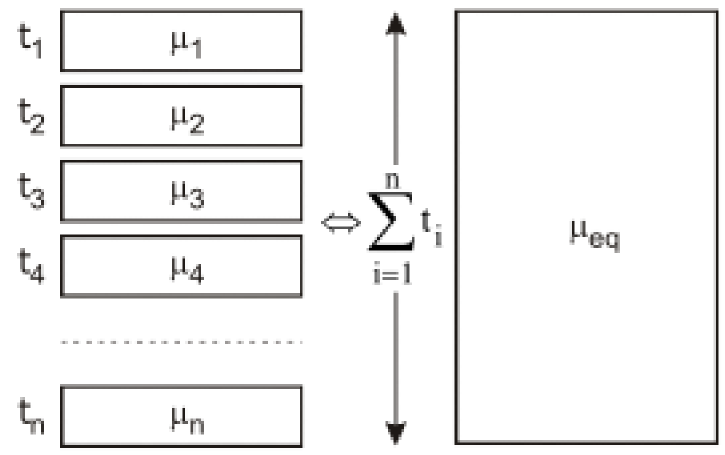


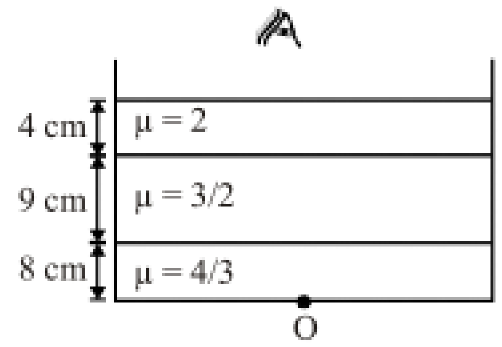
Net shift will be

$$(t_1 + t_2 + t_3 + ...+ t_n)\left[1-\frac{1}{\mu_{eq}}\right] = t_1\left[1-\frac{1}{\mu_1}\right] + t_2\left[1-\frac{1}{\mu_2}\right] + t_3\left[1-\frac{1}{\mu_3}\right]$$

$$\Rightarrow \qquad \frac{t_1 + t_2 + t_3 + ...t_n}{\mu_{eq}} = \frac{t_1}{\mu_1} + \frac{t_2}{\mu_2} + \frac{t_3}{\mu_3} + ...\frac{t_n}{\mu_n}$$

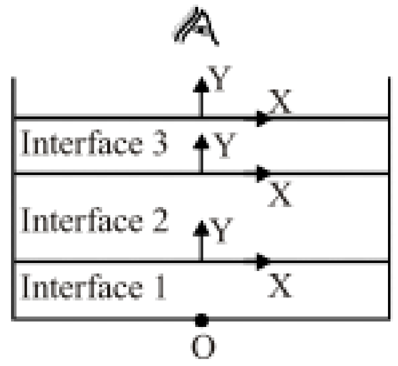
Illustration :

A tank contains three layers of immiscible liquids (figure). The first layer is of water with refractive index 4/3 and thickness 8 cm. The second layer is an oil with refractive index 3/2 and thickness 9cm while the third layer is of glycerine with refractive index 2 and thickness 4 cm. Find the apparent depth of the bottom of the container.



Sol. I : Method of interfaces :

A ray of light from the object undergoes refraction at three interfaces. (1) Water-oil, (2) Oil-glycerine (3) Glycerine-air. The coordinate system for each of the interfaces is shown in figure.



Water-Oil Interface :

$d_1 = -8cm,\ \mu_1 = 4/3,\ \mu_2 = 1.5$

As $\frac{\mu_1}{d_1} = \frac{\mu_2}{d_2}$ *or* $d_2 = \frac{\mu_2}{\mu_1}d_1$ *we get* $d_2 = -9cm$

Oil-Glycerine Interface :

$d_1 = -(9 + 9) = -18cm,\ \mu_1 = 1.5,\ \mu_2 = 2$

As $\frac{\mu_1}{d_1} = \frac{\mu_2}{d_2}$ *or* $d_2 = \frac{\mu_2}{\mu_1}d_1$ *we get* $d_2 = -24\ cm$

Glycerine-Air Interface :

$d_1 = -(4 + 24) = -28\ cm,\ \mu_1 = 2,\ \mu_2 = 1$

As $\frac{\mu_1}{d_1} = \frac{\mu_2}{d_2}$ *or* $d_2 = \frac{\mu_2}{\mu_1} d_1$ *we get* $d_2 = -14$ *cm*

Thus the final image is 14cm below the glycerine - air interface.

Sol II : Method of elements :

The system now comprises three slabs - one of water, one of oil and one of glycerine. As discussed in this Section and given by equation, the net shift of the system in the sum of the individual shifts each of the slabs asuming they were in air.

Therefore,

Net shift $$s = t_1\left[1 - \frac{1}{\mu_1}\right] + t_2\left[1 - \frac{1}{\mu_2}\right] + t_3\left[1 - \frac{1}{\mu_3}\right]$$

$$s = 8\left[1 - \frac{3}{4}\right] + 9\left[1 - \frac{2}{3}\right] + 4\left[1 - \frac{1}{2}\right]$$

$$= 7cm.$$

The direction of the shift is in the direction of the incident rays which is upwards. Therefore, final position of the object is = (4 + 9 + 8) – 7 = 14cm below the glycerine-air interface.

Illustration :

In a tank a 4 cm thick layer of water $\left(\mu = \frac{4}{3}\right)$ *floats on a 6 cm thick layer of an organic liquid (μ = 1.48). Wiewing at normal incidence how far below the water surface does the battom of tank appear to be ?*

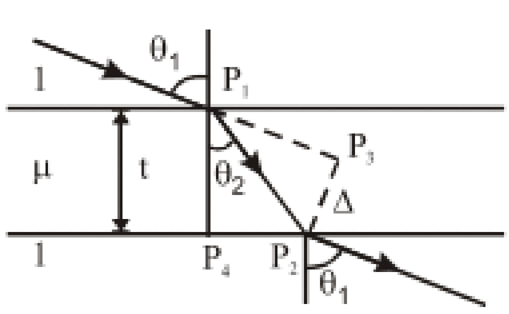
Sol. $$d_{Ap} = \frac{d_1}{\mu_1} + \frac{d_2}{\mu_2}$$

$$= \frac{6}{1.48} + \frac{4}{4/3}$$

$$d_{Ap} = 7.05 \text{ cm}$$

Sifting in path :

When a ray passes through a slab placed in a medium then after refraction the emergent ray is parallel to the incident ray but it seems that it has translated some distance (Δd, called shifting in path)



In triangle $P_1 P_2 P_4$

$$P_1 P_2 = \frac{t}{\cos\theta_2}$$

In triangle $P_1 P_2 P_3$

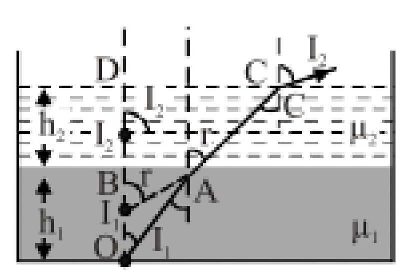
$$\Delta = P_2 P_3 = P_1 P_2 \sin(\theta_1 - \theta_2) = \frac{t}{\cos\theta_2} \sin(\theta_1 - \theta_2)$$

Elemenating θ_2 (using Snell's law), we get

$$\Delta = \frac{t}{\cos\theta_2} \sin(\theta_1 - \theta_2) = t\sin\theta_1 \left[1 - \frac{\mu_1 \cos\theta_1}{\sqrt{\mu_2^2 - \mu_1^2 \sin^2\theta_1}}\right]$$

Practice Exercise

Q.1 An object O is located at the bottom of a tank containing two immiscible liquids and is seen vertically from above. The lower and upper liquids are of depths h_1 and h_2 respectively and of refractive indices μ_1 and μ_2 respectively. Locate the position of the image of the object O from the surface.



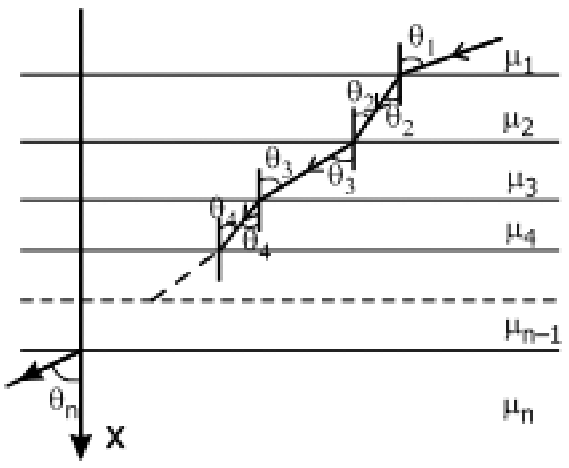
Q.2 A light is incident at 40° on a glass plate of refractive index $\mu = 1.3$ and width $h = 1$ cm, and emerges from the other side of it. Find the linear displacement of the light ray caused by refraction.

Answers

Q.1 $\frac{h_2}{\mu_2} + \frac{h_1}{\mu_1}$ Q.2 $x = 0.58$ cm

Snell's law for a number of parallel surface

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2 = \ldots\ldots\ldots = \mu_n \sin\theta_n$$



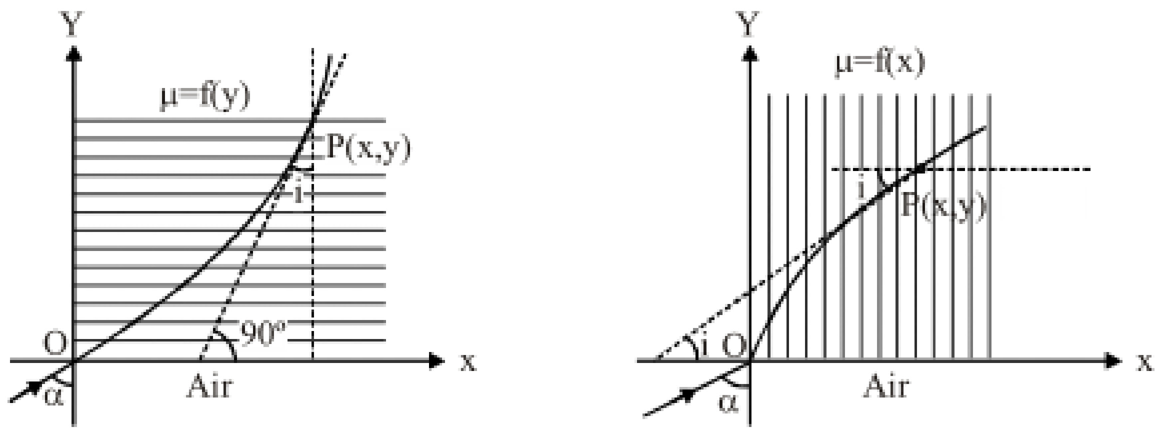
$$\mu_x \sin\theta_x = \text{const.}$$

i.e., the product of refractive index of a medium and sine of angle made by light ray in that medium with normal is constant for refraction through many parallel surfaces.

If $\mu_1 = \mu_n \Rightarrow \theta_1 = \theta_n$

i.e., the ray in 1st medium is parallel to the ray in the nth medium.

Continuous variation of refractive



$$\mu \sin\theta = \text{constant}$$

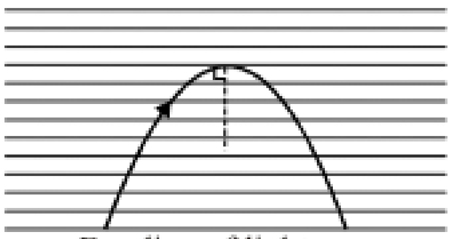
$$\frac{dy}{dx} = \text{slope of tangent}$$

$= \cot i$, if $\mu = f(y)$

$= \tan i$, if $\mu = f(x)$

In situation where there is continuous varation of refractive index, critical angle is approximately 90° and bending of ray takes place if angle of incidence approaches 90° while travelling successively from denser

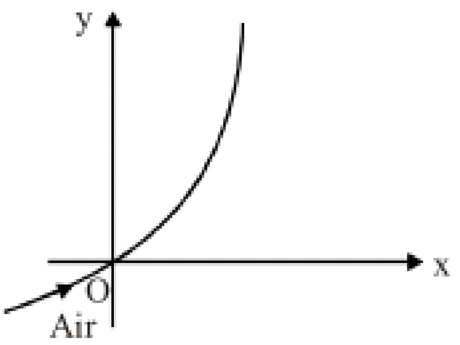
to rarer layers.



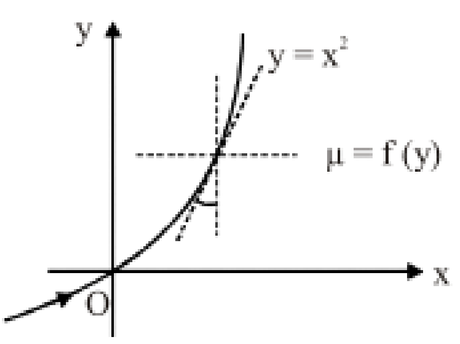
Bending of light ray

Illustration :

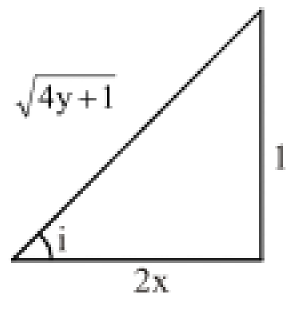
Find the variation of R. I. assuming it to be function of y such that a ray entering at origin at grazing incidence follows a parabolic path $y = x^2$ as shown in figure.



Sol. $\frac{dy}{dx} = \tan(90 - i) = \cos i = 2x$



Using Snell's law :



$$1 \cdot \sin\frac{\pi}{2} = \mu \sin i$$

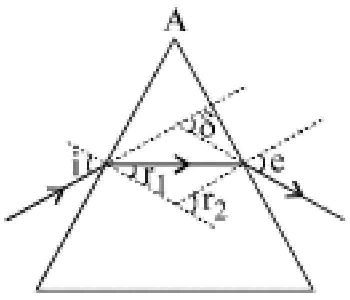
or $$\sin i = \frac{1}{\mu}$$

or $$\mu = \text{coses } i = \sqrt{4x^2+1}$$

or $$\mu = \sqrt{4y+1}$$

REFRACTION THROUGH PRISM

Prism is a transparent medium whose refracting surfaces are not parallel but are inclined to each other.



A prism deviates a light ray

1. Basic Terms

(i) Angle of prism or reflecting angle (A)

The angle between the faces on which light is incident and from which it emerges.

(ii) Angle of deviation (δ)

It is the angle between the emergent and the incident ray. It other words, it is the angle through which incident ray turns in passing through a prism.

$$\delta = (i - r_1) + (e - r_2)$$

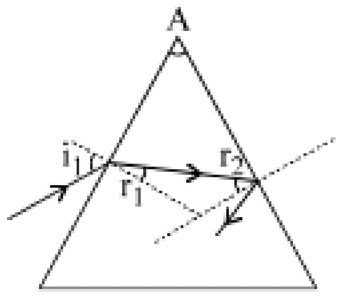
or $$\delta = i + e - (r_1 + r_2)$$

or $$\delta = i + e - A$$

2. Condition of no emergence

A ray of light incident on a prism of angle A and refractive index μ will not emerge out of a prism (whatever may be the angle of incidence) if $A > 2\theta_c$, where θ_c is the critical angle.

i.e. $$\mu > \frac{1}{\sin\,(A/2)}$$

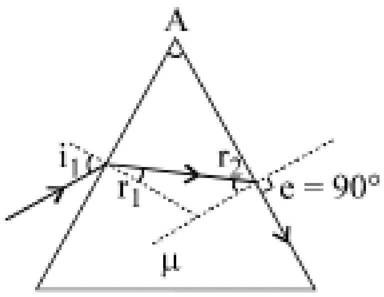


Condition of no emergence

3. Condition of grazing emergence

By the condition of grazing emergence we mean the angle of incidence i at which the angle of emergence becomes e = 90°.

$$i = \sin^{-1}\left[\sqrt{\mu^2 - 1}\,\sin A\; - \;\cos\;A\right]$$



Grazing emerging ray in a prism

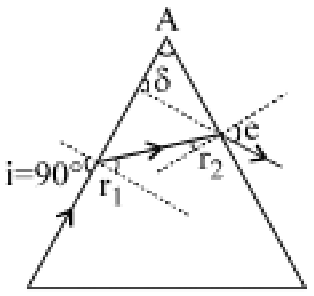
Note : That the light will emerge out of a given prism only if the angle of incidence is greater than the condition of grazing emergence.

4. Condition of maximum deviation

Maximum deviation occurs when the angle of incidence is 90°.

$$\delta_{max} = 90° + e - A$$

where $e = \sin^{-1}[\mu \sin(A - \theta_c)]$



Condition of maximum derivation

5. Condition of minimum deviation

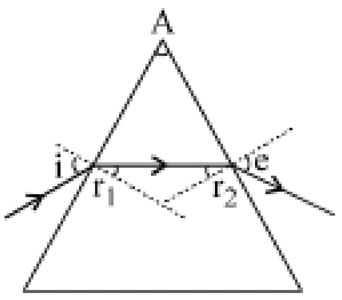
The minimum deviation occurs when the angle of incidence is equal to the angle of emergence, i.e

$$i = e$$

$$\delta_{min} = 2i - A$$

Using Snell's law

$$\mu = \frac{\sin\left[\frac{\delta_{min} + A}{2}\right]}{\sin\left[\frac{A}{2}\right]}$$

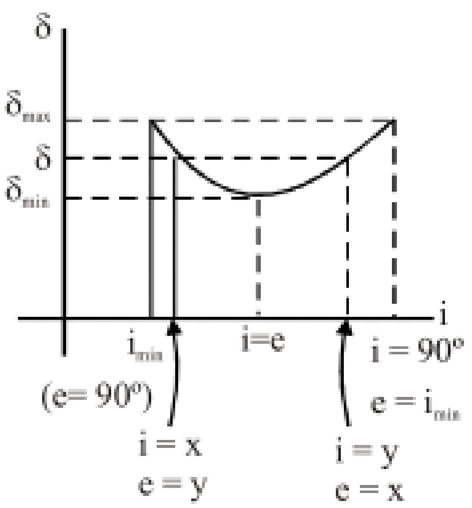


Light ray passes through a prism symmetrically in the condition of the minimum deviation

Note : That in the condition of minimum deviation the light ray passes through the prism symmetrically, i.e. the light ray in the prism becomes parallel to its base.

Characteristic of a prism

(a) Variation of δ versus i (shown in diagram).
For one δ (except δ min) there are two values of angle of incidence.
If i and e are interchanged then we get the same value of δ because of reversibility principle of light.



(b) There is one and only one angle of incidence for which the angle of deviation is minimum.

(c) When $\delta = \delta_{min}$, the angle of minimum deviation, then $i = e$ and $r_1 = r_2$, the ray passes symmetrically w.r.t. the refracting surfaces. We can show by simple calculation that $\delta_{min} = 2i_{min} - A$ where i_{min} = angle of incidence for minimum deviation, and $r = A/2$

$$\therefore \quad n_{rel} = \frac{\sin\left[\frac{A + \delta_m}{2}\right]}{\sin\left[\frac{A}{2}\right]}, \text{ where } n_{rel} = \frac{n_{prism}}{n_{surroundings}}$$

Also $\delta_{min} = (n - 1)A$ (for small values of $\angle A$)

Illustration :

A prism with angle $A = 60°$ produces a minimum deviation of $30°$. Find the refractive index of the material.

Sol. *We know that*

$$\mu = \frac{\sin\left(\frac{A+\delta_{min}}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

Here $\quad A = 60°,\ \delta_{min} = 30°$

$$\therefore \quad \mu = \frac{\sin\left(\frac{90+30}{2}\right)}{\sin\left(\frac{60}{2}\right)} = \frac{\sin 45°}{\sin 30°} = \sqrt{2}$$

Thin prisms

In thin prisms the distance between the refracting surfaces is negligible and the angle of prism (A) is very small.

Since $A = r_1 = r_2$, therefore, if A is small then both r_1 and r_2 are also small, and the same is true for i_1 and i_2.

According to Snell's law $\quad \sin i_1 = \mu \sin r_1 \quad$ or $\quad i_1 = \mu r_1$

$\sin i_2 = \mu \sin r_2 \quad$ oqr $\quad i_2 = \mu r_2$

Therefore, deviation

$$\delta = (i_1 - r_1) + (i_2 - r_2)$$
$$\delta = (r_1 + r_2) + (i_2 - r_2)$$
$$\delta = A(\mu - 1)$$

Note : That the deviation for a small angled prism is independent of the angle of incidence.

Illustration :

A thin prism of angle $A = 6°$ produces a deviation $\delta = 3°$. Find the refractive index of the material of prism.

Sol. *We know that* $\delta = A(\mu - 1)$

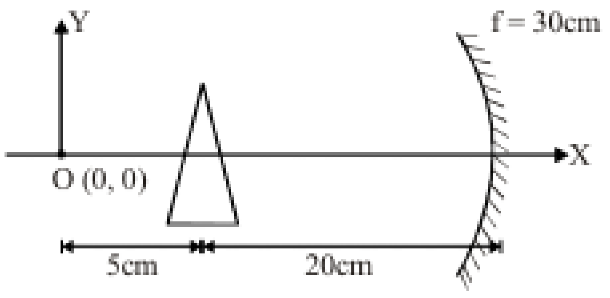
or $\quad \mu = 1 + \dfrac{\delta}{A}$

Here $\quad A = 6°,\ \delta = 3°$, *therefore*

$$\mu = 1 + \frac{3}{6} = 1.5$$

Illustration :

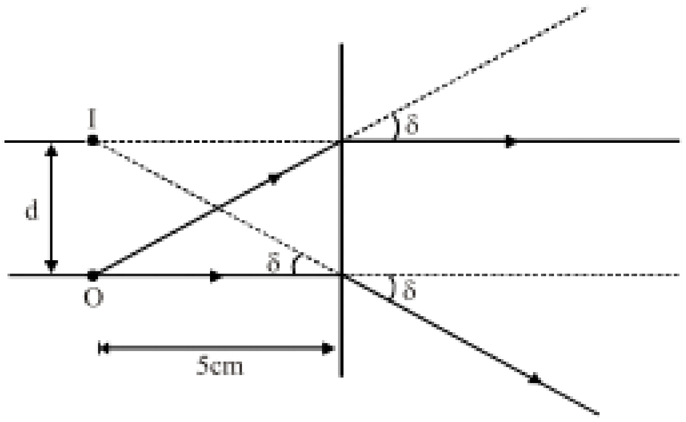
Find the co-ordinates of image of the point object 'O' formed after reflection from concave mirror as shown in figure assumming prism to be thin and small in size of prism angle 2°. Refractive index of prism materal is 3/2.



Sol. *Consider image formation through prism. All incident rays will be deviated by*

$$\delta = (\mu - 1)A = \left(\frac{3}{2} - 1\right)2° = 1° = \frac{\pi}{180} \text{ rad}$$

Now as prism is thin so object and imae will be in same plane as shown in figure.



It is clear $\quad \frac{d}{5} = \tan\delta \approx \delta \qquad (\because \delta \text{ is very small})$

or $\quad d = \frac{\pi}{36}$ cm

Now this image will act as an object for concave mirror.

$u = -25$ cm, $f = -30$ cm

$\therefore \quad v = \frac{uf}{u - f} = 150$ cm

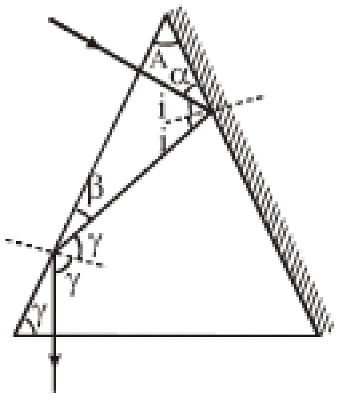
Also, $\quad m = \frac{-v}{u} = +6$

$\therefore \quad$ *Distance of image from principal axis* $= \frac{\pi}{36} \times 6 = \frac{\pi}{6}$ cm

Hence, co-ordinates of image formed after reflection from concave mirror are $\left(175\text{cm}, \frac{\pi}{6}\text{cm}\right)$

Illustration :

The cross-section of the glass prism has the form of an isosceles triangle. One of the equal faces is silvered. A ray of light incident normally on the other equal face and after being reflected twice, emerges through the base of prism along the normal. Find the angle of the prism.



Sol. *From the figure,*

$$\alpha = 90^\circ - A$$

$$i = 90^\circ - \alpha = A \qquad \ldots(i)$$

Also $\beta = 90^\circ - 2i = 90^\circ - 2A$

and $\gamma = 90^\circ - \beta = 2A$

Thus, $\gamma = r = 2A$

From geometry,

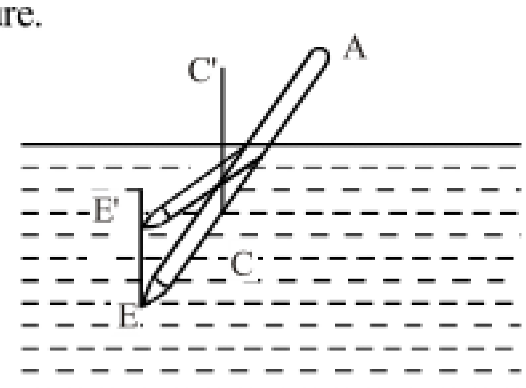
$$A + \gamma + \gamma = 180^\circ.$$

or $A = \frac{180}{5} = 36^\circ$.

Some interesting Facts related to refraction and total internal reflection

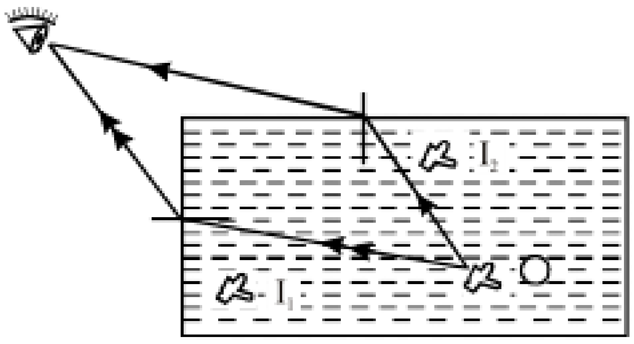
Bending of an object

When a point object in a denser medium is seen from a rarer medium it appears to be at a depth (d/μ). so if a linear object is dipped inclined to the surface of a liquid, (say water) actual depth will be different for its different points and so apparent depth. Due to this the object appears to be inclined from its actual position or BE as shown in figure.



Visibility of two images of an object

When an object is in a glass container and is seen from a level higher than that of liquied in the container as shown in figure , two images I_1 and I_2 of object O can be seen simultaneously-one due to refraction at the upper surface while the other at the side surface.



The sun is oval shaped at the time of rising and setting.

In the moring or eveing, the sunin at the horzon and refactive index in the atmosphere of the earth decreases with height. Due to this, light reaching earth's atmophere from different parts of vertical diameter of the sun enters at different heights in earth's atmophere and so travels in media of different refractive indices at the same instant and hence bends unequally.

Due to this unequal bending of light from vertical diameter, the image of the sun gest distorted and it apperars oval and larger. Howver at noon when the sun is overhead, then due to normal incidence there will be no bending and the sun will appear circular.

Similarly you can explain **Sun rises before it actually rises and sets after it actually sets.**

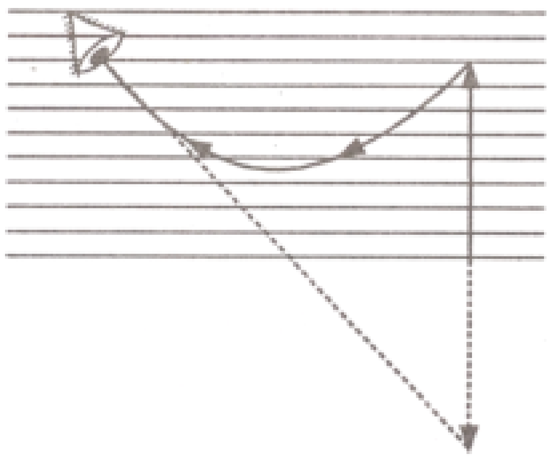
Stars twinkle.

Stars are self - luminous distant object, so only a few rays of light reach the eye through the atmosphere. However, due to fluctuations in refractive index of atmosphere the refraction becomes irregular and the light sometimes reaches the eye and sometimes it does not. This gives rise to twinkling of stars. If from moon or free space we look at a star this effect will not take place and star light will reach the eye continuously.

Tree appear inverted in deserts (mirage)

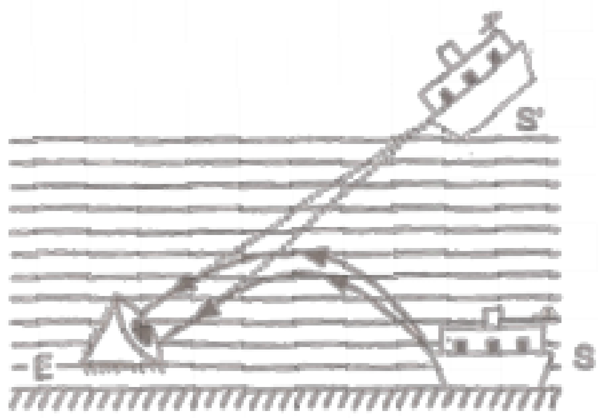
It is an optical illusion created due to the phenomenon of total internal reflection. This is seen in hot region. In hot areas like deserts surface of earth is very hot. So, air in the lower regions of atmosphere is hot as compared to that in higher regions. This result in variation of density with height and it increases as we go up. In this situation atmosphere can be assumed to be made of large number of thin layers of air.

A beam of light starting from an object say a tree and travelling downward finds itself going from denser to rater medium. Therefore, its angle of incidence at consecutive layers goes on increasing gradually till it surpasses the critical value and is reflected back due to total-internal reflection. A virtual image of the object is seen by eye at E. Due to the disturbance of air, the mirage is wavy in nature, thus giving an illusion for the presence of water which is actually not there. This effect is also called inferior mirage.



Ships appear above in the air in cold countries (looming)

This effect occurs when the density of air decreases much more rapidly with increasing height than it does under normal conditions. This situation sometimes happens in cold region particularly in the vicinity of the cold surface of sea or of a lake. Light rays starting from an object S (say a ship) are curved downward and on entering the eye the rays appear to come from S′, thus giving an impression that the ship is floating in air. This effect is also called superior mirage.

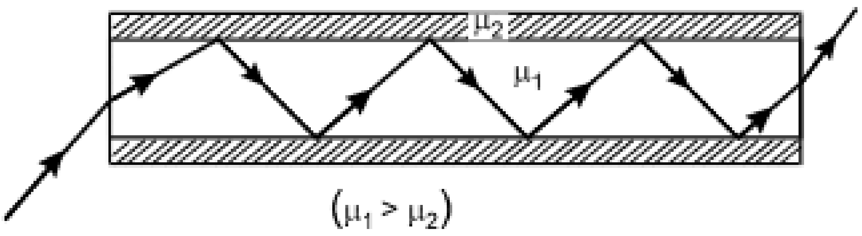


An eye placed inside water sees the external world within a cone and rest surface of water appears as a vast sheet of mirror. Radius is $r = \dfrac{h}{\sqrt{\mu^2 - 1}}$

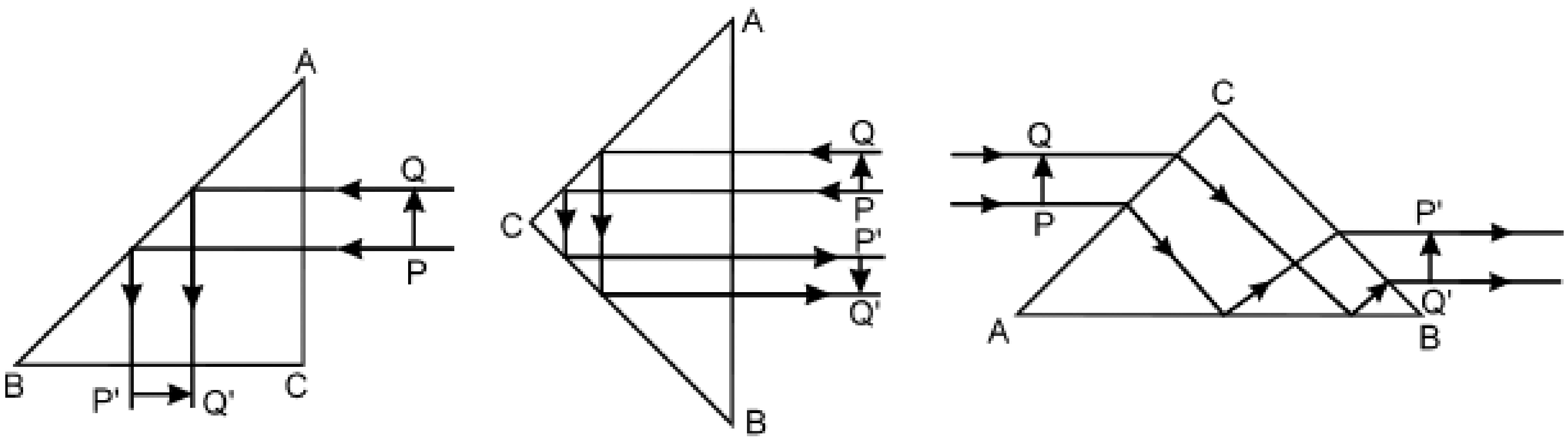
Diamond and glass both shine if cut to the special shape, but diamond shine more than glass piece cut to same shape.

Optical fibre

In optical fibre a fine material of high refractive index is coated with a material of relatively low refractive index. When a light is incident in it, it suffers a no. of total internal reflection and comes out of it. It is used as light pipe.



reflecting prisms.



Dispersion of Light

When a beam of light (containing several wavelengths) falls on one face of a prism, it splits into its constituent colours. This phenomenon of splitting of light into its constituent colours is called dispersion of light and the band of colours obtained on a screen is called spectrum. The cause of dispersion is variation of refractive index with wavelength of light. An approximate empirical relation as proposed by Cauchy is given by

$$\mu(\lambda) = A + \frac{B}{\lambda^2}$$

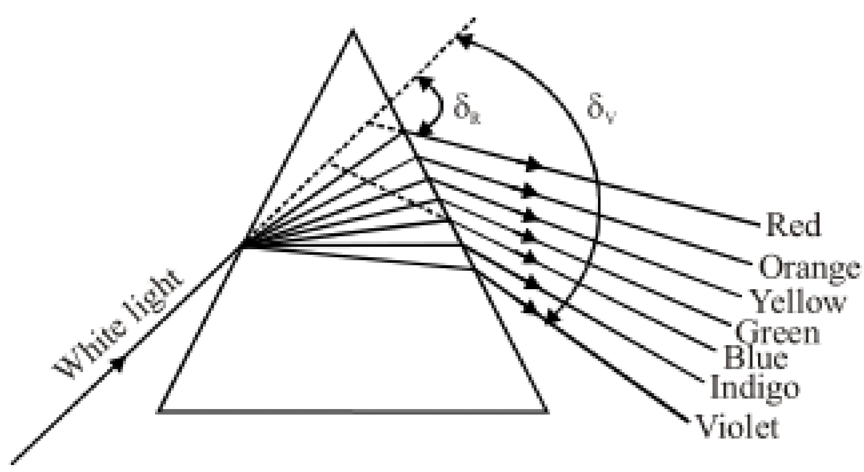
where A and B are known as Cauchy's constant. The value of A and B depends on material of prism.

We know that $\lambda_{Red} > \lambda_{Violet}$

$\therefore \quad \mu_{Red} < \mu_{Violet}$

Hence, $\quad \delta_{Red} < \delta_{Violet}$

The difference in the deviations suffered by two colours in passing through a prism gives the angular dispersion for these colours. The angle between the violet and red colours is known as angular dispersion. We know that for small angle of prism, deviation is given by



$$\delta = (\mu - 1)A$$

$\therefore \quad \delta_V$ = Deviation in violet colour $= (\mu_V - 1)A$

δ_R = Deviation in red colour $= (\mu_R - 1)A$

Hence, Angular Dispersion (A.D) $= \delta_V - \delta_R$

$$= (\mu_V - \mu_R)A$$

It is clear from above relation that angular dispersion depends upon (i) the nature of material of the prism and (ii) the angle of the prism. This is also defined as the rate of change of angle of deviation with wavelngth i.e., A.D. $= \dfrac{d\delta}{d\lambda}$.

Dispersive power of a prism is defined as the ratio between angular dispersion to mean deviation produced by the prism.

ω = Dispersive power

$$= \frac{\delta_V - \delta_R}{\delta_Y} = \frac{\mu_V - \mu_R}{\delta_Y - 1} = \frac{d\mu}{\mu_Y - 1}$$

Where $d\mu$ denotes the difference between the refractive indices of material of prism for violet and red light. It is also defined as dispersion per unit deviation. Yellow colour is taken as mean colour.

Also, $\quad \mu_Y = \dfrac{\mu_V + \mu_R}{2}$ or $\dfrac{\mu_B + \mu_R}{2}$

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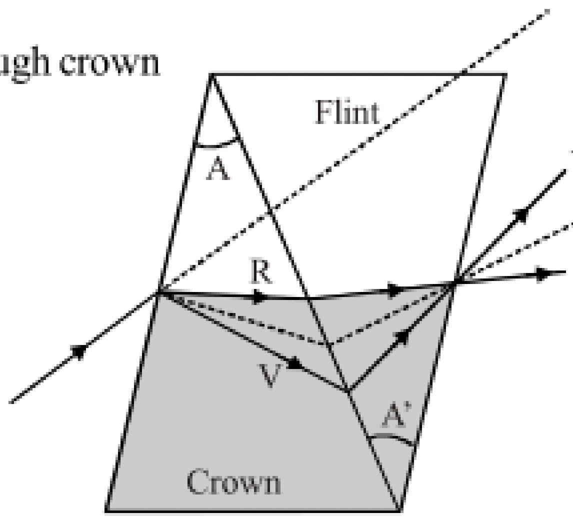
Dispersion without Deviation

Let us consider a crown glass prism combined with a flint glass prism in position as shown in figure. Let A and A' be the angles of crown glass prism and flint glass prism respectively. Let μ_0, μ and μ_r be the refractive indices of the crown glass for violet, yellow and red colours respectively.

Let μ_0', μ' and μ_r' be the corresponding values for the flint glass prism.

Let δ and δ' be the deviations suffered by yellow light through crown glass prism and flint glass prism respectively.



If the combination does not produce any deviation, then

$$\delta + \delta' = 0$$

or $\quad (\mu - 1)A + (\mu' - 1)A' = 0$

or $\quad (\mu' - 1)A' = -(\mu - 1)A$

or $\quad A' = -\left(\dfrac{\mu - 1}{\mu' - 1}\right)A$

This is the condition for no devition. The negative sign indicates that the two prisms are to be placed in position as shown in figure.

Net angular dispersion $= [(d_v - d_r) + (d'_v - d'_r)] = (\mu_v - \mu_r)A + (\mu'_v - \mu'_r)A'$

$$= A\left[(\mu_v - \mu_r) + (\mu_v' - \mu_r')\frac{A'}{A}\right]$$

$$= A\left[(\mu_v - \mu_r) + \left(\frac{\mu - 1}{\mu' - 1}\right)(\mu_v' - \mu_r')\right]$$

$$= (\mu - 1)A\left[\frac{\mu_v - \mu_r}{\mu - 1} - \frac{\mu_v' - \mu_r'}{\mu' - 1}\right] = \delta(\omega - \omega')$$

Here ω and ω' are the dispersive powers of crown glass and flint glass respectively. As the dispersive powers ω and ω' are not equal, in such a combination there will be resultant dispersion and the final dispersed beam is parallel to the incident beam.

Since $\omega' > \omega$ therefore the net angular dispersion is negative. This explains why the order of colours in the spectrum due to combination is opposite to that in the crown glass prism.

Deviation without dispersion :

For the combination of prism shown in figure, if there is to be no angular dispersion, then

$$(\delta_v - \delta_r) + (\delta_v' - \delta_r') = 0$$

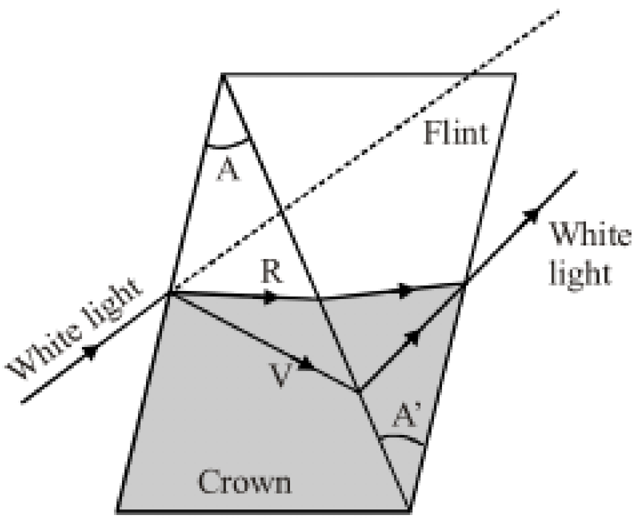
or $\quad (\mu_v - \mu_r)A + (\mu_v' - \mu_r')A' = 0$

or $\quad (\mu_v' - \mu_r')A' = -(\mu_v - \mu_r)A$

or $\quad A' = -\left(\dfrac{\mu_v - \mu_r}{\mu_v' - \mu_r'}\right)A$

This is the condition for achromatism i.e., the condition for no dispersion. This condition can be written in another form as given below.

From equation (1),



$$(\mu - 1)A\,\frac{\mu_v - \mu_r}{\mu - 1} + (\mu' - 1)A\,\frac{\mu_v' - \mu_r'}{\mu' - 1} = 0$$

or $\quad \delta\omega + \delta'\omega' = 0$

or $\quad \dfrac{\omega}{\omega'} = -\dfrac{\delta'}{\delta}$

Since, $\omega' > \omega$, $\quad \therefore \delta > \delta'$

or $\quad (\mu - 1)A > (\mu' - 1)A'$

But, $\quad (\mu - 1) < (\mu' - 1) \quad \therefore A > A'$

So, the crown glass prism should have a larger angle than the flint glass prism.

Net deviation $= \delta + \delta'$

$$= (\mu - 1)A + (\mu' - 1)A' = (\mu - 1)A\left[1 + \frac{(\mu'-1)A'}{(\mu-1)A}\right]$$

$$= (\mu - 1)A\left[1 - \frac{\mu'-1}{\mu-1}\,\frac{\mu_v - \mu_r}{\mu_v' - \mu_r'}\right] \qquad \text{(Using equation (2))}$$

$$= (\mu - 1)A\left[1 - \frac{\mu_v - \mu_r}{\mu - 1} \times \frac{\mu_v - \mu_r}{\mu - 1} \times \frac{\mu' - 1}{\mu_v' - \mu_r'}\right] = \delta\left(1 - \frac{\omega}{\omega'}\right)$$

Since $\omega' > \omega$, therefore the net deviation is in the direction of the deviation produced by crown glass prism.

Remark :

A parallel sided glass slab can be looked upon as the combination of two prisms producing no deviation and no dispersion.

Practice Exercise

Q.1 A ray of light undergoes deviation of 30^o when incident on an equilateral prism of refractive index $\sqrt{2}$. What is the angle subtended by the ray inside the prism with the base of the prism?

Q.2 A thin prism P_1 with angle 4^o and made from glass of refractive index 1.54 is combined with another prism P_2 made from glass of refractive index 1.72 to produce dispersion without deviation. What is the angle of the prism P_2?

Q.3 A ray of light is incident at an angle of 60^o on one face of a prism which has an angle of 30^o. The ray emerging out of the prism makes an angle of 30^o with the incident ray. Show that the emergent ray is perpendicular to the face through which it emerges and calculate the refractive index of the material of the prism.

Q.4 A glass prism of angle 72^o and index of refraction 1.66 is immersed in a liquid of refractive index 1.33. Find the angle of minimum deviation for a parallel beam of incident light passing through the prism.

Answers

Q.1 0 Q.2 3^o Q.3 $\sqrt{3}$ Q.4 22.37^o

formation of image by a Spherical refracting Surface

Pole (vertex) : Point of intersection of principal axis and the refracting surface.

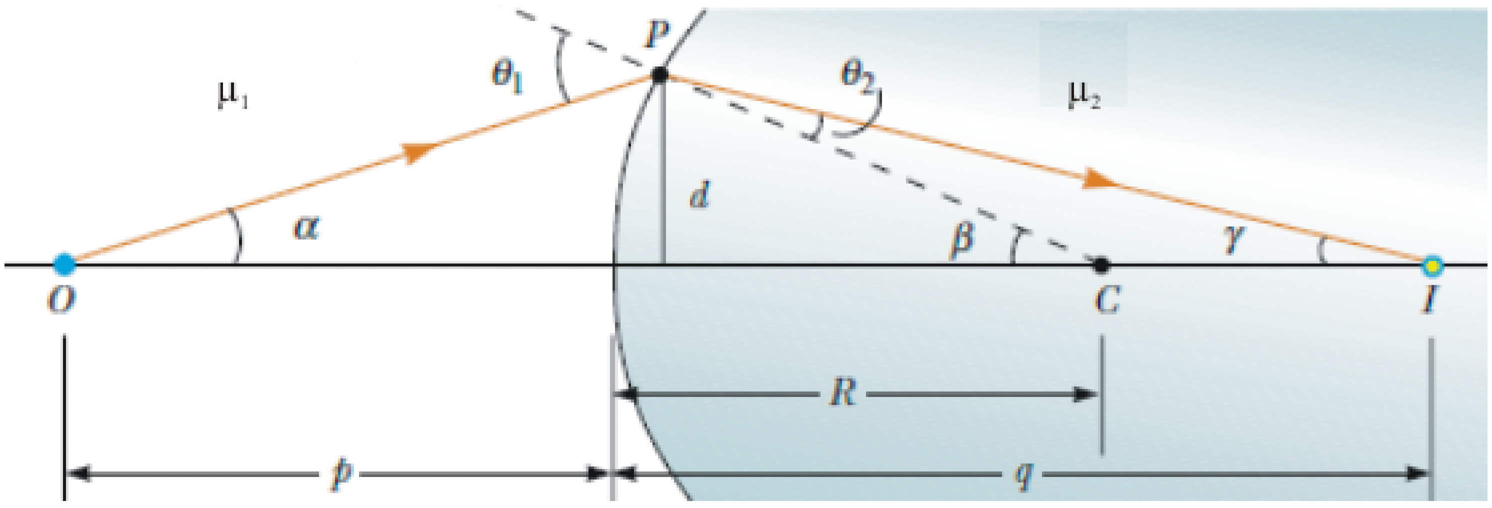
Focal Point (F) : It is an axial point having the property that any incident ray traveling parallel to the axis will after refraction, proceed toward, or appear to come from this point.

Focal length (f) : The distance of the focus from the vertex of the refracting surface is called focal length. There is a great significance of the sign of focal length as it is able to state whether the spherical refracting surface is converging or diverging as in the chart.

Sign of f	Nature of the system
+ ve	converging
– ve	diverging

Relation between position of object distance, image distance and radius of the spherical refracting surface :

In this section we describe how images are formed when light rays are refracted at the boundary between two transparent materials. Consider two transparent media having indices of refraction μ_1 and μ_2, where the boundary between the two media is a spherical surface of radius R (Fig.). We assume that the object at O is in the medium for which the index of refraction is μ_1. Let us consider the paraxial rays leaving O. As we shall see, all such rays are refracted at the spherical surface and focus at a single point I, the image point.Figure shows a single ray leaving point O and refracting to point I.



Snell's law of refraction applied to this ray gives

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

Because θ_1 and θ_2 are assumed to be small, we can use the small-angle approximation $\sin\theta \approx \theta$ (with angles in radians) and say that

$$\mu_1\theta_1 = \mu_2\theta_2$$

Now we use the fact that an exterior angle of any triangle equals the sum of the two opposite interior

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angles. Applying this rule to triangles OPC and PIC in Figure gives

$$\theta_1 = \alpha + \beta$$

$$\beta = \theta_2 + \gamma$$

If we combine all three expressions and eliminate θ_1 and θ_2, we find that

$$\mu_1 \alpha + \mu_2 \gamma = (\mu_2 - \mu_1)\beta$$

From Figure, we see three right triangles that have a common vertical leg of length d. For paraxial rays (unlike the relatively large-angle ray shown in Fig.), the horizontal legs of these triangles are approximately p for the triangle containing angle α, R for the triangle containing angle β, and q for the triangle containing angle γ). In the small-angle approximation, $\tan\alpha \approx \alpha$, so we can write the approximate relationships from these triangles as follows:

$$\tan\alpha \approx \alpha = \frac{d}{p} \qquad \tan\beta \approx \beta = \frac{d}{R} \qquad \tan\gamma \approx \gamma = \frac{d}{q}$$

We substitute these expressions into above Equation and divide through by d to give

$$\frac{\mu_2}{p} + \frac{\mu_1}{q} = \frac{\mu_2 - \mu_1}{R}$$

Let

u= object distance (with sign convention)

v= image distance (with sign convention)

R= Radius of curvature (with sign convention)

then

$p = -u \qquad q = +v \qquad R = +R$

putting the values of p, q and R in above equation we get

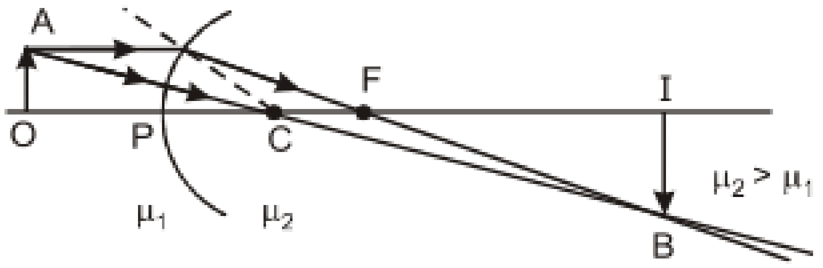
$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

Rules for image tracing for a linear, transverse extended object :

The basic rule is same as that in mirrors. Briefly,

(i) Draw a ray parallel to the principal axis which after refraction will be along the line passing through F.

(ii) Draw a ray incident along the line through centre it will pass undeviated.



Transverse Magnification :

Snell's law of refraction applied to this ray gives

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

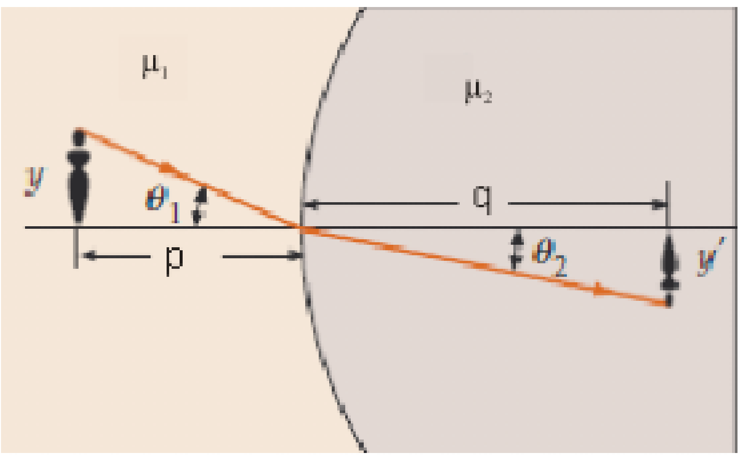
Because θ_1 and θ_2 are assumed to be small, we can say that

$$\sin\theta_1 \approx \theta_1 \approx \tan\theta_1 = \frac{y}{p}$$

$$\sin\theta_2 \approx \theta_2 \approx \tan\theta_2 = \frac{-y'}{q}$$

From the above equations we get

$$\mu_1 \frac{y}{p} = \mu_2 \frac{-y'}{q}$$



$$\Rightarrow \quad \frac{y'}{y} = -\frac{\mu_1}{\mu_2}\frac{q}{p}$$

Transverse magnification is given by

$$m_T = \frac{y'}{y} = -\frac{\mu_1}{\mu_2}\frac{q}{p}$$

Let

u= object distance (with sign convention)

v= image distance (with sign convention)

then

p=-u q=+v R=+R

putting the values of p and q in above equation we get

$$m_T = \frac{\mu_1 v}{\mu_2 u}$$

Longitudinal Magnification

for small longitudinal object,

$$m_L = \frac{dv}{du} = \frac{\mu_1}{\mu_2}\frac{v^2}{u^2}$$

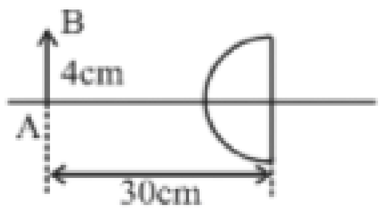
Optic Power :

It represents the conversing ability of an element. For a single spherical refracting surface it is defined as

$$P = \frac{\mu_2 - \mu_1}{R}$$

Illustration :

A linear object of length 4 cm is placed at 30 cm from the plane surface of hemispherical glass of radius 10 cm. The hemispherical glass is surrounded by water. Find the final position and size of the image.



Sol. *For 1st surface*

$\mu_1 = \frac{4}{3},\ \mu_2 = \frac{3}{2}, u = -20$ **cm**, and $R = +10$ cm,

using $$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{(\mu_2 - \mu_1)}{R}$$

$$\Rightarrow \frac{\left(3/2\right)}{v} - \frac{\left(4/3\right)}{(-20)} = \frac{\left(3/2 - 4/3\right)}{10}$$

$$\Rightarrow v = -30\,\text{cm}$$

Using $$\frac{A'B'}{AB} = \frac{\mu_1 v}{\mu_2 u} \Rightarrow \frac{A'B'}{(4\text{cm})} = \frac{\left(4/3\right)(-30)}{\left(3/2\right)(-20)}$$

$$\Rightarrow A'B' = 5.3\,\text{cm.}$$

$A'B'$ behaves as the object for plane surface

$$\mu_1^{'} = \frac{3}{2},\ \mu_2^{'} = \frac{4}{3} \text{ and } R = \infty,\ u' = -40$$

$$\Rightarrow \frac{\mu_2^{'}}{v'} = \frac{\mu_1^{'}}{u'} \Rightarrow \frac{\left(4/3\right)}{v'} = \frac{\left(3/2\right)}{(-40)}$$

Solving it we will get, $v' = -25.4$ cm

Now using, $$\frac{A''B''}{A'B'} = \frac{\left(\mu_1^{'} v\right)}{\left(\mu_2^{'} u\right)}$$

$$\Rightarrow \frac{A''B''}{(5.3)} = \frac{\left(3/2\right)(-35.4)}{\left(4/3\right)(-40)} \Rightarrow A''B'' = 5.3$$

The final images in all the above cases are shown in figure.

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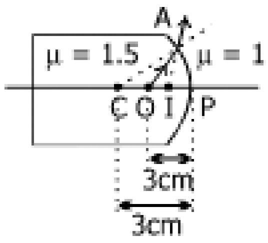
Practice Exercise

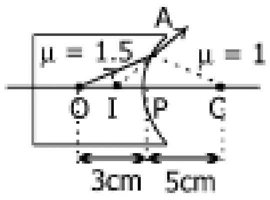
Q.1 A parallel beam of light travelling in water ($\mu = 4/3$) is refracted by a spherical air bubble of radius 2 mm situated in water. Assuming the light rays to be paraxial (a) Find the position of the image due to refraction at the first surface and the position of final image. (b) Draw a ray diagram showing the positions of both the images.

Q.2 A point source of light is placed in air at a distance 2R from the centre of a glass sphere of radius of curvature R and refractive index 1.5. Obtain the position of the intermediate and final images.

Q.3 An air bubble in glass ($\mu = 1.5$) is situated at a distance 3 cm from a spherical surface of diameter 10 cm as shown in **figure**. At what distance from the surface will the bubble appear if the surface is (a) convex (b) concave.



(A)



(B)

Q.4 If a mark of size 0.2 cm on the surface of a glass sphere of diameter 10 cm and $\mu = 1.5$ is viewed through the diametrically opposite point, where will the image be seen and of what is size?

Answers

Q.1	(a) 0.6 cm and 0.1 cm left of first surface	Q.2	∞ and 2R
Q.3	(a) –2.5 cm (b) –1.66 cm	Q.4	0.6 cm

Refraction through lens

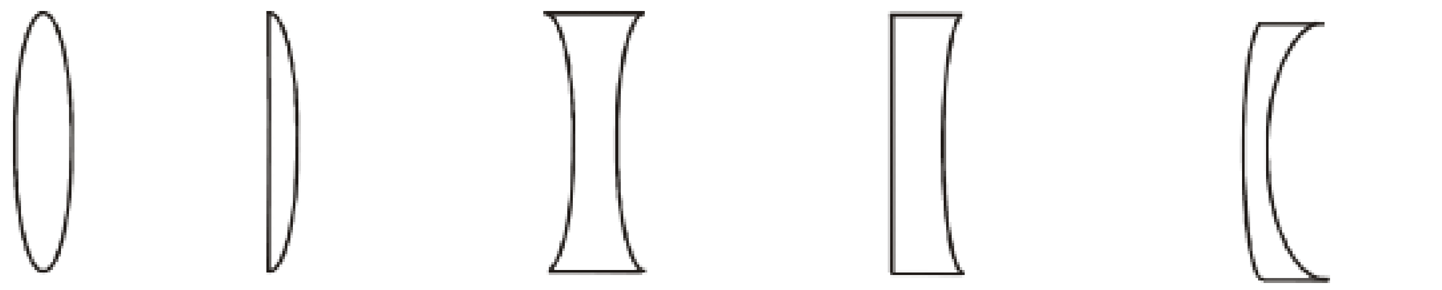
Lens :

A lens is an optical system bounded by two or more than two refracting surfaces having common axis.

Refraction through a simple thin lens :

In simple lens if the surfaces are very close to each other then the lens is called simple thin lens if they are separated by non negligible distance then called simple thick lens.

Thin lens is classified (Geometrically) to following categories.



Biconvex **Plane convex** **Biconcave** **Plano concave** **Convexo concave**

But according to its action a lens may be of two types.

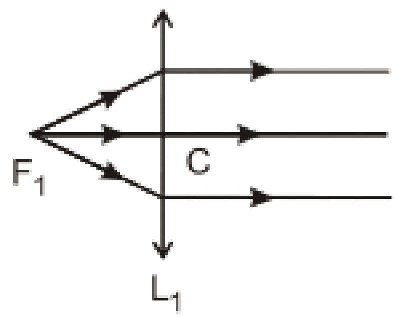
Simple Thin Spherical lens

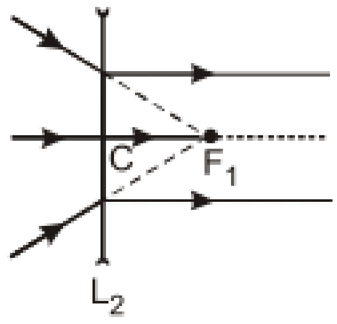
- **Converging lens or convex lens** (If it converges Parallel beam)
- **Diverging lens or concave lens** (If it diverges parallel beam)

Some basic terms related to thin spherical lens :

Optic Centre (C) : If a light ray is incident on lens is such that after refraction from lens the emergent ray is parallel to the incident ray then the point at which refracted ray intersects the principal axis is called optic centre of that lens.

Primary Focal Point (F_1) : The position of object if image is at infinity is called primary focal point

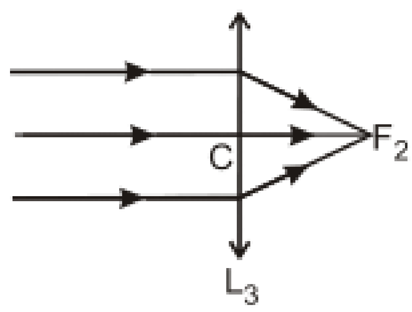


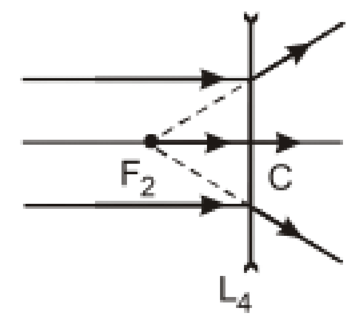


Where L_1 = converging lens

L_2 = diverging lens

Secondary focal point (F_2) : The position of image if object is at infinity is called secondary focal point





Where L_3 = converging lens

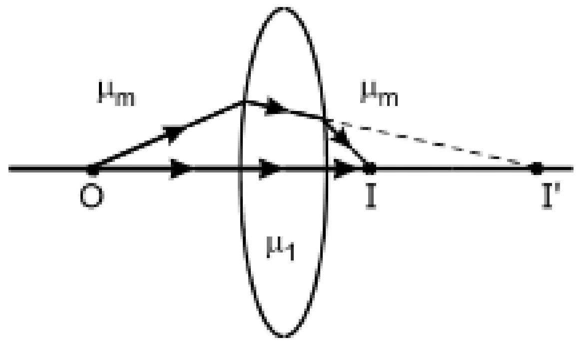
L_4 = diverging lens

focal length f : The distance between optical centre and secondary focal point is termed as focal length.

Note :	**Sign of f**	**Nature of the lens**
	+ ve	converging
	– ve	diverging

Quantitative Discussion of a thin spherical lens placed in a Medium for an axial point object :

Consider a thin spherical lens ($m = m_l$) is placed in a medium ($m = m_m$) as shown in figure.
Let a point object O is placed on the principal axis of the lens. For the first refracting surface O is an object point and the corresponding image point it I_1. Now I_1 will act as object point for second surface which again form an image at I. For lens we can say that O is object point and I is image point.



At first surface, $$\frac{\mu_l}{v'} - \frac{\mu_m}{u} = \frac{\mu_l - \mu_m}{R_1} \quad \text{... (i)}$$

At second surface, $$\frac{\mu_m}{v} - \frac{\mu_l}{v'} = \frac{\mu_m - \mu_l}{R_2} \quad \text{... (ii)}$$

Combining (i) and (ii) we get.

$$\frac{1}{v} - \frac{1}{u} = \left(\frac{\mu_l}{\mu_m} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{... (iii)}$$

Calculation of f : If $u = \infty$ $\Rightarrow v = f$

$$\frac{1}{f} = \left(\frac{\mu_l}{\mu_m} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{... (iv)}$$

This formula is known as **lens maker formula.**
from (iii) and (iv)

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

This formula is known as **lens formula**

Power of a thin spherical lens :

It is defined as sum of the power of individual surfaces

$$P = P_1 + P_2 = \frac{\mu_l - \mu_m}{R_1} + \frac{\mu_m - \mu_l}{R_2}$$

$$P = (\mu_l - \mu_m)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{\mu_m}{f}$$

Note : The sign of P and f are same.

Mathematical Analysis to state whether the lens is converging or diverging :

Biconvex :

$$\frac{1}{f}=\left(\frac{\mu_l}{\mu_m}-1\right)\left(\frac{1}{+R_1}-\frac{1}{-R_2}\right)=\left(\frac{\mu_l}{\mu_m}-1\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

If	$\mu_l > \mu_m$	$\mu_l < \mu_m$	$\mu_l = \mu_m$
f =	+ve	− ve	∞
P =	+ ve	− ve	0
Nature→	Converging	diverging	neither converging nor diverging

Plano convex :

$$\frac{1}{f}=\left(\frac{\mu_l}{\mu_m}-1\right)\left(\frac{1}{\pm\infty}-\frac{1}{-R}\right)=\left(\frac{\mu_l}{\mu_m}-1\right)\frac{1}{R}$$

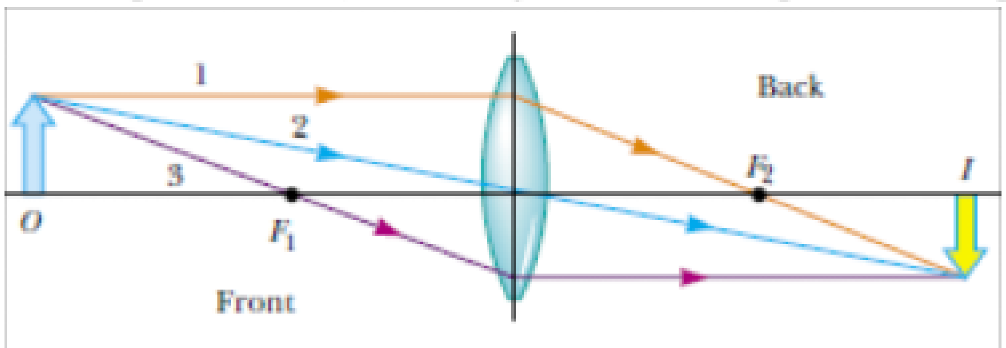
If	$\mu_l > \mu_m$	$\mu_l < \mu_m$	$\mu_l = \mu_m$
f =	+ve	− ve	∞
P =	+ ve	− ve	0
Nature→	Converging	diverging	neither converging nor diverging

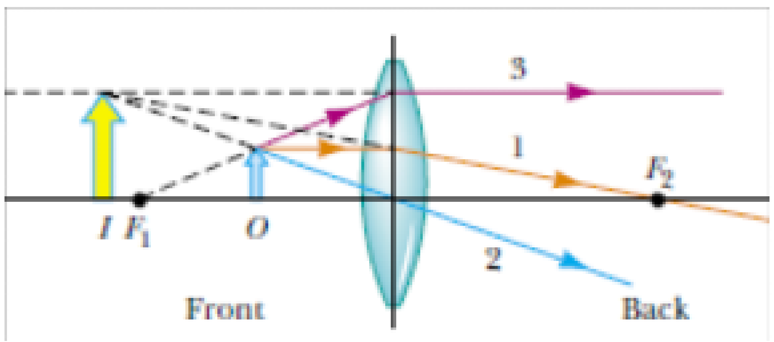
Extended objects :

Way of image tracing

To locate the image of a converging lens (Fig.), the following three rays are drawn from the top of the object:

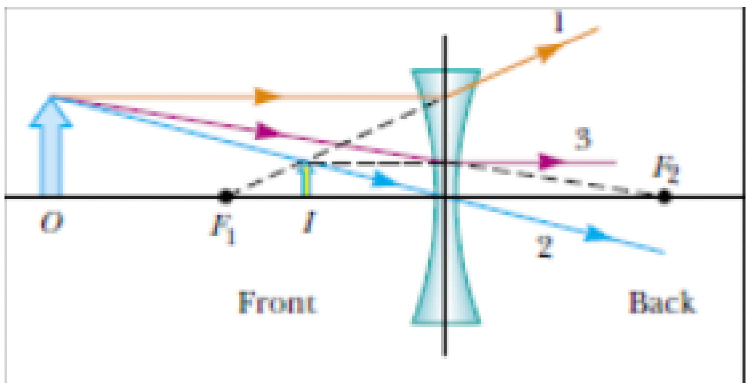
- Ray 1 is drawn parallel to the principal axis. After being refracted by the lens, this ray passes through the focal point on the back side of the lens.
- Ray 2 is drawn through the center of the lens and continues in a straight line.
- Ray 3 is drawn through the focal point on the front side of the lens (or as if coming from the focal point if lul< f) and emerges from the lens parallel to the principal axis.





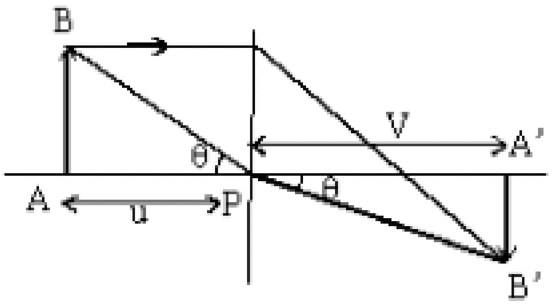
To locate the image of a diverging lens (Fig.), the following three rays are drawn from the top of the object:

- Ray 1 is drawn parallel to the principal axis. After being refracted by the lens, this ray emerges directed away from the focal point on the front side of the lens.
- Ray 2 is drawn through the center of the lens and continues in a straight line.
- Ray 3 is drawn in the direction toward the focal point on the back side of the lens and emerges from the lens parallel to the principal axis.



Magnification :

Transverse magnification :



From $\Delta' S \;\; PAB$ and

$$\frac{AB}{AP} = \frac{A'B'}{A'P} \quad \Rightarrow \quad \frac{+h_{object}}{-u} = \frac{-h_{image}}{+v}$$

$$m_T = \frac{h_{image}}{h_{object}} = \frac{v}{u} = \frac{f}{f+u}$$

Longitudinal magnification : For small linear longitudinal object

Using

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

Differentiating w.r.to. u we get

$$\frac{dv}{du} = \frac{v^2}{u^2} = m_T^2$$

$$m_L = m_T^2$$

Illustration :

What if the object moves right up to the lens surface, so that $u \to 0$? Where is the image ?

Sol. *In this case because u << R, where R is either of the radii of the surface of the lens, the curvature of the lens can be ignored and it should appear to have the same effect as a plane piece of material. This would suggest that the image is just on the front side of the lens, at $v = 0$. We can verify this mathematically by rearranging the thin lens equation.*

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u}$$

If we let $u \to 0$, the second term on the right become verty large compared to the first and we can neglect 1/f. The equation becomes

$$\frac{1}{v} = -\frac{1}{u}$$

$$v = -u = 0$$

Thus v is on the front side of the lens (because it has the oposite sign as v), and just at the lens surface.

Illustration

Plot graphs of image distance as a function of object distance for a lens for which the focal length is 10 cm if the lens is

(a) convex (b) concave

Sol. (a) (b)

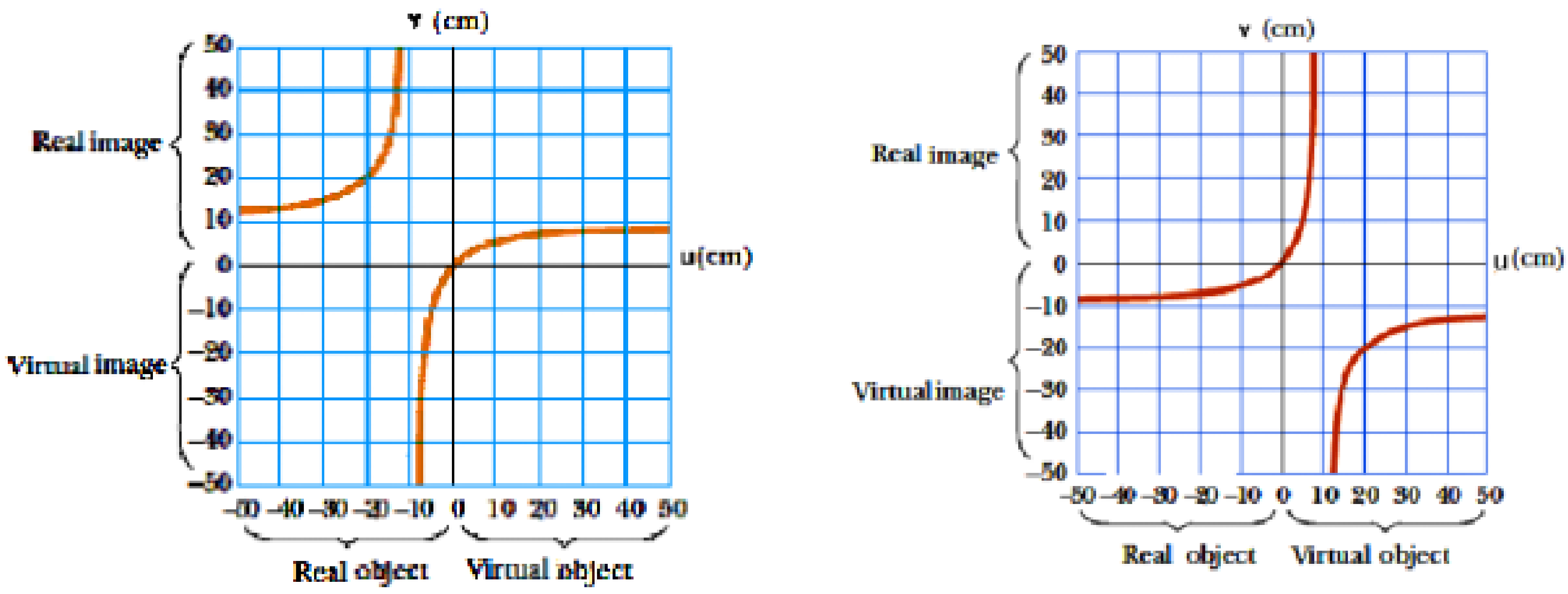


Illustration :

A thin equiconvex lens of refractive index 3/2 and radius of curvature 30 m is put in water (refractive index $=\frac{4}{3}$). Find its focal length.

Sol.
$$\frac{1}{f}=\left(\frac{\mu_1}{\mu_2}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\frac{1}{f}=\left(\frac{3/2}{4/3}-1\right)\left(\frac{1}{0.3}+\frac{1}{0.3}\right)$$

or $$\frac{1}{f}=\left(\frac{9}{8}-1\right)\left(\frac{2}{0.3}\right)$$

or $$\frac{1}{f}=\frac{1}{8}\times\frac{2}{0.3}$$

or $$f=1.20\,\text{m}.$$

Illustration :

A lens of focal length f projects m times magnified image of an object on a screen. Find the distance of the screen from the lens.

Sol. *Image will be real.*

We know that

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$$

$$\Rightarrow \quad \frac{v}{f}=1-\frac{v}{u}$$

$$\Rightarrow \quad \frac{v}{f}=1+m \qquad [\because\ u \text{ is negative}]$$

$$\Rightarrow \quad v=f(m+1).$$

Practice Exercise

Q.1 A glass convex lens of refractive index (3/2) has got a focal length equal to 0.3 m. Find the focal length of the lens if it is immersed in water of refractive index (4/3).

Q.2 A projector lens marks an image of an object on a screen 6 m from the lens. If the magnification is 24, what is the focal length of the lens?

Q.3 A thin glass (refractive index 1.5) lens has optical power of -5 D in air. Find optical power in a liquid medium with refractive index 1.6 .

Q.4 Find the relation between n_1 and n_2 if the behaviour of a light ray is as shown in the figure aside .

Q.5 A point object is located at a distance of 15 cm from the front surface thick bi–convex lens. The lens is 10 cm thick and radii of its front and back surfaces are 10 cm and 25 cm respectively. How far beyond the back surface of this lens ($m = 1.5$) is the image formed?

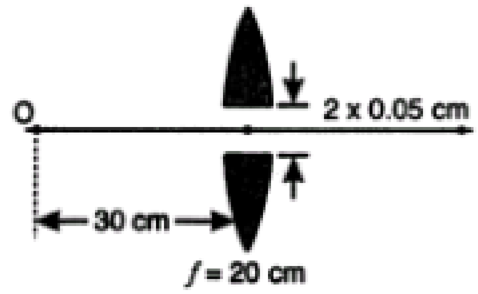
Q.6 An equiconvex lens of refractive index (3/2) and focal length 10 cm in air is held with its axis vertical and its lower surface immersed in water ($m = 4/3$), the upper surface being in air. At what distance from the lens, will a vertical beam of parallel light incident on the lens be focused?

Q.7 A magnifying lens has a focal length of 10 cm. (a) Where should the object be placed if the image is to be 30 cm from the lens? (b) What will be the magnification?

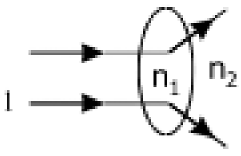
Q.8 A converging beam of light forms a sharp image on a screen. A lens is placed in the path of the beam, the lens being 10 cm from the screen. It is found that the screen has to be moved 8 cm further away from the lens to obtain a sharp image. Find the focal length and nature of lens.

Q.9 An object 25 cm high is placed in front of a convex lens of focal length 30 cm. If the height of image formed is 50 cm, find the distance between the object and the image?

Q.10 A point object O is placed at a distance of 0.3 m from a convex lens (focal length 0.2 m) cut into two halves each of which is displaced by 0.0005 m as shown in figure. Find the position of the image. If more than one image is formed, find their number and distance between them.



Answers

Q.1 1.2 m   Q.2 0.24 metre   Q.3 1 D   Q.4 $n_2 > n_1$ 

Q.5 200cm   Q.6 20 cm   Q.7 (a) –7.5 cm (b) 4   Q.8 –22.5 cm

Q.9 Case 1 (If the image is inverted i.e., real) $\rightarrow$ 90 cm ; Case 2 (If the image is erect i.e., virtual) $\rightarrow$ 15 cm

Q.10 0.3 cm

Note:

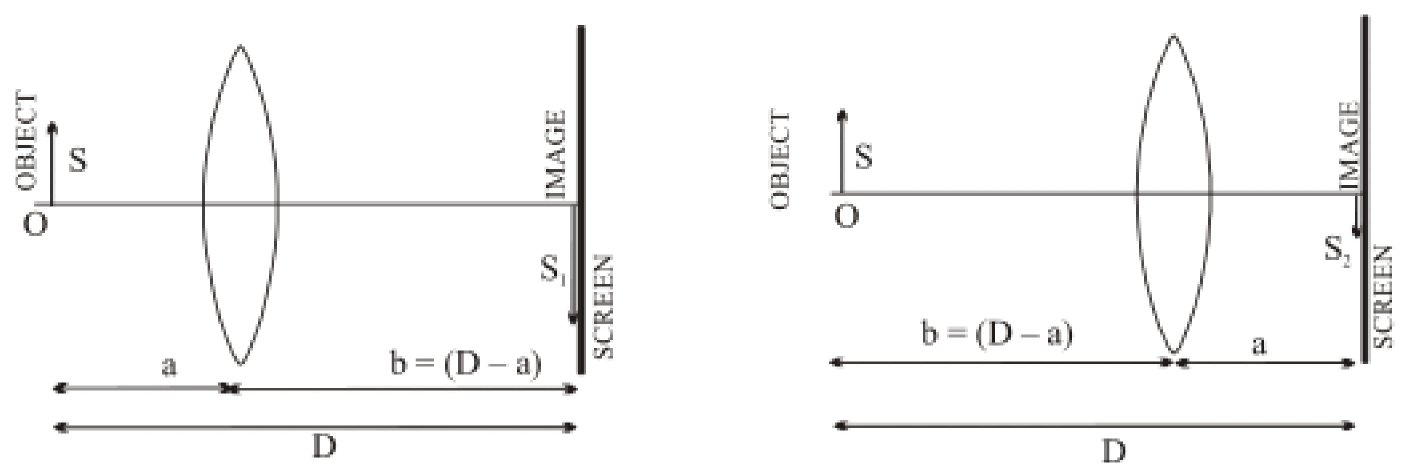
All the formula derived earlier is valid only for thin lens and applicable when lens is thin as well as medium on both sides of the lens is same. If the lens is thick or medium on both sides of the lens is different then we have to work with each surface step by step.

Displacement method :

This is a laboratory method to find the focal length of convex lens. In displacement method, in two different situation real image of a real object are formed on the screen for given position of object and screen by displacing the lens.

If a thin converging lens of focal length 'f' is placed between an object and a screen fixed at a distance D apart and if $D > 4f$, then there are two positions of the lens at which a sharp image of the object is formed on the screen.

If the object is at a distance 'u' from the lens, the distance of image from the lens $v = (D - u)$. So from lens formula,



$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

we have, $$\frac{1}{(D-a)} - \frac{1}{-a} = \frac{1}{f}$$

i.e. $$a^2 - Da + Df = 0$$

So that $$a = \frac{1}{2}\left[D \pm \sqrt{D(D-4f)}\right] \qquad \text{... (i)}$$

Now there are three possibilities

(a) If $D < 4f$: a will be imaginary, so physically no position of lens is possible.

(b) If $D = 4f$: In this situation $a = D/2 = 2f$. So only one position is possible and in this situation,
$b = D - a = 4f - 2f = 2f\ (= a)$

(c) If $D > 4f$: In this situation both the roots of equation (i) will be real.

i.e. $a_1 = a = \frac{1}{2}\left[D-\sqrt{D(D-4f)}\right]$

and $a_2 = D - a = \frac{1}{2}\left[D+\sqrt{D(D-4f)}\right]$ ... (ii)

So if $d > 4f$, there are two positions of lens at distance a_1 and a_2 from the object for which real image is formed on the screen.

Let us assume $a > b$

$\Rightarrow$ $a = \frac{1}{2}\left[D-\sqrt{D(D-4f)}\right]$

and $b = \frac{1}{2}\left[D+\sqrt{D(D-4f)}\right]$

Calculation of focal length of lens :

The displacement of the lens will be

$$x = b - a = \sqrt{D^2 - 4Df}$$

$$\Rightarrow f = \frac{D^2 - x^2}{4D}$$

Note : (a) $m_1 m_2 = 1 \Rightarrow \left(\frac{-S_1}{S_0}\right)\left(\frac{-S_2}{S_0}\right) = 1 \qquad \Rightarrow S_o = \sqrt{S_1 S_2}$

(b) $m_2 - m_1 = \frac{v'}{u'} - \frac{v}{u} = \frac{+a}{-b} - \frac{+b}{-a} = \frac{b}{a} - \frac{a}{b} = \frac{b^2 - a^2}{ab} = \frac{(a+b)(b-a)}{ab} = \frac{Dx}{Df} = \frac{x}{f}$

Illustration :

For two positions of a converging lens between an object and a screen which are 96 cm apart, two real images are formed. The ratio of the lengths of the two images is 4.84. Calculate the focal length of the lens.

Sol. $a+b=96$ ---------- (i)

Here $\frac{m_1}{m_2} = 4.84$

$\Rightarrow m_1^2 = 4.84 \qquad \Rightarrow m_1 = -2.2 \qquad \Rightarrow \frac{+b}{-a} = -2.2$

$\Rightarrow b = 2.2a$ ... (ii)

Solviing we get $a = 20$ cm gives $d = 76$ cm $\Rightarrow$ $x=b-a=56$cm

$\therefore \Rightarrow f = \frac{D^2 - x^2}{4D} = 20.625$ cm .

Practice Exercise

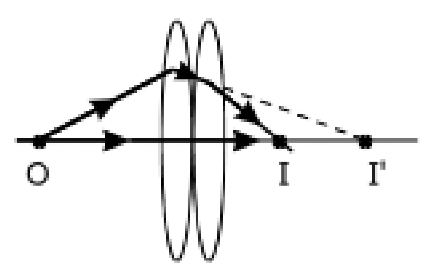
Q.1 A source and a screen are fixed in a place a distaqnce 'l' apart. A thin lens is placed betwen them at a position such that the source is focussed on the screen. For what ranges of lens focal lengths are there for (a) two (b) one (c) no such positions?

Q.2 An object is placed at a distance of 75 cm from a screen. Where should a convex lens of focal length 12 cm be placed so as to obtain a real image of the object?

Answers

Q.1 (a) $f < l/4$ (b) $f = l/4$ (c) $f > l/4$ Q.2 15 cm or 60 cm

Combination of two thin spherical lenses in contact



For first lens :

$$\frac{1}{v'} - \frac{1}{u} = \frac{1}{f_1}$$

For second lens :

$$\frac{1}{v} - \frac{1}{v'} = \frac{1}{f_2}$$

Combining, we get

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$\Rightarrow \frac{1}{f_{eq}} = \frac{1}{f_1} + \frac{1}{f_2} \qquad \text{i.e., } P_{eq} = P_1 + P_2$$

Similarly for n thin lenses in contact

$$\frac{1}{f_{eq}} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \ldots\ldots \frac{1}{f_n}$$

$$\Rightarrow \quad P_{eq} = P_1 + P_2 + P_3 + \ldots\ldots + P_n$$

Note :

(1) If two thin lenses of equal focal but of opposite nature (i.e. one convergent and other divergent) are put in contact, the resultant focal length of the combination will be.

$$\frac{1}{f_{eq}} = \frac{1}{f} + \frac{1}{-f} = 0$$

i.e., $f_{eq} = \infty$ and $P_{2eq} = 0$

i.e., the system will be have as a plane glass plate.

(2) If two lenses of same nature are put in contact, then as (f_1 and f_2 are magnitude of focal lengths)

$$\frac{1}{f_{eq}} = \frac{1}{f_1} + \frac{1}{f_2}$$

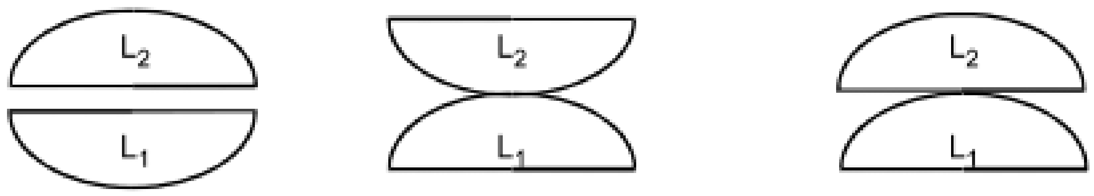
$$\frac{1}{f_{eq}} > \frac{1}{f_1} \text{ and } \frac{1}{f_{eq}} > \frac{1}{f_2}$$

The resultant focal length will be lesser than individual.

(3) If two thin lenses of opposite nature with different focal lengths are put in contact, the resultant focal length will be of same nature as that of the lens of shorter focal length but its magnitude will be more than that of shorter focal length.

(4) If a lens of focal length f is divided into two equal parts as shown in figure and each part has a focal length f' then each part will have focal length 2 times.

Now if these part are put in contact as in figure (A), (B) or (C) the resultant focal length of the combination will be equal to initial value.



(a) (b) (c)

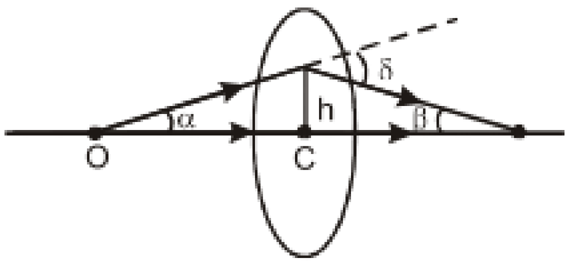
Practice Exercise

Q.1 Two plano–concave lenses of glass of refractive index 1.5 have radii of curvature of 20 and 30 cm. They are placed in contact with the curved surfaces towards each other and the space between them is filled with a liquid of refractive index (4/3). Find the focal length of the system.

Answers

Q.1 72 cm

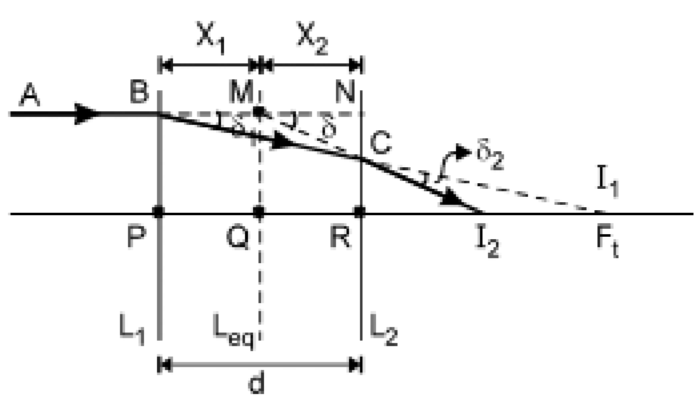
Deviation produced by thin lens.



$$\delta = \alpha + \beta = \frac{h}{-u} + \frac{h}{v} = h\left(\frac{1}{v} - \frac{1}{u}\right) = \frac{h}{f}$$

Combination of two thin spherical lenses separated by some distance

Let L_1 and L_2 are two thin lenses separated at distance 'd' from each other and focal lengths of the lenses are f_1 and f_2 respectively. We have to calculate position, focal length, power of the equivalent lens. The deviation produced by the Leq is the sum of deviations produced by L_1 & L_2.



Let a ray parallel to principal axis of the lens is coming and it is deviated by the first lens by angle d_1 and would from image at F_1 or say l_1 is the absence of L_2 but L_2 deviates it again by angle d_2 before forming image at F_1.

Produce AB by a dotted line and also l_2C. They intersect each other at a point M. Again produce BM such that it intersects L_2 at a point say N.

Draw BP, MQ and NR perpendicular to the principal axis.

Let $BP = h_1$ & $CR = h_2$

$= NR$

$= MQ$

Using $d = d_1 + d_2$ and geometry we get

$$\frac{1}{f_{eq}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \quad \text{i.e.} \quad P_{eq} = P_1 + P_2 - dP_1P_2$$

and position of the equivalent lens :

$$\frac{df_{eq}}{f_1} \text{ left of the second lens.}$$

Note :

The above formula is applicable for only parallel incident rays (object is situated at ∞).

Illustration :

A convergent lens of 6 diopters is combined with a diverging lens of –2 diopter. Find the power and focal length of the combination.

Sol. *Here* $P_1 = 6$ *diopter,* $P_2 = -2$ *diopter*

Power of the combination is given by

Using the formula $P = P_1 + P_2 = 6 - 2 = 4$ *diopters*

$$f = \frac{1}{P} = \frac{1}{4} = 0.25\,\text{m} = 25\,\text{cm}$$

Practice Exercise

Q.1 A convex lens A of focal length 20 cm and a concave lens B of focal length 5 cm are kept along the same axis with a distance d between them. What is the value of d if a parallel beam of light incident on A leaves B as a parallel beam?

Answers

Q.1 15 cm

Lens mirror combination

Concept of image forming at object itself :

In some unique situations the image is formed at the same point as the object. For a lens, if we apply the len formula we get

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

or

$$\frac{1}{u}-\frac{1}{u}=\frac{1}{f}$$

which is not possible.

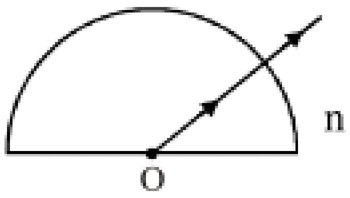
On the other hand, for the case of refraction across a single curved surface, the formula is

$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{(\mu_2-\mu_1)}{R}$$

which reduces to

$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{(\mu_2-\mu_1)}{R}$$

The solution for which is u = R. Therefore the object and image coincide when the object is placed at the center of curvature (figure.)



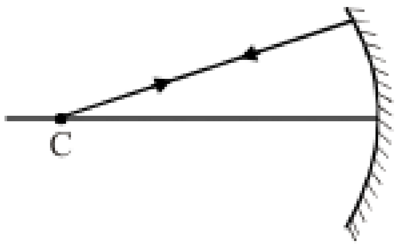
Similarly, in the case of mirrors, the formula is

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{R}$$

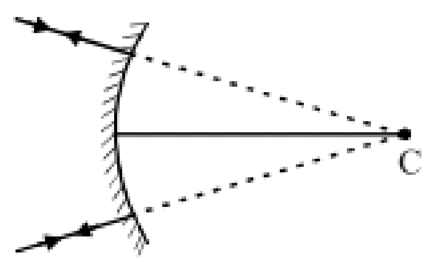
which simplifies to

$$\frac{1}{u}+\frac{1}{u}=\frac{2}{R}$$

and the solution once again is u = R. Therefore for mirrors and refraction across a single curved surface, we can say that object and image will coincide only when the object is kept at the center of curvature.



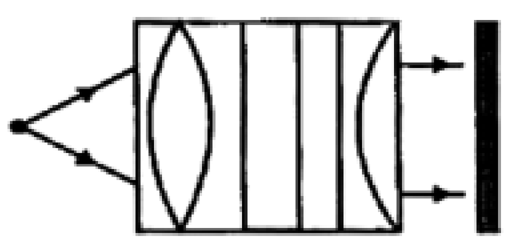
In problems in optics, we will usually have a train of optical elements with the stipulation that the image is formed on the object itself. In such cases, there will have to be a mirror at the end of the optical train and the rays have to be incident normally on the mirror in order to retrace their paths. Three kinds of mirrors are possible.



Case-1 :

A plane mirror at the end of the optical train

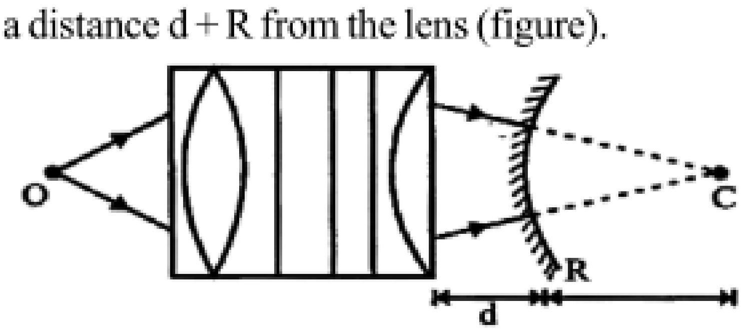
In this situation, the ray of light emerging from the system just before it impinges on the mirror has to be parallel so as to strike the mirror normally. Thus, the image after the last lens must be formed at infinity.



Case-2:

A concvex mirror of radius of curvature R

Here, the bundle of rays must convege on to the centre of curvature which is in front of the concave mirror. If the distance of the last lens from the mirror is d, we can say that the image formed from the last lens must be at a distance d + R from the lens (figure).



Case-3:

A concave mirror of radius of curvature R

Here the bundle of rays must converge on to the center of curvature of the convex of the convex mirror. If the distance of the last lens from the mirror isd, we can say that the image formed from the last lens must be at a distance (d – R) from the lens (figure)

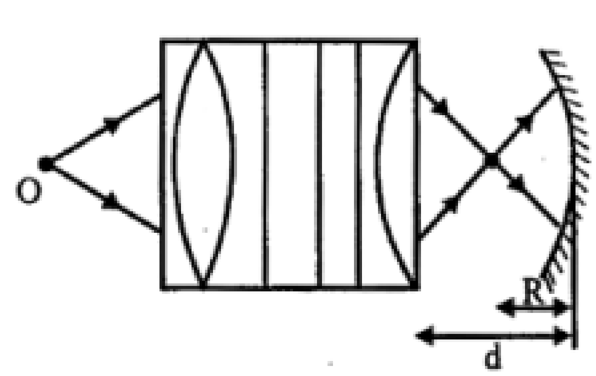


Illustration :

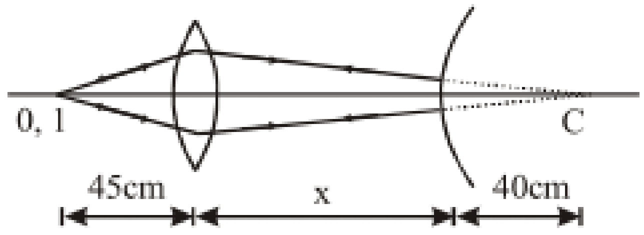
An object is placed at a distance of 45 cm from a converging lens of focal length 30 cm. A mirror of radius of curvature 40 cm is to be placed on the other side of lens so that the object coincides with its image.

Find the position of the mirror if it is

(a) convex ?

(b) concave ?

Sol. *(a) The object and image will coincide only if the light ray retraces its path and it will occur only when the ray normally strike at the mirror. In other words, the centre of curvature of the mirror and the rays incident on the mirror are collinear.*



The rays after refraction from lens must be directed towards the centre of curvature of mirror at C. If x is the separation, then for the lens

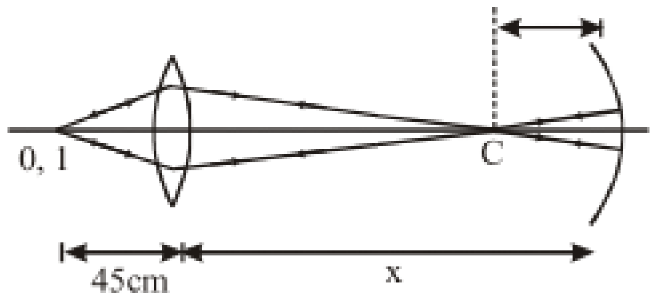
$$u = -45\text{cm}, v = x + 40, f = 30\,\text{cm}$$

Using lens formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

or $\frac{1}{x+40} - \frac{1}{-45} = \frac{1}{30}$

or $x = \frac{45(30)}{45-30} - 40 = 50\,\text{cm}$

(b)



In case of concave mirror, the refracted rays through the lens meets at C, the centre of curvature (C) of the mirror.

Using lens formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

$$u = -45\ cm,\ v = x - 40,\ f = 30\ cm$$

$$\frac{1}{x-10} - \frac{1}{-45} = \frac{1}{30}$$

or $x - 10 = \frac{(45) \times 30}{45-30}$

or $x = 90 + 40 = 130\,\text{cm}$.

Practice Exercise

Q.1 A concave lens of focal length 20 cm is placed 15 cm in front of a concave mirror of radius of curvature 26 cm and further 10 cm away from the lens an object is placed . The principal axis of the lens and the mirror are coincident and the object is on this axis. Find the position and nature of the image.

Q.2 A point object is placed at a distance of 12 cm on the axis of a convex lens of focal length 10 cm. On the other side of the lens, a convex mirror is placed at a distance of 10 cm from the lens such that the image formed by the combination coincides with the object itself. What is the focal length of convex mirror?

Q.3 An object 2 cm high is placed in front of a double convex lens of focal length 12.5 cm. On the other side of the lens a concave mirror of focal length 10 cm is placed at a distance of 45 cm. If the separation between the object and the mirror is 70 cm, calculate the location, nature and magnification of the image.

Answers

Q.1 Final image will be inverted, real and 8 times of the object Q.2 25 cm

Q.3 25 cm, inverted, real and of same size as object, –1

Effect of Silvering A surface of a Thin spherical Lens

If a surface of a thin spherical lens is silvered it behaves like a mirror and we can calculate is focal length. Let a thin spherical lens is polished at the right face. The radii of curvature of the left and right faces are R_1 and R_2.

When a ray of light becomes incident on this silvered lens it will be first refracted by the lens, then reflected from mirror and again refracted by the lens .

Hence power of equivalent mirror can be written as

$$P_{eq} = P_{lens} + P_{mirror} + P_{lens}$$

$$\Rightarrow \qquad P_{eq} = 2P_{lens} + P_{mirror}$$

$$\Rightarrow \qquad \left(-\frac{1}{f_{eq}}\right) = 2\left(\frac{1}{f_{lens}}\right) + \left(-\frac{1}{f_{mirror}}\right)$$

$$\Rightarrow \qquad \frac{1}{f_{eq}} = -\frac{2}{f_{lens}} + \frac{1}{f_{mirror}}$$

Illustration :

One face of an equiconvex lens of focal length 60 cm made of glass $(\mu = 1.5)$ *is silvered. Does it behave like a concave mirror or convex mirror?*

Sol. *here* $f_l = +60$ *cm (converging lens)*

$f_m = -30$ *(converging mirror)*

$$\Rightarrow \quad \frac{1}{f_{eq}} = -\frac{2}{f_{lens}} + \frac{1}{f_{mirror}} = -\frac{2}{+60} + \frac{1}{-30}$$

$$\therefore \quad f_{eq} = -15$$

The positive sign indicates that the resulting mirror is converging or concave.

Practice Exercise

Q.1 A thin hollow equiconvex lens, silvered at the back, converges a parallel beam of light at a distance of 0.2 m in front of it. Where will it converge the same light if filled with water having $m = 4/3$?

Q.2 A plano convex lens of refractive index 1.5 and radius of curvature 30 cm is silvered at the curved surface. Now this lens has been used to form the image of an object. At what distance from this lens, an object be placed in order to have a real image of the size of the object ?

Q.3 A pin is placed 10 cm infront of a convex lens of focal length 20 cm, made of material having refractive index 1.5. The surface of the lens further away from the pin is silvered and has a radius of curvature 22 cm. Determine the position of the final image. Is the image real or virtual?

Answers

Q.1 –12 cm Q.2 20 cm Q.3 –11 cm and real

Optical defects of mirrors & lenses

In the formation of image we have considered :

(i) Incident rays are paraxial

(ii) Incident rays are monochromatic

But practically all these points are not perfectly correct hence image is defected. The defects (aberrations) are classified broadly in two parts.

Aberrations

Chromatic aberration

When the rays used in the formation of image having more than one wavelength then the defect in image only due to presence of more than one wavelength.

Monochromatic aberration

When the rays taken in image formation are monochromatic, then the defect produced is called monochromatic aberration. This defect is also called seidel aberration.

Spherical aberration

The defect in image produced in the formation of image of an axial point object (of monochromatic light) by a spherical mirror or lens is called spherical aberration.The image of an object in point object formed by a spherical mirror or by a spherical lens is usually blurred. This defect of image is called spherical aberration

Methods to reduce spherical aberration :

A. For mirrors : By using a proper surface e.g., paraboloidal surface for parallel incident beam.

B. For lenses : In lenses spherical aberration cannot be completely vanished. It can be minimized only.

(i) By using stops.

(ii) By using crossed lens.

Note : for minimum spherical aberration.

$$\frac{R_1}{R_2} = \frac{2\mu^2 - \mu - 4}{\mu(2\mu + 1)}$$

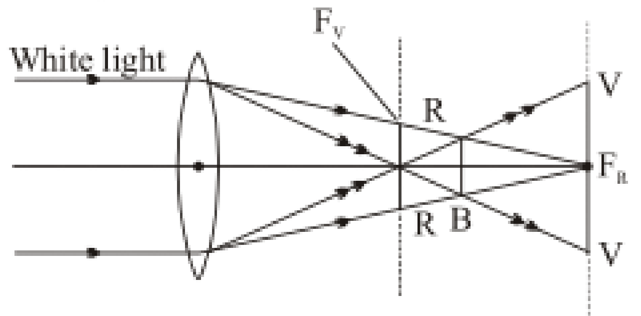
(iii) By using combination of lenses, $d = f_1 - f_2$.

Chromatic aberration

The image of an object in white light formed by a lens is usually coloured and blurred. This defect of image is called chromatic aberration and arises due to the fact that focal length of a lens is different for different colours. For a single lens,

$$\frac{1}{f} = (\mu - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

and an μ of lens in maximum for violet while minimum for red, violet is focused nearest to the lens while red farthest from it as shown in figure.



As a result of this in case of convergent lens, at F_V centre of image will be violet and focused while sides red and blurred while at F_R reverse is the case, i.e., centre will be red and focused while sides violet and blurred. The difference between f_V and f_R is a measure of longitudinal chromatic aberration, i.e.,

$$\text{L.C.A.} = f_R - f_V = -df$$

with $\quad df = f_V - f_R \qquad \text{... (1)}$

However, as for a single lens,

$$\frac{1}{f} = (\mu - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \qquad \text{... (2)}$$

i.e.,
$$-\frac{df}{f^2} = d\mu\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \qquad \text{... (3)}$$

So dividing equation (3) by (2)

$$-\frac{df}{f} = \frac{d\mu}{(\mu-1)} = \omega\left[\text{as } \omega \frac{d\mu}{(\mu-1)}\right] \qquad \text{... (4)}$$

And hence, from equation (1) and (4),

$$\text{L.C.A.} = -df = \omega f \qquad \text{... (5)}$$

Now, as for a single lens neither f nor ω can be zero, we cannot zero, we cannot have a single lens free from chromatic aberration.

Condition of Achromatism

In case of two thin lenses in contact

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} \qquad \text{i.e.,} \qquad -\frac{dF}{F^2} = -\frac{df_1}{f_1^2} - \frac{df_2}{f_2^2}$$

The combination will be free from chromatic aberration if $dF = 0$

i.e.,
$$\frac{df_1}{f_1^2} + \frac{df_2}{f_2^2} = 0$$

which in the light of equation (5) reduces to

$$\frac{\omega_1 f_1}{f_1^2} + \frac{\omega_2 f_2}{f_2^2} = 0$$

i.e.,
$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0 \qquad \text{... (6)}$$

This condition is called condition of achromatism (for two thin lenses in contact) and the lens combination which satisfies this condition achromatic lens. From this condition, i.e., from equation (6) it is clear that in case of achromatic doublet :

(1) The two lenses must be of different materials.

Since, if $\omega_1 = \omega_2$, $\frac{1}{f_1} + \frac{1}{f_2} = 0$, i.e., $\frac{1}{F} = 0$ or $F = \infty$

i.e., combination will not behave as a lens, but as a plane glass plate.

(2) As ω_1 and ω_2 are positive quantities, for eq. (6) to hold, f_1 and f_2 must be of opposite nature, i.e. if one of the lenses is convex the other must be concave.

(3) If the achromatic combination is convergent,

$f_C < f_D$ and as $-\frac{f_C}{f_D} = \frac{\omega_C}{\omega_D}$, $\omega_C < \omega_D$

i.e., a convergent achromatic doublet, convex lens has lesser focal length and dispersive power than divergent one.

(ii) For lenses separated by a distance : $d = \frac{\omega_2 f_1 + \omega_1 f_2}{\omega_1 + \omega_2}$

Practice Exercise

Q.1 An optical doublet is formed from two lenses A and B made of glass of different refractive indices μ_A, μ_B respectively. Lens A has two convex sides of radius of curvature R and lens B has one flat side and one concave side of radius of curvature R. What is the power of the doublet? For red, yellow and blue wavelengths, the refractive index μ_A is 1.50, 1.51 and 1.52 respectively whereas μ_B is 1.60, 1.62 and 1.64 respectively. What is the difference in power of the doublet for these three wavelengths?

Q.2 Two lenses, one made of crown glass and the other of flint glass, are to be combined so that the combination is achromatic for the blue and red light and acts as a convex lens of focal length 35 cm. Calculate the focal length of the components if for –

Crown glass	$\mu_Y = 1.5175$ and $(\mu_B - \mu_R) = 0.00856$
Flint glass	$\mu_Y = 1.6214$ and $(\mu_B - \mu_R) = 0.01722$

Q.3 An equiconvex lens of crown glass and an equiconcave lens of flint glass make an achromatic system. The radius of curvature of convex lens is 0.54 m and the refractive indices for the crown glass are $\mu_R = 1.53$ and $\mu_V = 1.55$, find the dispersive power of flint glass.

Q.4 A telescope objective of focal length 60 cm is made of two thin lenses, one of crown glass of refractive index 1.52 and other of flint glass refractive index 1.66. One surface of the flint glass is plane. Calculate the radii of curvature of both the lenses which form the achromatic doublet if dispersive powers of crown and flint glass are 0.0151 and 0.0302 respectively.

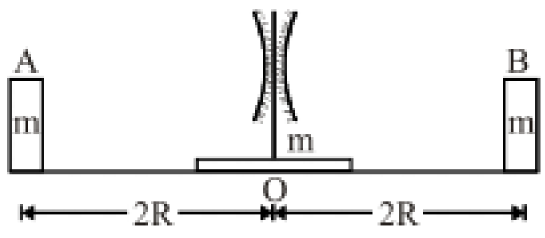
Answers

Q.1 $\frac{1}{R}[2\mu_A - \mu_B - 1]$, 0 Q.2 $f_C = 14.17$ cm & $f_D = 23.8$ cm Q.3 0.055

Q.4 The radii of curvature of convex lens (of crown glass) are 25.74 cm and 39.6 cm respectively while of concave lens (of flint glass) are 39.6 cm and ∞ respectively

Solved Example

Q.1 Two concave mirrors of equal radii of curvature R are fixed on a stand facing opposite directions. The whole system has a mass m and is kept on a frictionless horizontal table (figure).



Two block A and B, each of mass m, are placed on the two sides of the stand. At $t = 0$, the separation between A and the mirror is 2R and also the separation between B and the mirror is 2R. The block B moves towards the mirror at a speed v. All collisions which take place are elastic. Taking the original position of the mirrors standard system to be $x = 0$ and x-axis along AB, find the position of the images of A and B at

(a) $t = \frac{R}{v}$, (b) $t = \frac{3R}{v}$ (c) $t = \frac{5R}{v}$

Sol. (a) At $t = \frac{R}{v}$

For block A, $u = -2R$

$$\therefore \quad \frac{1}{v} + \frac{1}{-2R} = \frac{2}{-R}$$

or $v = \frac{-2R}{3}$

For block B : The distance travels by block B in time $\frac{R}{v}$ is R

Thus $u = -R$

$$\frac{1}{v} + \frac{1}{-R} = \frac{2}{-R}$$

or $v = R$

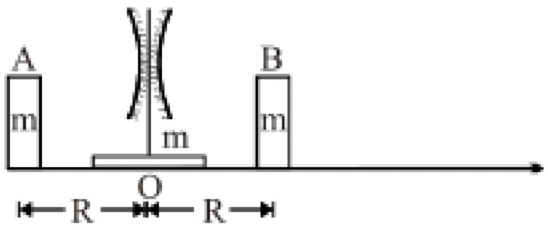
The x-coordinate of the image of the block with respect to the mirror will be +R.

(b) At $t = \frac{3R}{v}$

The block B will collide with the stand after time $\frac{2R}{v}$

After collision block B becomes at est and mirror starts moving with the same velocity v. In the remaining time R/v, the distance moved by the mirror is R.

The position of blocks and mirror are shown in figure.



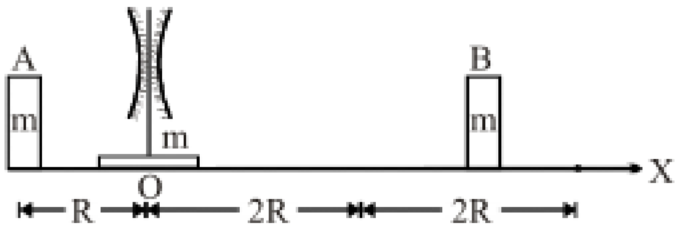
At this time the blocks lie at the centre of curvature of the respective mirrors. Their images will form at the centres of cuvature. So their co-ordinates are :

For block A, $x = -R$

For block B, $x = +R$

(c) At $t = \frac{5R}{v}$.

The block B will collide to the mirror after a time $\frac{2R}{v}$. Thereafter mirror starts moving towards block **A** with velocity v. At $t = \frac{4R}{v}$, the mirror will collide with block A and stops after collision. The positions of blocks and mirror are shown in figure.



For block A : Its image will form on the same place. Therefore the positions of the blocks are

$$x_A = -3R$$

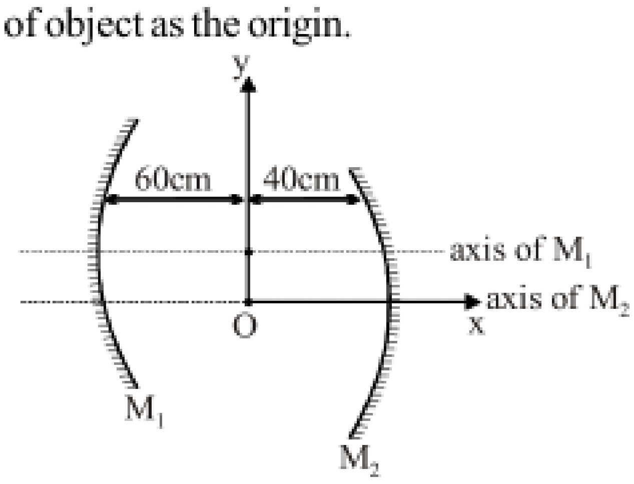
For block B : $u = -2R$

$$\frac{1}{v} + \frac{1}{-2R} = \frac{2}{-R}$$

$$v = -\frac{2R}{3}$$

The co-ordinates of B are $-\left(2R - \frac{2R}{3}\right) = \frac{-4R}{3}$

Q.2 Two concave mirrors each of radius of curvature 40 cm are placed such that their principal axes are parallel to each other and at a distance of 1cm to each other. Both the mirrors are at a distance of 100 cm to each other. Consider first reflection at M_1 and then at M_2, find the coordinates of the image thus formed. Take locatioin of object as the origin.



Sol. Using mirror formula for first reflection :

$$\frac{1}{f}=\frac{1}{v}+\frac{1}{u}\Rightarrow\frac{1}{-20}=\frac{1}{v}+\frac{1}{-70}$$

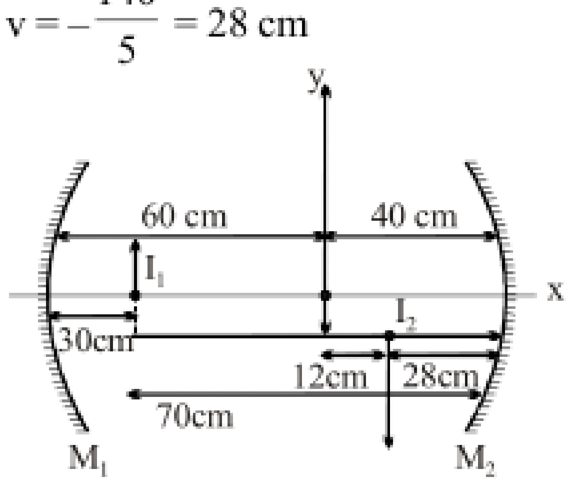
$$\Rightarrow\quad\frac{1}{v}=\frac{1}{60}-\frac{1}{20}\Rightarrow v=-30\text{ cm}$$

Using mirror formula for second reflection

$$\frac{1}{f}=\frac{1}{v}+\frac{1}{u}\Rightarrow\frac{1}{-20}=\frac{1}{v}+\frac{1}{-70}$$

$$\Rightarrow\quad\frac{1}{v}=\frac{1}{70}\quad\frac{1}{20}=\frac{2-7}{140}$$

$$\Rightarrow\quad v=-\frac{140}{5}=28\text{ cm}$$



Height of $I_2 \Rightarrow m=\left(\frac{-30}{-60}\right)=\frac{I_1}{-1}$

$\Rightarrow I_1=\frac{1}{2}$ cm

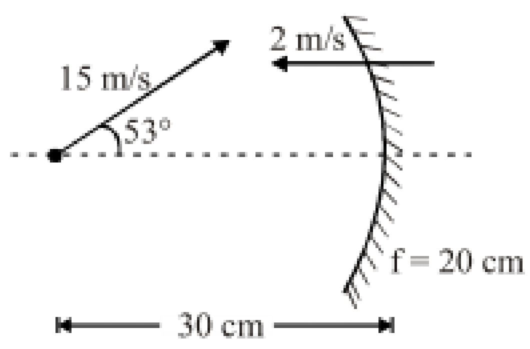
Height of first image from x-axes $=1+\frac{1}{2}=\frac{3}{2}$ cm

Height of $I_2 \Rightarrow m=\left(\frac{-28}{-70}\right)=\frac{2I_2}{3}$

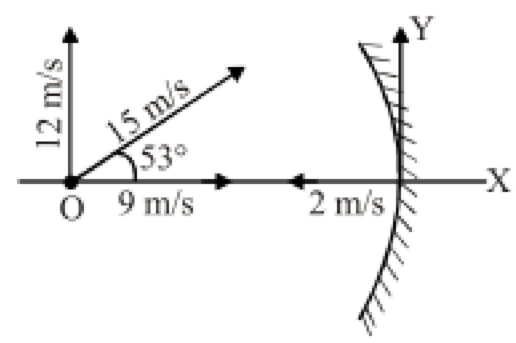
$\Rightarrow I_2=\frac{3\times 28}{2\times 70}$ I2 = 0.6 cm

Co-ordinate of $I_2=(12-0.6)$

Q.3 Find the velocity of image in situation as shown in figure.



Sol. $\vec{V}_O$ = Velocity of obeject = $(9\hat{i} + 2\hat{j})\,m/s$



$\vec{V}_m$ = Velocity of mirror $= -2\hat{i}$ m/s

$$m = \frac{f}{f-u} = \frac{-20}{-20-(-30)} = -2$$

For velocity component parallel to optical axis

$$(\vec{V}_{I/m})_{||} = -m^2\,(\vec{V}_{O/m})_{||}$$

$$(\vec{V}_{I/m})_{||} = (-2)^2\,11\,\hat{i} = -44\,\hat{i} \text{ m/s}$$

For velocity component perpendicular to optical axis

$$(\vec{V}_{I/m})_{\perp} = (\vec{V}_{O/m})_{\perp}$$

$$= (-2)\,12\,\hat{j} = -24\,\hat{j} \text{ m/s}$$

$\therefore$ $\vec{V}_{I/m}$ = Velocity of image w.r.t. mirror

$$= (\vec{V}_{I/m})_{||} + (\vec{V}_{I/m})_{\perp}$$

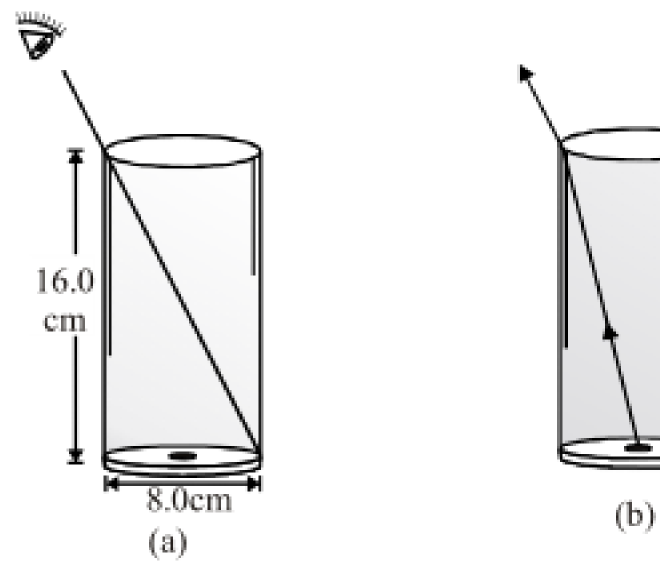
$$= (-44\,\hat{i} - 24\,\hat{j})\,m/s$$

Also, $\vec{V}_{I/m} = \vec{V}_I - \vec{V}_m$

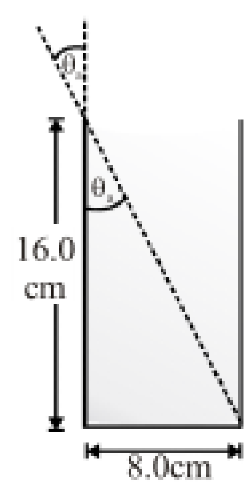
or $\vec{V}_I = (-44\hat{i} - 24\hat{j}) - 2\hat{i}$

$$= (-46\hat{i} - 24\hat{j})\,m/s$$

Q.4 You sight along the rim of a glass with vertical sides so that the top rim is lined up with the opposite edge of the bottom. The glass is a thin-walled, hollow cylinder 16.0 cm high with a top and bottom of the glass diameter of 8.0 cm. While you keep your eye in the same position, a friend fills the glass with a transparent liquid, and you then see a dime that is lying at the center of the bottom of the glass. What is the index of refraction of the liquid ?



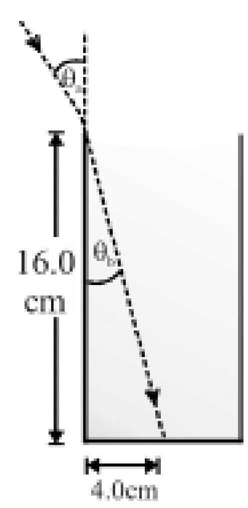
Sol. Use geometry of find the angles of incidence and refraction. Before the liquid is poured in the ray along your line of sight has the path shown in figure.



$$\tan\theta_a = \frac{8.0\text{cm}}{16.0\text{cm}} = 0.500$$

$$\theta_a = 26.57^o$$

After the liquid is poured in, θ_a is the same and the refracted ray passes throught the center of the bottom of the glass as shown in figure.



$$\tan\theta_b = \frac{4.0\text{cm}}{16.0\text{cm}} = 0.250$$

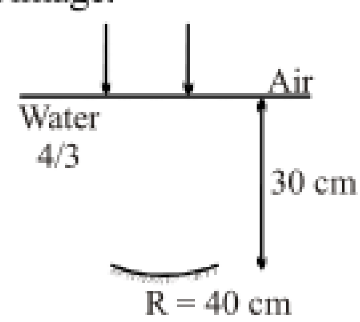
$$\theta_b = 14.04^{\circ}$$

Use Snell's law to find n_b the refractive index of the liquid:

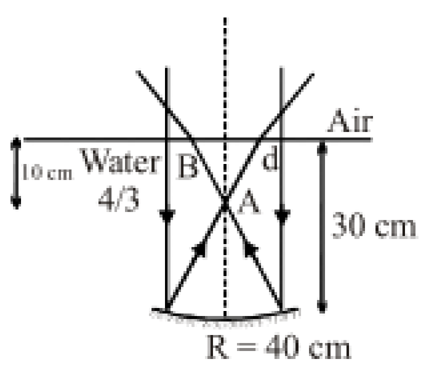
$$n_a \sin\theta_a = n_b \sin\theta_b$$

$$n_b = \frac{n_a \sin\theta_a}{\sin\theta_b} = \frac{(1.00)(\sin 26.57^{\circ})}{\sin 14.04} = 1.84$$

Q.5 A concave mirror is placed inside water with its shining surface upwards and principal axis of concave mirror. Find the position of final image.

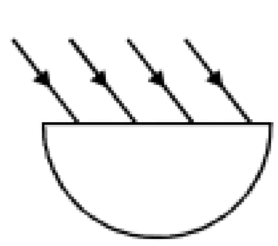


Sol. The incident rays will pass undeviated through the water surface and strike the mirror parallel to its principal axis. Therefore for the mirror, object is at ∞. Its image A (in figure) will be formed at focus which is 20 cm from the mirror. Now for the interface between water and air, d = 10 cm.



$$\therefore \qquad d' = \frac{d}{\left(\frac{n_w}{n_a}\right)} = \frac{10}{\left(\frac{4/3}{1}\right)} = 7.5 \text{ cm}$$

Q.6 Rays of light fall on the plane surface of a semicylinder of refractive index $n = \sqrt{2}$, at angle 45° in the plane normal to the axis of cylinder. Discuss the condition that the rays do not suffer total internal reflection.

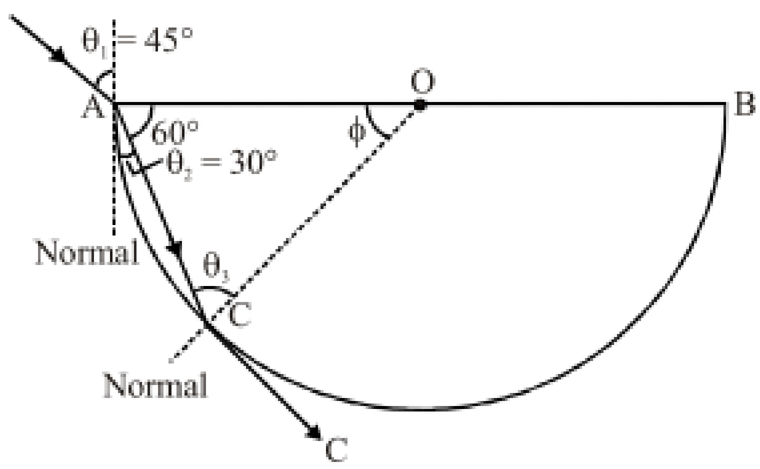


Sol. First we consider a ray incident at A. From Snell's law,

$$1 \sin 45^{\circ} = \sqrt{2} \sin\theta_2$$

$$\sin\theta_2 = \frac{1}{2}$$

$$\theta_2 = 30^{\circ}$$



Let the angle $\phi = \angle AOC$ denote the position of the point C on the curved surface.

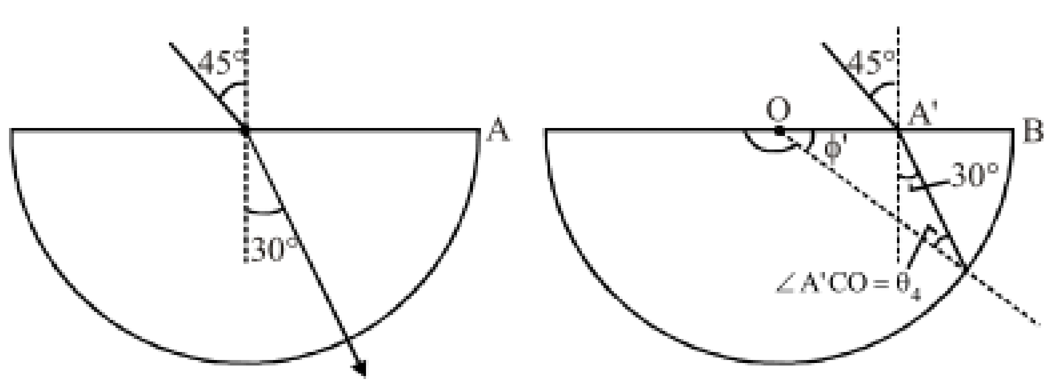
$$\angle CAO = 60°$$

The critical angle for glass to air interface can be determined from Snell's law.

$$n \sin C = 1 \sin 90°$$

$$\sin C = \frac{1}{n} = \frac{1}{\sqrt{2}}$$

$$C = 45°$$



If total internal reflection has to take place at the curved surface, angle θ_3 must be greater than the critical angle, $C = 45°$.

As $\theta_3 = 180° - \phi - 60°$, therefore, for no total internal reflection,

$$180° - \phi - 60° < 45°$$

or $\quad \phi > 75°$

When the ray falls at O, the refracted ray will move radially out, without deviation. The normal rays do not suffer deviation. Next we consider a ray to the right of O.

For no total internal reflection,

$$\angle A'CO < 45°$$

In $\Delta OA'C$, $\angle OAA' + \angle A'CO + \angle COA' = 180°$

$$120° + \theta_4 + (180 + \phi) = 180°$$

$$\theta_4 = \phi - 120°$$

Thus, $\quad \phi - 120° < 45°$

$$\phi < 165°$$

Hence for rays to transmit through curved surface,

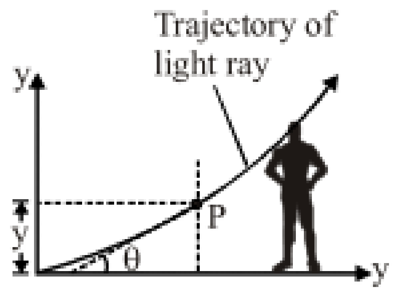
$$75° < \phi < 165°$$

Q.7 Due to a vertical temperature gradient in the atmosphere the index of refraction varies. Suppose index of refraction varies as $n = n_0\sqrt{1+ay}$ where n_0 is the index of refraction at the surface and $a = 2.0\times10^{-6}\ m^{-1}$. A person of height $h = 2.0$ m stands on a level surface. Beyond what distance he cannot see the runway ?

Sol. Let O be the distant object just visible to the man. Let P be a point on the trajectory of the ray. From figure, $\theta = 90 - i$.

The slope of tangent at point P is $\tan\theta = dy/dx = \cot i$. From Snell's law, $n \sin i$ = constant

At the surface $n = n_0$ and $i = 90°$



$$n_0 \sin 90° = n \sin i = (n_0\sqrt{1+ay})\ \sin i$$

$$\sin i = \frac{1}{\sqrt{1+ay}}$$

$$\cot i = \frac{dy}{dx} = \sqrt{ay}$$

$$\int_0^y \frac{dy}{\sqrt{ay}} = \int_0^x dx$$

$$x = 2\sqrt{\frac{y}{a}}$$

On substituting $y = 2.0$ m and $a = 2\times10^{-6}\ m^{-1}$, we have

$$x_{max} = 2\sqrt{\frac{2}{2\times10^{-6}}} = 2000\ m$$

Q.8 A ray of light passes through an equilateral prism such that the angle of incidence and the angle of emergence are both equal to 3/4th of the angle of prism. Find the angle of minimum deviation .

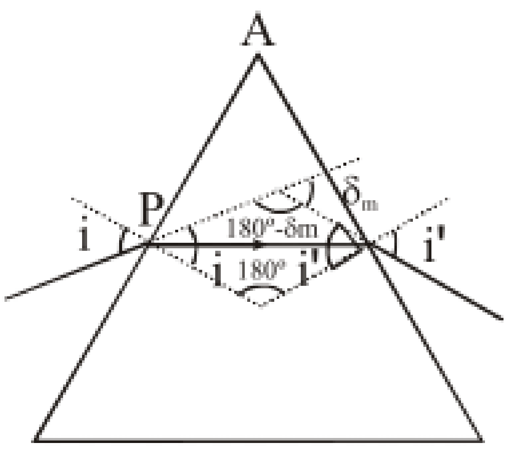
Sol. Givne $A = 60°$

$$i = i' = \frac{3}{4}A = 45°$$

$\because$ $i + i' = A + \delta$

or $90° = 60° + \delta$

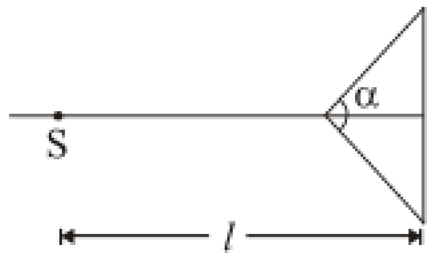
$\therefore$ $\delta = 30°$

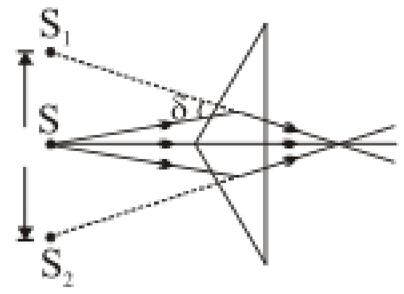


Note that $i = i'$ is the condition for minimum deviation.

Hence $\delta = 30° = \delta_{min}$.

Q.9 A thin biprism (figure) of obtuse angle $\alpha = 178°$ is placed at a distance $l = 20$ cm from a slit. How many images are formed and what is the separation between them ? Refractive index of he material $\mu = 1.6$.





Sol. Two image are formed by the two thin prisms-one above the axis and the oher below the axis by the same distance. The fracting angle of each thin prism $= \frac{\pi - \alpha}{2} = \frac{1}{2}(\pi - \alpha)$ where α is the obtuse angle in radian.

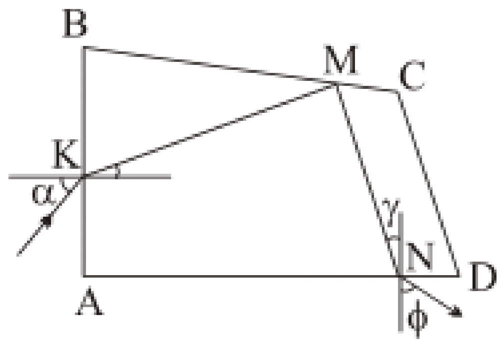
Then δ (deviation of a ray) $= (\mu - 1)\, \frac{1}{2}\, (\pi - \alpha)$

$\therefore \frac{d}{2} = l\delta$

or $\quad d = 2l(\mu - 1)\, \frac{1}{2}(\pi - \alpha)$

Here $\quad d = (1.6 - 1) \times 0.20 \left(\pi - 178 \times \frac{\pi}{180}\right) = 0.6 \times 0.20 \times \pi \times \frac{1}{90} = 0.004 \text{ m} = 4 \text{ mm}$

Q.10 The faces of prism ABCD made of glass with a refraction index n form dihedral angle: $\angle A = 90°$, $\angle B = 75°$, $\angle C = 135°$ and $\angle D = 60°$ (the Abbe prism). A beam of light falls on face AB and after complete internal reflection from face BC escapes through face AD. Find the angle of incidence α of the beam onto face AB if a beam that has passed through the prism is perpendicular to the incident beam.

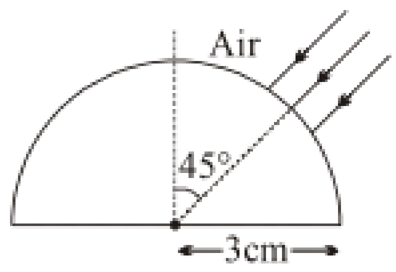


Sol. According to the initial condition, the incident beam and the beam that has passed through the prism are mutually perpendicular. Therefore, $\angle\phi = \angle\alpha$ and also $\angle\gamma = \angle\beta$ (figure). The sum os the angles of the quadrangle AKMN is 360°. Therefore, $\angle KMN = 90°$ and beam KM is incident on to face BC at an angle of 45°. If we know the angles of triangle KBM, it is easy to find that $b = 30°$. In confomity with the law of refraction, $\frac{\sin\alpha}{\sin\beta} = n$

Hence, $\quad \sin\alpha = 0.5\, n$ and arc sin $0.5\, n$

Since full internal reflection at an angle of 45° is observed only when $n \geq \sqrt{2}$, the angle a is within $45° \leq a \leq 90°$.

Q.11 Shows a transparent hemisphere of radius 3.0 cm made of ia material of refractie index 2.0 :



(a) A narrow beam of parallel rays is incident ofthe hemisphere as shown in figure. Are the rays totally reflected at plane surface ?

(b) Find the image formed by refraction at the first surface.

(c) Find the image formed by the reflection or by refraction at the plane surface.

Sol. (a) The critical angle for material air interface

$$\sin C = \frac{1}{\mu} = \frac{1}{2}$$

$\therefore$ C = 30°

The rays are incident normally on the spherical surface, so they pass undeviated and then incident of plane face at an angle 45°. As the angle of incidence is greater than critical angle (30°), so rays get totally reflected.

(b) For spherical surface :

$$u = \infty$$

We have $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$

or $\frac{2}{v} - \frac{1}{\infty} = \frac{2-1}{R}$

$\therefore$ $v = 2R$

Thus the image will form on diametrically opposite point.

(c) Some of the rays get totally reflected and so they will form the image at I_2.

Q.12 (a) A ray of light suffers an internal reflection inside a water drop. Find the condition for minimum deviation, the angle of incidence at minimum deviation and the value of minimum deviajtion.

(b) A source and a screen are held fixed at a distance l from each other] A thin lens is placed between them such that the source is foucused on the screen. For, what values of focal length of the lens there are one, two or no positions for the lens ?

Sol. $D(\text{deviation}) = (i - \theta) + (\pi - 2\theta) + (i - \theta) = \pi + 2i - 4\theta$

$$\frac{dD}{di} = 2 - 4\,\frac{d\theta}{di}$$

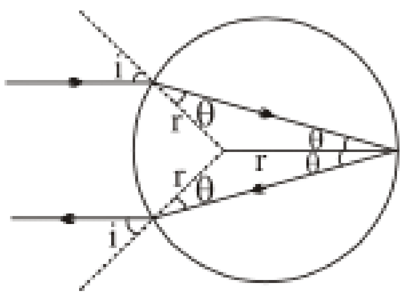
By Snell's law, $\mu \sin\theta = \sin I$

Differentiating w.r.t. 'i' we get,

$$\mu \cos\theta \,\frac{d\theta}{di} = \cos i$$

$$\Rightarrow \quad \frac{dD}{di} = 2 - 4\left[\frac{\cos i}{\mu\cos\theta}\right]$$

When D is minimum $\dfrac{dD}{di} = 0$



$$\Rightarrow \quad 2 - \frac{4\cos i}{\mu\cos\theta} = 0$$

$$2 = \frac{4\cos i}{\mu\cos\theta}$$

$$\mu\cos\theta = 2\cos i$$

$$\mu \cdot \sqrt{1-\sin^2\theta} = 2\cos i$$

$$\mu \cdot \sqrt{1-\frac{\sin^2 i}{\mu^2}} = 2\cos i$$

$\Rightarrow \quad \mu^2 - \sin^2 i = 4\cos^2 i = 4[1 - \sin^2 i]$

$\Rightarrow \quad \mu^2 = 4 - 3\sin^2 i$

$$\sin i = \sqrt{\frac{4-\mu^2}{3}}$$

$$\Rightarrow \quad i = \sin^{-1}\sqrt{\frac{4-\mu^2}{3}}$$

$$D_{min} = \pi + 2\sin^{-1}\sqrt{\frac{4-\mu^2}{3}} - \sin^{-1}\frac{1}{\mu}\sqrt{\frac{4-\mu^2}{3}}$$

Q.13 A lens has a power of +5 diopter in air. What will be its power if completely immersed in water ? Given

$$\mu_g = \frac{3}{2};\ \mu_w = \frac{4}{3}$$

Sol. Let f_a and f_w be the focal lengths of the lens in air water respectively, then

$$P_a = \frac{1}{f_a} \qquad \text{and} \qquad P_w = \frac{\mu_w}{f_w}$$

$$f_a = 0.2\text{ m} = 20\text{ cm}$$

Using lensmaker's formula

$$P_a = \frac{1}{f_a} = (\mu_g - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \qquad \text{...(i)}$$

$$\frac{1}{f_w} = \left(\frac{\mu_g}{\mu_w} - 1\right)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

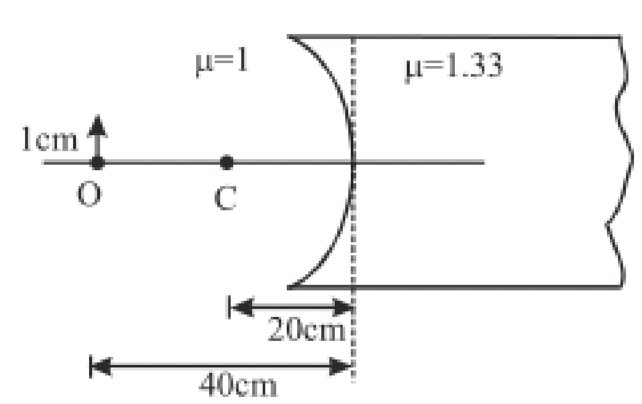
$$\Rightarrow \qquad P_w = \frac{\mu_w}{f_w} = (\mu_g - \mu_w)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \qquad \text{...(ii)}$$

Dividing equation (ii) by equation (i), we get,

$$\frac{P_w}{P_a} = \frac{(\mu_g - \mu_w)}{(\mu_g - 1)} = \frac{1}{3}$$

or $$P_w = \frac{P_a}{3} + \frac{+5}{3}\text{D}$$

Q.14 For the optical arrangement shown in the figure.



Sol. *According to Cartesian sign convention*

$$u = -40\ cm, \quad R = -20\ cm$$

$$\mu = 1,\ \mu_2 = 1.33$$

Applying equation for refraction through spherical surface, we get

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

$$\frac{1.33}{v} - \frac{1}{-40} = \frac{1.33-1}{-20}$$

After solving, $v = -32\ cm.$

The magnification is $m = \frac{h_2}{h_1} = \frac{\mu_1 v}{\mu_2 u}$

$$\therefore \quad \frac{h_2}{1} = -\frac{1(32)}{1.33(-40)} \quad or \quad h_2 = 0.6\,\text{cm}$$

The positive sign shows that the image is erect.

Q.15 A glass slab of thickness 3cm and refractive index 1.5 is placed in front of a concave mirror of focal length 20 cm. Where should a point object be placed if it is to image on to itself? The glass slab and the concave mirror are shown in figure.

Sol. Let the distance of the object from the mirror be x. We know that the slab simply shifts the object. The shift being equal to

$$s = t\left[1 - \frac{1}{\mu}\right] = 1\text{cm}$$

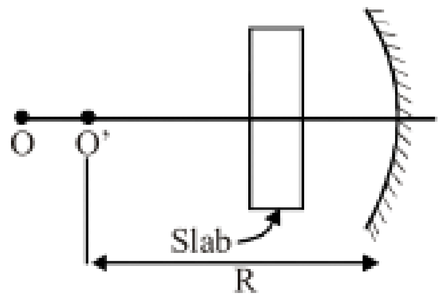
The direction of shift is towards the concave mirror.

$\therefore$ the apparent distance of the object from the mirror is $x - 1$.

If the rays are to retrace their paths, the object should appear to be at the center of curvature of the mirror.

$$\therefore x - 1 = 2f = 40\ \text{cm}$$

or $\quad x = 41$ cm from the mirror.



Q.16 An thin equiconvex lens of glass ($\mu = 1.5$) having a focal length of 30 cm in air is placed at a distance of

10 cm from a plane mirror, which in turn, is placed with its plane perpendicular to the optic axis of the lens. Water ($\mu = 4/3$) fills the space between the lens and the mirror. A parallel beam of light is incident on to the lens parallel to the principal axis.

(a) Find the position of the final image w.r.t. the optical centre of the lens.

(b) If the mirror is rotated by 1°, as shown in the figure, find the displacement of the image.

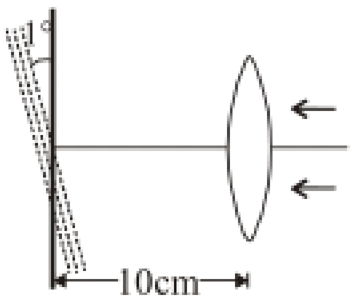
Sol. (a) The focal length of the glasslens is 30 cm

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{2(\mu - 1)}{R}$$

The radius of curvature, R = 30 cm

The object distance, m = ∞.

Now we apply Gauss's law at surface S_1 ans S_2.



$$\frac{3/2}{v_1} - \frac{1}{u} = \frac{\frac{3}{2} - 1}{30} \qquad .... (1)$$

$$\frac{\frac{4}{3}}{v} - \frac{\frac{3}{2}}{v_1} = \frac{-1/6}{-30} \qquad ... (2)$$

$\Rightarrow v = 60$ cm

After reflection from the mirror, the light rays appear to converge to a point 40 cm to the right of the convex lens. This serves as a virtual object for the lens : u = + 40 cm

$$\frac{\frac{3}{2}}{v_1} - \frac{\frac{4}{3}}{+40} = \frac{\frac{1}{6}}{30}$$

$$\frac{1}{v} - \frac{\frac{3}{2}}{v_1} = \frac{-\frac{1}{2}}{-30}$$

$\Rightarrow v = 18$ cm to the right of the convex lens.

(b) If the mirror is rotated by 1°, the reflected ray rotaties by 2°. The virtual object for the lens formed by the reflection from the mirror is displaced by:

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$$\Delta y_1 = 50 \times \frac{2\pi}{180}\text{cm}$$

The magnification due to the refraction at the two surfaces of the lens is

$$m = m_1 m_2 = \left(\frac{v}{\mu_3} \bigg/ \frac{v_1}{\mu_2}\right) \times \left(\frac{v_1}{\mu_2} \bigg/ \frac{u}{\mu_1}\right) = \left(\frac{v}{\mu_3} \bigg/ \frac{u}{\mu_1}\right) = \frac{18}{1} \bigg/ \frac{40}{(4/3)} = \frac{18}{30}$$

The displacement of the final image is

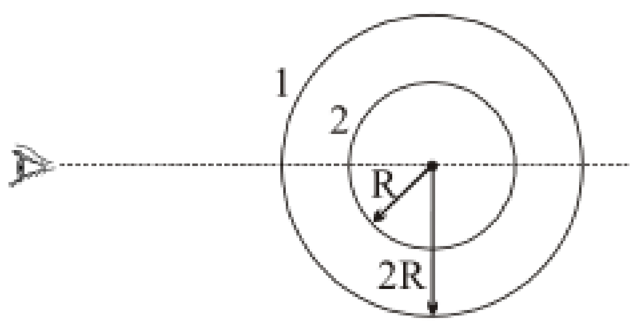
$$\frac{18}{30} \times 50 \times \frac{2\pi}{180}\text{cm} = \frac{\pi}{3}\text{cm}$$

Q.18 A hollow sphere of glass of inner and outer radii R and 2R respectively has a small mark on its inner surface. This mark is observed from a point outside the sphere such that the centre of the sphere lies in between. Prove that the mark will appear nearer than it really is, by a distance $\frac{(\mu-1)R}{(3\mu-1)}$, where R is the radius of the inner surface.

Sol. Refraction at surface 2,

$$\frac{\mu}{v} + \frac{1}{2R} = \frac{\mu - 1}{-R} \qquad \text{or} \qquad \frac{\mu}{v} = -\frac{1}{R}\left[(\mu-1) + \frac{1}{2}\right] = \frac{1}{R}\left[\frac{2\mu-1}{2}\right] = \frac{-(2\mu-1)}{2R}$$

or $$v = -\left[\frac{2\mu R}{2\mu - 1}\right]$$



For surface 1

$$u = -\left(R + \frac{2\mu R}{2\mu-1}\right) = -\left(\frac{4\mu-1}{2\mu-1}\right)R$$

$$\frac{1}{v} + \frac{\mu(2\mu-1)}{(4\mu-1)R} = \frac{1-\mu}{-2R}$$

$$\frac{1}{v} = -\frac{1}{R}\left[\frac{1-\mu}{2} + \frac{\mu(2\mu-1)}{(4\mu-1)}\right] = -\frac{1}{R}\left[\frac{4\mu - 1 - 4\mu^2 + \mu + 4\mu^2 - 2\mu}{2(4\mu-1)}\right] = -\frac{1}{R}\left[\frac{3\mu-1}{2(4\mu-1)}\right]$$

or $$v = -\frac{2R(4\mu - 1)}{(3\mu - 1)}$$

$\therefore$ Distance between the final image and object,

$$3R - \frac{2R(4\mu - 1)}{(3\mu - 1)} = R\left[\frac{9\mu - 3 - 8\mu + 2}{3\mu - 1}\right] = \frac{(\mu - 1)R}{(3\mu - 1)}$$

Wave Optics

Introduction

In Gemetrical Optics we studied light rays passing through a lens or reflecting from a mirror to describe the formation of image. In this chapter, we are concerned with wave optics or physical optics, the study of interference & diffraction, These phenomena cannot be adequately explained with the ray approximation used. We now learn how treating light as waves rather then as rays leads to a satisfying description of such phenomena.

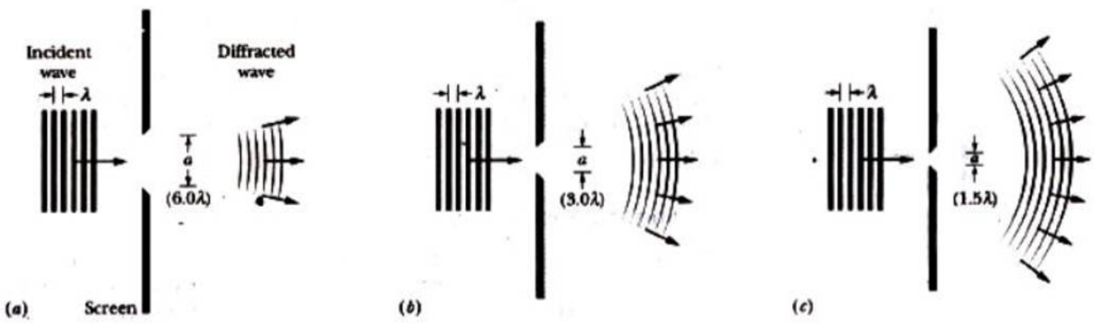
Rainbow shows all the seven colors of visible light, but that is due to dispersion. Whereas, an oil film floating on water also shows seven colors, but the color in the oil film is due to interference of light in it. You might have seen the same in case of soap bubble.

In previous chapter, we considered light to be travelling in a straight line path. However, we know that light is an electromagnetic wave. Thus, it should exhibit wave characteristic as well. And how a soap film shows seven colors can be explained using this wave phenomenon.

Diffraction

In the next section we shall discuss the experiment that first proved that light is a wave. To prepare for that discussion, we must introduce the idea of diffraction of waves, a phenomenon that we explore much more fully in later stage. Its essence is this: *If a wave encounters a barrier that has an opening of*

more fully in later stage. Its essence is this: *If a wave encounters a barrier that has an opening of*
dimensions similar to the wavelength, the part of the wave that passes through the opening will flare (spread) out -will diffract-into the region beyond the barrier. Diffraction occurs for waves of all types, not just light waves;



A plane wave going through a small opening becomes more like a spherical wave on the other side. Thus, the wave bends at the edges. Also, if the dimensions of the obstacle or the opening is much larger than the wavelength, the diffraction is negligible and the rays go along straight lines.

You must have observed the above phenomenon, if there is a small hole on a wall, light coming from it spreads in a larger area.

In the case of light, the wavelength is around 380-780 nm. The obstacles or openings encountered in normal situations are generally of the order of millimeters or even larger. Thus, the wavelength is several thousands times smaller than the usual obstacles or openings. The diffraction is almost negligible and the

light waves propagate in straight lines and cast shadows of the obstacles. The light can then be treated as light rays which are straight lines drawn from the source and which terminate at an opaque surface and which pass through an opening undeflected. This is the Geometrical optics approximation and majority of the phenomena in normal life may be discussed in this approximation.

Principle of Superposition

We have seen that wave are very different from particles. One of the important differences between waves and particles is that we can explore the possibility to two or more waves combining at one point in the same medium. Particles can be combined to form extended objects, but the particles must be at different locations. In contrast, two waves can both be present at the same location.

If two or more traveling waves are moving through a medium, the resultant value of the wave function at any point is the algebraic sum of the values of the wave functions of the individual waves.

Two traveling waves can pass through each other without being destroyed or even altered. For instance, when two pebbles are thrown into pond and hit the surface at different locations, the expanding circular surface wave from the two locations do not destroy each other but rather pass through each other. The resulting complex pattern can be viewed as two independent sets of expanding circles.

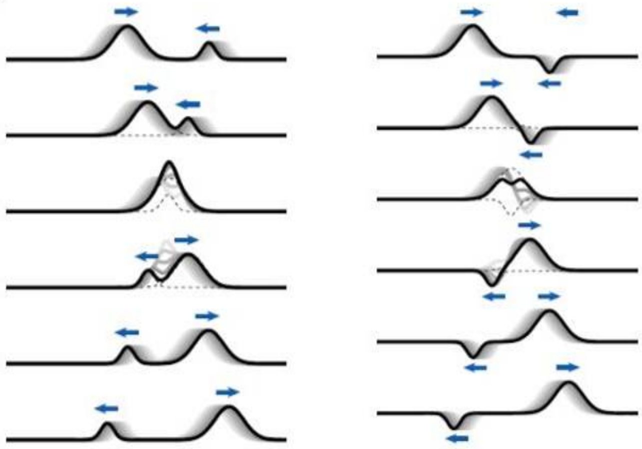
Pictorial representation of the superposition of two pulses is given. The wave function for the pulse

Pictorial representation of the superposition of two pulses is given. The wave function for the pulse moving to the right is y_1, and the wave function for the pulse moving to the left is y_2. The pulses have the same speed but different shapes, and the displacement of the elements of the medium is in the positive y direction for both pulses. When the waves begin to overlap, the wave function for the resulting complex wave is given by $y_1 + y_2$. When the crests of the pulses coincide, the resulting wave given by $y_1 + y_2$ has larger amplitude than that of the individual pulses. The two pulses finally separate and continue moving in their original directions. Notice that the pulse shapes remain unchanged after the interaction, as if the two pules had never met !

The combination of separate waves in the same region of space to produce a resultant wave is called interference. For the two pulses shown in figure, the displacement of the elements of the medium is in the positive y direction for both pulses, and the resultant pulse (created when the individual pulses overlap) exhibits an amplitude greater than that of either individual pulse. Because the displacements caused by the two pulses are in the same direction, we refer to their superposition as constructive interference.

Now consider two pulses traveling in opposite directions on a taut string where one pulse is inverted relative to the other as illustrated in figure. When these pulses begin to overlap, the resultant pulse is given by $y_1 + y_2$, but the values of the function y_2 are negative. Again, the two pulses pass through direction, however, we refer to their superposition as destructive interference.

The superposition principle is the centerpiece of the waves in interference model. In many situations, both in acoustics and optics, waves combine according to this principle and exhibit interesting phenomena with practical application.



Theory of Interference

Consider a homogeneous medium in which there are two point sources of sinusoidal spherical waves, S_1 and S_2 with the same period T. Let E_1 and E_2 be the optical disturbances arriving from the two sources at a point P. These disturbances can be written as

$$E_1 = A_1 \sin \omega t$$
$$E_2 = A_2 \sin (\omega t + \phi)$$

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$$E_1 = A_1 \sin \omega t$$
$$E_2 = A_2 \sin (\omega t + \phi)$$

Let the amplitudes A_1 and A_2 depend on the strengths of the sources and on the distance of the sources from P. From principle of superposition the resultant optical disturbance at P is a sinusoidal function of angular frequency ω and amplitude A given by

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \phi \text{ (from superposition)} \qquad ....(ii)$$

We know that $I \propto A^2$. Hence, the distribution of light intensity in the region of space surrounding the sources is given by :

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\ \cos \phi \qquad ...(iii)$$

Intensity will be maximum when $\cos\phi = 1$

$$\therefore \qquad I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$$

Intensity will be minimum when $\cos\phi = -1$

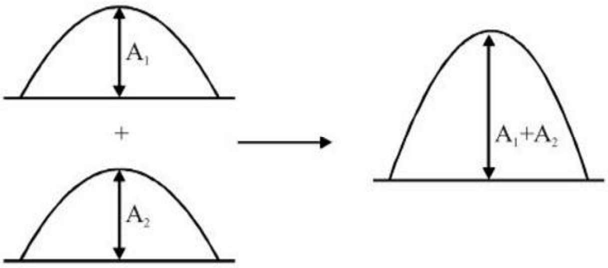
$$\therefore \qquad I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$$

Where I_1 and I_2 are the intensities observed when one or the other source is present alone and I is the intensity observed when both sources are present simultaneously. We see that the resultant intensity I is greater or smaller than the sum of the two separate intensities, $I_1 + I_2$, depending on whether the third term on the right side of equation is positive or negative. This term represents the effect of interference.

Intensity maxima are found at points where the two waves are in phase; and minima are found at points where two waves are out of phase. Thus, interference phenomena have a considerable effect on the local distribution of light intensity in the space surrounding the source. They do not, however, change the space average of the intensity, which remains equal to the space average of $I_1 + I_2$ as is required by the principle of conservation of energy. We see immediately that this is true when we note that the average value over space of the third term in equation (iii) is zero.

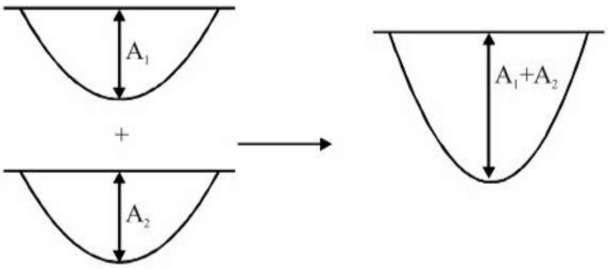
Now question arises; how these maxima and minima occur ?

Intensity maxima occurs where amplitudes of two interfering waves add to give the maximum value. i.e. when maximum positive value, of one wave appears simultaneously with the maximum positive value of the other.

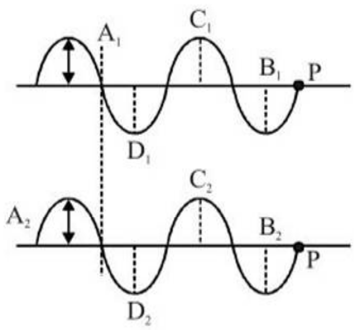


or the negative extreme of one coincides with negative exterme of the other wave.

or the negative extreme of one coincides with negative exterme of the other wave.



To obtain a maxima at a point continuously for a long time, we must obtain the wave at that point in same phase i.e. the crest must always appear with crest, and trough with lengths are same, as shown in figure.



at point P, a continous maxima will appear if B_1 and B_2 reach there simultaneously, also C_1 , C_2 and D_1 , D_2 must follow same. As the velocity of light depends only on the medium and is therfore same for both waves; the above condition can be achieved only if

$$B_1P = B_2P$$
$$C_1P = C_2P$$
i.e. $$\lambda_1 = \lambda_2$$

$$\Rightarrow \quad \lambda_1 = \frac{c}{f_1} \quad ; \quad \lambda_2 = \frac{c}{f_2}$$

$$\therefore \quad f_1 = f_2$$

Such waves for which frequency is same are called coherent waves and corresponding sources are coherent sources.

Coherent and Incoherent Sources

Why do we not commonly see interference effects with visible light? With light from a source such as the Sun, an incandescent bulb, or a fluorescent bulb, we do not see regions of constructive and destructive interference; rather, the *intensity* at any point is the sum of the intensities due to the individual waves. Light from anyone of these sources is, at the atomic level, by electronic transitions from one energy level to another which can not be externally controlled. *Hence two independent sources identical in all respects can not be coherent.*

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\ \cos\phi \qquad \text{...(i)}$$

Waves from independent sources are incoherent; they do not maintain a fixed phase relationship with each other (i.e. ϕ varies with time). We cannot accurately predict the phase (for instance, whether the wave is at a maximum or at a zero) at one point given the phase at another point. Incoherent waves have

Waves from independent sources are incoherent; they do not maintain a fixed phase relationship with each other (i.e. ϕ varies with time). We cannot accurately predict the phase (for instance, whether the wave is at a maximum or at a zero) at one point given the phase at another point. Incoherent waves have *rapidly fluctuating* phase relationships. It means average of third term of equation (i) is zero. Therefore, the result is an averaging out of interference effects, so that the total intensity (or power per unit area) is just the sum of the intensities of the individual waves.

Only the superposition of coherent waves produces sustained interference. Coherent waves must be locked in with a fixed phase relationship. *Coherent* and *incoherent* waves are idealized extremes; all real waves fall somewhere between the extremes. The light emitted by a laser can be highly coherent-two points in the beam can be coherent even if separated by as much as several kilometers.

Important points about Coherent source :

The sources which produce sustained, i.e. observable interference are called coherent sources. In case of interference as $I = [I_1 + I_2 + 2\sqrt{I_1 I_2}\ \cos\phi]$, interference will be sustained if the phase difference ϕ at a given point does not vary with time. If the interfering wave are :

$y_1 = A_1 \sin(\omega_1 t - k_1 x_1 + \phi)$ and $y_2 = A_2 \sin(\omega_2 t - k_2 x_2 + \phi_2)$

$\phi = (\omega_1 - \omega_2)\, t + (k_2 x_2 - k_1 x_1) + (\phi_1 - \phi_2)$

So ϕ will not vary with time if :

(a) $(\phi_1 - \phi_2) = \phi_0$ is constant, i.e., the initial phase difference between the wave does not vary with time and

(b) $(\omega_1 - \omega_2)\, t = 0$, i.e., $f_1 = f_2$. But for a wave as $v = f\lambda$, $f_1 = f_2$ will also means $\lambda_1 = \lambda_2$, i.e., $k_1 = k_2$ [as $k = 2\pi / \lambda$].

i.e., the two wave are of same frequency and wavelength. So two sources will be coherent if and only if they produce wave of same frequency (and hence wavelength) and have a constant initial phase difference. So in case of two coherent sources.

$$\phi = \frac{2\pi}{\lambda}(\Delta x) + \phi_0 \quad \text{with } \phi_0 = (\phi_1 - \phi_2)$$

Now as in general emission of light from atoms is random, rapid and independent of each other, ϕ_0 cannot remain constant with time and hence two independent light source identical in all respects cannot be coherent.

Illustration:

Two coherent monochromatic light beams of intensities I and 4I are superposed. Find the maximum and minimum possible intensities in the resulting beam.

Sol.

$$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = \left(\sqrt{I} + \sqrt{4I}\right)^2 = 9I$$

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 = \left(\sqrt{I} - \sqrt{4I}\right)^2 = I$$

Illustration :

In a Young's double slit experiment, the amplitude of intensity variation of the two sources is found to be 3% of the average intensity. Find the ratio of intensities of the two interference sources.

Sol.

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Sol.

Let a_1 and a_2 be the amplitudes of vibrations from the two sources, then

$$\frac{I_{max}}{I_{min}} = \frac{(a_1+a_2)^2}{(a_1-a_2)^2} = \frac{a_1^2\left(1+\frac{a_2}{a_1}\right)^2}{a_1^2\left(1-\frac{a_2}{a_1}\right)^2} \text{ or } \frac{I_{max}}{I_{min}} = \frac{\left(1+\frac{a_2}{a_1}\right)^2}{\left(1-\frac{a_2}{a_1}\right)^2}$$

It is given that the amplitude of intensity variation of the two sources is found to be 3% of the average intensity, It means if average intensity is 100I then maximum intensity is 103I and minimum is 97I

$$I_{max} = 103I \quad \text{and} \quad I_{min} = 97I$$

$$\therefore \quad \frac{I_{max}}{I_{min}} = \frac{103}{97}$$

Substituting in equation

$$\frac{103}{97} = \frac{\left(1+\frac{a_2}{a_1}\right)^2}{\left(1-\frac{a_2}{a_1}\right)^2}$$

$$\therefore \frac{\left(1+\frac{a_2}{a_1}\right)}{\left(1-\frac{a_2}{a_1}\right)} = \sqrt{\frac{103}{97}} = 1.03 \text{ or } \frac{a_1}{a_2} = \frac{1}{0.0148}$$

If I_1 and I_2 are the inensitites produced by the two sources,

$$\frac{I_1}{I_2} = \frac{a_1^2}{a_2^2} \quad \text{or} \quad \frac{I_1}{I_2} = \left(\frac{1}{0.0148}\right)^2 = 4565 \text{ Ans.}$$

Practice Exercise

Q.1 Two coherent monochroratic beams are superposed. The minimum and maximum intensities in the resulting interference pattern are found to be 4I and 16I respectively. Find the initial inensities of the two sources.

Q.2 Two coherent monochromatic light beams of intensities 4I and 16I are superposed. Find the maximum and minimum possible intensities in the resulting beam.

Q.3 In the above question if the phase difference between the two beams, at a point is $\pi/2$. Find the resultant intensity at that point.

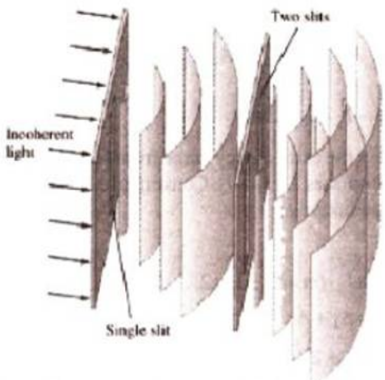
Answers

Answers

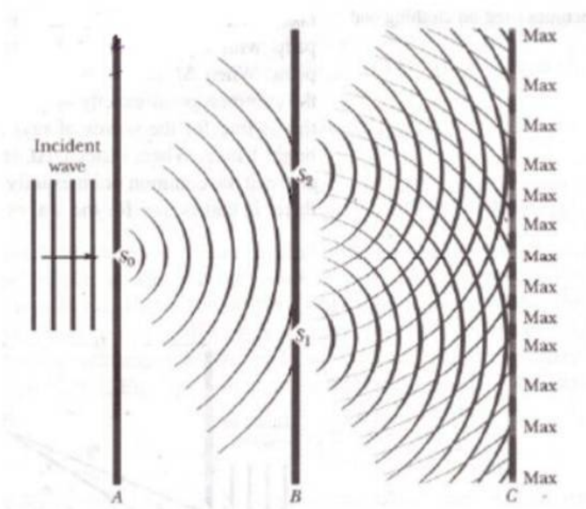
Q.1 9I and I Q.2 36I and 4I Q.3 20I

Young's Double Slit Experiement :

Thomas Young (1773-1829) performed the first visible-light interference experiments using a clever technique to obtain two coherent light sources from a single source. When a single narrow slit is illuminated, the light wave that passes through the slit diffracts or spreads out. The single slit acts as a single coherent source to illuminate two other slits. These two other slits then act as sources of coherent light for interference.



Young's technique for illuminating two slits with coherent light. The single slit on the left serves as a source of coherent light.



In Young's interference experiment, incident monochromatic light is diffracted by slit S_0, which then acts as a point source of light that emits semicircular wavefronts. As that light reaches screen *B,* it is diffracted by slits S_1 and.S_2, which then act as two point sources of light. The light waves traveling from slits S_1 and S_2 overlap and undergo interference, forming an interference pattern of maxima and minima on viewing screen C. This figure is a cross section; the screens, slits, and interference pattern extend into and out of the page. Between screens *Band* C, the semicircular wavefront's centered on S_2 depict the waves that would be there if only S_2 were open. Similarly, those centered on S_1 depict waves that would be there if only S_1 were open. Points of interference maxima form visible bright rows-called *bright bands, bright fringes,* or (loosely speaking) *maxima* that extend across the screen

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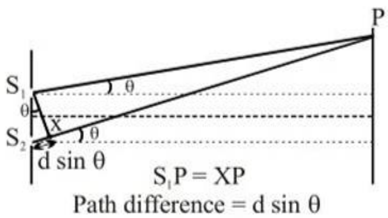
Intensity of Two Source Interference :

We now obtain an expression for the distribution of intensity of two coherent sources that are in phase. The wave function in this case is the electric field. We assume that the slits are narrow enough for diffraction to spread light from each slit uniformly over the screen. Thus, the amplitude of the fields at any point on the screen will be equal. At a given point of the screen the fields due to S_1 and S_2 are

$$E_1 = E_0 \text{ Sin } (\omega t) ; \qquad E_2 = E_0 \sin (\omega t + \phi)$$

where the phase difference ϕ depends on the path difference $\Delta x = r_2 - r_1$. Since one wavelength λ corresponds to a phase change of 2π, a distance δ corresponds to a phase change ϕ given by $\phi / 2\pi = \Delta x / \lambda$. If the screen is far from the slits, $\delta \approx d \sin\theta$, therefore



$$\phi = \frac{2\pi\delta}{\lambda} = \frac{2\pi d \sin\theta}{\lambda}$$

The resultant field is found from the principle of superposition :

$$E = E_1 + E_2 = E_0 \sin (\omega t) + E_0 \sin (\omega t + \phi)$$

By using the trigonometric identity $\sin A + \sin B = 2 \sin [(A + B)/2] \cos [(A - B)/2]$, we obtain

$$E = 2E_0 \cos\left(\frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right)$$

The amplitude of the resultant wave is $2E_0 \cos(\phi/2)$. The intensity of a wave is proportional to the square of the amplitude, so from equation of wave we have

$$I = 4I_0 \cos^2\left(\frac{\phi}{2}\right)$$

where $I_0 \alpha E_0^2$ is the intensity due to a single source. The maxima occur when $\phi = 0, 2\pi, 4\pi, ... = 2m\pi$. At these points $I = 4I_0$; that is the intensity is four times that of a single source. The minima ($I = 0$) occur when $\phi = \pi, 3\pi, 5\pi, ... = (2m + 1)\pi$.

Fig.(a) shows the waves emitted by sources S_1 and S_2. The waves from the source start in phase and arrive in phase, leading to constructive interference at the point.
The distances travelled by waves differ by any integer number of wavelengths.

$$x_2 - x_1 = \lambda, 2\lambda, 3\lambda, ..., n\lambda.$$

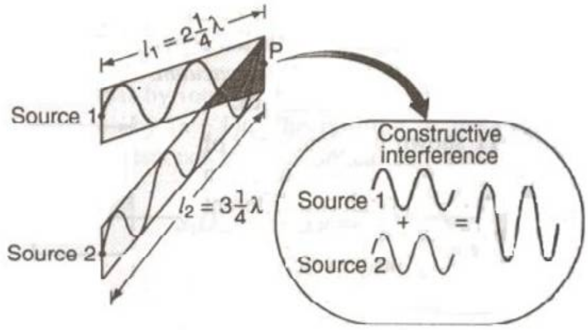
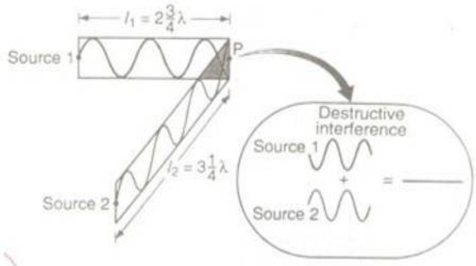


Fig. (b) shows two waves starting in phase but arriving in opposite phase.

Fig. (b) shows two waves starting in phase but arriving in opposite phase.
the distance travelled by waves differ by odd integer number of wavelengths.

$$x_2 - x_1 = \frac{\lambda}{2}, \frac{3}{2}\lambda, \frac{5}{2}\lambda, ...\left(n - \frac{1}{2}\right)\lambda$$



(i) If amplitudes of waves arriving at point P on the screen are different then resultant intensity is given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\ \cos\delta$$

Also, $$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2, \quad \text{when} \quad \cos\delta = 1$$

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2, \quad \text{when} \quad \cos\delta = -1$$

(ii) $$\frac{I_{max}}{I_{min}} = \left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2 = \left(\frac{A_1+A_2}{A_1-A_2}\right)^2 = \left(\frac{r+1}{r-1}\right)^2$$

where $$r = \frac{A_1}{A_2} = \sqrt{\frac{I_1}{I_2}}$$

(iii) The phenomenon of interference is based on conservation of energy. There is no destruction of energy in the interference phenomenon. The energy which apparently disappears at the minima, has actually been transferred to the maxima where the intensity is greater than that produced by the two beams acting separately.

$$I_{av} = \frac{\int_0^{2\pi} I d\delta}{\int_0^{2\pi} d\delta} = \frac{1}{2\pi}\int_0^{2\pi}(I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta = I_1 + I_2$$

$\because$ $$\int_0^{2\pi} \cos\delta \; d\delta = 0$$

as the average value of intensity is equal to the sum of individual intensities, therefore the energy is not destroyed but merely redistributed in the interference pattern.

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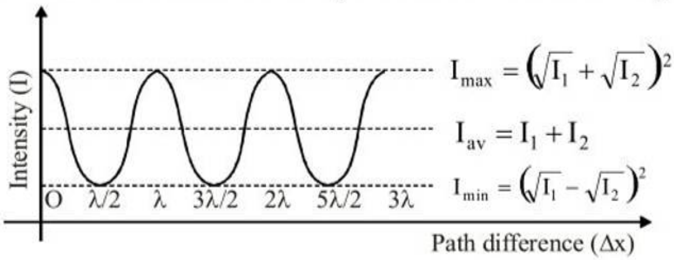
(iv) All maxima are equally spaced and equally bright. This is true for minima as well. Also interference maxima and minima are alternate. The intensity distribution in interference pattern is shown in figure.



(v) The fringe visibility (v) is given by

$$v = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} = \frac{2A_1A_2}{A_1^2 + A_2^2}$$

when $I_1 = I_2$

or $A_1 = A_2$

$I_{min} = 0$

hence, $v = 1$ (best visibility)

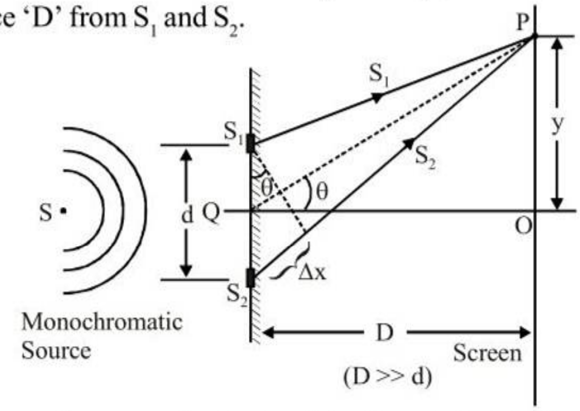
(vi) Path difference (Δx) and phase difference (ϕ) are related as given below :

λ path difference = 2π phase difference

or $\Delta x = \dfrac{\lambda}{2\pi}\phi$

Maxima and Minima

The experiment set up for Young's double slit experiment is shown in figure. Light after passing through a pin hole 'S' is allowed to fall on thin slits 'S_1' and 'S_2' placed symmetrically w.r.t. 'S'. A screen is placed at a distance 'D' from S_1 and S_2.



Geometric construction for describing Young's double-slit experiment

Let 'P' be the point, at which we want to investigate the intensity. Two rays S_1P and S_2P starting from S_1 and S_2 reach P and interfere with each other.

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If Δx is the path difference between two rays,

$$\Delta x = \sqrt{D^2 + \left(y + \frac{d}{L}\right)^2} - \sqrt{D^2 + \left(y - \frac{d}{L}\right)^2}$$

$$= D\left(1 + \frac{\left(y + \frac{d}{L}\right)^2}{D^2}\right)^{1/2} - D\left(1 + \frac{\left(y - \frac{d}{L}\right)^2}{D^2}\right)^{1/2}$$

For small value of 'y' << D [using binomial approximation]

$$= D\left(1 + \frac{1}{2}\left(\frac{y + d/L}{D}\right)^2\right) - D\left(1 + \frac{1}{2}\left(\frac{y - d/L}{D}\right)^2\right)$$

$$= \left(\frac{y + d/L}{2D}\right)^2 - \left(\frac{y - d/L}{2D}\right)^2$$

$$= \frac{yd + yd}{2D} = \frac{2yd}{2D} = \frac{yd}{D} = d \text{ tna } \theta$$

(a) Maxima : Point 'P' will be a bright spot if the path difference Δx is integral multiple of λ.

$$\Delta x = \frac{yd}{D} = n\lambda$$

$$y_n = \frac{nD\lambda}{d} \qquad \text{where, } n = 0, 1, 2, 3, \ldots\ldots$$

Thus, bright spots are obtained at distances, $0, \frac{\lambda D}{d}, \frac{2\lambda D}{d}, \frac{3\lambda D}{d}$ .... from O.

(b) Minima : Point 'P' will be a dark spot if the path difference 'Δx' is an odd multiple of $\frac{\lambda}{2}$.

i.e., if $\qquad \frac{yd}{D} = \frac{(2n+1)\lambda}{2}$

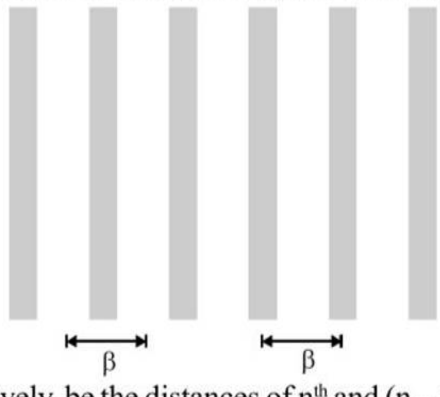
$$\therefore \qquad y_n = \frac{(2n+1)\lambda D}{2d} \quad \text{where, } n = 0, 1, 2, 3, \ldots\ldots\ldots$$

Thus, dark spots are obtained at distances, $\frac{\lambda D}{2d}, \frac{3\lambda D}{2d}, \frac{5\lambda D}{2d}$ .... from O.

(c) Fringe width (β)

(c) Fringe width (β)

It is the distance between two consecutive bright or dark fringes.



Let y_n and y_{n-1} respectively, be the distances of n^{th} and $(n-1)^{th}$ bright fringe from O,

$$\beta = [y_n - y_{n-1}] = n\frac{\lambda D}{d} - (n-1)\frac{\lambda D}{d} = \frac{\lambda D}{d}(n - n + 1)$$

or $\qquad \beta = \frac{\lambda D}{d}$

Similarly, it can be proved that distance between two consecutive dark fringes, β is given by

$$\beta = \frac{\lambda D}{d}$$

$$\therefore \qquad \beta = \beta = \lambda D / d$$

Hence, the bright and dark fringes are equally spaced.

Illustration :

A beam of light consisting of two wavelengths 6500 Å and 5200 Å is used to obtain interference fringes in a Young's double slit experiment.

(i) Find the distance of the third bright fringe on the screen from the central maximum fo the wavelength 6500Å.

(ii) What is the least distance from the central maximum when the bright fringes due to both the wavelengths coincide ?

The distance between the slit is 2 mm and the distance between the plane of the slits and the screen is 120 cm.

Sol.

Given,
$$\lambda_1 = 6500\ \text{Å} = 6.5 \times 10^{-7}\ m$$
$$\lambda_2 = 5200\ \text{Å} = 5.2 \times 10^{-7}\ m$$
$$d = 0.2\ cm = 2 \times 10^{-3}\ m$$
$$D = 120\ cm = 1.2\ m$$

(i) For n^{th} bright spot $y_n = n\dfrac{\lambda D}{d}$

Here, $n = 3$ *and* $\lambda = \lambda_1 = 6.5 \times 10^{-7}\ m$

$$y_3 = \frac{3 \times 6.5 \times 10^{-7} \times 1.2}{2 \times 10^{-3}} = 1.17 \times 10^{-3}\ m \qquad Ans.$$

(ii) Since $\lambda_2 < \lambda_1$, fringe width for λ_2 is smaller. If two bright fringes due to λ_1 and λ_2 are to coincide, then minimum distance from the central spot will be where n^{th} order bright spot due to λ_1 and $(n + 1)^{th}$ bright spot due to λ_2 coincide.

$$n\frac{\lambda_1 D}{d} = \frac{(n+1)\lambda_2 D}{d}$$
$$n \times 6.5 \times 10^{-7} = (n + 1) \times 5.2 \times 10^{-7} \qquad or\ n = 4$$
$$y = \frac{n\lambda D}{d} = \frac{4 \times 6.5 \times 10^{-7} \times 1.2}{2 \times 10^{-3}} = 1.56 \times 10^{-3}\ m\ Ans.$$

Important Points about YDSE

(i) If whole apparatus is immersed in liquid of refractive index μ then,

$$\beta = \frac{\lambda D}{\mu d} \text{ i.e., fringes width decreases}$$

(ii) Some times in numerical problems, angular fringe width (ω) is given which is defined as angular separation between two consecutive maxima or minima

$$\omega = \frac{\beta}{D} = \frac{\lambda}{d}$$

In medium, other than air or vacuum,

$$\omega = \frac{\lambda}{\mu d}$$

(iii) $\Delta x = \frac{yd}{D}$ is valid when angular position of maxima or minima is less than $\frac{\pi}{6}$. However $\Delta x = d \sin\theta$ is valid for larger values of θ provided $d << D$.

(iv) Central bright fringe (CBF) is a point on screen where path difference is zero. In above case CBF is formed at O. But in many situation it may not be located symmetrically w.r.t. slits.

(v) If white light is used instead of monochromatic light then, interference pattern consists of white central bright fringe surrounded by few coloured fringes and then uniform illumination due to overlapping of interference pattern on each wavelength.

(vi) If the interference experiment is performed with bichromatic light, the bright fringes of two wavelength will be coincident for the first time under following condition.

$$Y = n\,(\beta)_{Longer} = (n+1)\,\beta_{Shorter} \quad \text{or} \quad n\lambda_{Longer} = (n+1)\,\lambda_{Shorter}$$

(vii) In many numerical problems we have to calculate number of maxima or minima. We know that for maximum.

$$\sin\theta = \frac{n\lambda}{d} \qquad \text{or} \qquad n = \frac{d\sin\theta}{\lambda}$$

$$\therefore \quad n\,\frac{d}{\lambda} \not> 1 \qquad\qquad (\because\ \sin\theta \le 1)$$

$$n_{highest} = \left[\frac{d}{\lambda}\right]$$

Where n highest $\Rightarrow$ Highest order of maxima on one side.

Suppose in some question $\frac{d}{\lambda}$ works out to be 2.3 so, permissible values of n are $0, \pm 1, \pm 2$. Hence, total 5 maxima will be obtained on screen.

Illustration :

In a Young's double slit experiment for interference of light, the slits are 0.2 cm apart and are illuminated by yellow light (λ = 600 nm). What would be the fringe width on a screen placed 1 m from the plane of slits if the whole system is immersed in water of refractive index 4/3?

Sol.

Given, $\lambda = 600\ nm = 6 \times 10^{-7}\ m$

$D = 1m$

$d = 0.2\ cm = 2 \times 10^{-3}\ m$

As the apparatus is dipped in water ($\mu = 4/3$)

$$\lambda_{new} = \frac{\lambda}{\mu} = \frac{6\times10^{-7}}{4/3} m = 4.5 \times 10^{-7} m$$

$$\beta = \frac{\lambda_{new}D}{d} = \frac{4.5\times10^{-7}\times1}{2\times10^{-3}} = 2.25 \times 10–4 m = 0.225 mm$$

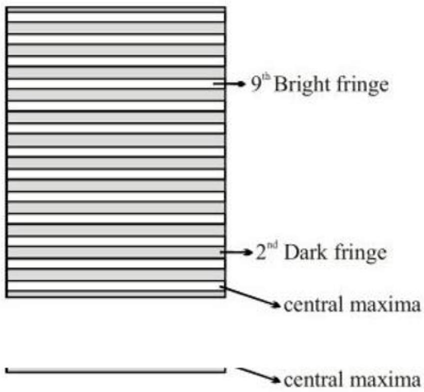
Illustration :

In Young's double slit experiment the slits are 0.5 mm apart and the interference is observed on a screen at a distance of 100 cm from the slit. It is found that the 9th bright fringe is at a distance of 7.5 mm measured from the second dark fringe from the centre of the fringe pattern on same side. Find the wavelength of the light used.

Sol. Given, $d = 0.5 \text{ mm} = 5 \times 10^{-4} m$

$D = 1m$

Distance between 9^{th} bright fringe and 2^{nd} dark fringe is 7.5 mm



$$i.e \frac{9\lambda D}{d} - \frac{3}{2}\frac{\lambda D}{d} = 7.5 \text{ mm}$$

$$7.5 \frac{\lambda D}{d} = 7.5 \times 10^{-3} m$$

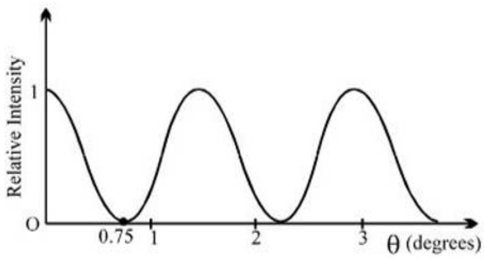
$$\lambda = \frac{10^{-3}\times d}{D} = \frac{10^{-3}\times5\times10^{-4}}{1} = 5 \times 10^{-7} m$$

D 1

$= 5000$ Å

Illustration :

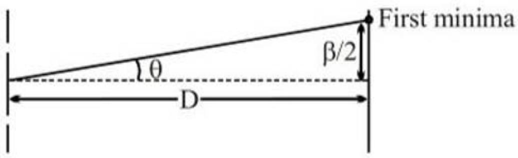
Light of wavelength 520 nm passing through a double slit, produces interference pattern of relative intensity versus deflection angle θ as shown in the figure. find the separation d between the slits.



Sol. $\lambda = 520 \times 10^{-9} m$

θ at which first minima occurs is $0.75°$

$$\theta = \frac{0.75\times\pi}{180} \text{ radians}$$



$$\therefore \theta = \frac{\beta}{2D}$$

$$\frac{0.75\times\pi}{180} = \frac{\lambda D/2d}{D}$$

$$\frac{\lambda}{2d} = \frac{0.75\pi}{180}$$

$$d = \frac{520\times10^{-9}\times180}{2\times0.75\pi} = 1.99 \times 10^{-2}\ mm$$

Illustration :

The distance between two slits in a YDSE apparatus is 3mm. The distance of the screen from the slits is 1m. Microwaves of wavelength 1 mm are incident on the plane of the slits normally. Find the distance of the first maxima on the screen from the central maxima. Also find the total number of maxima on the screen.

Sol. $d = 3 \times 10^{-3}\ m$

$\lambda = 10^{-3}\ m$

$D = 1m$

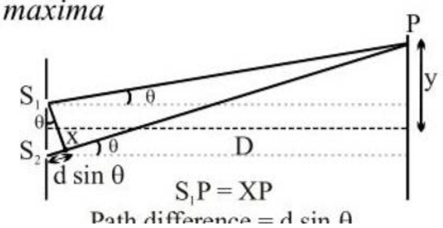
(i) *Distance between first maxima and central maxima*

$\Delta x = \lambda$

$\Delta x = d \sin\theta$

$$\sin\theta = \frac{10^{-3}}{3\times10^{-3}} = \frac{1}{3}$$

3×10^{-3} 3



Path difference = d sin θ

$\sin\theta$ is not very small ;

$\therefore$ *approximation $\sin\theta = \theta = \tan\theta$ can't be used*

$$\tan\theta = \frac{1}{\sqrt{8}}$$

$$\frac{y}{D} = \frac{1}{\sqrt{8}}$$

$$y = \frac{1}{\sqrt{8}}\ m$$

(ii) $\Delta x = d\sin\theta$;

$\therefore (\Delta x)_{max} = d$

For calculating number of maxima, compare $d = n\lambda$

$$n = \frac{d}{\lambda}$$

If n is an integer, then nth maxima will not be visible,

$\therefore$ *total no. of maxima = (n – 1) (above the central maxima) + 1 (central maxima) + (n – 1) (below the central maxima = 2n – 1*

If n is not an integer then, [n] + 1 + [n] = 2[n] + 1

Here in this question, $n = \frac{d}{\lambda} = 3$

$\therefore$ *number of maxima = 2 × 3 – 1 = 5*

Alternative :

Total number of maxima

for maxima, $\Delta x = n\lambda = d \sin\theta$

$$\sin\theta = \frac{n\lambda}{d}$$

for $\sin\theta = 1$; $\theta = \pi/2$, *such fringe can't be obtained on screen*

$$\therefore \quad -1 < \frac{n\lambda}{d} < 1$$

$$-1 < \frac{n \times 10^{-3}}{3 \times 10^{-3}} < 1$$

$$\therefore \quad n = -2, -1, 0, 1, 2$$

Total number of maxima = 5

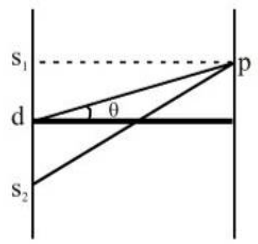
Illustration :

In a Young's double slit experiment, the separation between the slits is d, distance between the slit and screen is D (D >> d). In the interference pattern, there is a maxima exactly in front of each slit. Find the possible wavelength(s) used in the experiment.

slit. Find the possible wavelength(s) used in the experiment.

Sol.

Path difference is $\frac{dy}{D}$



$$S_2P - S_1P = \frac{d \times y}{D} = \frac{d \times (d/2)}{D} = \frac{d^2}{2D}$$

$$\frac{d^2}{2D} = n\lambda$$

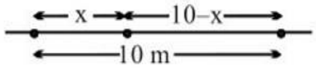
$$\lambda = \frac{d^2}{2nD},\ n = 1, 2, \ldots\ldots\ldots\ldots$$

$$\lambda = \frac{d^2}{2D}, \frac{d^2}{4D}, \frac{d^2}{6D}$$

Illustration :

One radio transmitter A operating at 60.0 MHz is 10.0 m from another similar transmitter B that is 180° out of phase with transmitter A. How far must an observer move from transmitter A toward transmitter B along the line connecting A and B to reach the nearest point where the two beams are in phase?

Sol.



As the two beams are out of phase, initially,
For two beams to be in phase

$$(10 - x) - x = (2n - 1)\,\frac{\lambda}{2}$$

$$10 - 2x = (2n - 1)\,\frac{\lambda}{2}$$

$$\nu = 60 \times 10^{6}\ Hz$$

$$\lambda = \frac{c}{\nu} = \frac{3\times10^{8}}{6\times10^{7}}\ m = 5\ m$$

$$\therefore\ 10 - 2x = (2n - 1) \times \frac{5}{2}$$

$$x = 5 - (2n - 1)\,\frac{5}{4}$$

$$x = \frac{25}{4} - \frac{5n}{2}$$

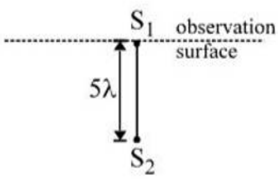
for x_{min} ; $n = 2$

25 5

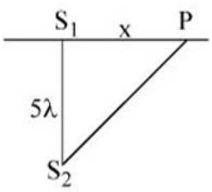
$$x = \frac{25}{4} - 5 = \frac{5}{4}\ m$$

Illustration :

Two microwave coherent point sources emitting waves of wavelength λ are placed at 5λ distance apart. The interference is being observed on a flat non-reflecting surface along a line passing through one source, in a direction perpendicular to the line joining the two sources (refer figure) Considering λ as 4 mm, calculate the positions of maxima and draw shape of interference pattern. Take initial phase difference between the two sources to be zero.



Sol.



$$S_2P - S_1P = n\,\lambda \text{ for maxima}$$

$$\therefore \qquad \sqrt{(5\lambda)^2 + x^2} - x = n\lambda \qquad\qquad n = 1,2,3,4,5$$

$$25\lambda^2 + x^2 = (n\lambda + x)^2 = (n\lambda)^2 + x^2 + 2n\lambda x$$

$$\lambda^2\,[25 - n^2] = 2n\,\lambda\,x$$

$$\therefore \qquad \text{Maxima occur at } x = \lambda\,\frac{(25-n^2)}{2n} \text{ where } n = 1,2,3,4$$

$$x = \frac{24\lambda}{2},\ \frac{\lambda(25-4)}{4},\ \frac{\lambda(25-9)}{6},\ \frac{\lambda(25-16)}{8}$$

$$= \frac{24\lambda}{2},\ \frac{21\lambda}{4},\ \frac{16\lambda}{6},\ \frac{9\lambda}{8}$$

Putting $\lambda = 4$ *m.m.*

$$\therefore \qquad x = 48,\ 21,\ \frac{64}{6},\ \frac{36}{8},\ 0 \text{ m.m.} = 48,\ 21,\ \frac{32}{3},\ \frac{9}{2},\ 0 \text{ m.m.}$$

Pattern will look like this 

Illustration :

Illustration :

In a YDSE apparatus, d = 1mm, λ = 600nm and D = 1m. The slits individually produce same intensity on the screen. Find the minimum distance between two points on the screen having 75% of the maximum intensity.

Sol. Given, $\lambda = 600 \text{ nm} = 6 \times 10^{-7}$ m

$D = 1m$

$d = 1 \text{ mm} = 10^{-3}$ m

$$I = I_{max} \cos^2 \frac{\phi}{2}$$

$$\phi = \frac{2\pi}{\lambda} \times (\Delta x)$$

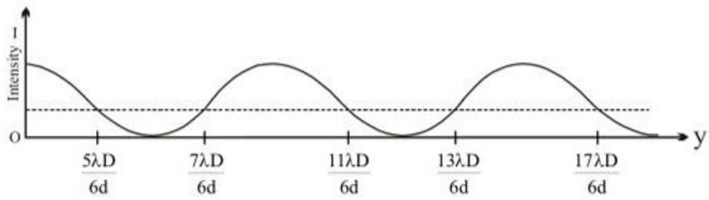
$$\frac{\phi}{2} = \frac{\pi}{\lambda} \times (\Delta x) \qquad\qquad (\Delta x = \frac{dy}{D})$$

$$I = I_{max} \cos^2 \frac{\pi dy}{\lambda D} \qquad \Rightarrow \qquad 0.75\, I_{max} = I_{max} \cos^2 \frac{\pi dy}{\lambda D}$$

$$\cos \frac{\pi dy}{\lambda D} = \pm\sqrt{\frac{3}{4}}$$

$$\frac{\pi dy}{\lambda D} = n\pi \pm \frac{\pi}{6}$$

$$y = \frac{n\lambda D}{d} \pm \frac{\lambda D}{6d}$$



Points at which intensity is 75% of maximum are $\frac{5\lambda D}{6d}, \frac{7\lambda D}{6d}, \frac{11\lambda D}{6d}, \frac{13\lambda D}{6d}, \frac{17\lambda D}{6d}$ ......

Let minimum distance between two points having intensity 75% of the I_{max} *be* Δy

$$\Delta y = \frac{2\lambda D}{6d} = 0.2\ mm$$

Practice Exercise

Q.1 In YDSE experiment the distance between slits is $d = 0.25$ cm and the distance of screen $D = 120$ cm from slits. If the wavelength of light used is $\lambda = 6000$Å and I_0 is the intensity of central maximum, at what distance from the centre, the intensity will be $\frac{I_0}{2}$?

Q.2 In a YDSE experiment, I_0 is given to be the intensity of the central bright fringe and β is the fringe width. Then, find the intensity at a distance y from Central Bright Fringe.

Q.3 Two narrow slits emitting light in phase are separated by a distance of 1.0 cm. The wavelength of the light

Q.3 Two narrow slits emitting light in phase are separated by a distance of 1.0 cm. The wavelength of the light is 5.0×10^{-7} m. The interference pattern is observed on a screen placed at a distance of 1.0 m (a) Find the separation between the consecutive maxima. (b) Find the separation between the sources which will give a separation of 1.0 mm between the consecutive maxima.

Q.4 If YDSE is performed with monochromatic light of wavelength λ, the distance between the slits is d and distance between slits and screen is D.
(a) Find the distance between second and fifth maxima.
(b) Find the distance between second and tenth minima.
(c) Find the distance between second minima and fifth maxima.

Q.5 In Young's double slit arrangement, a monochromatic source of wavelength 6000 Å is used. The screen is placed at 1m from the slits. Fringes formed on the screen, are observed by a student sitting close to the slits. The student's eye can distinguish two neighbouring fringes if they subtend an angle more than 1 minute of arc. Calculate the maximum distance between the slits so that the fringes are clearly visible.

Q.6 In the **above question** find the position of 3rd maxima and 5th minima.

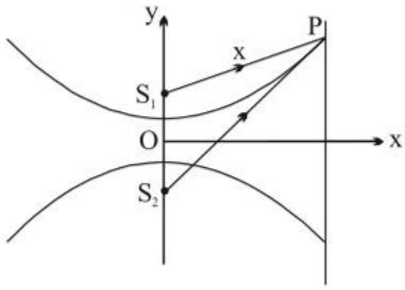
Answers

Q.1 7.2×10^{-5} m Q.2 $I_0 \cos^2(\pi y/\beta)$ Q.3 (a) 0.05 mm (b) 0.50 mm

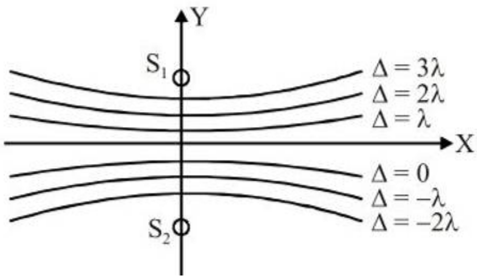
Q.4 (a) $\frac{3\lambda D}{d}$ (b) $\frac{8\lambda D}{d}$ (c) $\frac{7\lambda D}{2d}$ Q.5 $\frac{6.48}{\pi}$ mm Q.6 $\frac{\pi}{0.036}$ mm, $\frac{\pi}{0.024}$ mm

Shape of Interference Fringes in YDSE

We discuss the shape of fringes when two pinholes are used instead of the two slits in YDSE.
Fringes are locus of points which move in such a way that its path difference from the two slits remains constant.



$$S_2P - S_1P = D = \text{constant} \qquad \text{........(1)}$$



If $\Delta = \pm \dfrac{\lambda}{2}$, the fringe represents 1st minima.

If $\Delta = \pm \dfrac{3\lambda}{2}$ it represents 2nd minima

If $\Delta = 0$ it represents central maxima,

If $\Delta = 0$ it represents central maxima,

If $\Delta = \pm \lambda$, it represents 1st maxima etc.

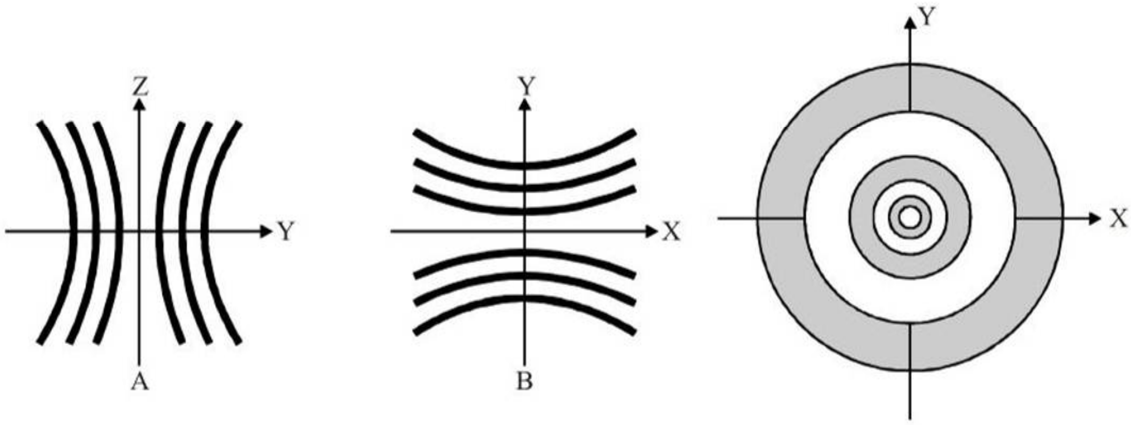
Equation (1) represents a hyperbola with its two foci at S_1 and S_2

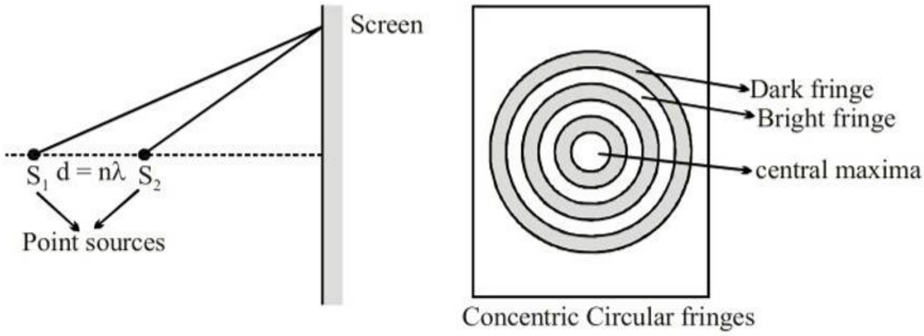
The interference pattern which we get on screen is the section of hyperboloid of revolution when we revolve the hyperbola about the axis S_1S_2.

(A) If the screen is $\perp$ to the X-axis, i.e. in the YZ plane, as is generally the case, fringes are hyperbolic with a straight central section.

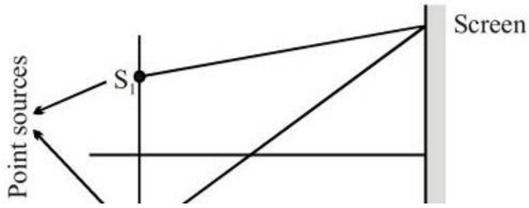
(B) If the screen is in the XY plane, again fringes are hyperbolic.

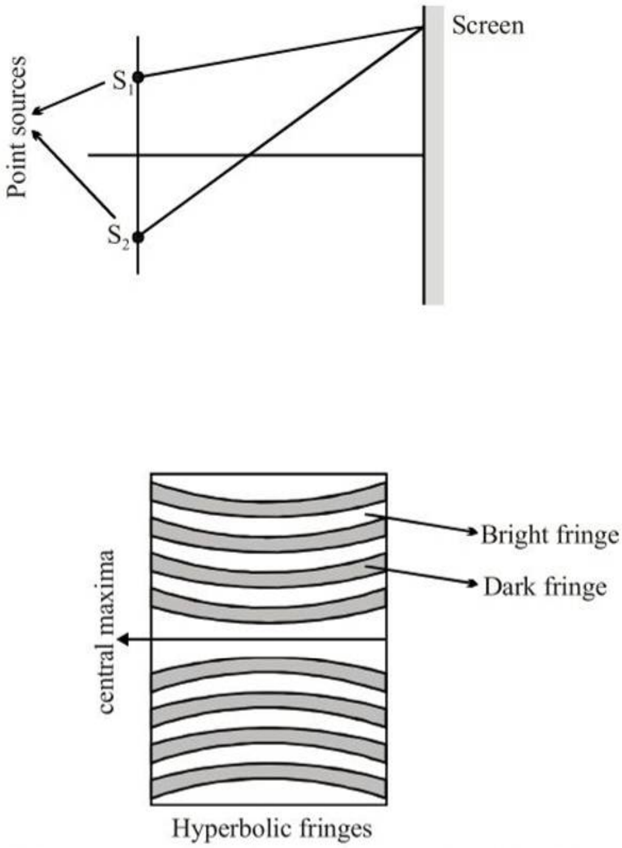
(C) If screen is $\perp$ to Y-axis (along S_1S_2) i.e. in the XZ plane, fringes are concentric circles with center on the axis S_1S_2 ; the central fringe is bright if $S_1S_2 = n\lambda$ and dark if $S_1S_2 = (2n-1)\dfrac{\lambda}{2}$.





Shape of the pattern when the interference takes place due to waves produced by two point sources (where the line of sources is perpendicular to the screen).





Shape of the pattern when the interference takes place due to waves produced by two point sources (where the line of sources is parallel to the screen).

YDSE with white light :

The central maxima will be white because all wavelengths will constructively interfere here. However slightly below (or above) the position of central maxima fringes will be coloured for example if P is a point on the screen such that

$$S_2P - S_1P = \frac{\lambda_{violet}}{2} = 190 \text{ nm},$$

completely destructive interference will occur for violet light. Hence we will have a line devoid of violet colour that will appear reddish. And if

$$S_2P - S_1P = \frac{\lambda_{red}}{2} \simeq 350 \text{ nm},$$

completely destructive interference for red light results and the line at this position will be violet. The coloured fringes disappear at points far away from the central white fringe ; for these points there are so many wavelengths which interfere constructively, that we obtain a uniform white illumination for example if

$$S_2P - S_1P = 3000 \text{ nm},$$

then constructive interference will occur for wavelengths $\lambda = \frac{3000}{n}$ nm. In the visible region these wavelength are 750 nm (red), 600 nm (yellow), 500 nm (greenish–yellow), 430 nm (violet). Clearly

n

wavelength are 750 nm (red), 600 nm (yellow), 500 nm (greenish–yellow), 430 nm (violet). Clearly

such a light will appear white to the unaided eye.

Thus with white light we get a white central fringe at the point of zero path difference, followed by a few coloured fringes on its both sides, the color soon fading off to a uniform white.

In the usual interference pattern with a monochromatic source, a large number of identical interference fringes are obtained and it is usually not possible to determine the position of central maxima. **Interference with white light is used to determine the position of central maxima in such cases.**

Geometrical path and optical path

Actual distance travelled by light in a medium is called geometrical path (Δx). Consider a light wave given by the equation.

$$E = E_0 \sin(\omega t - kx + \phi)$$

If the light travels by Δx, its phase changes by $k\Delta x = \frac{\omega}{v}\Delta x$, where ω, the frequency of light does not depend on the medium, it depends only on the source, but v, the speed of light depends on the medium as $v = \frac{c}{\mu}$.

Consequently, change in phase,

$$\Delta\phi = k\Delta x = \frac{\omega}{C}(\mu\Delta x)$$

It is clear that a wave travelling a distance Δx in a medium of refractive index μ suffers the same phase change as when it travels a distance $\mu\Delta x$ in vacuum. i.e. a path length of Δx in medium of refractive index μ is equivalent to a path length of $\mu\Delta x$ in vacuum.

The quantity $\mu\Delta x$ is called the optical path length of light, Δx_{opt}. And in terms of optical path length, phase difference would be given by,

$$\Delta\phi = \frac{\omega}{C}\Delta x_{opt} = \frac{2\pi}{\lambda_0}\Delta x_{opt}$$

where λ_0 = wavelength of light in vacuum.
However in terms of the geometrical path length Δx,

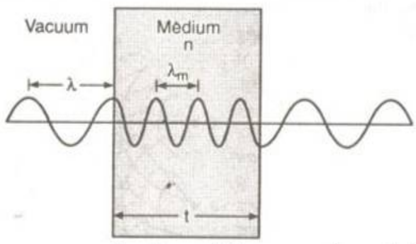
$$\Delta\phi = \frac{\omega}{C}(\mu\Delta x) = \frac{2\pi}{\lambda}\Delta x$$

where λ = wavelength of light in the medium $\left(\lambda = \frac{\lambda_0}{\mu}\right)$.

Equivalent optical path length

Equivalent optical path length

When a beam of light travels from one medium to another its speed changes but its frequency does not.



Designating the wavelength in vacuum by λ and the wavelength in the material by λ_m we have

$$\frac{\lambda}{\lambda_m} = \frac{C}{v} = n$$

or $$\lambda_m = \frac{\lambda}{n}$$

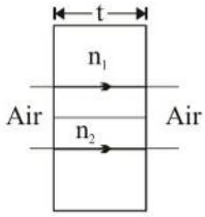
The wavelength is shorter in the medium than in vacuum. If the light beam passes through a thickness t of a medium,

$$\text{Number of wavelength in slab} = \frac{t}{\lambda_m} = \frac{t}{\lambda/n} = \frac{nt}{\lambda}$$

which shows that a thickness t of the medium has as many wavelengths as there are in a length nt of vacuum. Therefore in terms of wavelengths, a thickness t in a medium of refractive index n is equivalent to a path length nt in vacuum. The quantity nt is called equivalent optical path length.

Optical path length interms of wave lengths

In fig. shown two light rays of identical wavelength and initially in phase in air travel through two different media of refractive indices n_1 and n_2, same thickness t. The wavelengths of the waves will be different in the two media ; so the two waves will no longer be in phase when they emerge.



Number of wavelengths in medium 1,

$$N_1 = \frac{t}{\lambda_1} = \frac{t}{\lambda / n_1} = \frac{n_1 t}{\lambda}$$

Number of wavelength in medium 2,

$$N_2 = \frac{t}{\lambda_2} = \frac{t}{\lambda / n_2} = \frac{n_2 t}{\lambda}$$

To find a new phase difference we subtract the number of wavelengths of the waves in the two media ; assuming $n_2 > n1$, we have

To find a new phase difference we subtract the number of wavelengths of the waves in the two media ; assuming $n_2 > n1$, we have

$$N_2 - N_1 = (n_2 - n_1)\,\frac{t}{\lambda}$$

Phase difference corresponding to difference of one wavelength is 2π ; hence phase difference corresponding to $N_2 - N_1$ is

$$\Delta\phi = \frac{2\pi}{\lambda}\,(n_2 - n_1)\,t$$

In other words we can say that the equivalent optical paths of wave in media 1 and 2 are $n_1 t$ and $n_2 t$ respectively. Thus the path difference, $\Delta x = n_2 t - n_1 t$.

Illustration :

Light of wavelength λ in air enters a medium of refractive index μ. Two points in this medium, lying along the path of this light, are at a distance x apart. Find the phase difference between these points.

Sol. *Phase difference* $= \dfrac{2\pi}{\lambda} \times$ *path difference*

Now $\lambda' = \dfrac{\lambda}{\mu}$

$$\Delta\phi = \frac{2\pi\mu x}{\lambda}$$

Changes observed in the Interference Pattern

Case - I

If the space between the main slit and double slit is completely filled with two uniform medium of refractive index μ_1 and μ_2 (as shown in figure).

If the mediums above and below the perpendicular line bisecting the two slits are different, then the fringe pattern shifts towards the side of denser medium by $y = \frac{(\mu_2 - \mu_2)lD}{d}$ but the fringe width does not change.

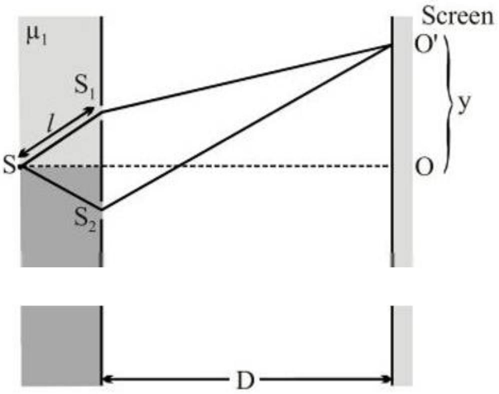
Let μ_1 and μ_2 be the refractive indices of the two media with $\mu_1 > \mu_2$. At the position of central maxima, the optical path differene between the two interfering waves is zero, so if O' is the new position of central maxima, then,

$$SS_1 \text{ (in } \mu_1) + S_1O' \text{ (in air)} = SS_2 \text{ (in } \mu_2) + S_2O' \text{ (in air)}$$

$$\Rightarrow \quad \mu_1 l + S_1O' = \mu_2 l + S_2O'$$

$$\Rightarrow \quad S_2O' - S_1O' = (\mu_1 - \mu_2)l$$

(Where $\mu_1 l$ and $\mu_2 l$ are the optical path lengths)



But $\quad S_2O' - S_1O' = \frac{yd}{D}$

So, we have,

$$\frac{yd}{D} = (\mu_1 - \mu_2)l \Rightarrow \qquad y = \frac{(\mu_1 - \mu_2)lD}{d}$$

Hence the central maxima (fringe pattern) shifts towards the side of denser medium by a distance,

$$y = \frac{(\mu_1 - \mu_2)lD}{d}$$

for n^{th} maxima

$$[SS_2 \text{ (in } \mu_2) + S_2P \text{ (in air)}] - [SS_1 \text{ (in } \mu_1) + S_1P \text{ (in air)}] = n\lambda$$

$$\Rightarrow \quad [\mu_2 l + S_2P] - [\mu_1 l + S_1P] = n\lambda$$

$$\Rightarrow \quad S_2P - S_1P = (\mu_1 - \mu_2)l + n\lambda$$

But $\quad S_2P - S_1P = \frac{yd}{D}$

$$\frac{yd}{D} = (\mu_1 - \mu_2)l + n\lambda$$

$$y = \frac{(\mu_2 - \mu_2)lD}{d} + \frac{n\lambda D}{d}$$

This means, all fringes are shifted by same distance.

$$y = \frac{(\mu_1 - \mu_2)lD}{d}$$

CASE - II

If the main slit is moved upward or downward parallel to the double slits.
then the fringe width does not change but the fringe pattern shifts in the direction opposite to that of the movement of the main slit

For the position of central maxima

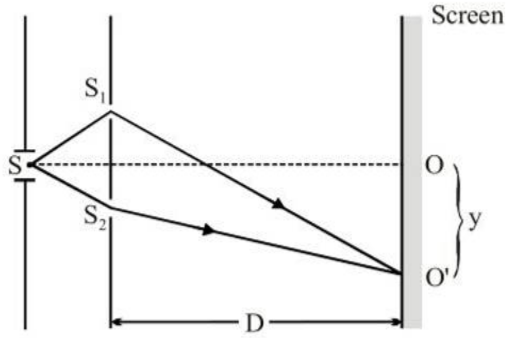
$\Delta x = 0$

$\Rightarrow \quad SS_1 + S_1O' = SS_2 + S_2O'$

$\Rightarrow \quad S_1O' - S_2O' = SS_2 - SS_1$

$\Rightarrow \quad \frac{yd}{D} = (SS_2 - SS_1)$

$\Rightarrow \quad y = \frac{(SS_2 - SS_1)D}{d}$



Similarly, for n^{th} maxima, we can prove the shift is same.

So the fringe pattern shifts by a distance $\frac{(SS_2 - SS_1)D}{d}$ in the direction opposite to the direction of motion of the main slit.

CASE - III

CASE - III

If the light waves from infinity reach the two slits as shown in the figure.
Let the final position of central maxima be O'.

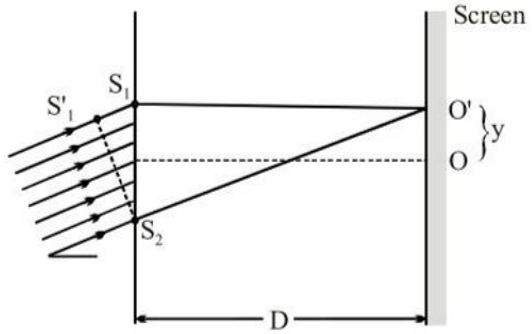
$S_1S_1' + S_1O' = S_2O'$ (Here S'_1 and S_2 are in same phase)

$S_2O' - S_1O' = S_1S_1'$

But $\quad S_2O' - S_1O' = \frac{yd}{D}$

$\frac{yd}{D} = S_1S_1'$

$y = \frac{(S_1S_1')D}{d}$



If the parallel wave make an angle θ with perpendicular bisector of $S_1 S_2$ then from the figure,

$S_1S_1' = d \sin\theta$ so, $y = \frac{(d \sin\theta)D}{d}$ or, $y = D \sin\theta$

CASE - IV

If a transparent film of refractive index μ and thickness 't' is introduced in front of any of the slits.

The central maxima shifts the side of the slit in which the slab is introduced, to compensate the path length

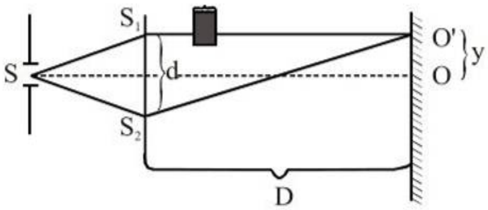
For the position of central maxima,

$$(S_1O' - t) \text{ in air} + t(\text{in } \mu) = S_2O' \text{ in air}$$

$$S_1O' - t + \mu t = S_2O' \text{ (optical path length)}$$

or $$S_2O' - S_1O' = (\mu - 1)t$$

$$\frac{yd}{D} = (\mu - 1)t \Rightarrow y = \frac{(\mu-1)tD}{d}$$



All the fringes shift by the same distance

Notice that this shift is in the direction of the slit before which the glass slab is placed. If the glass slab is placed before the upper slit, the fringe pattern gets shifted upwards and if the glass slab is placed before the lower slit the fringe pattern gets shifted downwards.

Note: Interference with white light is used to determine the position of central maxima in such cases.

Illustration :

One slit of a double slit experiment is covered by a thin glass plate of refractive index 1.4 and the other by a thin glass plate of refractive index 1.7. The point on the screen, where central bright fringe was formed before the introduction of the glass sheets, is now occupied by the 5th bright

One slit of a double slit experiment is covered by a thin glass plate of refractive index 1.4 and the other by a thin glass plate of refractive index 1.7. The point on the screen, where central bright fringe was formed before the introduction of the glass sheets, is now occupied by the 5th bright

fringe. Assuming that both the glass plates have same thickness and wavelength of light used is 4800 Å, find their thickness.

Sol. $\lambda = 4.8 \times 10^{-7}$ m

Optical path difference at the centre of the screen,

$$\Delta x = 5\lambda$$

$$\Delta x = |[D - t + 1.4t] - [D - t + 1.7t]|$$

$$\therefore \quad 5\lambda = 0.3\,t$$

$$t = \frac{5\lambda}{0.3} = \frac{5 \times 4.8 \times 10^{-7}}{0.3} = 8 \times 10^{-6} \text{ m}$$

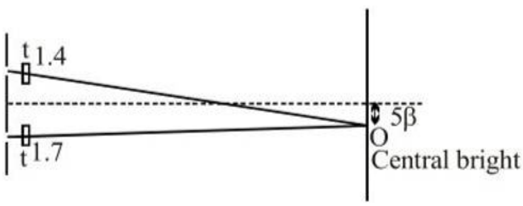
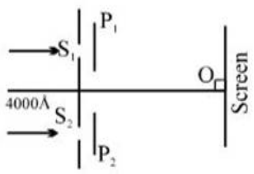


Illustration:

P_1, P_2 are transparent plates having equal thickness 20μm and refractive indices $\mu_1 = 1.6$, $\mu_2 = 1.5$. P_1 transmits 75% whereas P_2 transmits 50% of energy incident. Without P_1 and P_2 intensity at O, $I_0 = 4I$. Find intensity at O after placing P_1 and P_2. S_1, S_2 are identical slits.



Sol. $S_2P - S_1P = (\mu_1 - 1)t - (\mu_2 - 1)t$
$= 0.1\ t = 2 \times 10^{-6}\ m$

$$\Delta\phi = \frac{\Delta x \times 2\pi}{\lambda} = \frac{2\times 10^{-6} \times 2\pi}{4000\times 10^{-10}} = 10\pi$$

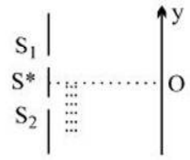
$I_1 = 0.75I \qquad I_2 = 0.5I$

$$I = 0.75I + 0.5I + 2\left(\sqrt{0.75\times 0.5}\right)I\cos\Delta\phi$$

$$= \frac{5I}{4} + I\sqrt{\frac{3}{2}}$$

Illustration :

The Young's double slit experiment is done in a medium of refractive index 4/3. A light of 600 nm wavelength is falling on the slits having 0.45 mm separation. The lower slit S_2 is covered by a thin glass sheet of thickness 10.4 μm and refractive index 1.5. The interference pattern is observed on a screen placed 1.5 m from the slits as shown



(a) *Find the location of the central maximum (bright fringe with zero path difference) on the y-axis.*

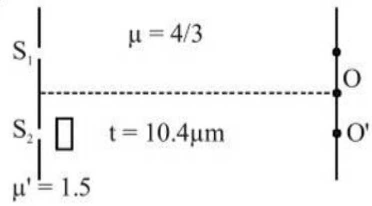
(b) *Find the light intensity at point O relative to the maximum fringe intensity.*

(c) *Now, if 600 nm light is replaced by white light of range 400 to 700 nm, find the wavelengths of the light that form maxima exactly at point O .*

(c) *Now, if 600 nm light is replaced by white light of range 400 to 700 nm, find the wavelengths of the light that form maxima exactly at point O .*

[All wavelengths in this problem are for the given medium of refractive index 4/3. Ignore dispersion

Sol. (a) *For central maxima, optical path difference, $\Delta x = 0$*



$d = 0.45\ mm,\ D = 1.5\ m$

$\Delta x = [\mu(S_2O' - t) + \mu' t] - \mu S_1O' = 0$

$O = (\mu' - \mu)t - \mu(S_1O' - S_2O')$

$$\left(1.5 - \frac{4}{3}\right)(10.4 \times 10^{-6}) = \frac{4}{3} \times y\frac{d}{D}$$

$y = 13/3\ mm$ *(below the centre)*

(b) *Optical path difference at the centre of the screen*

$$\Delta x = \frac{4}{3} \times (D - t) + 1.5t - \frac{4}{3}(D)$$

$$= 1.5t - \frac{4}{3}t = \frac{t}{6}$$

$$\Delta\phi = \frac{2\pi}{\lambda_{air}} \times \Delta x \qquad \{\lambda_{air} = \mu \times \lambda_{water}\}$$

$$= \frac{\pi t}{3\lambda\mu}$$

$$\frac{\Delta\phi}{2} = \frac{\pi t}{6\lambda\mu} = \frac{13\pi}{6}$$

$$I = I_0 \cos^2\frac{\phi}{2};\ I = I_0 \times \cos^2\left(\frac{13\pi}{6}\right)$$

(c) *Path difference in water medium at the centre of screen*

$$\Delta x = (D - t) + \frac{1.5t}{4/3} - D = \frac{3\times1.5}{4}\ t - t = \frac{t}{8}$$

$$\text{for maxima, } \frac{t}{8} = n\lambda\ \frac{10.4\times10^{-6}}{8} = n\times\lambda$$

we now have to calculate the wavelengths for which, centre of the screen is a maxima.
for λ_{min} *we get* n_{max} *and for* λ_{max} *we get* n_{min} *the integral values of n that lie between these two values of n will give the required* λ.

$$\therefore n_{min} = \frac{10.4\times10^{-6}}{8\times700\times10^{-9}} = 1.85$$

$$n_{max} = \frac{10.4\times10^{-6}}{8\times400\times10^{-9}} = 3.25$$

$$n_{max} = \frac{10.4\times10}{8\times400\times10^{-9}} = 3.25$$

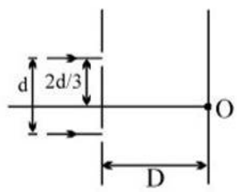
$$\therefore n = 2, 3$$

$$\therefore \lambda_1 = \frac{10.4\times10^{-6}}{8\times2} = 650\ nm \quad \Rightarrow \quad \lambda_2 = \frac{10.4\times10^{-6}}{8\times3} = 433.33\ nm$$

Practice Exercise

Q.1 A Young's double slit apparatus has slits separated by 0.28 mm and a screen 48 cm away from the slits. The whole apparatus is immersed in water and the slits are illuminated by the red light (l = 700 nm in vacuum). Find the fringe-width of the pattern formed on the screen.

Q.2 In the figure shown if a parallel beam of white light is incident on the plane of the slits then the find distance of the white spot on the screen from O. [Assume d << D, λ << d]

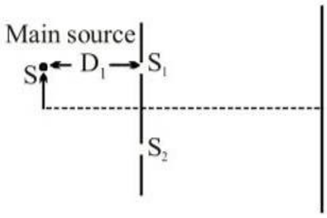


Q.3 In the above question if the light incident is monochromatic and point O is a maxima, then the wavelength of the light incident cannot be

(A) $d^2/3D$ (B) $d^2/6D$

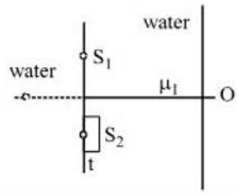
(C) $d^2/12D$ (D) $d^2/18D$

Q.4 In YDSE, th main source is displaced by a distance d = 2mm from the initial symmetric position, (parallel to the plane of the slits), as shown in figure. Given the distance of source from slits, D1 = 2m. d = 6mm, D = 3m. Find the displacement of fringe pattern.



Q.5 A plate of thickness t made of a material of refractive index μ is placed in front of one of the slits in a double slit experiment. What should be the minimum thickness t which will make the intensity at the centre of the fringe pattern zero ? Wavelength of the light used is λ. Neglect any absorption of light in the plate.

Q.6 A young's double slit experiment is conducted in water (μ_1) as shown in the figure, and a glass plate of thickness t and refractive index μ_2 is placed in the path of S_2. Wavelength of light in water is λ. Find the magnitude of the phase difference between waves coming from S_1 and S_2 at 'O'.



Answers

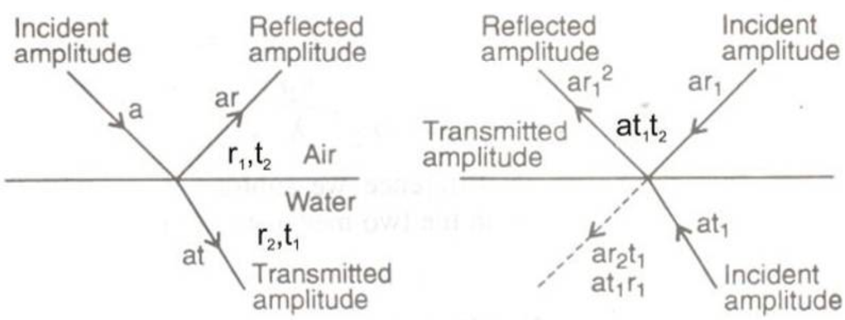
Answers

Q.1 0.90 mm Q.2 d/6 Q.3 A Q.4 3 mm

Q.5 $\frac{\lambda}{2(\mu-1)}$ Q.6 $\left|\left(\frac{\mu_2}{\mu_1}-1\right)t\right|\frac{2\pi}{\lambda}$

The phase change on reflection

A ray of light is incident on air-water interface ; let the amplitude reflection and transmission coefficients be r_1 and t_1 respectively. The amplitudes of reflected and transmitted waves are ar_1 and at_1 respectively. From the principle of reversibility of light, the system retraces its whole previous motion. The wave of amplitude ar_1 gives a reflected wave of amplitude ar_1t_1. The wave of amplitude at_1



gives a reflected wave of amplitude ar_2t_1 and transmitted wave amplitude at_1t_2.

So $$ar_1^2 + at_1t_2 = a$$

$$t_1t_2 = 1 - t_1^2 \qquad \text{... (1)}$$

Further, the waves of amplitudes at_1r_2 and ar_1t_1 must cancel each other.

$$at_1r_2 + ar_1t_1 = 0$$

$$r_2 = -r_1 \qquad \text{... (2)}$$

equation shows a difference of phase of π between the two cases ; a reversal of sign means a displacement in the opposite sense. If there is no change of phase on reflection from above, there must be a phase change of π from below and vice-versa.

When light gets reflected from a denser medium there is an abrupt phase change of π ; no phase change occurs when reflection takes place from rarer medium.

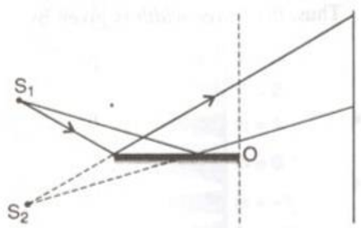
At the position of central maxima,the path differene between the two interfering waves is zero, so if O' is the new position of central maxima, then,

The Lloyd's mirror experiment

In this arrangement the light reflected from a long mirror and the light coming directly from the source

The Lloyd's mirror experiment

In this arrangement the light reflected from a long mirror and the light coming directly from the source without reflection produced interference on a screen. An important feature of this experiment lies in the fact that when the screen is placed in contact with the end of the mirror the edge O of the reflecting surface comes at the centre of a dark fringe instead of a bright fringe. The direct beam does not suffer any phase change. This means that the reflected beam undergoes a phase change of π radian. Hence at a point P on the screen the conditions for minima and maxima are

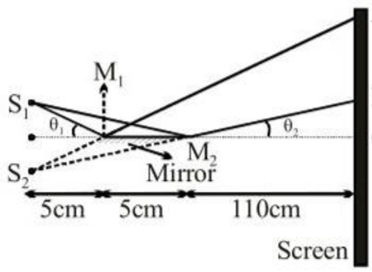


$$S_2P - S_1P = n\lambda,\ n = 0, 1, 2, 3 \text{ .... [minima]}$$

$$S_2P - S_1P = \left(n + \frac{1}{2}\right)\lambda \qquad \text{[maxima]}$$

Illustration :

A Lloyd's mirror of length 5 cm is illuminated with monochromatic light of wavelength $\lambda = 6000$ Å from a narrow slit 1 mm from its place and 5 cm in its plane from its near edge. Find the fringe width on a screen 120 cm from the slit and width of interference on the screen.



Sol. *In plane mirror, object and image distances are equal.*

So, d = 2mm = 0.2 cm ; $\lambda = 6000$ Å $= 6000 \times 10^{-8}$ cm ; $D = 120$ cm

$$\therefore \text{Fringe width } \beta = \frac{\lambda D}{d} = \frac{6\times10^{-5}\times120}{0.20} = 0.036 \text{ cm}$$

The width of the fringe pattern is AB. From the figure.

$$\tan\theta_1 = \frac{0.1}{5} = \tan\theta_2 = \frac{0.1}{10}$$

$$\therefore \text{Fringe width } \beta = \frac{\lambda D}{d} = \frac{6\times10^{-5}\times120}{0.20} = 0.036 \text{ cm}$$

The width of the fringe pattern is AB. From the figure.

$$\tan\theta_1 = \frac{0.1}{5} = \tan\theta_2 = \frac{0.1}{10}$$

In right angled triangle AM_1O and BM_2O

$$\tan\theta_1 = \frac{0.1}{5} = \frac{OA}{M_1O}$$

or $$OA = 115 \times \frac{0.1}{5} \text{ cm}$$

and $$\tan\theta_2 = \frac{0.1}{10} = \frac{OB}{OM_2}$$

or $$OB = 110 \times \frac{0.1}{10} \text{ cm}$$

$$\therefore \text{Width of fringe pattern} = OA - OB = \frac{115\times0.1}{5} - \frac{110\times0.1}{10} = 1.2 \text{ cm}$$

Billet's Split-lens

This device consists of two halves of a convex lens placed close together to from two real or virtual images S_1 and S_2 of the narrow slit S illuminated by a monochromatic source of light. S_1 and S_2 now act in the same way as the double slit in Young's experiment. The distance between S_1 and S_2 can be charged by adjusting the space between the two halves of the convex lens, a number of interference bands of varying widths can be obtained and observed in the overlapping region.

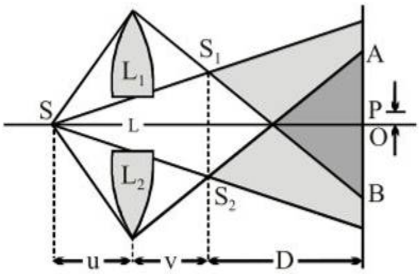


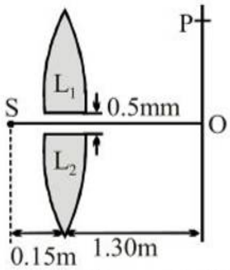
Illustration :

In figure showsn, S is a monochromatic point source emitting light of wavelength λ = 500 nm. A thin lens of circular shape and focal length 0.10 m is cut into two identical halves L_1 and L_2 by a plane passing through a diameter. The two halves are placed symmetrically about the central axis SO with a gap of 0.5 nm. The distance along the axis from S to L_1 and L_2 is 0.15 m, while that from L_1 and L_2 to O is 1.30 m. the screen at O is normal to SO.

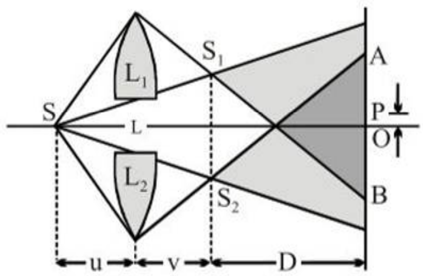
SO with a gap of 0.5 nm. The distance along the axis from S to L_1 and L_2 is 0.15 m, while that from L_1 and L_2 to O is 1.30 m. the screen at O is normal to SO.

(i) If the third intensity maximum occurs at the point P on the screen, find distance OP.

(ii) If the gap between L_1 and L_2 is reduced from its original value of 0.5 mm, will the distance OP increase, decrease or remain the same ?



Sol. *(i) As shown in figure each part of the lens will form image of S which will act as coherent sources. From lens equation, we can write*



$$\frac{1}{v} - \frac{1}{-15} = \frac{1}{10}$$

or $v = 30\ cm$

$$m = \frac{-v}{u} = -2$$

Also, $d = 3 \times 0.5\ mm = 1.5\ mm$

$D = 1.30 - 0.30 = 1\ m$

Now, from the theory of interference the distance y of a point P on the screen is given by

$$y = \frac{D}{d}(\Delta x)$$

and as point is third maximum

$$\Delta x = 3\lambda$$

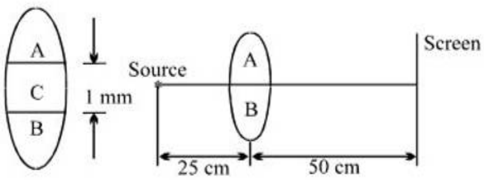
So, $$y = \frac{D}{d}(3\lambda)$$

or $$y = \frac{5\times10^{-7}}{0.5\times10^{-3}} \times 10^{-3}\ m = 1\ mm$$

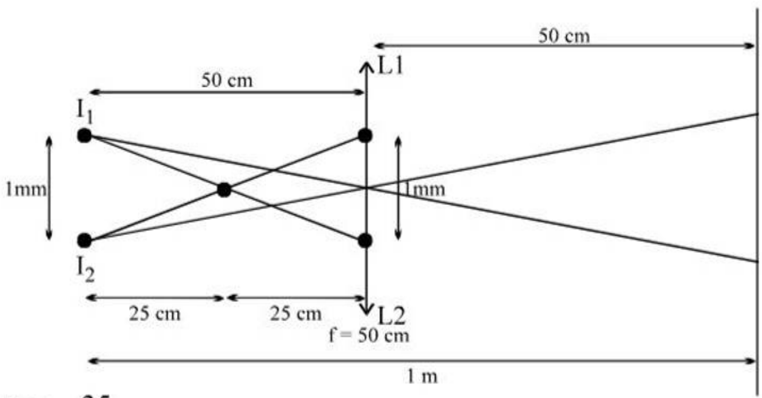
(ii) *If gap between* L_1 *and* L_2 *is reduced then d will decrease. As* $\beta = \frac{D\lambda}{d}$ *and* $OP = 3\beta$, *therefore OP will increase.*

Illustration :

A convex lens of focal length 50 cm is cut along the diameter into two identical halves A and B and in the process a layer C of the lens thickness 1 mm is lost. Then the two halves A and B are put together to form a composite lens. Now infront of this composite lens a source of light emitting wavelength $\lambda = 6000\ \mathring{A}$ *is placed at a distance of 25 cm as shown in the figure. Behind the lens there is a screen at a distance 50 cm from it. Find the fringe width of the interference pattern obtained on the screen.*



Sol.



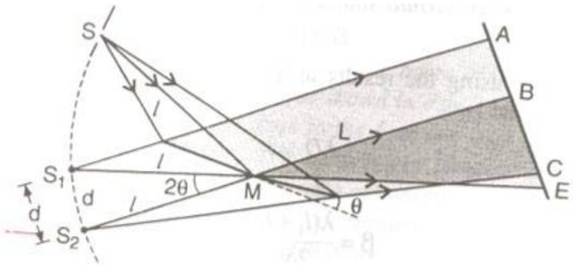
$u = -25$

$$\frac{1}{v} - \frac{1}{-25} = \frac{1}{50} \Rightarrow \frac{1}{v} = -\frac{1}{50} \Rightarrow v = -50$$

$$\beta = \frac{6\times10^{-7}\times1}{10^{-3}} = 6\times10^{-4} = 0.6\ mm$$

Fresnel's Mirrors

Figure shows Fresnel's bimirrors apparatus to produce interference by division of the wavefront. Light from a slit S is reflected by two plane mirrors slightly inclined to each other.



The mirrors produce two virtual images S_1 and S_2 of the slit, the interference fringes are observed in the region BC, where the reflected beams overlap. If θ is the angle between the planes of the mirrors, then S_1 and S_2 subtend angle 2θ at the point of intersection M between the mirrors.

If l is the distance between the slit and the mirrors intersection and L is the distance between the screen and the mirrors intersection, then the separation between the images S_1 and S_2 is

$$d = l(2\theta) = 2l\theta$$

and $$D = l + L$$

$$d = l(2\theta) = 2l\theta$$

and $$D = l + L$$

Thus, the fringe width is given by

$$\beta = \frac{\lambda D}{d}$$

or $$\beta = \frac{\lambda(l + L)}{2l\theta}$$

or $$\beta = \frac{\lambda}{2\theta}\left[1 + \frac{L}{l}\right]$$

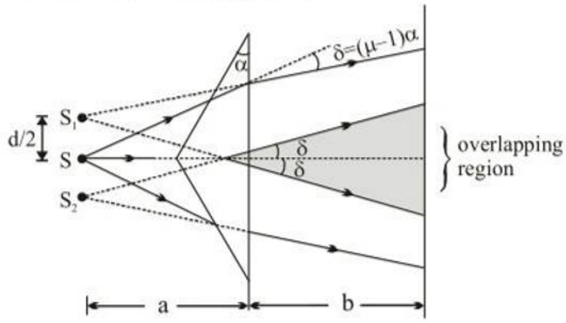
Fresnel's Biprism

Figure shows the Fresnel's biprism experiment schematically. The thin prism P refracts light from the slit source S into two beams. When a screen is placed as shown in the figure, the interference fringes are observed only in the region shown.

If A is the angle of refraction of the thin prism and μ is the refractive index of its medium, then the angle of deviation produced by the prism is

$$\delta = A(\mu - 1)$$

(i) In numerical problems 'd' is calculated as given below :



$$\frac{d}{2} = a \tan \delta \approx a\,\delta = a\,(\mu - 1)\,\alpha$$

$$d = 2a\,(\mu - 1)\,\alpha$$

'd' can also be calculated using lens displacement method and it is given by

$$d = \sqrt{d_1 d_2}$$

where d_1 and d_2 are the distances between images of S_1 and S_2 in two positions of a convex lens placed between the biprism and the screen,

(ii) The expression for fringe width is same as in YDSE

i.e., $$\beta = \frac{D\lambda}{d} = \frac{(a+b)\lambda}{2a(\mu-1)\alpha}$$

and interference pattern consists of alternate bright and dark fringes.

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and interference pattern consists of alternate bright and dark fringes.

If source is at infinity i.e., $a \to \infty$ then $\beta = \dfrac{\lambda}{2(\mu-1)\alpha}$

(iii) Let L = length of overlapping region from figure we have

$$\frac{L}{b} = \frac{d}{a} \quad \text{or} \quad L = \frac{bd}{a}$$

Also, No = No. of fringes

$$= \frac{\text{Length of interference pattern}}{\text{Fringe width}} = \frac{L}{\beta}$$

Illustration :

Interference bands are produced by a Fresnel's biprism in the focal plane of a reading microscope. The focal plane is 100 cm distant from the slit. A lens is inserted between the biprism and microscope and gives two images of the slit for two position of lens. In one, separation between them is 4.05 nm and in order 2.90 mm. If sodium light is used, find the distance between interrecne bands. 'λ' for sodium light = 5886 × 10⁻⁸ cm.

Sol. *Here* $\lambda = 5886 \times 10^{-8}$ *cm ;* $D = 100$ *cm ;* $d_1 = 4.05$ *mm* $= 0.405$ *cm*

$d_2 = 2.90$ *mm* $= 0.290$ *cm*

$$d = \sqrt{d_1 d_2} = \sqrt{0.405 \times 0.290}$$

$$\beta = \frac{\lambda D}{d} = \frac{5886 \times 10^{-8} \times 100}{\sqrt{0.405 \times 0.290}} = 0.017 \textit{ Ans.}$$

Illustration :

In a biprism experiment with sodium light, bands of width 0.0195 cm are observed at 100 cm from the slit. On introducing a convex lens 30cm away from the slit, two images of the slit are seen 0.7 cm apart, at 100 cm distance from the slit. Calculate the wavelength of sodium light.

Sol. $\beta = \frac{\lambda D}{d}$ or $\lambda = \frac{\beta d}{D}$

Here, $\beta = 0.0195$ *cm ;* $D = 100$ *cm*

For a convex lens $\frac{I}{O} = \frac{v}{u}$, $v + u = 100$ *cm*

$u = 30$ *cm or* $\frac{0.7}{O} = \frac{70}{30}$ cm *or* $O = 0.30$ *cm*

i.e., Distance between the two coherent sources

$d = O = 0.30$ *cm*

$\therefore \quad \lambda \frac{0.0195 \times 0.30}{100} = 5850 \times 10^{-8}$ *cm or* $\lambda = 5850$ *Å*

Thin film interference

When light passes the boundary between two transparent media, some light is reflected at the boundary. As shown in the figure some light is reflected from first interface and some from second interface. If we consider a monochromatic incident light the two reflected waves are also monochromatic and coherent because they arise from the same monochromatic incident light wave via amplitude division. These waves interfere, since they are superposed along the same normal line.

The phase difference between two interfering waves is due to :

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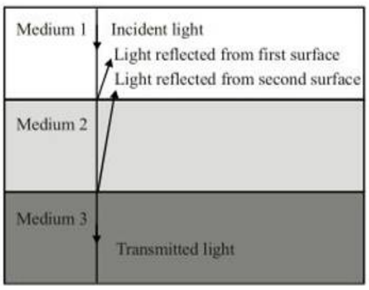
The phase difference between two interfering waves is due to :

(1) Optical path difference (due to distances travelled),

(2) Reflection from a denser medium.



The second factor is irrelevant for reflection at rarer medium.

Three situations may arise :

(1) Neither wave experiences a phase change upon reflection.

(2) Both the waves suffer a phase change upon reflection.

In either of these two cases the phase change due to reflection is irrelevant ; no difference in phase results due to reflection.

In either of these cases phase change is determined solely from optical path difference.

Condition for constructive interference :

$$2\mu t = m\lambda$$

Condition for destructive interference :

$$2\mu t = \left(m + \frac{1}{2}\right)\lambda$$

where m = 0, 1, 2, ...

(3) One of the reflected waves experiences a phase change of π radian upon reflection and the other waves does not.

It is material which wave suffers a phase change ; the conclusions in the previous case are first reversed.

Condition for destructive interference :

$$2\mu t = m\lambda$$

Condition for constructive interference :

$$2\mu t = \left(m + \frac{1}{2}\right)\lambda$$

where m = 0, 1, 2, ...

Illustration :

Many people's glasses appear to be a blue-green colour when viewed under reflected light. A thin film of index of refraction n = 1.35 is applied to the outside surface of the glass so that the film/

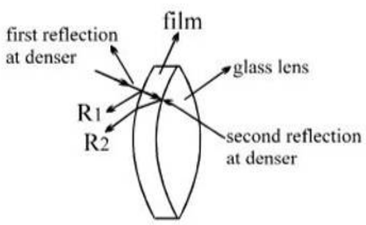
where m = 0, 1, 2, ...

Illustration :

Many people's glasses appear to be a blue-green colour when viewed under reflected light. A thin film of index of refraction n = 1.35 is applied to the outside surface of the glass so that the film/
glass interface does not reflect any red light incident near normal of wavelength λ = 630 nm. What thickness must the film layer be in order to achieve this? Take the index of refractions of air and glass to be 1.0 and 1.6 respectively.

(A) 157.5 nm *(B) 315.0 nm* *(C) 233.3 nm* *(D) 116.7 nm*

Sol.

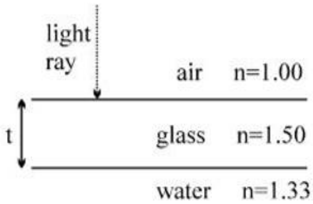


$$\Delta X_{net} = 2\mu t = \lambda/2$$

$$t = \lambda/4\mu = \frac{(630\times10^{-9})}{4\times1.35} = 116.7\ nm$$

Illustration :

A light ray is incident normal to a thin layer of glass. Given the figure, what is the minimum thickness of the glass that gives the reflected light an orangish color (λ_{air} = 600 nm)?



(A) 50 nm *(B) 100 nm* *(C) 150 nm*
(D) 200 nm *(E) 500 nm*

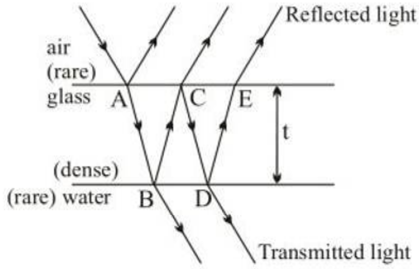
Sol. *For reflected light to have orangish color, rays from A, C, E must be out of phase for l = 600 nm*

or $\delta = (2n + 1)\,\pi$

or $2\mu_g t\,(2n + 1)\,\dfrac{\lambda}{2}$

i.e. $t = (2n+1)\dfrac{\lambda}{4\mu_g}$

or $t_{min} = \dfrac{\lambda}{4\mu_g} = 100\ nm$



Interference due to reflected light :

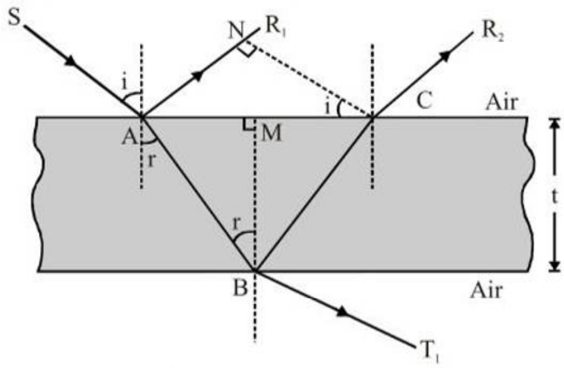
Consider a transparent film of thickness t and refractive index μ. A ray SA incident on the upper surface of the film is partly reflected along AR_1 and partly refracted along AB. At B part of it is reflected along BC and finally emerges out along CR_2. The difference in path between the two rays. AR_1 and CR_2 is

Interference due to reflected light :

Consider a transparent film of thickness t and refractive index μ. A ray SA incident on the upper surface of the film is partly reflected along AR_1 and partly refracted along AB. At B part of it is reflected along BC and finally emerges out along CR_2. The difference in path between the two rays. AR_1 and CR_2 is calculated as given below :

Let CN and BM be perpendicular to AR_1 and AC. As the paths of the rays AR_1 and CR_2 beyond CN are equal. The path difference between them is

$$\Delta x = \text{Path ABC in film} - \text{Path AN in air}$$
$$= \mu(AB + BC) - AN = 2\mu\, AB - AN$$



Now, $AB = BC = \dfrac{AB}{BM} \times BM = BM \sec r = t \sec r$

and $AN = \dfrac{AN}{AC} \cdot AC = AC \sin i = 2\, AM \sin i$

$$= 2\frac{AM}{BM}\,BM\sin i = 2\,(\tan r)\,t\left(\frac{\sin i}{\sin r}\sin r\right)$$

$$= 2\mu t\,\frac{\sin^2 r}{\cos r} = 2\mu t\sec r\sin^2 r$$

Then, $\Delta = 2\mu AB - AN = 2\mu t\sec r - 2\,\mu t\sec r\sin^2 r$

$= 2\,\mu t\sec r\,(1 - \sin^2 r) = 2\,\mu t\cos r$

The ray AR_1 having suffered a reflection at the surface of denser medium undergoes a phase change π or path diff. of $\frac{\lambda}{2}$.

At B the reflection takes place when the ray is going from a denser to rarer medium and there is no phase change.

Hence, the effective path difference between AR_1 and CR_2 is given by

$$\text{Path Diff. } (\Delta x) = 2\,\mu t\cos r - \frac{\lambda}{2}$$

(i) If the path difference $\Delta x = n\lambda$ where $n = 0, 1, 2, 3, 4$ etc., constructive interference takes place and the film appears bright.

$\therefore \quad 2\mu t\cos r - \frac{\lambda}{2} = n\lambda$

(i) If the path difference $\Delta x = n\lambda$ where $n = 0, 1, 2, 3, 4$ etc., constructive interference takes place and the film appears bright.

$\therefore \quad 2\mu t\cos r - \frac{\lambda}{2} = n\lambda$

or $\quad 2\mu t\cos r = (2n+1)\frac{\lambda}{2}$

(ii) If the path difference $\Delta x = (2n+1)\frac{\lambda}{2}$ where $n = 0, 1, 2, \ldots\ldots\ldots$ etc., destructive interference takes place and the film appears dark.

$\therefore \quad 2\mu t\cos r - \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2}$ or $2\mu t\cos r = n\lambda$

Remarks :

(i) If the thickness of the film is very small as compared to the wavelength of light used, so that $2\mu t\cos r$ can be neglected, then the total path difference between AR_1 and CR_2 will reduce to $\frac{\lambda}{2}$. Thus two rays will interfere destructively and darkness will result.

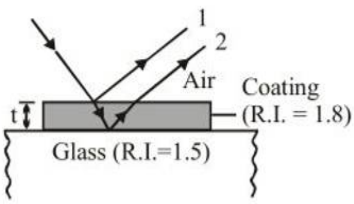
(ii) It should be remembered that the interference pattern will not be perfect because the intensities of the ray AR_1 and CR_2 will not be the same.

Illustration :

A glass plate of refractive index 1.5 is coated with a thin layer of thickness t and refractive index 1.8. Light of wavelength λ travelling in air is incident normally on the layer. It is partly reflected at the upper and the lower surfaces of the layer and the two reflected rays interfere. Write the condition for their constructive interference. If λ = 648 nm, obtain the least value of t for which the rays interfere constructively.

Sol. *The ray reflected from upper surface suffer a phase change of π due to reflection, at denser media, so the condition of sonstructive interference for normal incidence is given by*

$$2\mu t+\frac{\lambda}{2}=n\lambda \qquad or \qquad 2\mu t=\frac{(2n-1)\lambda}{2}$$



For minimum value of t, $n = 1$

$$t_{min}=\frac{\lambda}{4\mu}=90nm$$

Illustration :

White light may be considered to have λ from 400 0Å to 7500 Å. If an oil film has thickness 10^{-4}

$$t_{min}=\frac{\lambda}{4\mu}=90nm$$

Illustration :

White light may be considered to have λ from 400 0Å to 7500 Å. If an oil film has thickness 10^{-4} cm, deduce the wavelength in the visible region for which the reflection along the normal direction will be (i) weak (ii) strong. Take μ of oil as 14.

Sol. *Here* $r = 0° ;\ \mu = 1.4 ;\ t = 10^{-4}$ cm

$\therefore$ $2\ \mu t = 2 \times 1.4 \times 10^{-4}$ cm $= 2.8 \times 10^{-4} \times 10^{8}$ Å $= 28000$ Å

(i) *Condition for weak reflection (destructive interference) is given by*

$2\ \mu t = n\lambda$

or $$\lambda=\frac{2\mu t}{n}=\frac{28000}{n}$$

The value of n should be selected such that λ lies between 4000 Å and 7500 Å. This will be possible if

$$\lambda=\frac{28000}{4}=7000\,\text{Å} \qquad (for\ n = 4)$$

$$\lambda=\frac{28000}{5}=5600\,\text{Å} \qquad (for\ n = 5)$$

$$\lambda=\frac{28000}{6}=4667\,\text{Å} \qquad (for\ n = 6)$$

$$\lambda = \frac{28000}{7} = 4000\,\mathring{A} \qquad \textit{(for n = 7)}$$

The other values of n are not allowed as for those value of n, λ does not lie within the given wavelength range of 4000 Å to 7500 Å. Hence, all above values of λ cause weak reflection.

(ii) *For strong reflection (constructive interference), we have*

$$2\,\mu t = (2n + 1)\left(\frac{\lambda}{2}\right)$$

$$\therefore \qquad \lambda = \frac{2\times 2\mu t}{2n+1} = \frac{2\times 28000}{2n+1} = \frac{56000}{2n+1}$$

The possible values of λ in this case are given by

$$\lambda = \frac{56000}{9} = 6222\,\mathring{A} \qquad \textit{(for n = 4)}$$

$$\lambda = \frac{56000}{11} = 5091\,\mathring{A} \qquad \textit{(for n = 5)}$$

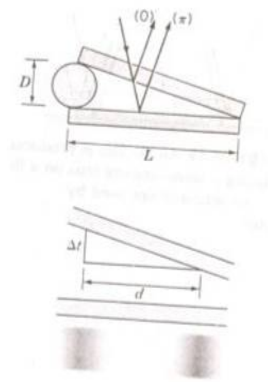
$$\lambda = \frac{56000}{13} = 4300\,\mathring{A} \qquad \textit{(for n = 6)}$$

Hence, only the above of n will cause strong reflection because the range will not be within desired wavelengths, if n is different.

desired wavelengths, if n is different.

Fringes of equal thickness

Soap bubbles and oil films on a road do not have uniform thickness of the film at any given point determines whether the reflected light has a maximum or minimum intensity. When white light is used, each wavelength has its own fringe pattern. At a given point of the film, one wavelength may be enhanced and / or another wavelength suppressed. This is the source of the colors in soap bubbles an oil films on the road.



A wedge-shaped film of air may be produced by placing a sheet of paper or a hair between the ends of two glass plates, as in fig. With flat plates, one sees a series of bright and dark bands, each characteristic of a particular thickness. If the plates are not flat, the fringes are not straight ; each is locus of points with the same thickness. If one plate is known to be flat, the fringes display the irregularities of the other, as shown in figure. The pattern shows where the plate needs to be polished for it to be made "optically flat."

Illustration :

A wedge-shaped film of air is produced by placing a fine wire of diameter D between the ends of two flat glass plates of length L = 20 cm, as in fig. When the air film is illuminated with light of wavelength λ = 550 nm, there are 12 dark fringes per centimeter. Find D.

Sol. *A indicated in fig. only one of the reflected ray suffers a phase inversion. At the thin end of the wedge, where the thickness is less than $\lambda/4$, the two rays interfere destructively. This region is dark in the reflected light. The condition for destructive interference in the reflected light is*

$$2t = m\lambda \qquad m = 0, 1, 2, ...$$

the change in thickness between adjacent dark fringes is $\Delta t = \lambda / 2$. The horizontal spacing between fringes d = 1/12 cm = 8.3×10^{-4} m. From figure we see that $D / L = \Delta t / \delta$, so

$$\Delta = \frac{\lambda L}{2d} = \frac{(5.5\times10^{-7}\,m)(0.2m)}{16.6\times10^{-4}\,m}$$

Thus $D = 6.6 \times 10^{-5}$ m

Newton's Rings

When a lens with a large radius of curvature is placed on a flat plate, as to fig. a thin film of air is formed. When the film is illuminated with monochromatic light, circular fringes, called newton's rings, can be with the unaided eye or with a low power microscope (figure). An important feature of Newton's rings is the dark central spot. Newton tried polishing the surfaces to get rid of it, The dark spot was also initially puzzling to Young. It implied that the light wave suffers a phase inversion on reflection at a medium with a higher refractive index. Young tested this idea by placing oil of sassafras between a lens of crown glass and a plate of flint glass. The refractive index of the oil is between the values for these two glasses. Since

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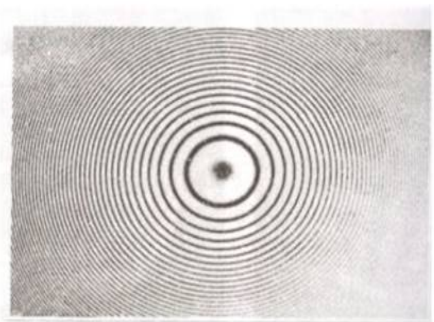
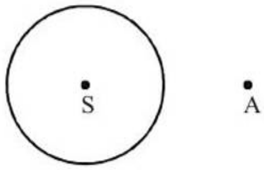
both reflections occur at a medium with a higher refractive index, they should both suffer a phase inversion and therefore be in phase be in phase. This is precisely what happened : The central spot became bright and undoubtedly gave Young much satisfaction.



Huygen's Principle

Huygens, the Dutch physicist and astronomer of the seventeenth century, gave a beautiful geometrical description of wave propagation. We can guess that he must have seen water waves many times in the canals of his native place Holland. A stick placed in water and oscillated up and down becomes a source of waves. Since the surface of water is two dimensional, the resulting wavefronts would be circles instead of spheres. At each point on such a circle, the water level moves up and down.

Huygens, considered light to be a mechanical wave moving in a hypothetical medium which was named as ether. If we consider a surface σ enclosing a light source S, the optical disturbance at any point beyond σ must reach after crossing σ. The particles of the surface σ vibrate as the wave from S reaches there and these vibrations cause the layer beyond to vibrate. We can thus assume that the particles on σ act as new sources of light waves emitting spherical waves and the disturbance at a point A (figure 17.1) beyond σ is caused by the superposition of all these spherical waves coming from different points of σ. Huygens called the particles spreading the vibration beyond them as *secondary sources* and the spherical wavefronts emitted from these secondary sources as the *secondary wavelets.*



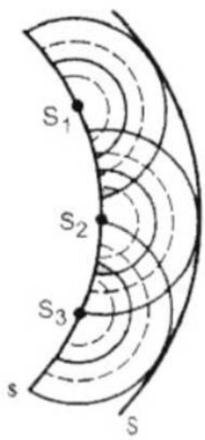
Huygens' principle may be stated in its most general form as follows :

Various points of an arbitrary surface, when reached by a wavefront, become secondary sources of

Huygens' principle may be stated in its most general form as follows :

Various points of an arbitrary surface, when reached by a wavefront, become secondary sources of light emitting secondary wavelets. The disturbance beyond the surface results from the superposition of these secondary wavelets.

Consider a spherical surface σ with its centre at a point source Semitting a pulse of light The optical disturbance reaches the particles on σ at time t =0 and lasts for a short interval in which the positive and negative disturbances are produced. These particles on σ then send spherical wavelets which spread beyond σ At time t, each of these wavelets has a radius υt In figure the solid lines represent positive optical disturbance and the dashed lines represent negative optical disturbance. The sphere Σ is the geometrical envelope of all the secondary wavelets which were emitted at time t = 0 from the primary wavefront σ.



It is clear that at the points just inside Σ, only the positive disturbances of various secondary wavelets are meeting. The wavelets, therefore, interfere constructively at these points and produce finite disturbance. For points well inside Σ, some of the wavelets contribute positive disturbance and some others, centred at a nearby point of σ produce negative disturbance. Thus, the resultant disturbance is zero at these points. The disturbance which was situated at σ at time t = 0 is, therefore, confined to a surface Σ at time t. Hence, the secondary wavelets from σ superpose in such a way that they produce a new wavefront at the geometrical envelope of the secondary wavelets.

This allows us to state the method of Huygens construction as follows :

Huygens construction

Various points of an arbitrary surface, as they are reached by a wavefront, become the sources of secondary wavelets. The geometrical envelope of these wavelets at any given later' instant represents the new position of the wavefront at that instant.

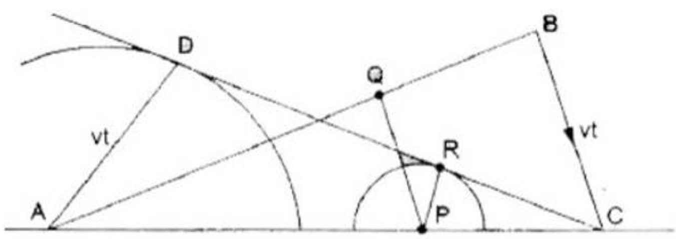
The method is quite general and although it was developed on the notion of mechanical waves it is valid for light waves. The surface used in the Huygens construction may have any arbitrary shape, not necessarily a wavefront itself. If the medium is homogeneous, (i.e., the optical properties of the medium are same

everywhere) light moves forward and does not reflect back. We assume, therefore, that the secondary

everywhere) light moves forward and does not reflect back. We assume, therefore, that the secondary wavelets are emitted only in the forward direction and the geometrical envelope of the wavelets is to be taken in the direction of advancement of the wave. If there is a change of medium, the wave may be reflected from the discontinuity just as a wave on a string is reflected from a fixed end or a free end. In that case, secondary wavelets on the backward side should also be considered.

Reflection of Light

Let us suppose that a parallel light beam is incident upon a reflecting plane surface σ such as a plane mirror. The wavefronts of the incident wave will be planes perpendicular to the direction of incidence. After reflection, the light returns in the same medium. Consider a particular wavefront *AB* of the incident light wave at *t* = 0 (figure). We shall construct the position of this "Yavefront at.time *t*.



To apply Huygens construction, we use the reflecting surface σ for the sources of secondary wavelets.

As the various points of (J are reached by the wavefront AB, they become sources of secondary wavelets. Because of the change of medium, the wavelets are emitted both in forward and backward directions. To study reflection, the wavelets emitted in the backward directions are to be considered.

Suppose, the point A of σ is reached by the wavefront AB at time t = O. This point then emits a secondary wavelet. At time t, this wavelet becomes a hemispherical surface of radius υt centred at A. Here υ is is the speed of light. Let C be the point which is just reached by the wavefront at time t and hence the wavelet is a point at C itself. Draw the tangent plane CD from C to the hemispherical wavelet originated from A. Consider an arbitrary point P on the surface and let AP/AC = x. Let PQ be the perpendicular from P to AB and let PR be the perpendicular from P to CD. By the figure,

$$\frac{PR}{AD} = \frac{PC}{AC} = \frac{AC - AP}{AC} = 1 - x$$

or, $\quad PR = AD\,(1 - x) = \upsilon t\,(1 - x) \qquad$ ...(i)

Also, $\quad \dfrac{QP}{BC} = \dfrac{AP}{AC} = x$

or $\quad QP = x\,BC = x\upsilon t.$

The time taken by the wavefront to reach the point P is, therefore,

$$t_1 = \frac{QP}{\upsilon} = xt$$

The point P becomes a source of secondary wavelets at time t_1. The radius of the wavelet at time t,originated from P is, therefore,

$$a = u\,(t - t_1) = u\,(t - xt) = ut\,(1 - x). \qquad \text{...(ii)}$$

By (i) and (ii), we see that PR is the radius of the secondary wavelet at time t coming from P. As CD is perpendicular to PR, CD touches this wavelet. As P is an arbitrary point on σ all the wavelets originated from different points of σ touch CD at time t. Thus, CD is the envelope of all these wavelets at time t. It is, therefore, the new position of the wavefront AB. The reflected rays are perpendicular to this wavefront CD.

In triangles ABC and ADC :

$$AD = BC = \upsilon t,$$

AC is common.

and $\quad \angle ADC = \angle ABC = 90^o$

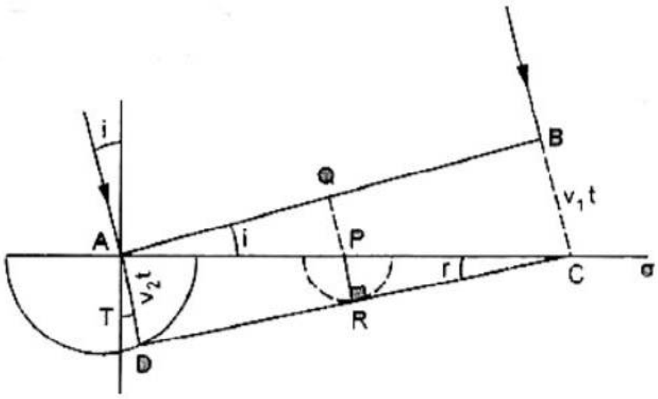
Thus, the triangles are congruent and

$$\angle BAC = \angle DCA \qquad \text{...(iii)}$$

Now, the incident ray is perpendicular to AB and the normal is perpendicular to AC. The angle between the incident ray and the normal is, therefore, equal to the angle between AB and AC. Thus, $\angle BAC$ is equal to the angle of incidence.

Similarly, $\angle DCA$ represents the angle of reflection and we have proved in (iii) that the angle of incidence equals the angle of reflection. From the geometry, it is clear that the incident ray, the reflected ray and the normal to the surface AC lie in the plane of drawing and hence, are coplanar.

Refraction of Light



Suppose σ represents the surface separating two transparent media, medium 1 and medium 2 in which the speeds of light are υ_1 and υ_2 respectively. A parallel beam of light moving in medium 1 is incident

or $$\frac{\sin i}{\sin r} = \frac{\upsilon_1}{\upsilon_2}$$

which is called the Snell's law. The ratio υ, / υ, is called the refractive index of medium 2 with respect to

or $$\frac{\quad}{\sin r} = \frac{\quad}{\upsilon_2}$$

which is called the Snell's law. The ratio υ_1 / υ_2 is called the refractive index of medium 2 with respect to medium 1 and is denoted by μ_{21}. If the medium 1 is vacuum, μ_{21} is simply the refractive index of the medium 2 and is denoted by μ.

$$\mu_{21} = \frac{\upsilon_1}{\upsilon_2} = \frac{c/\upsilon_2}{c/\upsilon_1} = \frac{\mu_2}{\mu_1}$$

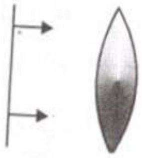
From the figure, it is clear that the incident ray, the refracted ray and the normal to the surface σ are all in the plane of the drawing, i.e., they are coplanar.

Suppose light from air is incident on water. It bends towards the normal giving $i > r$ From Snell's law proved above, $\upsilon_1 > \upsilon_2$. Thus, according to the wave theory the speed of light should be greater in air than in water. This is opposite to the prediction of Newton's corpuscle theory. If light bends due to the attraction of the particles of a medium then speed of light should be greater in the medium. Later, experiments on measurement of speed of light confirmed wave theory.

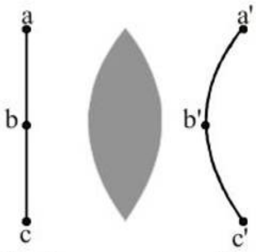
Thus, the basic rules of geometrical optics could be understood in terms of the wave theory of light using Huygens' principle.

Illustration :

If a plane wavefront is incident on a convex lens as shown in figure, how will transmitted wavefront appear.



Sol.



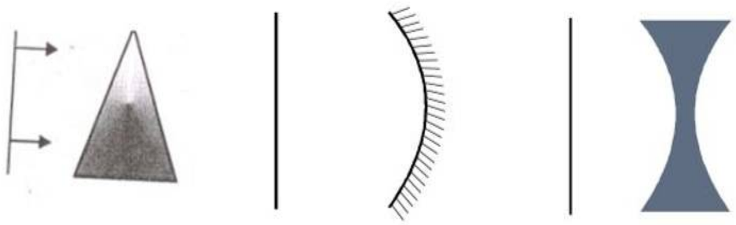
a, b and c lie on a wavefront i.e. they are in phase.

We have shown in figure that a reaches a', b reaches b' and c reaches c'.

a' and c' are ahead of b' because b has to travel more in denser medium in which velocity of light is less in comparison to air.

Practice Exercise

Q.1 If a plane wavefront is incident on an optical device as shown in figure, how will transmitted wavefront appear.



(A) (B) (C)

Answers

Ans. (A)

(B)

(C)

Why is interference observed only in thin films

Interference effects can be ignored with a thick film because it thickness is large. In this case the alternate bright and dark regions will be so close to each other that these will appear to merge into one another and interference pattern will not be visible.

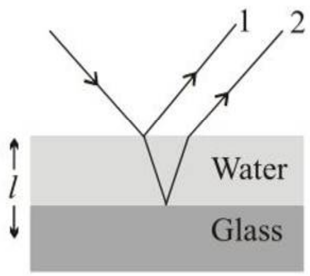
Further, when we see such a film in white light, the various complementary colors will be so close to each other that these merge into one another and make the appearance of the thick film white. To understand this let us consider film of water (refractive index, $\mu = 4/3$) 1cm thick on top of a glass surface (RI = 3/2).

Assume that it is illuminated from above (fig.). Then

$$\Delta x = \text{path difference between 1 and 2} = 2\mu_{\omega}t.$$

For constructive interference in the reflected waves

$$2m_{\omega}t = n\lambda$$



Because t = 1 cm is very large in comparison to the wavelength of visible light, the values of n in this equation will be large. For instance, if wavelength corresponding to orange-red light ($\lambda = 667$nm) is to be strongly reflected then

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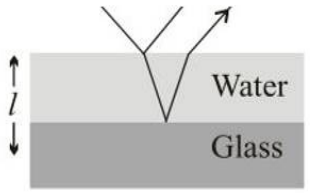


Because t = 1 cm is very large in comparison to the wavelength of visible light, the values of n in this equation will be large. For instance, if wavelength corresponding to orange-red light ($\lambda = 667$nm) is to be strongly reflected then

$$n = \frac{2\mu_{\omega}t}{\lambda} = \frac{(2)(1.33)(1\times10^{-2})}{6.67\times10^{-7}} = 40000$$

We may feel that this thick film would appear strongly orange red ; however, there will also be strong reflection of a slightly longer wavelength corresponding to n = 39,999 and of slightly shorter wavelength corresponding to n = 40,001. There wavelengths differ from each other by only one part in 40,000 or 667 nm/40,000 = 0.17nm. This is a much smaller wavelength difference than the eye can detect. Hence, no one wavelength appears to be reinforced more than any other in the light which is reflected from a thick film. If the film is illuminated by a white light, the reflected light appears white. For thin films, the integer n will be small, the difference between adjacent strongly reflected wavelengths will be substantial, and the preferential reflection of certain wavelengths will be easily observed by the eye.

SOLVED EXAMPLES

Q.1 In Young's double slit experiment using monochromatic light, the fringe pattern shifts by a certain distance on the screen when a mica sheet of refractive index 1.6 and thickness 1.964 micron is introduced in the path of one of interfering waves. The mica sheet is then removed and the distance between the slits and the screen is doubled. It is found that the distance between successive maxima (or minima) now is the same as the observed fringe shift on the introduction of mica sheet. Calculate the wavelength of the monochromatic light used in the experiment.

Sol. Shift 'Δy' in the fringe system is

$$\Delta y = \frac{\beta}{\lambda}(\mu - 1)\,t$$

when distance between slits and screen is doubled,

$$\beta' = \beta$$

Given $\quad \beta' = \Delta y$

$\therefore \quad \frac{\beta}{\lambda}(\mu - 1)t = 2\beta$

$\therefore \quad \lambda = \frac{(\mu - 1)t}{2}$

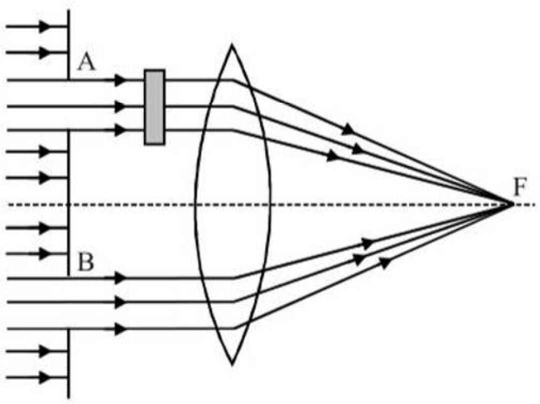
Here, $\quad \mu = 1.6,\ t = 1.964 \times 10^{-6}$ m

$$\lambda = \frac{(1.6 - 1) \times 1.965 \times 10^{-6}}{2}$$

$$\lambda = 0.3 \times 1.964 \times 10^{-6} \text{ m}$$

$$\lambda = 5892 \text{ Å} \qquad \text{Ans.}$$

Q.2 In a modified Young's double slit experiment, monochromatic uniform and parallel beam of light of wavelength 6000 Å and intensity $\left(\frac{10}{\pi}\right)$ Wm^{-2} incident normally on two circular operations A and B of radii 0.001 m and 0.002 m respectively. A perfectly transparent film of thickness 2000 Å and refractive index 1.5 for the wavelengths 6000 Ã is placed in front of aperture A (figure). Calculate the power(in watt) received all the focal spot F of the lens. The lens is symmetrically placed with respect to the aperture. Assume that 10% of the power received by each observer goes in the original direction and is brought to the focal spot.



Sol. Let I_1 and I_2 be the intensities at A and B

$$I_1 = I_2 = \frac{10}{\pi} Wm^{-2}$$

Area of cross-section of aperture A, $A_1 = \pi r_1{}^2 = \pi \times (0.001)^2 = \pi \times 10^{-6}\ m^2$

Area of cross-section of aperture B, $A_2 = \pi r_2{}^2 = \pi \times (0.001)^2 = \pi \times 4 \times 10^{-6}\ m^2$

Let P_1 and P_2 be the powers of incident radiations at A and B respectively.

$$P_1 = \frac{10}{\pi} \times \pi \times 10^{-6} = 10^{-5}\ W$$

$$P_2 = \frac{10}{\pi} \times 4\pi \times 10^{-6} = 4 \times 10^{-5}\ W$$

Induction of a transparent medium in one of the beams produces some path difference Δx.

Here, $\mu = 1.5$ and $t = 2000$Å

$\therefore$ $\Delta x = (1.5 - 1) \times 2000$ Å $= 0.5 \times 2000$ Å

or $x = 10^{-7}$ m

Let ϕ = phase difference between the two beams

$$\phi = \frac{2\pi}{\lambda} x$$

λ

or $$\phi = \frac{2\pi}{6000 \times 10^{-10}} \times 10^{-7} = \frac{\pi}{3} \text{ radian}$$

If a_1 and a_2 are the amplitudes of light from apertures A and B, net amplitude R at F is,

$$R^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos\phi$$

Power = Intensity × Area of cross-section = $I \times A^2$

or $P = KR^2 \times A^2 = K'R^2$

$P_1 = K'a_1{}^2$

and $P_2 = K'a_2{}^2$

Multiply equation by K' throughout

$$K'R^2 = K'a_1^2 + K'a_2^2 + 2\sqrt{K}a_1\sqrt{K'}a_2 \cos\phi$$

or $$P = P_1 + P_2 + 2\sqrt{P_1P_2}\ \cos\phi$$

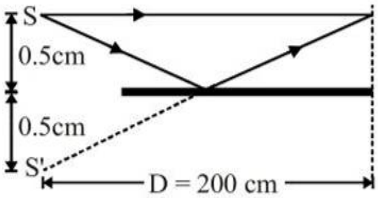
Substituting for P_1, P_2 and ϕ, we get

$$P = (10)^{-5} + 4 \times 10^{-5} + 2\sqrt{10^{-5} \times 4 \times 10^{-5}}\ \cos\left(\frac{\pi}{3}\right)$$

or $P = 10^{-5}(1 + 4 + 2)$

or $P = 7 \times 10^{-5}$ W Ans.

Q.3 A source of light of wavelength 5000 Å is placed at one end of a table 200 cm long and 5 mm above its flat well polished top. Find the fringe-width of the interference bands located on a screen at the end of the table.



Sol. Distance of source S from the table = 5 mm = 0.5 cm

Distance of S' from table = 0.5 cm

If 'd' is the distance between S and S'

$$d = 0.5 + 0.5 = 1 \text{ cm}$$

$$D = 200 \text{ cm}$$

$$\lambda = 5000 \text{ Å} = 5000 \times 10{-}8 \text{ cm} = 5 \times 10{-}5 \text{ cm}$$

Since, $\beta = \dfrac{\lambda D}{d}$

$$\therefore \beta = \frac{5\times10^{-5}\times200}{1} = 10^{-2} \text{ cm}$$

$\beta = 0.01$ cm   Ans.

Q.4 In the usual layout for interference fringes, two identical slits, each of width a are kept apart by d from centre of centre. find :

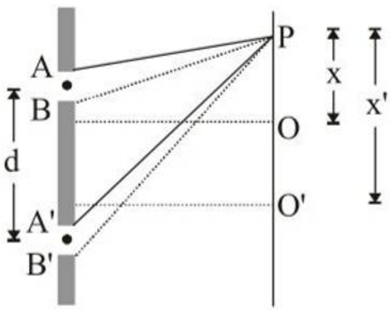
(a) the difference of path differences between rays from the bottom and top of slits i.e., $\delta\Delta = \Delta_b - \Delta_t$,

(b) the maximum value of a at which interference fringes continue to be sharp. Take D=distance between the screen and the slits.

the screen and the slits.

Sol (a) The rays from the top of the slits may be assumed to come from ideal sources with thier pole (equidistant point on the screen) at O. Then

$$\Delta_t = \frac{xd}{D}$$



Similarly, $\Delta_b = \dfrac{x'd}{D}$

$\therefore$ $\Delta_b - \Delta_t = \dfrac{(x'-x)d}{D}$

But $x' - x = a$

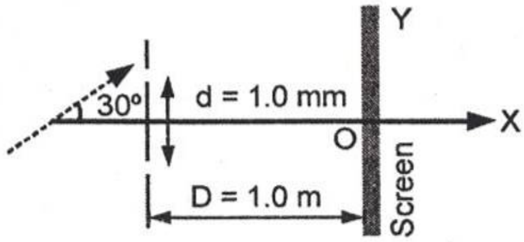
$\therefore$ $\delta\Delta = \Delta_b - \Delta_t = \dfrac{ad}{D}$

(b) If $\delta\Delta = \dfrac{\lambda}{2}$, the maximum from top edges will be superimposed on to the minimum from the bottom edges, owing to which the interference pattern will disappear completely.

$\therefore$ $\dfrac{\lambda}{2} = \dfrac{ad}{D} \Rightarrow \dfrac{\lambda D}{2d}$ Ans.

Q.5 A coherent parallel beam of microwaves of wavelength $\lambda = 0.5$ mm falls on a Young's double slit apparatus. The separation between the slits 1.0 mm. The intensity of microwaves is measured on a screen placed parallel to the plane of the slits at a distance of 1.0m from it as shown in the figure.

(a) If the incident beam falls normally on the double slit apparatus, find the y-coordinates of all the interference minima on the screen.

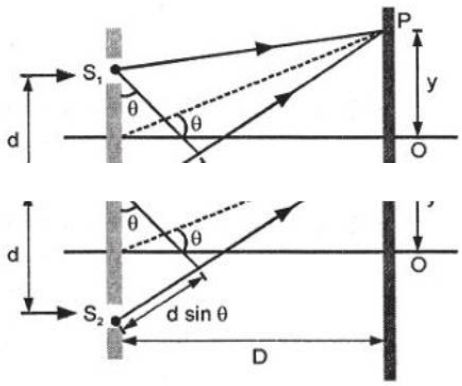


(b) If the incident beam makes an angle of 30° with the x-axis (as in the dotted arrow shown in fig.), find the y-coordinates of the first minima on either side of the central maximum.

Sol. (a) As shown in fig. the path difference between the two interfering waves reaching the point P of the screen will be $\Delta x = d \sin\theta$ and so the point P will beam interference minima if

$$\Delta x = \frac{(2n-1)\lambda}{2} \text{ with } n = 1, 2 \ldots.$$

So $\quad d \sin\theta = (2n-1)\dfrac{\lambda}{2}$



i.e.. $\quad \sin\theta = \dfrac{(2n-1)\lambda}{\quad} = \dfrac{(2n-1)}{\quad}$

i.e., $\quad \sin\theta = \dfrac{(2n-1)\lambda}{2d} = \dfrac{(2n-1)}{4}$

and as $\sin\theta \le 1$

i.e., $\quad n \le 2.5$

$\therefore \quad n = 1$ or 2

When $\quad n = 1,$

$$\sin\theta_1 = \frac{1}{4}$$

So that $\tan\theta_1 = \dfrac{1}{\sqrt{15}}$

When $\quad n = 2$

$$\sin\theta_2 = \frac{3}{4}$$

So that $\quad \tan\theta_2 = \dfrac{3}{\sqrt{17}}$

Now, the position of a point P on the screen which is at a distance D from the plane of slits will be given by

$$y = D\tan\theta = \tan\theta \qquad (\because D = 1\text{m})$$

So, the position of minima will be

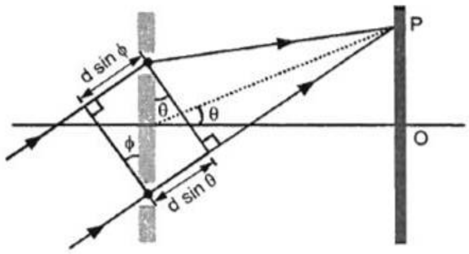
$$y_1 = \tan\theta_1 = \frac{1}{\sqrt{15}} = 0.258$$

and $\quad y_2 = \tan\theta_2 = \dfrac{3}{\sqrt{17}} = \dfrac{3}{2.6} = 1.13$ m

And as minima can be on either side of principal maxima, in the situation given there will be 4 minima at positions $\pm$ 2.258 m and $\pm$ 1.13 m on the screen.

(b) In this situation as shown in fig., the path difference between the interfering waves will be

$$\Delta x = [d \sin\theta - d \sin\phi]$$



For first minima, $d[\sin\theta - \sin\phi] = \pm \frac{\lambda}{2}$

i.e. $\sin\theta = \sin\phi \pm \frac{\lambda}{2d}$

Here, $\phi = 30°$; $\lambda = 0.5$ mm and $d = 1$mm,

$\therefore \quad \sin\theta = \frac{1}{2} \pm \frac{0.5}{2\times1}$

or $\quad \sin\theta = \frac{3}{4}$ or $\frac{1}{4}$,

or $\quad \tan\theta = \frac{3}{\sqrt{7}}$ or $\frac{1}{\sqrt{15}}$

So, the position of first minima on either side of central maxima in this situation will be

$$y = D\tan\theta = \tan\theta \ (\because D = 1\text{m}),$$

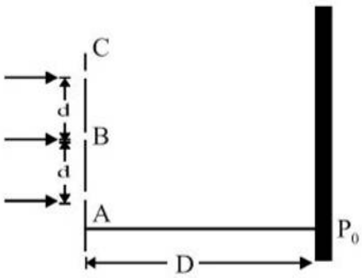
$\therefore \quad y = \frac{3}{\sqrt{7}}$ m and $\frac{1}{\sqrt{15}}$ m Ans.

So, the position of first minima on either side of central maxima in this situation will be

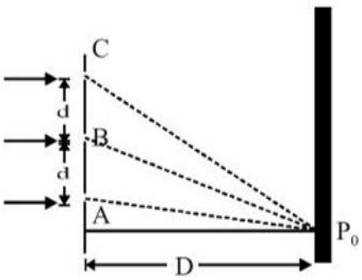
$$y = D\tan\theta = \tan\theta \ (\because D = 1\text{m}),$$

$\therefore \quad y = \frac{3}{\sqrt{7}}$ m and $\frac{1}{\sqrt{15}}$ m Ans.

Q.6 Figure shows three equidistant slits being illuminated by a monochromatic parallel beam of light. Let $BP_0 - AP_0 = \lambda/3$ and $D >> \lambda$. (a) Show that in this case $d = \sqrt{2\lambda D/3}$. (b) Show that the intensity at P_0 is zero.



Sol.



$$BP_0 - AP_0 = \frac{\lambda}{3}$$

or $\quad d\sin\theta = \frac{\lambda}{3}$

or $\quad d\tan\theta = \frac{\lambda}{3}$ $\qquad$ (For small angle $\tan\theta \approx \sin\theta \approx \theta$)

$\therefore \quad d\left(\frac{d/2}{D}\right) = \frac{\lambda}{3}$

or $\quad d = \sqrt{\frac{2F\lambda}{3}}$

(b) $\quad \Delta x_{A/B}$ = path difference between waves coming from A and B = $\frac{\lambda}{3}$

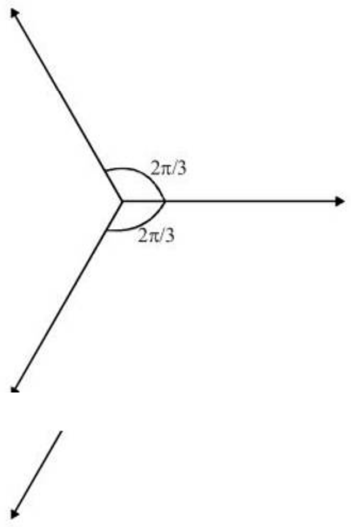
$\therefore \quad \phi_{A/B}$ = phase difference

$$= \frac{2\pi}{\lambda}\Delta x_{A/B} = \frac{2\pi}{3}$$

Similarly, $\quad \Delta x_{B/C} = \Delta x_{A/B} = \lambda/3$

$\therefore \quad \phi_{B/C} = 2\pi/3$

Now, phase diagram of the waves arriving at P_0 is as shown below :



$\therefore \quad$ Amplitude of resultant wave is zero

As intensity (I) α A^2

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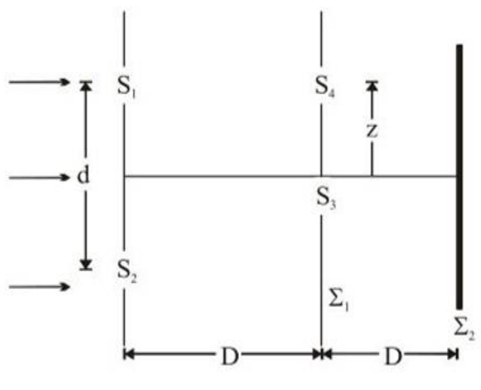
$\therefore \quad$ Intensity at P_0 will be zero.

Q.7 $\quad$ Consider the situation shown in figure. The two slits S_1 and S_2 placed symmetrically around the central line are illuminated by a monochromatic light of wavelength λ. The separation between the slits is d. The light transmitted by the slits falls on a screen Σ_1 placed at a distance D from the slits. The slit S_3 is at the central line and the slits S_4 is at a distance z from S_3. Another screen Σ_2 is placed a further distance D away from Σ_1. Find the ratio of the maximum to minimum intensity observed on Σ_2 if z is equal to



(a) $\frac{D\lambda}{2d}$ $\qquad$ (b) $\frac{\lambda D}{4d}$

Sol. (a) Let I is intensity due to slits S_1 and S_2 on screen S_1. Further, intensity at any point on screen Σ_1 is given by

$$I_P = 4I\cos^2\left(\frac{\phi}{2}\right)$$

At slit S_3, $\phi = 0$

$\therefore$ $I_{S_3} = 4I$

At slit S_4, $\Delta x = \frac{dz}{D} = \frac{\lambda}{2}$

$\therefore$ $\phi = \pi \quad \Rightarrow I_{S_4} = 0$

Now on screen Σ_2 $I_{max} = \left(\sqrt{I_{S_3}} + \sqrt{I_{S_4}}\right)^2 = 4I$

$$I_{min} = \left(\sqrt{I_{S_3}} - \sqrt{I_{S_4}}\right)^2 = 4I$$

$\therefore$ $\frac{I_{max}}{I_{min}} = 1$ Ans

(b) $z = \frac{D\lambda}{4d}$

(b) $z = \frac{D\lambda}{4d}$

$$I_{S_3} = 4I$$

At slit S_4, $\Delta x = \frac{dz}{D} = \frac{\lambda}{4}$

$\therefore$ $\phi = \frac{2\pi}{\lambda} \times \frac{\lambda}{4} = \frac{\pi}{2}$

$\therefore$ $I_{S_4} = 4I\cos^2\left(\frac{\pi}{4}\right) = 2I$

$\therefore$ $I_{max} = \left(\sqrt{I_{S_3}} + \sqrt{I_{S_4}}\right)^2 = \left(\sqrt{4I} + \sqrt{2I}\right)^2$

$$= I\,(2 + \sqrt{2})^2$$

Similarly, $I_{min} = \left(\sqrt{I_{S_3}} - \sqrt{I_{S_4}}\right)^2 = \left(\sqrt{4I} - \sqrt{2I}\right)^2 = I\left(2 - \sqrt{2}\right)^2$

$\therefore$ $\frac{I_{max}}{I_{min}} = \left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)^2$ Ans.

Photo Electric Effect

Photon theory

According to Planck's quantum theory, light consists of photons as energy packets having following properties :

(i) Each photon is of energy $E = h\nu = hc/\lambda$.
Where h is Planck's constant. Where $h = 6.63 \times 10^{-34}$ J-sec $= 4.14 \times 10^{-15}$ eV -sec

(ii) All photons travel in stright line with the speed of light in vacuum.

(iii) Photons are electrically netural.

(iv) Photons have zero rest mass.

(v) Photons are not deflected by electric and magnetic fields.

(vi) The equivalent mas of a photon while moving is given by

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2} = \frac{hc}{c^2\lambda} = \frac{h}{c\lambda}$$

(vii) Momentum of the photon

$$p = E/c = h\nu/c = h/\lambda.$$

(viii) Numbe of photons of wavelength λ emitted in t second from a lamp of power P is.

$$n = \frac{Pt\lambda}{hc}$$

Illustration :

Violet light (λ =4000 Å) of intensity 4 watt/m² falls normally on a surface of area 10 cm × 20 cm. Find

(a) the energy recived by the surface per second.

(b) the number of photons hitting the surface per second.

(c) If surface is tilted such that plane of the surface makes an angle 30° with light beam, find the number of photons hitting the surface per second.

Sol. *(a) Energy received per second per unit area*

$E = I A \cos\theta = 4 \times 0.02\ J \times \cos 0^o = 0.08\ J.$

(b) $n\, h\, (c/\lambda) = E$

$$\Rightarrow \quad n = \frac{0.08 \times 4000 \times 10^{-10}}{6.63 \times 10^{-34} \times 3 \times 10^8}$$

$$= \frac{32 \times 10^{17}}{19.89} = 1.609 \times 10^{17}$$

(c) $$n = \frac{IA\cos 60^o \times \lambda}{hc}$$

$$= \frac{1}{2} \times \frac{32 \times 10^{17}}{19.89} = 0.805 \times 10^{17}.$$

Pratice Exercise

Q.1 Visible light has wavelengths in the range of 400 nm to 780 nm. Calculate the range of energy of the photons of visible light.

Q.2 Calculate the number of photons emitted per second by a 10 W sodium vapour lamp. Assume that 60% of the consumed energy is converted into light. Wavelength of sodium light = 590 nm.

Answers

Q.1 2.56×10^{-19} J to 5.00×10^{-19} J   Q.2 1.77×10^{19}

Radiation Pressure

The electromagnetic wave transports not only energy but also momentum, and hence can exert a radiation pressure on a surface due to the absorption and reflection of the momentum.

Illustration :

A photon of wavelength 6630 Å is incident on a totally reflecting surface.Find the momentum delivered by the photon .

Sol. *The momentum of the incident radiation is given as* $P = \dfrac{h}{\lambda}$

When the light is totally reflected normal to the surface the direction of the ray is reversed. That means it reverses the direction of its momentum without changing its magnitude.
⇒ Change in momentum has a magnitude

$$\Delta P = 2P = \frac{2h}{\lambda}$$

$$\Rightarrow \Delta P = \frac{2(2.63\times 10^{-34}\,J\text{-}sec)}{(6630\times 10^{-34}\,m)} = 2\times 10^{-27}\ kgm/s$$

Illustration :

A parallel beam of monochromatic light of wavelength λ is incident normally on a surface. The intensity of the beam is I. Find the force exerted by the light beam on the surface if surface is
(i) perfectly reflecting
(ii) perfectly absorbing

Sol. *Energy incident per unit time = IA*

Momentum incident per unit time $= \dfrac{IA}{c}$

(i) momentum transported to the wall per unit time $= \frac{2IA}{c}$

$$\therefore F = \frac{2IA}{c}$$

$$\therefore pressure = \frac{2I}{c}$$

(ii) momentum transported to the wall per unit time $= \frac{IA}{c}$

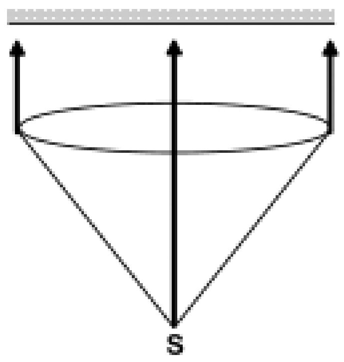
$$\therefore F = \frac{IA}{c}$$

$$\therefore pressure = \frac{I}{c}$$

Pratice Exercise

Q.1 A 100 W light bulb is placed at the centre of a spherical chamber of radius 20 cm. Assume that 60% of the energy supplied to the bulb is converted into light and that the surface of the chamber is perfectly absorbing. Find the pressure exerted by the light on the surface of the chamber.

Q.2 A totally reflecting, small plane mirror placed horizontally faces a parallel beam of light as shown in the figure. The mass of the mirror is 20 g. Assume that there is no absorption in the lens and that 30% of the light emitted by the source goes through the lens. Find the power of the source needed to support the weight of the mirror. Take $g = 10\ m/s^2$.

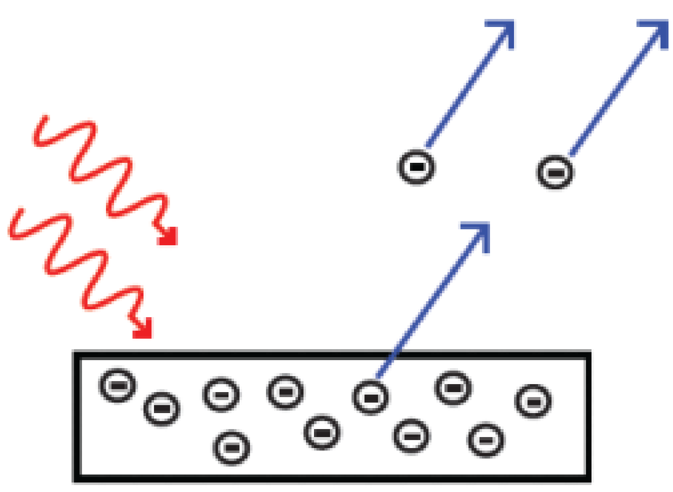


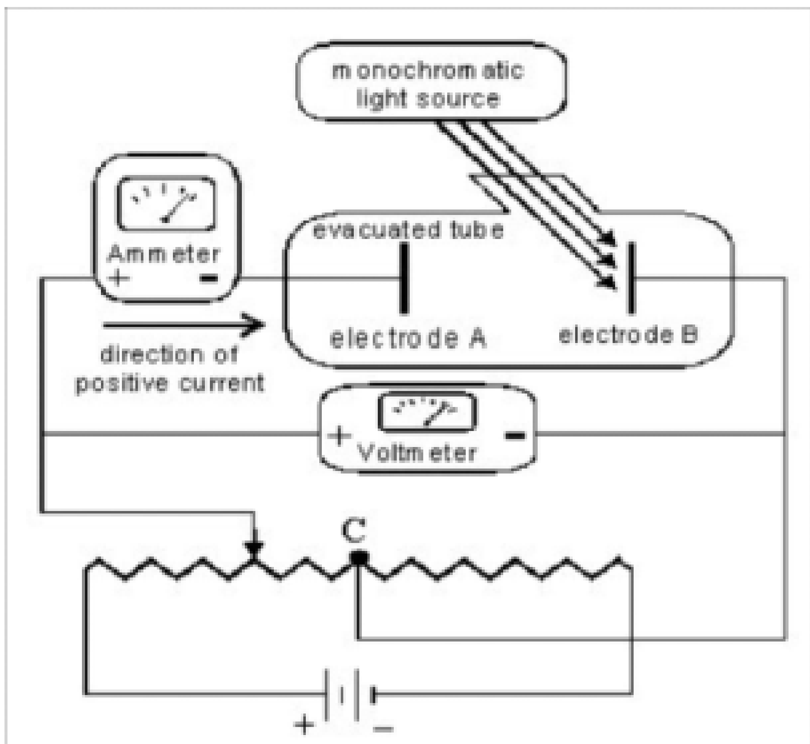
Answers

Q.1 4.0×10^{-7} Pa Q.2 100 MW

Photoelectric Effect

The photoelectric effect is process where electrons are ejected from a surface by the action of light (electromagnetic radiation). The process was discovered by Heinrich Hertz in 1887. Attempts to explain the effect by classical electromagnetic failed. In 1905, Albert Einstein presented an explanation based on the quantum concept of Max Planck.





Observation of the experiments on Photo-Electric Effect:

(i) The emission of photoelectrons is instantaneous.

(ii) the number of photoelectrons emitted per second is proportional to the intensity of the incident light.

(iii) The maximum velocity with which electrons emerge is dependent only on the frequency and not on the intensity of the incident light.

(iv) There is always a lower limit of frequency called threshold frequency below which no emission takes place, however high the intensity of the incident radiation may be.

Explanation

Einstein suggested that when a light beam is incident on a metal surface the free electrons of the metal absorb the entire energy of an incident photon during its collision with it. If this electron gets sufficient energy in this manner to do work against the surface adhesion of the given metallic surface and escape then it leaves the metal and a photoelectron is found.

For an electron to escape from a metallic surface by doing work against its attractive force and get out of the force field of the metallic surface, a minimum amount of energy is required to be supplied to electrons. This minimum energy required for an electron to escape from a metallic surface is called the work function of the given metal which is characteristic of the material and is hence different for different metals. Work function of a given metal is generally represented by the symbol ϕ.

The minimum frequency of light corresponding to which the energy of a photon is equal to the work function of a given metal is called the Threshold frequency of that metal and the corresponding wavelength is called Threshold wavelength.

$$h\nu_0 = \phi \text{ or } \nu_0 = \phi / h$$

Where v_0 is called threshold frequency.

So, $\frac{hc}{\lambda_0} = \phi$ or $\lambda_0 = \frac{hc}{\phi}$

Where λ_0 is called threshold wavelength.

Clearly, when a light beam of frequency less than v_0 or wavelength greater than λ_0 is incident then no photoelectrons can be emitted, no matter how high is the intensity of the incident beam.

Suppose, a photon transfers energy more than the work function of the given metal then the photoelectron may be ejected with a kinetic energy.

$$K_{max} = (hv - \phi)$$

or less than that because a part or all of the extra energy may be lossed during several collisions that the electron makes before emission.

If the frequency of the photon is v and threshold frequency for the metal is v_0 , then

$$K_{max} = h\,(v - v_0)$$

If the wavelength of the photon is λ and threshold wavelength for the metal is λ_0, then

$$K_{max} = hc\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_0}\right)$$

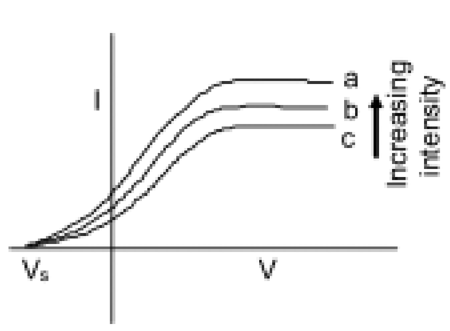
Stopping Potential

If the polarity of the battery is reversed and the applied potential is gradually increased, the photocurrent starts decreasing. This is because the electrons are retarded, and most of the elecrons are unable to reach the opposite electrode. It is observed that when the applied retarding potential is increased, the photocurrent eventually becomes zero. This potential is known as the stopping potential and depends only on the material of the photocathode and the frequency of light.

If V_s be the stopping potential, then

$$eV_S = hv - \phi$$

The stopping potential V_S depends only on the metal and does not depend on the intensity of incident light. a, b, c - different intensites.



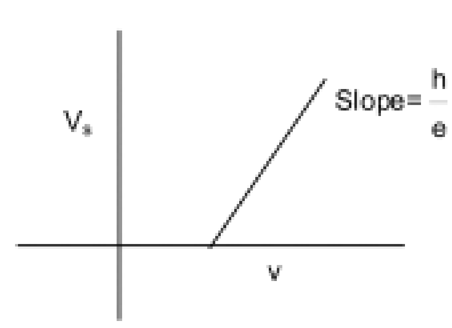


Illustration :

When a metallic surface is illuminated with monochromatic light of wavelength λ the stopping potential for photoelectric current is $3\,V_0$. When the same metallic surface is illuminated with a light of wavelength 2λ, the stopping potential is V_0. Find the threshold wavelength for the surface.

Sol. *Einstein's photoelectric equation:*

$$\frac{hc}{\lambda} = e\,(3\,V_0) + W \qquad \text{...(i)}$$

Here W = work function

$$\text{and } \frac{hc}{2\lambda} = e\,(V_0) + W \qquad \text{...(ii)}$$

Solving these equations

$$\Rightarrow \frac{hc}{\lambda} = \frac{3hc}{2\lambda} + W - 3\,W \quad \Rightarrow \quad 2W = \frac{hc}{2\lambda}$$

$$\Rightarrow \frac{hc}{\lambda_0} = \frac{hc}{4\lambda}$$

$\Rightarrow \lambda_0 = 4\,\lambda$ *When λ_0 = threshold wavelength*

Pratice Exercise

Q.1 The work function of a metal is 2.5×10^{-10} J. (a) Find the threshold frequency fro photoelectric emission. (b) If the metal is exposed to a light beam of frequency 6.0×10^{-14}Hz, what will be the stopping potential?

Q.2 The electric field associated with a light wave is given by

$$E = E_0 \sin [(1.57 \times 10^7\,m^{-1})\,(x - ct)]$$

Find the stopping potential when this light is used in an experiment on photoelectric effect with the emitter having work function 1.9 eV.

Q.3 A monochromatic light source of intensity 5 mW emits 8×10^{15} photons per second. This light ejected photoelectrons from a metal surface. The stopping potential for this setup is 2.0 V. Calculate the work function of the metal.

Q.4 A photographic film is coated with a silver bromide layer. When light falls on this films, silver bromide molecules dissociate and the field records the light there. A minimum of 0.6 eV is needed to dissociate a silver bromide molecule. Find the maximum wavelength of light that can be recorded by the film.

Q.5 A small metal plate (work function ϕ) is kept at a distance d from a singly ionized, fixed ion. A monochromatic light beam is incident on the metal plate and photoelectrons are emitted. Find the maximum wavelength of the light beam so that some of the photoelectrons may go round the ion along a circle.

Answers

Q.1 (a) 3.8×10^{14} Hz ; (b) 0.91 V   Q.2 1.2 V   Q.3 1.9 eV

Q.4 2070 nm   Q.5 $\frac{8\pi\varepsilon_0 dhc}{e^2 + 8\pi\varepsilon_0 \phi d}$

Wave Particle Duality

Electromagnetic radiation is an emission with a dual nature, i.e. it has both wave and particle aspects. In particular, the energy conveyed by an electromagnetic wave is always carried in packets whose magnitude is proportional to frequency of the wave. These packets of energy are called photons.
Energy of photon is E = h.f, where h is Planck's constant, and f is frequency of wave.

de-Broglie idea

As wave behaves like material particles, similarly matter also behaves like waves. According to him, a wavelength of the matter wave associated with a particle is given by $\lambda = \frac{h}{p} = \frac{h}{mv}$, where m is the mass and v is velocity of the particle.

If an electron id accelerated through a potential diference of V volt,

then $\frac{1}{2} m_e v^2 = eV$ or $V = \sqrt{\frac{2eV}{m_e}}$

$\therefore \quad \lambda = \frac{h}{m_e v} = \frac{h}{\sqrt{2eVm_e}}$

(It is assumed that the voltage V is not more than several tens of Kilovolt)

Illustration :

Find the ratio of de-Broglie wavelength of molecules of hydrogen and helium in two gas jars kept separately at temperature 27°C and 127°C respectively.

Sol. *de-Broglie wavelength* $\lambda = h/mv$

where the speed (r.m.s) of a gas particle at the given temperature (T) is given as

$$\frac{1}{2} mv^2 = \frac{3}{2} KT$$

$\Rightarrow v = \sqrt{\frac{3KT}{m}}$, *where K = Boltzmann's constant, m = mass of the gas particle and T = temperature of gas in K*

$$\Rightarrow mv = \sqrt{3mKT}$$

$$\Rightarrow \lambda = \frac{h}{mv} = \frac{h}{\sqrt{3mKT}} \qquad \therefore \frac{\lambda_H}{\lambda_{He}} = \sqrt{\frac{m_{He} T_{He}}{m_{He} T_H}}$$

$$= \sqrt{\frac{(4amu)(273 + 127^o)K}{(2amu)(273 + 27^o)K}} = \sqrt{\frac{8}{3}}$$

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Illustration : :

If the sationary proton and α - particle are accelerated through same potential difference. Find the ratio of de-Broglie wavelength.

Sol. *The gain in K.E. of a change particle after moving through a potential difference of V is given as qV, that is also equal to* $\frac{1}{2}mv^2$*, where v is the velocity of the charged particle.*

$$\frac{1}{2}mv^2 = qV \qquad \Rightarrow \qquad v = \sqrt{\frac{2qV}{m}}$$

$$\therefore mv = \sqrt{2mqV}$$

$$\Rightarrow \text{de-Broglie wavelength} = \lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mqV}}$$

$$\therefore \frac{\lambda_p}{\lambda_\alpha} = \sqrt{\frac{m_\alpha q_\alpha V_\alpha}{m_p q_p V_p}}$$

Putting $V_a = V_p$, $\frac{\lambda_p}{\lambda_\alpha} = \sqrt{\frac{(4)(2)}{(1)(1)}} = 2\sqrt{2}$

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Atomic Structure

Structure of matter always been an interesting area of research for physicists. Till 20^{th} century it was assumed that matter consists of indivisible small tiny particles called "atoms" But with the study and research it was found that the atom is divisible and made of other small particles called electron, proton and Neutron. So may physicists tried to explain the structure of atom but finally it was Neils Bohr whose explanation about the structure was well accepted. For simplicity they have taken hydrogen atom and then it can be extended to other H-like atoms too. Some of the historical models are also explained and them drawbacks :

Thomson's Atomic Model

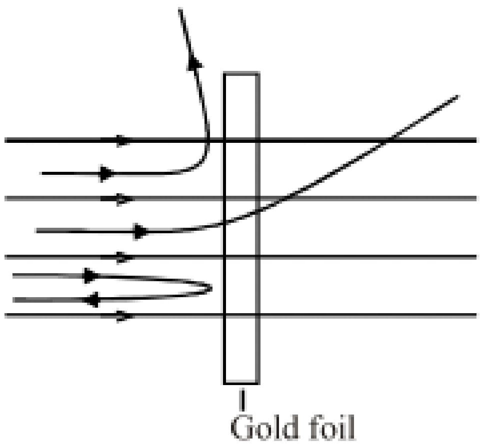
J.J. thomson found the charge to mass ratio of electron for atomic structure. He describes the atom as a region in which positive charge is spread out in space with electrons embedded throughout the region, much like the sceds in a water - melon. The atom as a whole then be electrically neutral.

Rutherford's Model

In 1911 Earnest, Rutherford and his students performed a critical experiment which showed that Thompson Model may not be correct. They bombarded highly energetic α -Particles be correct. (He^{++} Nucleus) onto a thin Gold foil. Following observation were there :

(i) Most of the Practices were passed through the foil as if it were an empty space.

(ii) Very few particles were even deflected backward completely reversing their direction.



(iii) Rest ones were deflected from 0^o to 180^o to their original direction of motion.

Rutherford concluded that most of the part of atom is empty and all the +ve charge is concentrated at the center in a very small volume he named it nucleus. Electrons are moving around sun. Hence this model was also referred as planetary model of the atom.

There were two basic difficulties with the model reason of characteristic radiation coming from atom. The second was according to maxwell theory of electromagnetic Radiation an orbiting electron i an accelerating charge hence it should emit EM radiations resulting in decrease of radius of orbit and finally it should fall on nucleus. But atom is an stable entity.

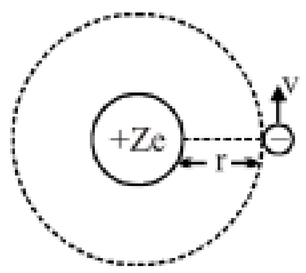
Bohr's Model of hydrogen Atom

The first successful picture of atom was given by Bohr. His model was successful in explaining the lines of EM radiations coming out from H_2 - gas. Although model is now considered obsolete and has been completely replaced by Quantum - Mechanical Theory but it was historically important to the development of Quantum mechanics.

To explain his model Bohr made some postulates:

(i) Electrons revolve around the nucleus in stationary circular orbits where centripetal acceleration is provided by the coulombic attraction of protons on electrons as :

$$\left(\frac{1}{4\pi \in_0}\right)\frac{(ze)(e)}{r^2} = \frac{mv^2}{r}$$



z = atomic number

m = mass of electron

r = radius of an orbit.

or $$\frac{ze^2}{4\pi \in_0 r} = mv^2 \qquad \text{...(i)}$$

(ii) Instead of the infinity of orbits which would be possible in classical mechanics, it is only possible for an electron to move in an orbit for which its orbital angular momentum L is an integral multiple of $\frac{h}{2\pi}$

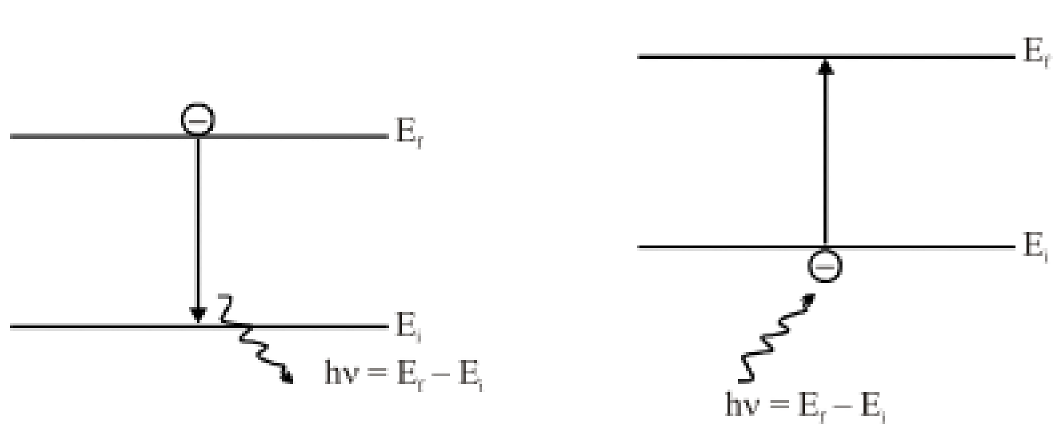
$$L = mvr = \frac{nh}{2\pi} \qquad \text{...(ii)}$$

h = Plank's constant

n = 1,2,3, .. (Quantized Penissible orbits)

(iii) The electron revolving in any one of these allowed orbits does not radiate. These non-radiating orbits are called stationary orbits.

(iv) Energy of electron changes only when then is a transition from higher orbit to lower or from lower to higher.



Photon of energy 'hv' is emitted when when there is a transition from higher to lower.

Calculating radius (r_n) and speed (v_n) of n^{th} orbit

from eq^n (i) and (ii)
we have

$$r_n = \frac{n^2h^2 \epsilon_0}{Ze^2\pi m} = (0.53\ \text{Å})\ \frac{n^2}{z}$$

and

$$v_n = \frac{ze^2}{2nh \epsilon_0} = (2.18 \times 10^6\ \text{m/sec})\ \frac{z}{n}$$

for H-atom,

Radius of 1st ordit , $r_1 = 0.53$ Å
speed of e^- in 1st orbit, $v_1 = 2.18 \times 10^6$ m/sec

Kinetic energy of electron in n^{th} orbit,

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\left\{\frac{ze^2}{2 \epsilon_0\ nh}\right\}^2$$

$$KE_n = \frac{1}{2}\left[\frac{mz^2e^4}{4 \epsilon_0^2\ n^2h^2}\right] \propto \frac{z^2}{n^2}$$

Potential energy of electron,
In the electric field of nucleus the PE of electron in n^{th} orbit is given by

$$U_n = \left(\frac{1}{4\pi \epsilon_0}\right)\frac{(ze)(-e)}{r}$$

$$U_n = -\left[\frac{mz^2e^4}{4 \epsilon_0^2\ n^2h^2}\right] \propto \frac{z^2}{n^2}$$

–ve sign indicates that electron is bound to the nucleus and some work in required to sperate it from the nucleus.

Expression for total energy of electron in n^{th} orbit,

$$E_n = KE_n + U_n = -\frac{1}{2}\left[\frac{mz^2e^4}{4 \epsilon_0^2\ n^2h^2}\right] \propto \frac{z^2}{n^2}$$

If we observe the relations carefully

then $\quad E_n = -KE_n = \frac{U_n}{2}$

Putting values of all the constants,

we get

$$E_n = (-13.6 \times 1.6 \times 10^{-19}\ \text{Joule})\ \frac{z^2}{n^2}$$

$$E_n = -(13.6\ \text{ev})\ \frac{z^2}{n^2}$$

So $\quad KE_n = +(13.6\ \text{ev})\ \frac{z^2}{n^2}$ and $U_n (27.2\ \text{ev})\ \frac{z^2}{n^2}$

From the general expression of total energy

$$E_n = -\left[\frac{me^4}{8 \epsilon_0^2 h^3 c}\right] hc \left(\frac{z^2}{n^2}\right)$$

or $\quad E_n = -(Rhc)\ \frac{z^2}{n^2}$ $\qquad$ Where $R = \frac{me^4}{8 \epsilon_0^2 ch^3}$ is called Rydberg cosntan

$R = 1.097 \times 10^7\ m^{-1}$

'Rhc' is called 1 Rydberg energy = 13.6 ev.

Energy levels of hydrogen atom (z = 1)

	————		$E_\infty = 0$
	————	$n = 5$	$E_5 = -0.54$
	————	$n = 4$	$E_4 = -0.85$ ev
2 nd excited state or 3rd energy	————	$n = 3$	$E_3 = -1.51$ ev
1st excited state or 2nd energy level	————	$n = 2$	$E_2 = -3.4$ ev
Ground state or 1st energy level	————	$n = 1$	$E_1 = -13.6$ ev

Students are adviced to remember these rules for H-atom.

Note :

(i) **Binding Energy of a state :** Energy require to remove electron from a particular quantum state is called BE of a that Particular state.

eg. BE of e^- of H-atom in $n = 4$ level is 0.85 ev

BE of 1st excited state of H-atom is 3.4 ev

BE of 1st excited state of He^+-atom is 13.6 ev

(ii) **Ionisation energy**

The energy required to remove an electron from ground state of the atom is called its ionisation energy.

eg. Ionisation energy of H-atom = 13.6 ev

Ionisation energy of He^+ = 54.4 ev

Ionisation energy of H-like atom = (13.6) z^2 ev

(iii) **Ionisation Potential**

The Potential difference through which an e^- must be accelerated to acquire this much energy (i.e. Ionisation energy) is called Ionisation Potential.

eg. Ionisation Potential of H-atom = 13.6 vol

$$\text{Ionisation Potential} = \frac{\text{Ionisation energy}}{e}$$

(iv) **Excitation energy :** The energy which must be provided to the e^- of atom so that it may go to a higher energy level is called excitation energy of that particular excited state.

for equation for H-atom,

Excitation energy of 1st excited state $\Delta E_1 = E_2 - E_1$
$= (-3.4) - (13.6)$
$= 10.2$ ev

Excitation energy of 2nd excited state $\Delta E_2 = E_3 - E_1$
$= (-1.51) - (13.6)$
$= 12.09$ ev

Excitation energy of 3st excited state $\Delta E_3 = E_4 - E_1$
$= (-0.85) - (-13.6)$
$= 12.75$ ev
and so on.

(v) **Excitation Potential :**

The Potential difference through which an e^- must be accelerated to acquire this much energy (i.e. excitation energy) is called excitation Potential.

$$\text{Excitation Potential} = \frac{\text{Excitation energy}}{e}$$

Illustration :

Find dependency of following physical quantities related with electron revolving in an orbit on Quantum number 'n' and on Atomic Number 'Z'.

(a) Equivalent current in n^{th} orbit

(b) Time period in n^{th} orbit

(c) Angular speed in n^{th} orbit

(d) Magnetic field at center due to revolving e^-

(e) Magnetic moment due to equivalent current.

Sol. *Equivalent current is charge crossing a point in unit time*

$$\text{hence } i = \frac{e}{T} = \frac{e\upsilon}{2\pi\pi} \propto \frac{z^2}{n^2}$$

$$\text{Time period } (T) = \frac{2\pi r}{\upsilon} \propto \frac{n^3}{z^2}$$

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Angular speed $(\omega) = \frac{\upsilon}{r} = \frac{2\pi}{T} \propto \frac{z^2}{n^3}$

$\vec{B}$ *at center of coil* $(\vec{B}) = \frac{\mu_0 i}{2r} \propto \frac{i}{r} \propto \frac{z^3}{n^5}$

Magnetic moment $(\vec{\mu}) = iA = i\,(\pi r^2) \propto ir^2 \propto n$

Illustration :

Which level of the doubly ionized lithium has the same energy as the ground state energy of the hydrogen atom. Find the ratio of the two radii of corresponding orbits.

Sol. *When excited atom makes transition from n = n to n = 2*

$$(13.6)z^2\left[\frac{1}{z^2} - \frac{1}{n^2}\right] = 10.2 + 17 = 27.2 \text{ ev}$$

When excited atom makes transition from n = n to n = 3

$$13.6\, z^2\left[\frac{1}{3^2} - \frac{1}{n^2}\right] = 4.25 + 5.95 = 10.2 \text{ ev}$$

Solving the above expression : z = 3, n = 6

Illustration :

Difference between n^{th} *and* $(n + 1)^{th}$ *Bohr's radius of H-atom is equal to its* $(n-1)^{th}$ *Bohr's Bohr's radius, find the value of n.*

Sol. *Given*

$$r_{n+1} - r_n = r_{n-1}$$

$$(0.53) = \frac{(\pi+1)^2}{z} - 0.53\,\frac{(n)^2}{z} = (0.53)\,\frac{(n-1)^2}{z}$$

$(n+1)^2 - n^2 = (n-1)^2$

Solving we get n = 4

Illustration :

A single electron orbits a stationary nucleus of charge Ze where Z is a constant and e is the electronic charge. It requires 47.2eV to excite the electron from the 2nd Bohr orbit to 3rd Bohr orbit. Find

(i) *the value of Z,*

(ii) *energy required to excite the electron from the third to the fourth orbit*

(iii) *the wavelength of radiation required to remove the electron from the first orbit to infinity*

(iv) *the kinetic energy, potential energy and angular momentum in the first Bohr orbit*

(v) *the radius of the first Bohr orbit.*

Sol. *We can find difference of energy of n = 2 and n = 3 as* $E_3 - E_2 = 47.2$

$$(13.6)\, z^2 \left[\frac{1}{4} - \frac{1}{9}\right] = 47.2$$

$$z = 5$$

Energy required to excite from n = 3 to n = 4, $\Delta E = E_4 - E_3 = 13.6\,(5)^2 \left[\frac{1}{9} - \frac{1}{16}\right] = 16.5$ *ev*

Ioniztion energy $= 13.6\,(z)^2 = 13.6\,(5)^5 = 340$ *ev*

corresponding wardength of photon $= \dfrac{12400}{340}\,\text{Å} = 36.5\,\text{Å}$

In first Bohr orbit : (n = 1)

$KE_1 = 340$ *ev*

$PE_1 = -680$ *ev*

$E_1 = -340$ *ev*

Radius of 1^{st} *Bohr orbit,* $r_1 = \dfrac{(0.53)n^2}{z} = \dfrac{0.53}{5} = 0.106\text{Å}$

Illustration :

Imagine a hypothetical atom in which mass of electron is 'm' but charge of e^- *becomes* $-3e$*. Assuming all other Parameters to be same compare the radius of 1st orbit this hypothetical atom an H-atom. Also find ratio of speed of this* e^- *in n-orbit with that of* e^- *of H-atom.*

Sol. *Radius* $(r_n) \propto \dfrac{1}{e}$ *for H-atom hence radius of 1st orbit of this hypothetical atom will be one-third of that found for H-atom.*

$\upsilon_n \propto e$ *for H-atom*

$\upsilon'_n \propto 3e$ *for Given atom*

hence $\dfrac{\upsilon_{n'}}{\upsilon_n} = 3$

Pratice Exercise

Q.1 Find the maximum Coulomb force that can act on the electron due to the nucleus in a hydrogen atom.

Q.2 A hydrogen atom emits ultraviolet radiation of wavelength 102.5 nm. What are the quantum numbers of the states involved in the transition?

Q.3 (a) Find the first excitation potential of He^+ ion. (b) Find the ionization potential of Li^{++} ion.

Q.4 Average lifetime of a hydrogen atom excited to n = 2 state is 10^{-8} s. Find the number of revolutions made by the electron on the average before it jumps to the ground state.

Q.5 Radiation coming from transitions n = 2 to n = 1 of hydrogen atoms falls on helium ions in n = 1 and n = 2 states. What are the possible transitions of helium ions as they absorb energy from the radiation?

Q.6 A beam of monochromatic light of wavelength λ z ejects photoelectrons from a cesium surface ($\phi = 1.9$ eV). These photoelectrons are made to collide with hydrogen atoms in ground state. Find the maximum value of λ for which (a) hydrogen atoms may be ionized, (b) hydrogen atoms may get excited from the ground state to the first excited state and (c) the excited hydrogen atoms may emit visible light.

Answers

Q.1	8.2×10^{-8} N	Q.2	1 and 3	Q.3	(a) 40.8 V	(b) 122.4 V	
Q.4	8.2×10^{6}	Q.5	n = 2 to n = 3 and n = 2 to n = 4				
Q.6	(a) 80 nm	(b) 102 nm	(c) 89 nm				

Excitation of atom

If we provide energy to the electron of atom then there i a possibility that is it is excited to higher energy level.

This process can be done in two ways :

- By Absorption of photons
- By collision with other atoms and electrons

By Absorption of photons

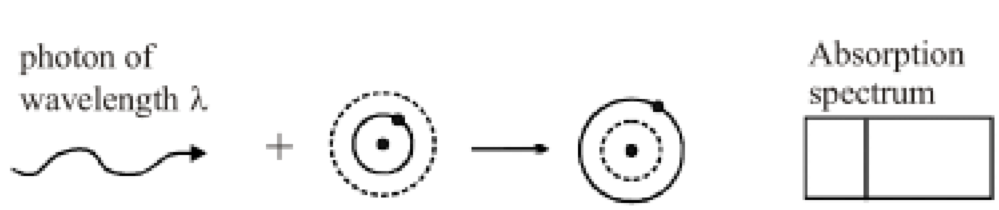
If an electron is to absorb a photon the energy $h\nu$ of photon must be equal to the energy difference ΔE between the initial energy level of the electron and a higher level.

It means if we consider the case of H-atom then this atom can absorb only certain specific energy photons which are 10.2 ev, 12.75 ev etc.

The electron of H-atom can not absorb photon of energy 11 ev but this electron can absorb any photon of energy greater then 13.6 eV or more specifically ionization energy of atom. After absorbing energy more than 13.6 eV, rest of the energy may appear as kinetic energy of the free electron.

Absorption spectrum :

When an atom absorb a photon whose energy is exactly equal to the difference of ground state and any of the excited states, this wavelength corresponds line of absorption spectrum.



Note :

- An atom will absorb energy from its ground state only.
- Wavelengths of absorption spectrum can be determined as

$$\frac{1}{\lambda} = RZ^2\left(\frac{1}{1^2} - \frac{1}{n^2}\right) \qquad n = 2,3,4, \ldots\ldots$$

- Number of lines in abortion spectrum between n = 1 and n = n level will be (n = 1)

By collision with other atoms and electrons

The electron of an atom may also absorb energy during collisions, and may be excited to a higher energy state. During collisions of atoms and electrons the loss of energy must be used to excite the atoms as at atomic level then is no significance of Thermal energy and rise of temperature. We can not estimate that what type of collision must occur but we can always analyze the possibilities during a collision.

(i) If loss of KE during collision is not sufficient to excite the atom then collision must be perfectly elastic.

(ii) The collision may be inelastic or perfectly inelastic only if loss of KE is exactly equal to any of the excitation energy of the atom.

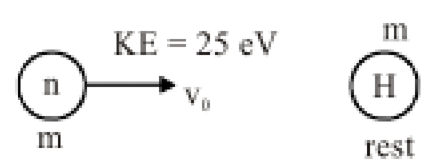
During a collision maximum loss of KE can be calculated using center of frame. i.e. KE with respect to COM can be lost.

$$KE_{system,COM} = \frac{1}{2}\mu V_{rel}$$

Where $\mu \rightarrow$ Reduced Mass of system

$V_{rel} \rightarrow$ relative velocity of the atoms

Consider an H-atom at rest and a neutron with KE = 25 eV is going to collide with the H-atom. Mass of neutron and H-atom can consider as same.



$$KE_{max\ lost} = \frac{1}{2}\mu v_{rel}^2$$

$$= \frac{1}{2}\left[\frac{m.m}{m+m}\right]v_0^2$$

$$= \frac{1}{2}\left[\text{KE of neutron}\right]$$

$$= 12.5 \text{ ev}$$

1st Possibility : Pertectly elastic collision and no excitation

• This is a possible case in every collision if no loss takes place, no excitation to will be there.

2nd Possibility : Inelastic collision, H-atom excited to n = 2 for this $\Delta E = 10.2$ ev is required and it is less than $KE_{max\ loss}$ hence a possible case.

3rd Possibility : In elastic collision, H - atom excited to n = 3 for this $\Delta E = 12.09$ ev $KE_{max\ lass}$ also passible

4th Possibility : In elestic collision, H-atom excited to n = 4 for this $\Delta E = KE_{max\ loss}$ ... .. not Possible

5th Possibility : Pertectly in eleastic collision.

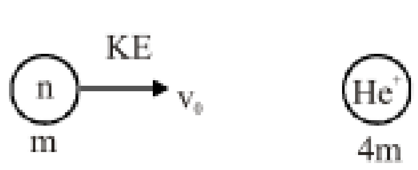
In this case $\Delta E = KE_{max\ loss} = 12.5$ev .... not Possible as 12.5 ev is not an excitation energy.

Note : In all the possibilites, apart from loss rest of the KE will shared by H-atom and newtron. Which can be calculated using momentum - consevation case.

Illustration :

Consider an He^+-atom at rest, a newtron collision with be atom such that He^+ - atom may be excited to 1st excited state. Find min KE of newtron required.

Sol.



$$KE_{max\ loss} = \frac{1}{2}\mu V_{rel}^2$$

$$= \frac{1}{2}\left[\frac{m.m}{m+m}\right]v_0^2$$

$$= \frac{4}{5}[KE \text{ of neutron}]$$

for min KE, $KE_{max\ loss} = \Delta E$ *(excitation energy)*

$$\frac{4}{5}(KE \text{ of newtron}) = 40.8 \text{ ev}$$

$$KE \text{ of newtron} = 51 \text{ ev}$$

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{4^2} - \frac{1}{n_2^2}\right)$$

Note : The collision of e^- is slightly different from that between atoms and neutrons because e^- is very tiny particles as compared to those hence it penetrates into the atom and can collide with e^- of atom in ground state. The collision of e^- with the e^- of atom will be perfectly elastic hence it may transfer any fraction of its KE to the e^- of atom.

For example an e^- moving with KE = 12 ev can excite the H-atom to 1^{st} excited state by transferring 10.2 ev KE to the e^- of H-atom.

Similarly An e^- moving with KE = 15 ev may ionize an H-atom.

Pratice Exercise

Q.1 State whether following statements are true/False :

(a) A neutron moving with KE = 20 ev collides with an H-atom at rest. The collision must be perfectly elastic.

(b) A neutron moving with KE = 30 ev collides with an H-atom at rest. The collision may be perfectly in elastic.

(c) An H-atom moving with KE = 25.5 ev collides with another H-atom at rest. The collision may be perfectly inelastic.

Q.2 A neutron having kinetic energy 12.5 eV collides with a hydrogen atom at rest. Neglect the difference in mass between the neutron and the hydrogen atom and assume that the neutron does not leave its line of motion. Find the possible kinetic energies of the neutron after the event.

Q.3 A neutron moving with a speed v strikes a hydrogen atom in ground state moving towards it with the same speed. Find the minimum speed of the neutron for which inelastic (completely or partially) collision may take place. The mass of neutron = mass of hydrogen = $1.67 \times^{-27}$ kg.

Answers

Q.1 (a) T (b) T (c) T Q.2 zero Q.3 3.13×10^4 m/s

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de-excitation of atom

Electrons excited to higher energy stay their only for 10^{-8} s then they make transition to any lower state by emitting photons and finally come to a ground state. Energy of photons emitted is equal to the difference of energy of the levels. While coming down they may emit photons of various wavelengths which corresponds to several spectral series.

Emission spectrum :

When an electron in excited state makes a transition to a lower state, a photon is emitted. Collection of these photon wavelengths in called emission spectrum.

Hydrogen atom (or hydrogen like hydrogen atoms) consists of only one electron but we get a number of spectral lines in its spectrum (emission).

1. **Lyman series** : The spectral lines of this series correspond to the transition of an electron from some higher energy state to the inner most orbit (n = 1 i.e. ground state).
For Lyman series, $n_1 = 1$, $n_2 = 2,3,4,......$

So, $$\frac{1}{\lambda_{\text{Lyman}}} = R\ Z^2\left(\frac{1}{1^2} - \frac{1}{n_2^2}\right)$$

for hydrogen atom, Z = 1

$$\frac{1}{\lambda_{\text{Lyman}}} = R\left(\frac{1}{1^2} - \frac{1}{n_2^2}\right)$$

2. **Balmer series** : The spectral lines of this series correspeond to the trasition of an electron from higher energy state to an oribit having n = 2,
For Balmer serise $n_1 = 2$, $n_2 = 3,4,5,$ ....
The wave numbers and the wavelengths of spectral lines constituting the Balmer series are given by,

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n_2^2}\right)$$

3. **Paschen series** : The spectral lines of this series correspand to the transition of an electron from some higher energy state to an orbit having n = 3.
For Paschen series, $n_1 = 3$, $n_2 = 4,5,6,$ ....
The wave numbers and the wavelengths of spectral lines constituting the paschen series are given by,

$$\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n_2^2}\right)$$

Paschen series is so named because it was discovered by paschen. Just like other series, this eries was first predicted by Bohr.

4. **Barcket series** : The spectral line of this series correspond to the transition of an electron from a higher energy state to the orbit having n = 4.
For this serie, $n_1 = 4$, $n_2 = 5,6,7,$ ......

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{6^2} - \frac{1}{n_2^2}\right)$$

5. **Pfund series** : The spectral line of this series correspond to the transition of an electron from a higher energy state to the orbit having n = 5.

For the series, $n_1 = 5$ and $n_2 = 6, 7, 8, \ldots\ldots..$

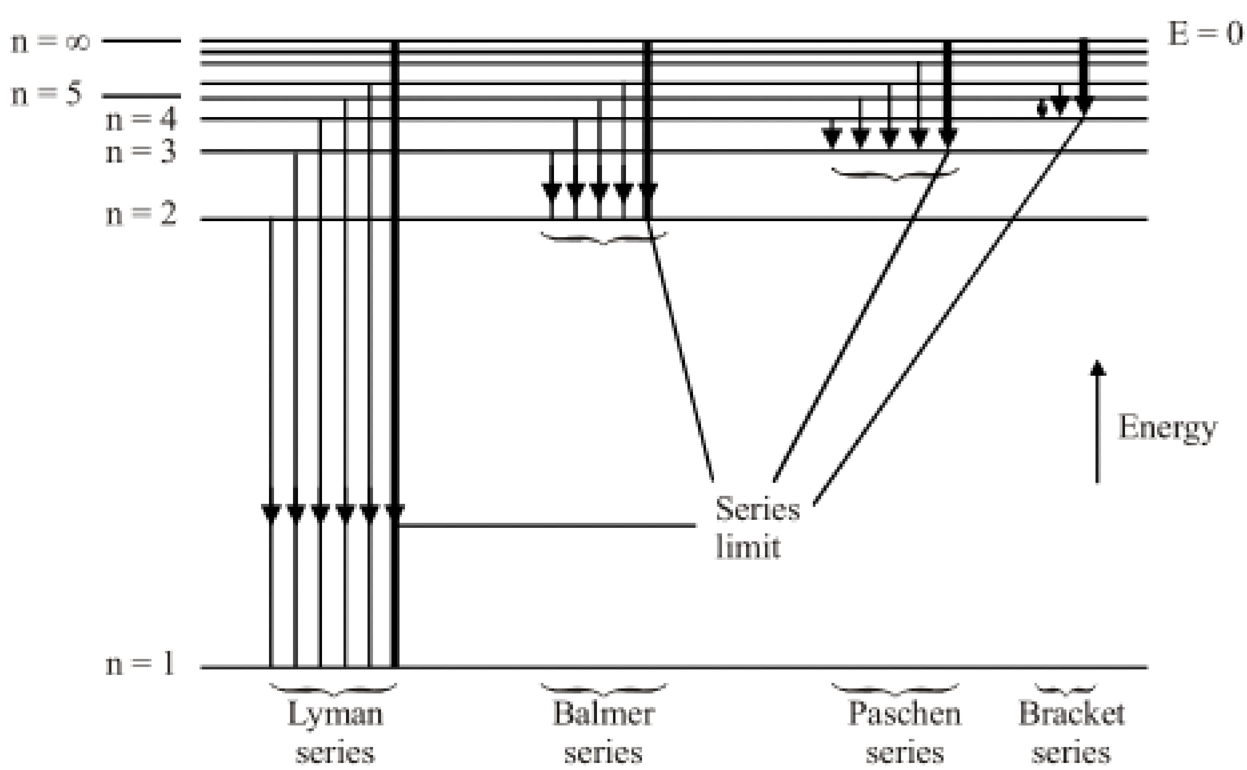
The wave number and the wavelength of the spectral lines constituting the Pfund series are given by.

$$\frac{1}{\lambda} = R_{\infty}\left(\frac{1}{5^2} - \frac{1}{n_2^2}\right)$$

6. **Humphery series** : For this series, $n_1 = 6$ and $n_2 = 7, 8, 9, \ldots\ldots\ldots\ldots$

For this series, $n_1 = 6$ and $n_2 = 7, 8, 9, \ldots\ldots..$

The wave number and the wave lengths of the spectral lines constiting the Humphery series are given by,



Spectral lines originate in transitions between energy levels. Shown are the spectral series of hydrogen. When n = ∞, the electron is free.

Note :

- While coming down to ground state a single atom in n = n state may emit a maximum (n – 1) photons.
- First line of a series corresponds to lowest energy photon emitted e.g. first line of blamer series corresponds to transition from n = 3 to n = 2.
- Series limit corresponds to maximum energy photon emitted e.g. series limit of blamer series corresponds to transition from n = ∞ to n = 2.
- Maximum number of lines in emission spectrum of a gas which is excited to a level n = n will be $^nC_2$ i.e. n (n – 1)/2.
- For an atom in quantum state n = n

  Max energy photon will be emitted for a transition from n = n to n = 1.

  Min energy photon will be emitted for a transition from n = n to n = n – 1.

Mass of the nucleus is comparable to mass of electron

If mass of the nucleus is comparable to mass of electron, then the electron and nucleus revolve in coplanar concentric circular paths of radii r_1 and r_2 about their common centre of mass. The electrostatic force of attraction provides them necessary centripetal force. They revolve with the same angular velocity and their sense of rotation is also same.

from the figure,

$$r_1 + r_2 = r_n \qquad \text{...(i)}$$

and $\quad Mvr_1 = mv_2$

or
$$\frac{r_1}{r_2} = \frac{m}{M} \qquad \text{...(ii)}$$

from equation (i) and (ii),

$$r_1 = \left(\frac{m}{m+M}\right) r_n \; ; \qquad r_2 = \left(\frac{m}{m+M}\right) r_n$$

Let their angular velocities be ω_1 and ω_2 and electrostatic force of attraction between them be F, then,

$$F = M\omega_1^2 r_1 \qquad \text{(for M)}$$

and
$$F = m\omega_2^2 r_2 \qquad \text{(for m)}$$

So, $\quad M\omega_1^2 r_1 = m\omega_2^2 r_2$ but $Mr_1 = mr_2$ [from equation (ii)]

So, $\quad \omega_1^2 = \omega_2^2$ or $\omega_1 = \omega_2 = \omega(\text{say})$

So they revolve with same angular velocity about their common centre of mass.

Now let us see why 'm' is replaced by the reduced mss μ when motion of nucleus is also to be considered. Centripetal force to the electron is provided by the electrastic force, so,

$$mr_2\omega^2 = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n^2} \qquad \text{or} \left(\frac{Mm}{M+m}\right) r_n^3\omega^2 = \frac{Ze^2}{4\pi\varepsilon_0}$$

or
$$\mu r_n^3\omega^2 = \frac{Ze^2}{4\pi\varepsilon_0} \qquad \text{...(iii)}$$

where,
$$\mu = \frac{1}{m} + \frac{1}{M} = \frac{Mm}{M+m}$$

Now, moment of inertia about the common centre of mass,

$$I = Mr_1^2 + mr_2^2 = M\left(\frac{m}{m+M}\right)^2 Mr_1^2 + mr_2^2 + m\left(\frac{M}{m+M}\right)^2 r_n^2 = \mu r_n^2$$

According to Boh's theory of equation of angular momentum,

$$I\omega = \frac{nh}{2\pi} \qquad \Rightarrow \qquad \mu r_n^2 \omega = \frac{nh}{2\pi} \qquad \text{... (iv)}$$

From equations (iii) and (iv), we get,

$$r_n = \frac{\varepsilon_0 n^2 h^2}{\pi \mu e^2 Z} \qquad ...(v)$$

Comparing this value with the value of r_n when nucleus was assumed be massive, we see that, 'm' has been replaced by μ. Futher electrical potential energy of the system.

$$U_n = \frac{Ze^2}{4\pi\varepsilon_0 r_n} \text{ and kinetic energy, } K_n = \frac{1}{2} I\omega^2 = \frac{1}{2}\mu r_n^2 \omega^2$$

Explanation of bohr quantisation rule from de-Broglie's Concept

Einstein suggested that light behaves both as a material particle as well as wave. de-Brogle extended Einstein's view and said that all forms of matter like electrons, protons, neutrons etc. also dual character. He futher said that wavelength (λ) associated with a particle of mass 'm' moving with velocity 'v' is given by,

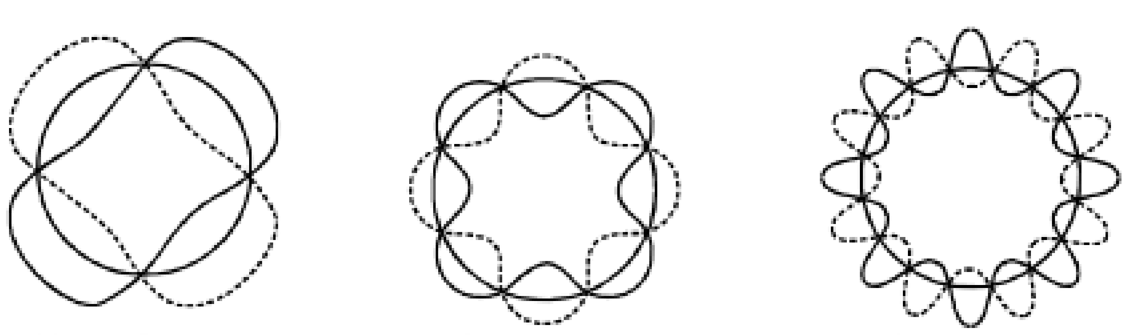
$$\lambda = \frac{h}{mv} = \frac{h}{p} \qquad \text{(where } \lambda \text{ is called de-Broglie's wavelength)}$$

Futher : If the K.E. of the moving particle is K, then, $\lambda = h/\sqrt{2mK}$

If a charged particle 'q' is accelerated through a potential difference ΔV then, $\lambda = h\sqrt{2mq(\Delta V)}$

An electron behaves as standing or stationary wave, which extends round the nucleuses in a circular orbit. If the two ends of the electron wave meet to given a regular series of crests and troughs, the electron wave is said to be in phase. i.e. there is constructive interference of electron waves and the electron motion has a character of standing wave or non-energy radiation motion.

Watever be the path of the electron wave round the nucleus, it is a necessary condition to get an electron wave in phase so that the circumference of the Bohr's orbit ($=2\pi r$) is equal to the whole number multiple to wavelength λ of electron wave i.e.



Circumference = 2 wavelengths | Circumference = 4 wavelengths | Circumference = 8 wavelengths

Figure show some modes of vibration of a wire loop. In each case a whole number of wavelengths fit into the circumference of the loop.

$$2\pi r = n\lambda \text{ or } \lambda = \frac{2\pi r}{n} \qquad ...(i)$$

Where 'n' is a whole number which denotes the number of wavelengths associated with an electron wave extending round the nucleus.

Now according to de - Broglie,

$$\lambda = \frac{h}{mv} \qquad \text{...(ii)}$$

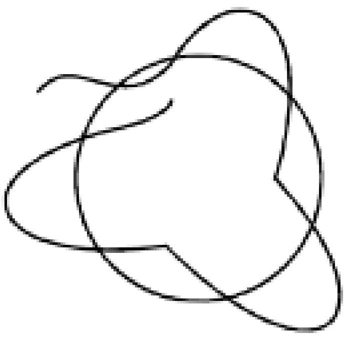


Figure shows a fractional number of wavelengths cannot persist because destructive interference will occur.

From equation (i) and (ii) we get,

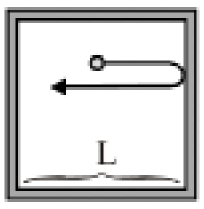
$$\frac{2\pi r}{n} = \frac{h}{mv} \text{ or } mvr = \frac{nh}{2\pi}$$

An electron revolving in a permitted orbit does not radiate energy, through it is accelerating, so the total energy of the electron remains constant. That is why the permitted orbits are also called stationary or non-radiating orbits.

Energy of a trapped electron

The energy of a trapped particle is quantized.

The simplest case is that of a particle that bounces back and forth between the walls of a box. We shall assume that walls of the box are infinitely hard, so the particle does not lose energy each time it strikes a wall and that its velocity is sufficiently small so that we can ignore relativistic conditions.



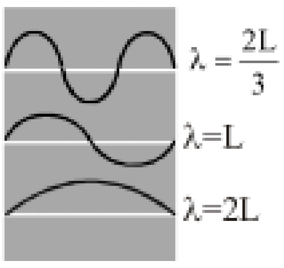
A particle confined to a box of width L.

From a wave points of view, a particle trapped in a box is like a standing wave in a string stretched between the box's walls. The possible de-Broglie wavelengths of the particle in the box therefore are determined by the width L of the box. The general formula for the permitted wavelength is given by.

$$\text{de Broglie wavelengths of trapped particle, } \lambda_n = \frac{2L}{n}, n = 1,2,3,........$$

Further for a matter wave de-Broglie wavelength,

$$\lambda_n = \frac{h}{mv_n} = \frac{h}{\sqrt{2mK_n}} \text{ (where } k_n \text{ is the kinetic energy of the particle)}$$



Wave functions of particle tapped in box of width L

Comparing the two expressions, we have,

$$\frac{h}{\sqrt{2mk_n}} = \frac{2L}{n} \text{ or } K_n = \frac{n^2h^2}{8mL^2}$$

Since the particle has no potential energy in this model, the only energies it can have, are,

Particle in a box, $E_n = \frac{n^2h^2}{8mL^2}$, n = 1,2,3, ...

Each permitted energy is called an energy level and the integer 'n' that specifies an energy level E_n is called its quantum number.

We can draw three general conclusions from the energy expression of a trapped electron. These conclusions apply to any particle confined to a certain region of space for instance, an atomic electron held captive by the attraction of the positively charge nucleus.

1. A trapped particle can not have an arbitrary energy, as free particle can have.
2. A trapped particle can not have zero energy.
3. Because Planck's constant is so small ($h = 6.63 \times 10^{-34}$ J sec), quantization of energy is conspicuous only when 'm' and L are also small. That is why we are not aware of energy quantization in our own experience.

X-RAY

Historical background

On Nov, 1895, Wilhelm Conrad Röntgen (accidentally) discovered an image cast from his cathode ray generator, projected far beyond the possible range of the cathode rays (now known as an electron beam). Further investigation showed that the rays were generated at the point of contact of the cathode ray beam on the interior of the vacuum tube, that they were not deflected by magnetic fields, and they penetrated many kinds of matter. Röntgen named the new form of radiation X-radiation (X standing for "Unknown").

What are X-Rays

X-rays are electromagnetic radiation of very short wavelength (0.1Å and 100Å) and high energy which are emitted when fast moving electrons or cathode rays strike a target of high atomic mass.

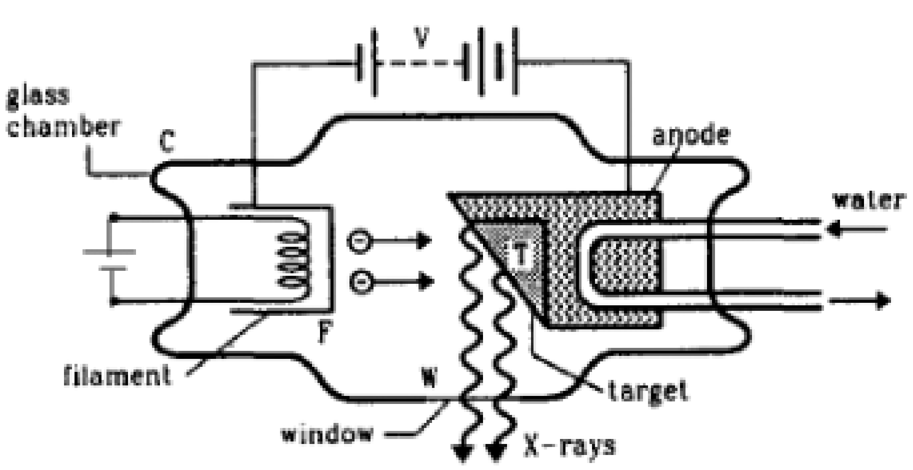
The X-Rays extends from the ultraviolet band to gamma rays band. Although their is very thin line between the X-ray and gamma rays but they are distinguished by the source of their generation. In the older days the X-Rays were distinguished from gamma rays by the wavelength and frequency but with time it has been observed that their exists X-Rays and gamma rays overlapping band and hence it is then understood that the distinction between them is source of generation.

Coolidge tube

William Coolidge invented the X-ray tube popularly called the Coolidge tube. His invention revolutionized the generation of X-rays and is the model upon which all X-ray tubes for medical applications are based.

The characteristic features of the Coolidge tube are its high vacuum and its use of a heated filament as the source of electrons. There is so little gas inside the tube that it is not involved in the production of x-rays, unlike the situation with cathode gas discharge tubes.

The operation of the Coolidge tube is as follows. Due to filament current the cathode filament is heated, it emits electrons. The hotter the filament gets, the greater the emission of electrons. A constant potential difference of several kilovolts is maintained between the filament and the target using a DC power supply so that the target is at a higher potential than the filament. These electrons are accelerated towards the anode with a very high speed and when the electrons strike the anode and emit x-rays. These X-rays are brought out of the tube through a window W made of thin mica or mylar or some such material which does not absorb X-rays appreciably. In the process, large amount of heat is developed, and thus an arrangement is provided to cool down the tube continuously by running water.



Note

(i) An increase in the filament current increase the number of electrons it emits. Larger number of electrons means an intense beam of X-rays is produced. This way we can control the quantity of X-rays i.e. Intensity of X-rays.

(ii) An increase in the voltage of the tube increase the kinetic energy of electrons ($eV = 1\ 2mv^2$). When such highly energetic beam of electrons are suddenly stooped by the target, an energetic beam of X-rays is produced. This way we can control the quality of X-rays i.e. penetration power of X-rays.

(iii) Based on penetrating power, X-rays are classified into two types. HARD-rays and SOFT-x-rays. The first one having high energy and hence high penetration power are HARD-X-rays and another one with low energy and hence low penetration power are SOFT-X-rays.

Continuous X-rays

When electron strikes target an electron loses a part of its kinetic energy and continues to move with the remaining energy until it hits another atom of the target. Part or whole of the energy lost by the electron is converted into a photon. This process is known as bremsstrahlung (breaking radiation)- as it leads to the electron getting decelerated by the target. There is existence of a minimum wavelength (or maximum frequency) in x-ray spectrum. This is called the cutoff wavelength or the threshold wavelength.

If the potential difference between the filament and the target is V, then the kinetic energy of the electron just before it hits the target is

$$K.E. = eV$$

Energy of the X-ray photon, $\Delta E = \dfrac{hc}{\lambda}$

ΔE is the fraction of K.E. of electron that gets converted into photon. l is the wavelength of the photon. Wavelength of the X-ray's photon

$$\lambda = \frac{hc}{\Delta E}$$

as

$$\Delta E \le eV \qquad \Rightarrow \qquad \lambda \ge \frac{hc}{eV}$$

$$\Rightarrow \qquad \lambda_{min} = \frac{hc}{eV}$$

This minimum wavelength does not depend on the target material and depend on the potential difference between the filament and the target.

Characteristic x-ray

When the high energy lectrons "knock off" the innermost electrons of the atoms of the target material causing a vacancy. This vacancy is filled by a electron that 'jump' from one of the outer shells. If the vacancy is created in K shell and electron from the L shell fills the vacancy then the emitted photon is a K_α X-ray. If a vacancy is creasted in the K-shell and an electron from the M shell fills this vacancy then the x-ray emitted is known as a K_β X-ray.

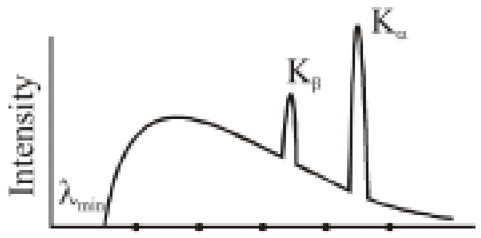
$$\lambda_{K_\alpha} = \frac{hc}{E_L - E_K}$$

$$\lambda_{K_\beta} = \frac{hc}{E_M - E_K}$$

Where E_K, E_L, E_M are the electron energy levels

These X-ray are known as characteristic X-ray as they depend on the target material (character of material), not on the applied voltage.

The adjoining graph shows the variation of the intensity of X-ray coming out of the tube with wavelengths. At some sharply defined wavelengths (K_α K_β) the intensity of the emitted radiation is very large.



Moseley's Law

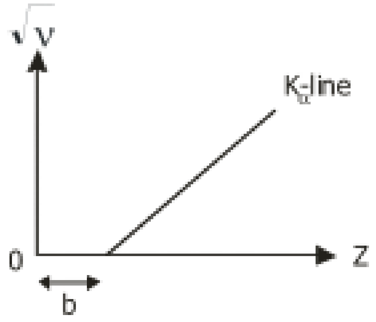
By measuring the wavelength associated with a particular line (now designated K_α), from the spectrum of each element, Moseley established that the lines from a large number of elements obeyed the relation

$$\sqrt{v} = a(Z - b)$$

Where a and b are constants with

$$a \approx \sqrt{\frac{3Rc}{4}} \quad (R = \text{Rydberg constant})$$

and $\quad b \approx 1$. (b is known as the screening constant)



Approximate explanation from Bohr's theory

Consider an atom from which an electron from the K shell been knocked out. Consider an electron from the L shell which is about to make a transition to the vacant site. If finds the nucleus of charge Ze screened by the spherical cloud of the remaining one electron in the K shell. If we neglect the effect of the outer electrons and the other L electrons, the electron making the transition finds a charge $(Z-1)$ e at the centre. One, therefore, may expect Bohr's model to given reasonable results if Z is replaced by $Z-b$

$$\Delta E = hv = Rhc\,(Z-b)^2\left(\frac{1}{1^2} - \frac{1}{2^2}\right)$$

$$\Rightarrow \quad \sqrt{v} = \sqrt{\frac{3Rc}{4}}\,(Z-b)$$

Properties of X-Ray

1. These are highly penetrating rays and can pass through several materials which are opaque to ordinary light.
2. They ionize the gas through which they pass. While passing through a gas, they knock out electrons from several of the neutral atoms, leaving these atoms with +ve charge.
3. They cause fluorescence in several materials. A plate coated with barium platinocyanide, ZnS (zinc sulphide) etc becomes luminous when exposed to X-ray.
4. They affect photographic plates especially designed for the purpose.
5. They are not deflected by electric and magnetic fields, showing that they not charge particles.

Illustration :

To decrease the cut-off wavelength of contimuous X-ray by 25%, find the % change potential difference across the X-ray tube

Sol. $\lambda = \dfrac{hc}{eV}, \dfrac{\lambda_1}{\lambda_2} = \dfrac{V_2}{V_1}, \lambda_1 = \dfrac{3}{4}\lambda_1$

$V_2 = \dfrac{4}{3}V_1, \dfrac{100}{3}$ % increases.

Illustration :

The wavelength of K_a X-rays produced by an X-ray tube is 0.76 Å. Find the atomic number of anticathode materials.

Sol. *For K_a X-ray line.*

$$\frac{1}{\lambda_a} = R(Z-1)^2\left[\frac{1}{1^2}-\frac{1}{2^2}\right] = R(Z-1)^2\left[1-\frac{1}{4}\right]$$

$$\Rightarrow \frac{1}{\lambda_a} = \frac{3}{4}R(Z-1)^2 \qquad ...(1)$$

With reference to given data,

$l_a = 0.76\ \text{Å} = 0.76 \times 10^{-10} m;\ R = 1.097 \times 10^7\ m$

Putting these values in equation (1)

$$(Z-1)^2 = \frac{4}{3} \times \frac{1}{0.76\times10^{-10}\times1.097\times10^7} \cong 1600$$

$\Rightarrow Z - 1 = 40 \Rightarrow Z = 41$

Pratice Exercise

Q.1 If the operating potential in an X-ray tube is increased by 1%, by what percentage does the cutoff wavelength decrease?

Q.2 When 40 kV is applied across an X-ray tube, X-ray is obtained with a maximum frequency of 9.7×10^{18} Hz. Calculate the value of Planck constant from these data.

Q.3 The K_β X-Ray of argon has a wavelength of 0.36 nm. The minimum energy needed to ionize an argon atom is 16 eV. Find the energy needed to knock out an electron from the K shell of an argon atom.

Q.4 A free atom of iron emits K_α X-rays of energy 6.4 keV. Calculate the recoil kinetic energy of the atom. Mass of an iron atom $= 9.3 \times 10^{-26}$ kg.

Answers

Q.1 approximately 1% Q.2 4.12×10^{-15} eV–s Q.3 3.47 keV
Q.4 3.9×10^{-4} eV

Nuclear Physics

It exists at the centre of an atom, containing entire positive charge and almost whole of mass. The electron revolve around the nucleus to form an atom. The nucleus consists of protons (+ve charge) and neutrons. A proton has positive charge equal in magnitude to that of an electron ($+ 1.6 \times 10^{-19}$ C) and a mass equal to 1840 C) and a mass equal to 1840 times that of an electron. A neutron has no charge and mass is approximately equal to that of proton.

Properties of a nucleus

(1) **Nuclear Mass :**

As we know that every nucleus contains protons and neutrons and so every nucleus has a definite mass.However, since the mass of electron is negligible so atomic mass is roughly equal to nuclear mass.

Atomic masses are measured in atomic mass unit (a.m.u.) defined as

$$1 \text{ amu} = 1.6604 \times 10^{-27} \text{ kg}$$

$$\Rightarrow \quad 1\text{u} = 931.478 \text{ MeV/c}^2$$

and its energy equivalent is 931.48 MeV

The number of protons in a nucleus of an atom is called as the atomic number (Z) of that atom. The number of protons plus neutrons (called as Nucleus) in a nucleus of an atom is called as mass number (A) of that atom.

A particular set of nucleons forming an atom is called as nuclide. It is represented as ${}_ZX^A$. The nuclides having same number of protons (Z), but different number of nucleons (A) are called as isotopes. The nuclide having same number of nucleons (A), but different number of protons (Z) are called as isobars. The nuclide having same number of neutrons (A–Z) are called as isotones.

(2) **Nuclear charge**

Since nucleus contain +vely charged protons (charge = 1.6×10^{-19} C) and neutrons (neutral) so every nucleus has a net +ve charge.

(3) **Nuclear radius**

A rough estimate of nuclear size suggests us that the radius of the nucleus of an atom having mass number 'A' is given by

$$R = R_0 A^{1/3}$$

Where R_0 is a constant found to be equal to

$$R_0 = 1.4 \times 10^{-15} \text{ m} = 1.4 \text{ fm}.$$

(4) **Nuclear Density**

In spite of the fact that nuclear radius depends on mass number of the atom but nuclear density is independent of mass number because if neutrons are supposed to be of almost the same mass as that of protons then the total mass of a nucleus is proportional to A. If each nucleon are

supposed to have a mass m then nuclear density is given by

$$\rho = \frac{mA}{\frac{4}{3}\pi R_0^3 A} = \frac{3m}{4\pi R_0^3} \qquad \textit{(Which is independent of A)}$$

(5) <u>Nuclear spin and magnetic moment</u>

Like orbital electrons in an atom, nucleons inside nucleus have well defined quantum states. Correspondingly they have angular momentum and hence a magnetic moment. Like electrons nucleons also have intrinsic angular momentum and 'magnetic' moment corresponding to their spin.

Nuclear Forces

If only the electrostatic and gravitational forces existed in the nucleus, then it would be impossible to have stable nuclei composed of protons and neutrons. The gravitational forces are much too small to hold the nucleons together compared to the electrostatic forces repelling the protons. Since stable atoms of neutrons and protons do exist, there must be another attractive force acting within the nucleus. This force is called the nuclear force.

<u>Properties of nuclear force</u>

(1) They are charge independent.The nuclear force between two proton is same as that between two neutrons or between a neutron and proton. This is known as charge independent character of nuclear forces.

(2) They may be repulsive may be attractive (Repulsive at exceedingly small separation between two nucleons appreciably smaller than 10^{-13} cm i.e. 10^{-15} m

(3) It is a short range force. Its radius of action is of the order of 10^{-13} cm.

(4) The nuclear force is of saturation character. Each nucleon in nucleus interacts with a limited number of nucleons.

(5) Nuclear force are much stronger than electromagnetic force or gravitational attractive forces.It is the strongest of all the forces. This is why it is called strong interaction.

(6) Nuclear force is spin dependent. If two interacting nucleons are having parallel spins then nuclear force operative between them is comparatively stronger and if their spins are antiparallel, nuclear interaction is comparatively weaker.

(7) Nuclear force is a non-central force. They can not be represented as directed along the st. line connecting the centres of the interacting nucleons. Its non central nature is due to the fact that it depends also on the orientation of the nucleon spins.

Mass defect

It is observed that the mass of a nucleus is slightly less than the sum of the masses of constituent nucleons. Suppose a nucleus consists of 'Z' protons and 'N' neutrons. Mass of a proton, a neutron and the resulting nucleus are respectively m_p, m_n and M then mass defect of the nucleus is given by

$$\Delta m = Zm_p + Nm_n - M$$

If A is the mass number of the nucleus

$$\Delta m = [Zm_p + (A - Z)m_n - M]$$

In terms of atomic masses we may also write mass defect as

$$\Delta m = [\ Zm(^1_1H) + Nm_n - m(^A_z X)]$$

Where $m(^1_1H) =$ mass of one hydrogen atom.

$m(^A_z X) =$ mass of atom having atomic no. Z and mass no. A

e.g.,

mass of ^1_1H = 1.00784 u

mass of neutron = 1.00 874 u

Expected mass of deuterium = 2.01654 u

but measured mass = 2.0141 u.

mass defect,

$\Delta m = 0.00244$ u.

Nuclear Stability

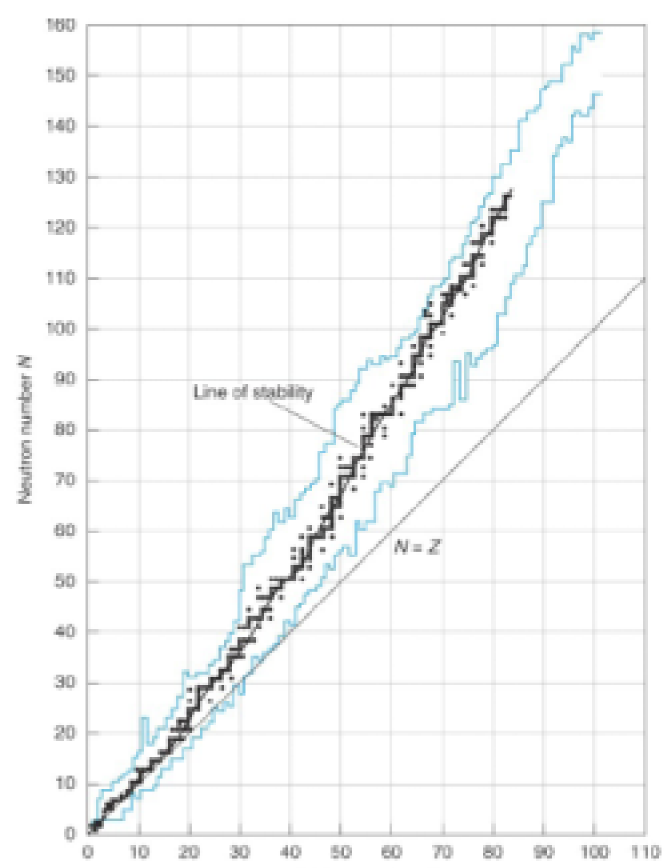


Figure shows pot of N vs. Z for known nuclides. The stable nuclides are indicated by the black dots. Non-stable nuclides decay by emission of particles, or electromagnetic radiation, in a process called radioactivity

Binding energy

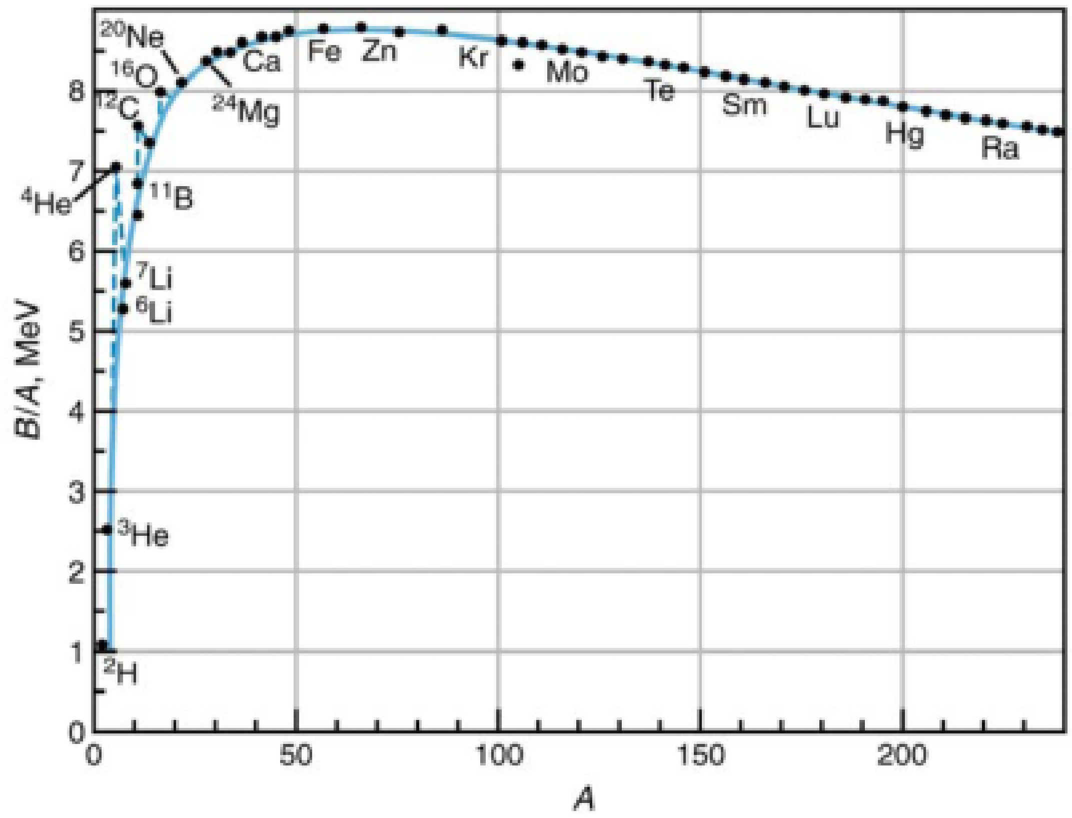
To break a nucleus into its constituent nuclei some energy is required to be supplied. This energy is called Binding Energy of the given nucleus or the energy equivalent of the missing mass of a nucleus is called the binding energy of the nucleus.

$BE = (\Delta m)c^2 = [Zm_p + (A-Z)m_n - M]c^2$,

$BE = \Delta m(\text{in amu})\times 931$ MeV   Where Δm = mass defect

Binding energy per nucleon is a measure of the stability of the nucleus. If there be n nucleons which is equal to A,

$$\frac{\text{Binding Energy}}{\text{Nucleon}} = \frac{\text{B.E.}}{\text{A}}$$



From the plot of B.E. / nucleons Vs mass number (A), we observe that :

(1) Binding energy per nucleon has low value for both heavy and light nuclei i.e. Heavy as well as light nuclei, both are unstable.B.E. / nucleons increases on an average and reaches a maximum of about 8.7 MeV for $A \equiv 50-80$. For more heavy nuclei, B.E. / nucleons decreases slowly as A increases. For the heaviest natural element U^{238} it drops to about 7.5 MeV. From above observation, if follows that nuclei in the region of atomic masses 50 – 80 are most stable.

(2) The intermediate nuclei have large value of binding energy per nucleon so they are more stable.

(3) Binding energy per nucleon increases rapidly upto mass number 20 but there are peaks corresponding to ${}^{4}_{2}He$, ${}^{12}_{6}C$, ${}^{16}_{8}O$ which indicates that these nuclei are more stable than neighbours. The reason is that they may be considered to posses magic numbers i.e. their mass number is divisible by 4 and these nuclei may have ${}^{4}_{2}He$ as their constituents.

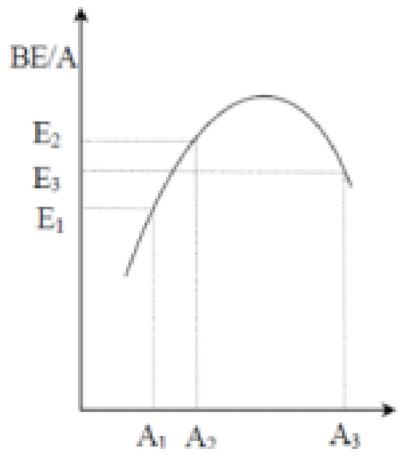
(4) The minimum value of the BE/Nucleon is in the case of deuteron that is 1.11 Mev.

(5) The maximum value of the BE/Nucleon is 8.79 Mev for the nuclide ${}^{56}_{26}Fe$ which is therefore the most stable nucleus.

Illustration :

Using the following plot of BE/nucleon vs mass number, mention the condition for which the energy is absorbed or released for the reaction

$${}_{Z_1}P^{A_1} + {}_{Z_2}Q^{A_2} \rightarrow {}_{Z_3}P^{A_3}$$



Sol. *Binding energy for reactant is $(xE_a + yE_b)$ and that for product is zE_c*

Case-I :

$$\text{if } (A_1E_1 + A_2E_2) > A_3E_3$$

Energy is absorbed.

Case-II :

$$\text{if } (A_1E_1 + A_2E_2) < A_3E_3$$

Energy is released.

Note

(i) If we split a heavy nucleus into two medium sized nuclei and total binding energy of new nuclei is greater then parent nuclei, then energy is released (Nuclear fission)

(ii) If two nuclei of small mass number combine to form a single medium size nucleus for which binding energy is greater than the constituent nuclei, then energy is released (Nuclear fusion)

Nuclear Fission

The breaking of a heavy nucleus into two or more fragments of comparable mass, with the release of tremendous energy is called as nuclear fission.

The most typical fission reaction occurs when slow moving neutrons strike ${}_{92}U^{235}$. The following nuclear reaction takes place.

$${}_{92}U^{235} + {}_{0}n^{1} \longrightarrow {}_{56}Ba^{141} + {}_{36}Kr^{92} + 3\,{}_{0}n^{1} + 200 \text{ MeV}$$

If more than one of the neutrons produced in the above fission reaction are capable of inducing a fission reaction (provided U^{235} is available), then the number of fission taking place at successive stages goes increasing at a very brisk rate and this generates a series of fission. This is known as chain reaction. The chain reaction takes place only if the size of the fissionable material (U^{235}) is greater than a certain size called the critical size.

If the number of fission in a given interval of time goes on increasing continuously, then a condition of explosion is created. In such cases, the chain reaction is known as uncontrolled chain reaction. This forms the basis of atomic bomb.

In a chain reaction, the fast moving neutrons are absorbed by certain substances known as moderators (like heavy water), then the number of fissions can be controlled and the chain reaction is such cases is known as controlled chain reaction. This forms the basis of a nuclear reactor.

Nuclear Fusion

The process in which two or more light nuclei are combined into a single nucleus with the release of tremendous amount of energy is called as nuclear fusion. Like a fission reaction, the sum of masses before the fusion (i.e. of bigger nucleus) and this difference appears as the fusion energy. The most typical fusion reaction is the fusion of two deuterium nuclei into helium.

$$_1H^2 + {}_1H^2 \longrightarrow {}_2He^4 + 21.6 \text{ MeV}$$

For the fusion reaction to occur, the light nuclei are brought closer to each other (with a distance of 10^{-14} m). This is possible only at very high temperature to counter the repulsive force between nuclei. Due to this reason, the fusion reaction is very difficult to perform. The inner core of sun is at very high temperature, and is suitable for fusion. In fact the source of sun's and other star's energy is the nuclear fusion reaction.

Conservation laws in nuclear reaction

Nuclear reaction processes have led to the formulation of useful conservation principles. The four principles of most interest in this module are discussed below.

(i) Conservation of electric charge implies that charges are neither created nor destroyed. Single positive and negative charges may, however, neutralize each other. It is also possible for a neutral particle to produce one charge of each sign.

(ii) Conservation of mass number does not allow a net change in the number of nucleons i.e. total number of protons and neutrons should also remain same on both sides of a nuclear reaction.. However, the conversion of a proton to a neutron and vice versa is allowed.

(iii) Conservation of mass and energy implies that the total of the kinetic energy and the energy equivalent of the mass in a system must be conserved in all decays and reactions. Mass can be converted to energy and energy can be converted to mass, but the sum of mass and energy must be constant. In nuclear reactions, sum of masses before reaction is greater than the sum of masses after the reaction. The difference in masses appears in form of energy following the Law of inter-conversion of mass & energy. The energy released in a nuclear reaction is called as Q Value of a reaction and is given as follows.

If difference in mass before and after the reaction is Δm amu (Δm = mass of reactants minus mass of products) then

$$Q \text{ value} = \Delta m\,(931) \text{ MeV}$$

(iv) Conservation of momentum is responsible for the distribution of the available kinetic energy among product nuclei, particles, and/or radiation. The total amount is the same before and after the reaction even though it may be distributed differently among entirely different nuclides and/or particles.

Illustration :

In the sun about 4 billion kg of matter is converted to energy each second. Find the power output of the sun in watt .

Sol. $\frac{m}{t} = 4\times10^{8}\ kgs^{-1}$

$E = mc^2$

$\Rightarrow \quad \frac{E}{t} = \left(\frac{m}{t}\right)c^2$

$\Rightarrow \quad \frac{E}{t} = 4\times10^{8}\times9\times10^{16}$

$\Rightarrow \quad \frac{E}{t} = 3.6\times10^{25}\ Js^{-1}$

$\Rightarrow \quad \frac{E}{t} = 3.6\times10^{25}\ W$

Illustration :

A neutron breaks into a proton and electron. Calculate the energy produced in this reaction in MeV. Mass of an electron = 9×10^{-31} kg, Mass of Proton = 1.6725×10^{-27} kg, Mass of neutron = 1.6747×10^{-27} kg. Speed of light = 3×10^{8} m/sec.

Sol. ${}_0n^1 \rightarrow {}_1H^1 + {}_{-1}e^0$

Δm= [Mass of neutron – (mass of proton + mass of electron)]

$= [1.6747 \times 10^{-27} - (1.6725 \times 10^{-27} + 9 \times 10^{-31})]$

$= 0.0013 \times 10^{-27}$ kg

$\therefore$ Energy released

$E = \Delta mc^2 = (0.0013 \times 10^{-27}) \times (3 \times 10^{8})^2 = 1.17 \times 10^{-13}$ joule

$= \frac{1.17\times10^{-13}}{1.6\times10^{-19}}\ eV = 0.73 \times 10^{6}\ eV = 0.73\ MeV$

Illustration :

The nuclei involved in the nuclear reaction $A_1 + A_2 \rightarrow A_3 + A_4$ have the binding energies E_1, E_2, E_3 and E_4. Find the energy released (Q value) of this reaction.

Sol. *Suppose M_1, M_2, M_3, M_4 are the rest masses of the nuclei A_1, A_2, A_3 and A_4 perticipating in the reaction*

$$A_1 + A_2 \longrightarrow A_3 + A_4 + Q$$

Here Q is the energy released. Then by conservation of energy.

$Q = (M_1 + M_2 - M_3 - M_4)c^2$

Now $\quad M_1c^2 = c^2\,(Z_1m_H + (A_1 - Z)\,m_n) - E_1$ etc. and

$Z_1 + Z_2 = Z_3 + Z_4$ (conservation of charge)

$A_1 + A_2 = A_3 + A_4$ (conservation of mass number)

Here $\quad Q = (E_3 + E_4) - (E_1 + E_2)$

Pratice Exercise

Q.1 Calculate the electric potential energy due to the electric repulsion between two nuclei of ^{12}C when they 'touch' each other at the surface.

Q.2 Find the binding energy of ${}_{26}Fe^{56}$. Atomic mass of ${}_{26}Fe^{56}$ is 55·9349 u and that of ^{1}H is 1·00783 u. Mass of neutron is 1·00867u.

Q.3 Calculate the Q-value in the following decay

$$^{25}Al \rightarrow {}^{25}Mg + e^{+} + \nu$$

Q.4 Find the maximum energy that a beta particle can have in the following decay

$$^{176}Lu \rightarrow {}^{176}Hf + e + \overline{\nu}$$

Atomic mass of ^{176}Lu is 175.942694 u and that of ^{176}Hf is 175.941420 u

Answers

Q.1 10.2 MeV. Q.2 492 MeV. Q.3 3.254 MeV Q.4 0.2806 MeV

Radioactivity

Among about 2500 known nuclides, fewer than 300 are stable. The others are unstable structures that decay to form other nuclides by spontaneously emitting particles and electromagnetic radiation, a process called radioactivity. The time scale of these decay processes ranges from a small fraction of a microsecond to billions of years. The substances which emit these radiations are called as radioactive substances. It was discovered by Henry Becquerel for atoms of Uranium. Later it was discovered that many naturally occurring compounds of heavy elements like radium, thorium etc also emit radiations.

At present, it is known that all the naturally occurring elements having atomic number greater than 82 are radioactive. For example some of them are ; radium, polonium, thorium, actinium, uranium, radon etc. Later on Rutherford found that emission of radiation always accompanied by transformation of one element (transmutation) into another. In actual radioactivity is the result of disintegration of an unable nucleus. Rutherford studied the nature of these radiations and found that these mainly consist of α, β, γ particles (rays).

α-Particles : (${}_{2}He^{4}$)

These carry a charge of +2e and mass equal to $4m_p$. These are nuclei of helium atoms. The energies of α-particles very from 5 MeV to 9 MeV ; their velocities vary from 0.01 – 0.1 times of c (velocity of light). They can be deflected by electric and magnetic field and have lower penetrating power but high ionising power.

β-Particles : ($_{-1}e^0$)

These are fast moving electrons having charge equal to e and mass $m_e = 9.1 \times 10^{-31}$ kg. Their velocities very from 1% to 99% of the velocity of light (c). They can also be deflected by electric and magnetic fields. They have low ionising power but high penetrating power.

γ-Radiations : ($_0\gamma^0$)

These are electro-magnetic waves of nuclear origin and of very short wavelength. They have no mass. They have maximum penetrating power and minimum ionising power. The energy released in a nuclear reaction is mainly emitted in from these γ-radiations.

Radioactive decays

α-decay

Nuclides decay is by emitting α-particles. α-particles are generally emitted by very heavy nuclei containing to many nucleons to remain stable. The emission of such a nucleon cluster as a whole rather than the emission of single nucleon is energetically more advantageous because of the particularly high binding energy of alpha-particles. The parent nucleus (Z,A) is transformed as

$$_zX^A \longrightarrow {}_2He^4 + {}_{z-z}Y^{A-4}$$

$$\Delta m = \left\{M\left[{}_2He^4\right] + M\left[{}_{z-2}Y^{A-4}\right]\right\} - \left\{M\left[{}_zX^A\right]\right\}$$

Note

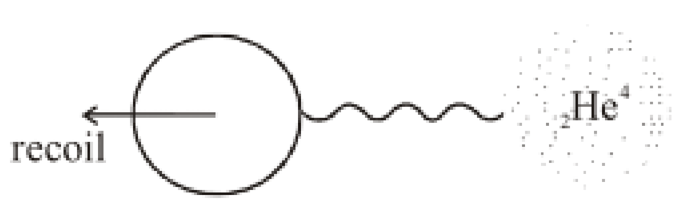
(i) Nuclear mass is different from atomic mass because nucleus is without electrons.

(ii) Released energy converts into kinetic energy

(iii) In nucleus, Atomic energy is 13.6 eV small atomic binding energy has been neglected.

(iv) Released energy is shared as kinetic energy by products and outgoing particles.

Calculation of Kinetic Energy



Momentum of $\alpha_{particle}$ + momentum of daughter nuclei = 0

$(m_\alpha \vec{v})$ $\qquad\qquad\qquad (\vec{p}_D)$

assuming parent nuclei to be at rest initially

$$\vec{p}_\alpha + \vec{p}_D = 0$$

$$|\vec{p}_\alpha| = |\vec{p}_D|$$

If Q is released energy or Q value of reaction.

$$K_\alpha + K_D = Q$$

$$\Rightarrow \qquad K_\alpha + \frac{p_D^{\,2}}{2m_D} = Q$$

$$\Rightarrow \qquad K_\alpha + \frac{p_\alpha^{\,2}}{2m_D} = Q$$

but

$$\Rightarrow \qquad K_\alpha + \frac{2K_\alpha \cdot m_\alpha}{2m_D} = Q$$

$$\Rightarrow \qquad K_\alpha \left[1 + \frac{m_\alpha}{m_D}\right] = Q$$

$$K_\alpha = \frac{m_D \times Q}{m_D + m_\alpha}$$

$$K_\alpha = \frac{(A-4)m}{4m+(A-4)m} Q$$

$$\Rightarrow \qquad K_\alpha = \left[\frac{A-4}{A}\right] Q$$

β decay

Another way in which nuclides decay radioactively is by the emission of β particles. .When neutron-proton ratio inside a nucleus is not suitable for it to be stable (either less or more) then β-decay takes place. Due to a special type of interaction called weak interaction a neutron gets converted into a proton and a proton gets converted into a neutron and a positron. Electrons or positrons are emitted from the nucleus just after their creation. This emission of electron or positron from nucleus is called β-decay. Emission of positron (of the order of MeV) is called β^+-decay and emission of electron (of the order of MeV) is called β^--decay.

(i) Negative β decay(β^- decay)

Neutron inside nucleus is transformed into proton .

$$n \longrightarrow p + e^- + \overline{\nu} \quad \text{(Antineutrino)}$$

$$_zX^A \longrightarrow {}_{z+1}Y^A + e + \overline{\nu} \ \text{+energy released.}$$

Equation corresponding to nuclear mass

$$\Delta m = M\left[{}_zX^A\right] - \left\{M\left[{}_{z+1}Y^A\right] + Me\right\}$$

Equation corresponding to atomic mass

$$\Delta m = M^*\left[{}_zX^A\right] - M^*\left[{}_{z+1}Y^A\right]$$

energy released

$$E = \Delta mc^2$$

(ii) Positive β decay(β⁺ decay)

Proton inside nucleus is transformed into neutron .

$$p \longrightarrow n + e^{+} \text{ (Positron)} + \nu \text{ (neutrino)}$$

Positron is anti-particle of electron. It is highly reactive.

$$_{z}X^{A} \longrightarrow {}_{z-1}Y^{A} + e^{+} + \nu \text{ +energy released.}$$

Equation corresponding to nuclear mass

$$\Delta m = M\left[{}_{z}A^{x}\right] - \left\{M\left[{}_{z-1}Y^{A}\right] + Me\right\}$$

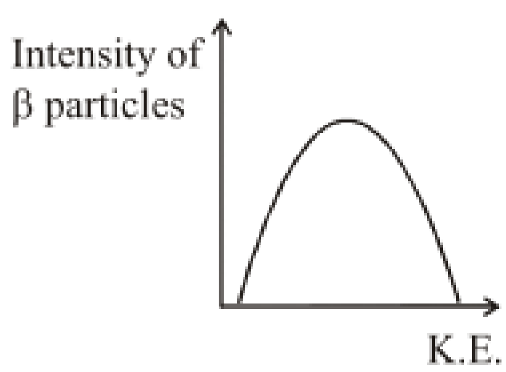
Equation corresponding to atomic mass

$$\Delta m = M^{*}\left[{}_{z}A^{x}\right] - \left\{M^{*}{}_{z-1}Y^{A} - 2Me\right\}$$

energy released

$$E = \Delta mc^{2}$$

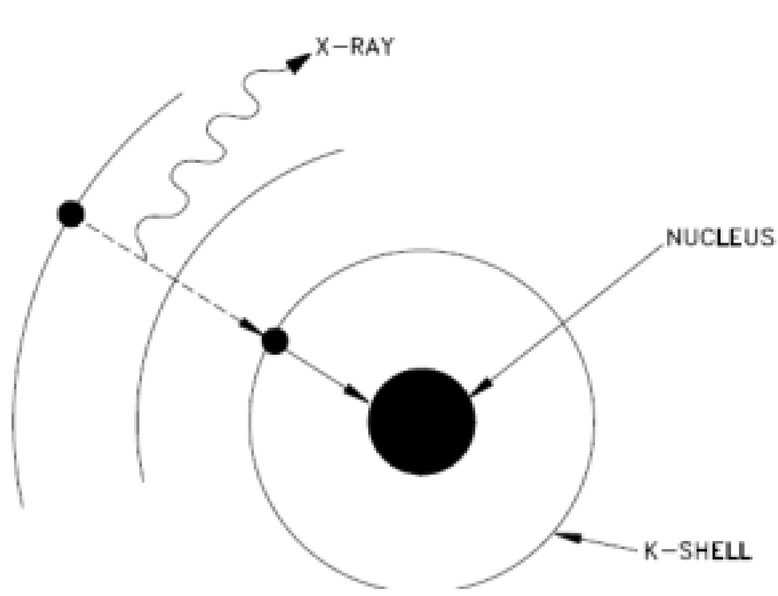
Experiments show that β-particles are emitted with continuous range of kinetic energy.



(iii) Electron capture

Nuclei having an excess of protons may capture an electron from one of the inner orbits which immediately combines with a proton in the nucleus to form a neutron. This process is called electron capture (EC). The electron is normally captured from the innermost orbit (the K-shell), and, consequently, this process is sometimes called K-capture.

The process is observed from the emission of the characteristic X-rays produced, when an orbiting electron from an outer shell makes a downward transition into a K shell vacancy. The X-rays are characteristic of daughter nuclei not of the parent because x-ray emission taken place 'K-capture'



$${}_{-1}e^{0} + {}_{4}Be^{7} \longrightarrow {}_{3}Li^{7} + \nu \qquad \text{(A simple K-capture equation)}$$

Neutrino and anti-neutrino

1. It has zero electric charge, hence shows no electromagnetic interaction.

2. Rest mass is possibly zero. Recent experiments show that mass of neutrino is less than $\left(\frac{7}{c^2}\text{ev}\right)$.

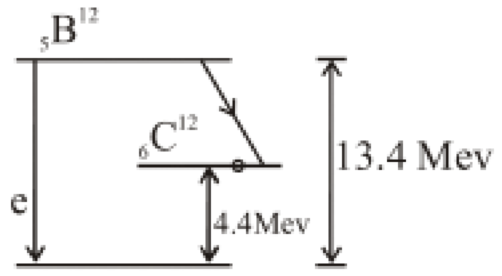
3. It travles with speed of light.

4. It has spin quantum number $1/2$. A spin of $1/2$ satisfies the law of conservation of angular momentum when applied to β-decay.

5. It shows very weak interactions with matter.

γ - Decay

As we know that like the discrete energy orbits of electrons in an atom. Nucleons in an atom inside the nucleus also have well defined energy state or discrete quantum state. After every α or β emission a nucleus is in the excited state correspondingly subsequent to every α-or β-emission a nucleus emits electromagnetic radiation (of the order of MeV) to come to ground state. The frequency or the wavelength of the emitted radiations lie in γ-region and is called γ-emission.



Group-Displacement Law

(i) When a nuclide emits one α-particle ($_2He^4$), its mass number (A) decreases by 4 units and atomic number (Z) decreases by two units.

$$_zX^A \longrightarrow {}_{Z2}Y^{A-4} + {}_2He^4 + \text{Energy}$$

(ii) When a nuclide emits a β^-particle, its mass number remains unchanged but atomic number increases by one unit.

$$_zX^A \longrightarrow {}_{z+1}Y^A + {}_{-1}e^0 + \bar{\nu} + \text{Energy} \qquad (\bar{\nu}\text{ is antineutrino})$$

(iii) When a nuclide emits a β^+ particle, its mass number remains unchanged but atomic number decreases by one unit.

$$_zX^A \longrightarrow {}_{z-1}Y^A + {}_{+1}e^0 + \nu + \text{Energy} \qquad (\nu\text{ is neutrino})$$

(iv) When a γ particle is produced, both atomic and mass number remain constant.

Rutherford-Soddy Law (Statistical Law)

The disintegration of a radioactive substance is random and spontaneous.

Radioactive decay is purely a nuclear phenomenon and is independent of any physical and chemical conditions.

The radioactive decay follows first order kinetics, i.e., the rate of decay is proportional to then number of undecayed atoms in a radioactive substance at any time t. If dN be the number of atoms (nuclei) disintegrating in time dt, the rate of decay is given as dN/dt. From first order kinetic rate law :

$$\frac{dN}{dt} = -\lambda N$$

where λ is called as decay or disintegration constant.

Let N_0 be the number of nuclei at time $t = 0$ and N_t be the number of nuclei after time t, then according to integrated first order rate law, we have :

$$N_t = N_0 \, e^{-\lambda t}$$

$$\Rightarrow \qquad \lambda t = ln \frac{N_0}{N_t} = 2.303 \log \frac{N_0}{N_t}$$

The half life ($t_{1/2}$) period of a radioacitve substance is defined as the time in which one-half of the radioactive substance is disintegrated. If N_0 be the number of nuclei at $t = 0$, then in ha half life T, the number of nuclei decayed will be $N_0/2$

$$N_t = N_0 e^{-\lambda t} \qquad \text{... (i)}$$

$$\Rightarrow \qquad \frac{N_0}{2} = N_0 e^{-\lambda T} \qquad \text{... (ii)}$$

from (i) & (ii)

$$\frac{N_t}{N_0} = \left(\frac{1}{2}\right)^{t/T} = \left(\frac{1}{2}\right)^{n} \qquad \text{n : number of half lives}$$

The half life (T) and decay constant (λ) are related as :

$$T = \frac{0.693}{\lambda}$$

The mean life (T_m) of a radioactive substance is equal to the sum of life times of all atoms divided by the number of all atoms and is given follows

$$T_m = \frac{\int t dN}{\int dN} = \frac{\int_0^\infty t\lambda e^{-\lambda t} dt}{\int_0^\infty \lambda e^{-\lambda t} dt}$$

$$T_m = \frac{1}{\lambda}$$

Activity of a Radioactive Isotope

The activity of a radioactive substance (or radioisotope) means the rate of decay per second or the number of nuclei disintegrating per second. It is generally denoted by A.

$$\Rightarrow \qquad A = -\frac{dN}{dt}$$

If at time t = 0, the activity of a radioactive substance be A_0 and afte time t = t sec, activity be A_0 then :

$$A_0 = -\left[\frac{dN}{dt}\right]_{t=0} = \lambda N_0$$

$$A_t = -\left[\frac{dN}{dt}\right]_{t=t} = \lambda N_1$$

$$\Rightarrow \qquad A_t = A_0^{-\lambda t}$$

Unit of activity

The activity is measured in terms of curie (Ci). 1 curie is the activity of 1gm of a freshly prepared sample of radium Ra^{226} ($t_{1/2}$ = 1602 yrs.)

1 curie 1Ci = 3.7×10^{10} dps (disintegration per second)

1 dps is also known as 1 bq (Becquerel) $\Rightarrow$ 1Ci = 3.7×10^{10} bq

Note

All the equations discussed above is valid only when the number of nuclii are very large

Survival probability and decay probability for a finite time interval

The probability of suravival (i.e. not decaying) in time t is

$$P_{survival} = \frac{N_0 e^{-\lambda t}}{N_0} = e^{-\lambda t}.$$

Hence the probability of decay is

$$P_{decay} = 1 - e^{-\lambda t}.$$

Successive disintegration and secular equilibrium

Suppose A→B→C ... ... (i.e., radioactive nucleus A decays to B and B decays to C)

Let number of radioactive nucleus A (Parent nucleus) at time t = 0 be N_0 and that of B = 0.

For A

$$\frac{dN_A}{dt} = -\lambda_A N_A$$

$$\Rightarrow \qquad N_A = N_0 e^{-\lambda_A t}$$

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For B

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B$$

$$\Rightarrow \quad \frac{dN_B}{dt} = \lambda_A N_0 e^{-\lambda_A t} - \lambda_B N_B$$

Multiplying both sides of this equation by $e^{\lambda_B t} dt$, we get

$$e^{\lambda_B t}.dN_B + \lambda_B N_B e^{\lambda_B t} dt = \lambda_A N_0 e^{(\lambda_B - \lambda_A)t} dt$$

$$\Rightarrow \quad \frac{d(N_B.e^{\lambda_B t})}{dt} = e^{\lambda_B t}\frac{dN_B}{dt} + N_B.\lambda_B e^{\lambda_B t}$$

$$\Rightarrow \quad d(N_B.e^{\lambda_B t)} = e^{\lambda_B t} dN_B + N_B \lambda_B e^{\lambda_B t} dt$$

$$\Rightarrow \quad \int d(N_B e^{\lambda_B t}) = \lambda_A N_0 \int e^{(\lambda_B - \lambda_A)t} dt$$

$$\Rightarrow \quad N_B e^{\lambda_B t} = \frac{\lambda_A N_0}{(\lambda_B - \lambda_A)} e^{(\lambda_B - \lambda_A)t} + C$$

Where C is constant of integration.

at t=0, N_B=0

$$\Rightarrow \quad C = \frac{-\lambda_A N_0}{(\lambda_B - \lambda_A)}$$

Hence,

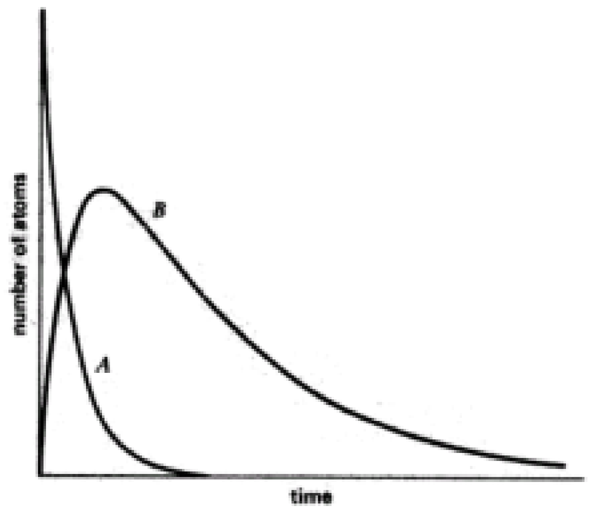
$$N_B e^{\lambda_B t} = \frac{\lambda_A N_0}{\lambda_B - \lambda_A}\left[e^{(\lambda_B - \lambda_A)t} - 1\right]$$

$$\Rightarrow \quad N_B = \frac{\lambda_A N_0}{\lambda_B - \lambda_A}\left[e^{-\lambda_A t} - e^{-\lambda_B t}\right]$$

Suppose the parent nucleus A is long lived i.e. the half life of the parent nucleus A is much larger in comparison to the half life of the daughter nucleus B

$\Rightarrow t_{1/2\,A} >> t_{1/2\,B} \qquad \Rightarrow \lambda_A << \lambda_B \quad \Rightarrow \lambda_B - \lambda_A \approx \lambda_B$

$\Rightarrow e^{-\lambda_B t}$ is negligible in comparison to $e^{-\lambda_A t}$



$$\Rightarrow \quad N_B = \frac{\lambda_A}{\lambda_B} N_0 e^{-\lambda_A t}$$

$$\Rightarrow \quad N_B = \frac{\lambda_A}{\lambda_B} N_A$$

$$\Rightarrow \quad N_A \lambda_A = N_B \lambda_B$$

i.e., after a time much longer in comparison to the half life of the daughter nucleus B but much shorter in comparison to the half life of parent nucleus A, we have $N_A\lambda_A = N_B\lambda_B$ This state is called secular equilibrium

Illustration :

The mean lives of an radio active substance are 1620 and 405 years for α-emission and β-emission respectively. Find out the time during which three fourth of a sample will decay if it is decaying both by α-emission and β-emission simultaneously.

Sol. *When a substance decarys by α and β emission simultaneously, the equivalent rate of disintegration λ_{eq} is given by :*

$$\lambda_{eq} = \lambda_\alpha + \lambda_\beta$$

where λ_α *= disintegration constant for α-emission only*

λ_β *= disintegration constant for β-emission only*

Mean life is given by : $T_{eq} = \dfrac{1}{\lambda_{eq}}$

$$\Rightarrow \quad \lambda_{eq} = \lambda_\alpha + \lambda_\beta = \frac{1}{T_{eq}} = \frac{1}{T_\alpha} + \frac{1}{T_\beta} = \frac{1}{1620} + \frac{1}{405} = 308 \times 10^{-3}$$

$$\lambda_{eq} t = 2.303 \log \frac{N_0}{N_t}$$

$$\Rightarrow \quad (3.08 \times 10^{-3})\, t = 2.303 \log \frac{100}{25}$$

$$\Rightarrow \quad t = 2.303 \times \frac{1}{3.08\times10^{-3}} \log 4 = 449.24 \text{ years}$$

Illustration :

Two radioctive materials A_1 and A_2 have have decay constants of 10 λ_0 and λ_0. If initially they have same number of nuclei, find the time after which the ratio of number their undecayed nuclei will be (1/e)

Sol. $\dfrac{N_A}{N_B} = \dfrac{e^{-10\lambda_0 t}}{e^{-\lambda_0 t}} = e^{-9\lambda_0 t} = \dfrac{1}{e} = e^{-1}$

$\Rightarrow 9\,\lambda_0\, t = 1$

or $t = \dfrac{1}{9\lambda_0}$

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Illustration :

The weight based ratio fo U^{238} and Pb^{226} in a sample of rock is 4 : 3. If the half life of U^{238} is 4.5×10^9 year, then find the age of rock.

Sol. *Let initial no. of U-atoms = N_0*

After time t, (age of rock), let no. of atoms remaining undecayed = N

$$\therefore \quad \frac{238N}{26(N_0 - N)} = \frac{4}{3}$$

$$\therefore \quad \frac{N_0}{N} = 1.79$$

$$t = \frac{T\log N_0/N}{\log 2}$$

$$= \frac{4.5\times10^9\times\log 1.79}{0.301}$$

Illustration :

A count rate-meter is used to measure the activity of a given sample. At one instant the meter shows 4750 counts per minutes. Five minutes later it shows 2700 counts per minutes. Find :

(a) decay constant *(b) the half lift of the sample*

Sol. *Initial activity = A_0 = dN/dt at t = 0*

Final activity = A_t = dN/dt at t = t

$$\left.\frac{dN}{dt}\right|_{t=0} = \lambda N_0 \textit{ and } \left.\frac{dN}{dt}\right|_{t=5} = \lambda N_t$$

$$\Rightarrow \quad \frac{4750}{2700} = \frac{N_0}{N_t}$$

$$\text{Using } \lambda t = 2.303 \log\frac{N_0}{N_t}$$

$$\Rightarrow \quad \lambda(5) = 2.303 \log\frac{4750}{2700}$$

$$\Rightarrow \quad \lambda = \frac{2.303}{5}\log\frac{4750}{2700} = 0.1129\ \text{min}^{-1}$$

$$\Rightarrow \quad t_{1/2} = \frac{0.693}{0.1129} = 6.14\ \text{min}$$

Illustration :

A small amount of solution containing Na^{24} radionuclide with activity $A = 2.0. \times 10^3$ disintegrations per second was injected in the bloodstream of a man. The activity of 1 cm^3 of blood sample taken $1 = 5.0$ hours later turned out to be $A' = 16$ disintegrations per minute per cm^3. The half-life of the radionuclide is $T = 15$ hours. Find the volume of the man's blood.

Sol. *Let V = volume of blood in the body of the human being. Then the total activity of the blood is A'V. Assuming all this activity is due to the injected Na^{24} and taking account of the decay of this radionuclide, we get*

$$VA' = Ae^{-\lambda t}$$

Now $\lambda = \dfrac{\ln 2}{15}$ *per hour,* $t = 5$ *hour*

Thus $V = \dfrac{A}{A'} e^{-\ln 2/3} = \dfrac{2.0 \times 10^3}{(16 / 60)} e^{-\ln 2/3}\ cc = 5.95$ *litre*

Pratice Exercise

Q.1 The half-life of ^{198}Au is 2.7 days, Calculate
(a) the decay constant,
(b) the average-life and
(c) the activity of 1.00 mg of ^{198}Au,
Take atomic weight of ^{198}Au to be 198 g/mol.

Q.2 A radioactive sample has 6×10^{18} active nuclei at a certain instant. How many of these nuclei will still be in the same active state after two half-lives

Q.3 The activity of a radioactive sample falls from 600 s^{-1} to 500 s^{-1} in 40 minutes. Calculate its half-life.

Q.4 The number of ^{238}U atoms in an ancient rock equals the number of ^{206}Pb atoms. The half-life of decay of ^{238}U is 4.5×10^9 y. Estimate the age of the rock assuming that all the ^{206}Pb atoms are formed from the decay of ^{238}U

Q.5 A radioactive nucleus can decay by two different processes. The half-life for the first process is t_1 and that for the second process is t_2• Find the effective half-life t of the nucleus.

Q.6 A radioactive sample decays with an average-life of 20 ms. A capacitor of capacitance 100 μF is charged to some potential and then the plates are connected through a resistance R. What should be the value of R so that the ratio of the charge on the capacitor to the activity of the radioactive sample remains constant in time?

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Q.7 Suppose, the daughter nucleus in a nuclear decay is itself radioactive. Let λ_p and λ_d be the decay constants of the parent and the daughter nuclei. Also, let N_p and N_d be the number of parent and daughter nuclei at time t. Find the condition for which the number of daughter nuclei becomes constant.

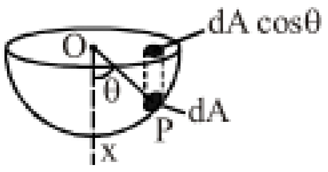
Answers

Q.1	(a) 2.9×10^{-6} s^{-1},	(b)3.9 days	(c) 240 Ci.		
Q.2	1.5×10^{18}	Q.3	152 min.	Q.4	4.5×10^{9} y
Q.5	$\frac{1}{t}=\frac{1}{t_1}+\frac{1}{t_2}$	Q.6	200 Ω.	Q.7	$\lambda_p N_p = \lambda_d N_d$

Solved Examples

Q.1 A point source of light is placed at the centre of curvature of a hemispherical surface. The radius of curvature is r and the inner surface is completely reflecting. Find the force on the hemisphere due to the light falling on it if the source emits a power W.

Sol. The energy emitted by the source per unit time, i.e. W fall on a area $4\pi r^2$ at a distance r in unit time. Thus, the energy falling per unit area per unit time is $\frac{W}{4\pi r^2}$. Consider a small area dA at the point P of the hemisphere (figure).



The energy falling per unit time on it, is

$$P = \frac{WdA}{4\pi r^2}$$

The corresponding momentum incident on this area per unit time is

$$= \frac{WdA}{4\pi r^2 c}$$

Suppose the radius OP through the area dA makes an angle θ with the symmetry axis OX. The force on dA is along this radius.

$$dF = \frac{2WdA}{4\pi r^2 c}$$

By symmetry, the resultant force on the hemisphere is along OX. The component of dF along OX is

$$dF \cos\theta = \frac{2WdA}{4\pi r^2 c} \cos\theta$$

$$= \frac{2W}{4\pi r^2 c} \text{ (projection the area dA on the plane containing the rim)}$$

The net force along OX is

$$F = \frac{2W}{4\pi r^2 c} \Sigma \text{ (projection the area dA on the plane containing the rim)}$$

$$= \frac{2W}{4\pi r^2 c} (\pi r^2) = \frac{W}{2c}$$

Q.2 Find the maximum kinetic energy of photo-electron liberated from the surface of lithium ($\phi = 2.39$ eV) by electromagnetic radiation whose electric component varies with time as $E = a(1 + \cos\omega t)\cos\omega_0 I$, where 'a' is a constant. $\omega = 6 \times 10^{14}$ rad/sec and $\omega_0 = 3.60 \times 10^{15}$ rad/s.

Sol. $E = a(1 + \cos\omega t)\cos\omega_0 t = a\cos\omega_0 t = \cos\omega_0 t + a\cos\omega t\cos\omega_0 t$

$$\Rightarrow E = a\cos\omega_0 t + \frac{1}{2} a\cos(\omega + \omega_0)t + \frac{1}{2} a\cos(\omega + \omega_0)t$$

This is a complex vibration consisting of harmonic components of frequencies ω_0, $(\omega + \omega_0)$ and $(\omega - \omega_0)$.

The highest angular frequency is $(\omega + \omega_0)$.

Now, $h\nu = \phi + k_{max}$

$$\text{So, } k_{max} = \frac{h}{2\pi}(\omega + \omega_0) - \phi$$

$$= \frac{6.6\times10^{-34}}{2\pi}(6\times10^{14} + 3.6\times10^{15}) - 2.39\times1.6\times10^{-19}$$

$$= 4.41\times10^{-19} - 3.82\times10^{-19} = 0.59\times10^{-19}\ \text{J} = 0.37\ \text{eV}$$

Q.3 Find the ratio of de-Broglie wavelength of an α-particle to that of a proton being subjected to the same magnetic field so that the radii of their paths are equal to each other, assuming that the field induction vector $\vec{B}$ is perpendicular to the velocity vectors of the a-particle and the proton.

Sol. When a charged particle of charge q, mass m enters perpendicularly to the magnetic induction $\vec{B}$ of a magnetic field, it will experience a magnetic force

$F = q(\vec{v}\times\vec{B}) = qvB \sin 90^\circ = qvB$ that will provide a centripetal acceleration $\frac{v^2}{r}$

$$\Rightarrow qvB = \frac{mv^2}{r}$$

$$\Rightarrow mv = qBr$$

$$\Rightarrow \text{The de-Broglie wavelength } \lambda = \frac{h}{mv} = \frac{h}{qBr}$$

$$\Rightarrow \frac{\lambda_{\alpha-\text{particle}}}{\lambda_{\text{proton}}} = \frac{q_p r_p}{q_\alpha r_\alpha}$$

$$\text{Since } \frac{r_\alpha}{r_p} = \text{I and } \frac{q_\alpha}{q_p} = 2$$

$$\Rightarrow \frac{\lambda_\alpha}{\lambda_p} = 1/2$$

Q.4 A particle of mass m moves along a circular orbit in a centrosymmetric potential field $U = \frac{kr^2}{2}$. Using Bohr's quantization condition. Find (a) radius of n^{th} orbit (b) Energy of n^{th} orbit

Sol. $F = -\frac{dU}{dr} = -kr$

so $kr = \frac{mv^2}{r}$ ...(i)

and $mvr = \frac{nh}{2\pi}$ ...(ii)

Solving we get $r = \left[\dfrac{h^2n^2}{4\pi mk}\right]^{\frac{1}{2}}$

Total energy $E_n = KE_n + PE_n$

$$= \frac{1}{2}mv^2 + \frac{kr^2}{2} = kr^2$$

$$= K\sqrt{\frac{n^2h^2}{4\pi^2 mK}} = \sqrt{\frac{n^2h^2K^2}{4\pi^2 mK}}$$

$$= \frac{nhK}{2\pi\sqrt{mK}}$$

Q.5 Compare the radii and energy of ground state of H-atom and p-atom considering the motion of nucleus.

Sol. If we consider the motion of nucleus mass of e^- in all the expressions will be replaced by μ. Where

$$\mu_H = \frac{mM}{m+M};$$

M = Mass of Proton or neutron

m = mass of electron.

and $\mu_D = \dfrac{m(2M)}{m+2M}$

hence $\mu_D > \mu_H$

radius of n^{th} orbit $r_n \propto \dfrac{1}{\mu}$ so $r_H > r_D$

Energy of n^{th} orbit $E_n \propto \mu$ so $E_\mu < E_D$.

Q.6 What lines of atomic hydrogen absorbing spectrum fall with in the ware length range from 94.5 nm to 130 nm.

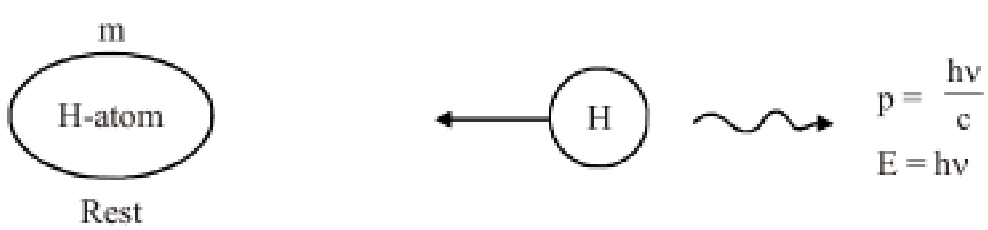
Sol. Absorbtion lines are always corresponding to lyman series.

Wave length of lyman series $\dfrac{1}{\lambda} = R\left[\dfrac{1}{12} - \dfrac{1}{n^2}\right]$

for n = 2 , λ_1 = 121 nm
for n = 3 , λ_2 = 102.2 nm
for n = 4 , λ_3 = 96.9 nm
for n = 5 , λ_4 = 94.64 nm
for n = 6 , λ_5 = 93.45 nm

hence 121.1 nm 102.2 nm 96.9 nm and 94.64 nm.

Q.7 A stationary H-atom emits a photon corresponding to the friest line of lyman series. What velocity does the atom acquire ? ($M_H = 1.67 \times 10^{-19}$ kg)

Sol. 

Applying momentum and energy conservation,

$$mv = \frac{h\nu}{C} \text{ and } \Delta E = \frac{1}{2} mv^2 + h\nu$$

when $\Delta E = 10.2 \times 1.6 \times 10^{-19}$ Joule

we get $\Delta E = \frac{1}{2} mv^2 + mvC$

$$\Delta E = mvC \left[\frac{V}{2C} + 1 \right] \approx mvC$$

$$v = \frac{DV}{mC} = \frac{10.2 \times 1.6 \times 10^{-19}}{1.6 \times 10^{-27} \times 3 \times 10^{8}}$$

$= 3.25$ m/sec

Q.8 The BE of an electron in ground state of the atom is equal to $E_0 = 24.6$ ev. Find the energy required to remove both electron from the atom.

Sol Ionisation energy of He^{+} atom $= 54.4$ ev

hence to remove both electrons from He-atom

we require $= 24.6 + 54.4 = 79$ ev

Q.9 An X-ray tube with a copper target is found the emit lines other than those due to copper. The k_a line of copper is known to have a wavelength 1.5405 Å and the other two K_a lines observed have wavelengths 0.7092 Å and 1.6578 Å. (Identify) the impurities (find the value of Z, atomic number).

Sol. According to Moseley's equation for K_a radiation

$$\frac{1}{\lambda} = R(Z-1)^2 \left[\frac{1}{1^2} - \frac{1}{2^2} \right] \text{ where } \lambda = \text{wavelength of copper}$$

Let λ_1 and λ_2 be the two other unknown wavelengths, then

$$\frac{\lambda_1}{\lambda} = \frac{(Z-1)^2}{(Z_1-1)^2} = \frac{0.7092}{1.5405}$$

Solving we get $Z_1 = 42$

Similarly

$$\frac{\lambda_2}{\lambda} = \frac{(28)^2}{(Z_2-1)^2} = \frac{1.6578}{1.5405}$$

Solving we get $Z_2 = 28$

Q.10 When 0.50 Å X-rays strike a material, the electrons from the k shell are observed to move in a circle of radius 23 mm in a magnetic field of 2×10^{-2}T. What is the binding energy of K-shell electrons ?

Sol. The velocity of the photoelectrons is found the F = ma:

$$e\upsilon B = m\frac{\upsilon^2}{R} \qquad \text{or} \qquad \upsilon = \frac{e}{m}BR$$

The kinetic energy of the photoelctrons is then

$$K = \frac{1}{2}m\upsilon^2 = \frac{1}{2}\frac{e^2B^2R^2}{m}$$

$$= \frac{1}{2}\frac{(1.6510^{-19}C)^2(2\times10^{-2}T)^2(23\times10^{-3}m)^2}{(9.1\times10^{-31}kg)} = 2.97\times10^{-15}\ J$$

$$\text{or } K = (2.97\times10^{-15}\ J)\frac{1keV}{1.6\times10^{-16}J} = 18.36\ eV$$

The energy of the incident photon is $E_\upsilon = \frac{hc}{\lambda} = \frac{12.4keV.Å}{0.50Å} = 24.8\ eV$

The binding energy is the difference between these two value:

$$BE = E\upsilon - K = 24\ keV - 18.6\ keV = 6.2\ keV$$

Q.11 Calculate the wavelength of the emitted characteristic X-ray from a tungsten (Z = 74) target when an electron drops from an M shall to a vacancy in the K shell.

Sol. Tungsten is a multiel atom. Due to the shielding of the nuclear charge by the negative charge of the inner core electrons, each electron is subject to an effective nuclaer charge Z_{eff} which is different for different shells.

Thus, the energy of an electron in the n[th] level of a multielectron atom is given by

$$E_n = \frac{13.6Z_{eff}^2}{n^2}eV$$

For an electron in the K shell (n = 1), $Z_{eff} = (Z-1)$.

Thus, the energy of the electron in the K shell is :

$$E_K = -\frac{(74-1)^2\times13.6}{1^2} \simeq -72500\ eV$$

For an electron in the M shell (n = 3), the nucleus is shielded by one electron of the n = 1 state and eight electrons of the n = 2 state, a total of nine electrons, so that $Z_{eff} = Z - 9$. Thus the energy of an electron in the M shell is :

$$E_M = \frac{(74-9)^2\times13.6}{3^2} \simeq -6380\ eV$$

Therefore, the emitted X-ray photon has an energy given by

$$h\upsilon = E_M - E_K = -6380\ eV - (-72500\ eV) = 66100\ eV$$

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or $\quad \dfrac{hc}{\lambda} = 66100 \times 1.6 \times 10^{-19}$ J

$\therefore \quad \lambda = \dfrac{hc}{66100 \times 1.6 \times 10^{-19}}$ m

$= \dfrac{(6.63 \times 10^{-34}) \times (3 \times 10^{8})}{66100 \times 1.6 \times 10^{-19}}$ m

$= 0.0188 \times 10^{-9}$ m.

Q.12 If the short series limit of the Balmer series for hydrogen is 3646Å, calculate the atomic no. of the element which gives X-ray wavelength down to 1.0 A. Identify the element.

Sol. The short limit of the Balmer series is given by

$$\overline{\upsilon} = 1/\lambda = R\left(\frac{1}{2^2} - \frac{1}{\infty^2}\right) = R/4$$

$\therefore \quad R = 4/\lambda = (4 / 3646) \times 10^{10}\ m^{-1}$

Futher the wavelengths of the k_0 series are given by the relation

$$\overline{\upsilon} = \frac{1}{\lambda} = R\,(Z-1)^2$$

or $\quad (Z-1)^2 = \dfrac{1}{R\lambda} = \dfrac{3646 \times 10^{-10}}{4 \times 1 \times 10^{-10}} = 911.5$

$\therefore \quad (Z-1) = \sqrt{911.5}$

$\cong 30.2$

or $\quad Z = 30.2 \cong 31$

Thus the atomic number of the element concerned is 31.

The element having atomic number Z = 31 is Gallium.

Q.13 In a nuclear reactor, fission is produced in 1 gm for U^{235} (235.0439 a.m.u.) in 24 hours by a slow neutron (1.0087 a.m.u.). Assuming that ${}_{35}Kr^{92}$ (91.8973 a.m.u.) and ${}_{56}Ba^{141}$ (140.9139 a.m.u.) are produced in all reactions and no energy is lost, write the complete reaction and calculate the total energy produced in kilowatt hour. Given 1 a.m.u. = 931 MeV.

Sol. The nuclear fission reaction is

$${}_{92}U^{235} + {}_{0}n^{1} \rightarrow {}_{56}Ba^{141} + {}_{36}Kr^{92} + 3\,{}_{0}n^{1}$$

The sum of the masses before reaction

$= 235.0439 + 1.0087 = 236.0526$ a.m.u.

The sum of the masses after reaction

$= 140.9139 + 91.8973 + (1.0087) = 235.8375$ a.m.u.

$\Rightarrow \quad \Delta m = 256.0526 - 235.8373 = 0.2153$ a.m.u.

energy released in the fission of U^{235} nucleus

$$E= 0.2153 \times 931 = 200 \text{ Mev}$$

Number of atoms in 1 gm

$$= \frac{6.02\times10^{23}}{235} = 2.56 \times 10^{21}$$

Energy released in fission of 1gm of U^{235}

$$E = 200 \times 2.56 \times 10^{2} = 5.12 \times 10^{23} \text{ MeV}$$
$$= (5.12 \times 10^{23}) \times (1.6 \times 10^{-13}) = 8.2 \times 10^{10} \text{ joule}$$
$$= \frac{8.2\times10^{10}}{3.6\times10^{6}} \text{kWh} = 2.28 \times 10^{4} \text{ kWh}$$

Q.14 A neutron collides elastically with an initially stationary deuteron. Find the fraction of the kinetic energy lost by the neutron (a) in a head-on collision; (b) in scattering at right angles.

Sol. (a) In a heat on collision $\sqrt{2mK} = p_d + p_n$

$$K = \frac{p_d^2}{2M} + \frac{p_n^2}{2m}$$

where p_d and p_n are the momenta of deuteron and neutron after the collision. Squaring

$$p_d^2 + p_n^2 + 2p_dp_n = 2mK$$

$$p_n^2 + \frac{m}{M}p_d^2 = 2mK$$

or since $p_d \neq 0$ in a head on collision

$$p_n = -\frac{1}{2}\left(1-\frac{m}{M}\right)P_d$$

Going back to energy conservation

$$\frac{p_d^2}{2M}\left[1+\frac{M}{4m}\left(1-\frac{m}{M}\right)^2\right] = K$$

So $$\frac{p_d^2}{2M} = \frac{4mM}{(m+M)^2}K$$

This is the energy lost by neutron. So the fraction of energy lost is

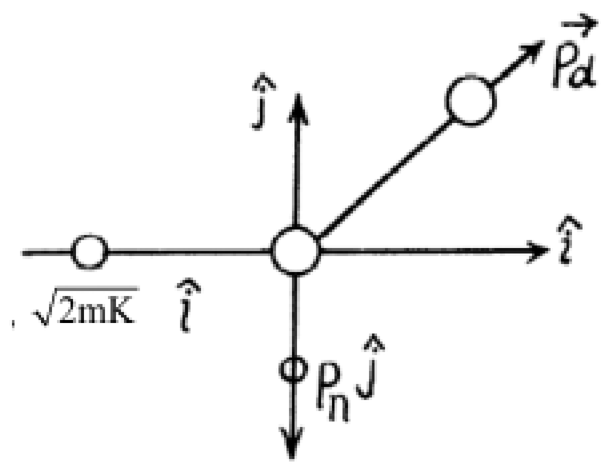
$$\eta = \frac{4mM}{(m+M)^2} = \frac{8}{9}$$

(b) In this case neutron is scattered by 90°. Then we have from the diagram

$$\vec{p}_d = p_n\hat{j} + \sqrt{2mK}\,\hat{i}$$

Then by energy conservation

$$\frac{p_n^2 + 2mK}{2M} + \frac{p_n^2}{2m} = K$$



or $$\frac{p_n^2}{2m}\left(1+\frac{m}{M}\right) = K\left(1-\frac{m}{M}\right)$$

or $$\frac{p_n^2}{2m}\left(1+\frac{m}{M}\right) = K\left(1-\frac{m}{M}\right)$$

The energy lost by neutron in then

$$K - \frac{p_n^2}{2m} = \frac{2m}{M+m}K$$

or fraction of energy lost is

$$\eta = \frac{2m}{M+m} = \frac{2}{3}$$

Q.15 A stationary Pb^{200} nucleus emits an α-particle with K.E., K = 5.77 MeV. Find the recoil velocity of daughter nucleus. What fraction of the total energy liberated in this decay is accounted for the recoil energy of the daughter nucleus ?

Sol. The momentum of the a-particle is given by,

$$P_d = P_\alpha = \sqrt{2m_\alpha K} \quad \text{...(i)}$$

Let the recoiled momentum of the dauhter nucleus be $P_d = m_d v_d$, where m_d and v_d are the mass and velocity of daughter nucles. Using the priciple of conservation of momentum we get,

$$P_d = P_\alpha = \sqrt{2m_\alpha K}$$

$$\Rightarrow \quad V_d = \frac{\sqrt{2m_\alpha K}}{m_d} \quad \text{...(ii)}$$

$$\Rightarrow \quad V_d = \frac{1}{196}\sqrt{\frac{2\times 4\times K}{m_p}} = \frac{2}{196}\sqrt{\frac{2K}{m_p}}$$

Where m_p is the mass of the proton.

$\Rightarrow \quad V_d = 3.39 \times 10^5$ m/s

Let the K.E. of the daughter nucleus be K' then,

$$\frac{K'}{K} = \frac{m_\alpha}{m_d}$$

As the momenta are same

$$\therefore \qquad \frac{K'}{K_t} = \frac{m_\alpha}{m_\alpha + m_d}$$

$$\Rightarrow \qquad K' = \frac{m_\alpha}{m_\alpha + m_d} K_t = \frac{4}{196+4} K_t$$

$$\Rightarrow \qquad K' = 0.02\ K_t$$

$$\Rightarrow \qquad \frac{K'}{K_t} = 0.02$$

Q.16 A P^{32} radionuclide with half-life T = 14.3 days is produced in a reactor at a constant rate $q = 2.7 \times 10^9$ nuclei per second. How soon after the beginning of production of that radionuclide will its activity be equal to $A = 1.0 \times 10^9$ dps ?

Sol. Production of the nucleus is governed by the equation

$$\frac{dN}{dt} = \underset{\substack{\uparrow \\ \text{supply}}}{g} - \underset{\substack{\uparrow \\ \text{decay}}}{\lambda N}$$

we see that N will approach a constant value $\frac{g}{\lambda}$. This can also be proved directly. Multiply by $e^{\lambda t}$ and write.

$$\frac{dN}{dt} e^{\lambda t} + \lambda e^{\lambda t} N = g e^{\lambda t}$$

Then $$\frac{d}{dt}(N e^{\lambda t}) = g e^{\lambda t}$$

or $$N e^{\lambda t} = \frac{g}{\lambda} e^{\lambda t} + \text{const.}$$

At t = 0 when the production is starteed, N = 0

$$0 = \frac{g}{\lambda} + \text{constant}$$

Hence $$N = \frac{g}{\lambda}(1 - e^{-\lambda t})$$

Now the activity is

$$A = \lambda N = g(1 - e^{-\lambda t})$$

From the problem

$$\frac{1}{2 \cdot 7} = 1 - e^{-\lambda t}$$

This gives $\lambda t = 0.463$

so $$t = \frac{0.463}{\lambda} = \frac{0.463 \times T}{0.693} = 9.5 \text{ days.}$$

Algebraically $$t = -\frac{T}{\ln 2} \ln\left(1 - \frac{A}{g}\right)$$

Q.17 A dose of 5mCi of P^{32} ($t_{1/2} = 14$ days) is a administered intravenously to a patient whose blood volume is 3.5 ℓts. At the end of 1 hour, it is assumed that the phosphorous is uniformly distributed. What would be the count rate/mℓ of the withdrawn blood if the counter measuring the activity had an efficiency of 10%:

(a) 1 hour after injection

(b) 28 days after injection

Sol. Let A_0 = initial activity A_t = activity at time t

According to question

$$A_0 = \frac{5 \times 10^{-5}}{35 \times 10^{3}} = 0.143 \times 10^{-5} \text{ Ci/m}\ell$$

and

$$\lambda = \frac{0.693}{14 \times 24} = 2.06 \times 10^{-3} \text{ /Hr}$$

Now using

$$\lambda t = 2.303 \log \frac{A_0}{A_t}$$

(a) After 1.0 Hr

$$\frac{2.06 \times 10^{-3} \times 1}{2.303} = \log \frac{0.143 \times 10^{-5}}{A_t}$$

$\Rightarrow$ $A_t = 1.42 \times 10^{-6}$ Ci/mℓ

$\Rightarrow$ Count rate = $(10/100) \times (1.42 \times 10^{-6}) \times 3.7 \times 10^{10}$ dps = 5280 dps

(b) After 28 days, i.e., after two half lives ($t_{1/2}$ = of P^{32} = 14 days) ;

$A_t = A_0 / 4 = 1.42 \times 10^{-6} /4$

$\Rightarrow$ Count rate = $(10/100) \times (1.42 \times 10^{-6} /4) \times 3.7 \times 10^{10}$ dps = 1322.75 dps

Q.18 In the chemical analysis of a rock, the mass ratio of two radioactive isotopes is found to be 100 : 1. The mean lives of the two isotopes are 4×10^9 and 2×10^9 years respectively. If it is assumed that at the time of formation the atoms of both the two isotopes were in equal proportion, calculate the age of the rock. Ratio of the atomic weights of two isotopes is 1.02 : 1.

Sol. Let two isotopes are A and B

$$\frac{m_A}{m_B} = 100 \;;\; \frac{A_A}{A_B} = \frac{1.02}{1}$$

$T_A = 4 \times 10^9$ years $T_B = 2 \times 109$ years [Also $\lambda = 1/T$]

Let ratio of nuclei of two isotops be :

$$\frac{N_{A0}}{N_{B0}} \text{ at } t = 0 \text{ and } \frac{N_{At}}{N_{Bt}} \text{ at } t = 1$$

For isotope A

$$\lambda_A t = 2.303 \log \frac{N_{A0}}{N_{At}}$$

Similarly for isotope B

$$\lambda_B t = 2.303 \log \frac{N_{B0}}{N_{Bt}}$$

On subtracting

$$(\lambda_A - \lambda_B)\, t = 2.303 \log \frac{N_{A0}/N_{B0}}{N_{At}/N_{Bt}}$$

$$\Rightarrow \quad (\lambda_A - \lambda_B)t = 2.303 \log \frac{\text{initial ratio}}{\text{final ratio}}$$

$$\Rightarrow \quad \left(\frac{1}{4\times10^9} - \frac{1}{2\times10^9}\right) t = 2.303 \log \frac{1}{100/1.02}$$

$$\Rightarrow \quad \text{age} = t = 1.83 \times 10^{10} \text{ years}$$

Q.19 A given sample contains two types of atoms A and B in the ratio 3 : 1. Atoms of type A undergo α-decay with a half life of 30 days to form 'B' while 'atoms of type B' undergo α-decay with a half life of 45 days to from 'C', which is stable. Calculate the time after which the activities of A and that of B are in the ratio 9 : 22

$$A \xrightarrow{T_{1/2}=30\text{days}} B \xrightarrow{T_{1/2}=45\text{days}} C$$

Sol. The radiactive decay series is given

$$A \xrightarrow{T_{\frac{1}{2}} = 30\text{days}} B \xrightarrow{T_{\frac{1}{2}} = 45\text{days}} C$$

Initially $N_A(0) : N_B(0) \backslash 3 : 1$

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$$\frac{dN_A}{dt} + \lambda_A N_A = 0$$

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B$$

$$\frac{dN_C}{dt} = \lambda_B N_B$$

$$N_A = N_A(0)\ e^{-\lambda_A t}$$

$$N_B = c_1 e^{-\lambda_B t} + \frac{\lambda_A N_A(0) e^{-\lambda_A t}}{-\lambda_A + \lambda_B}$$

Then we get, $c_1 = \frac{5}{2} N_0$

$$\therefore \qquad N_A(t) = \frac{3}{4} N_0 \left(\frac{1}{2}\right)^{\frac{1}{T_{1/2}}} = \frac{3}{4} N_0 \left(\frac{1}{2}\right)^{\frac{1}{30\text{days}}}$$

and $$N_B(t) = \left[\frac{5}{2} N_0 \left(\frac{1}{2}\right)^{\frac{1}{45\text{days}}} - \frac{9}{4} N_0 \left(\frac{1}{2}\right)^{\frac{1}{30\text{days}}}\right]$$

Now, $\frac{\lambda_A N_A}{\lambda_B N_B} = \frac{9}{22}$ i.e. $\frac{N_A}{N_B} = \frac{3}{11}$

or, $\left(\frac{1}{2}\right)^{-t/90} = 2$

or, $t = 90$ days